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ADIABATIC ASYMPTOTICS OF THE REFLECTION COEFFICIENT

V. S. BUSLAEV, M. V. BUSLAEVA, AND A. GRIGIS

§1. Introduction

1.1. Adiabatic asymptotics. We shall consider a special case of the standard singular Sturm–Liouville problem

$$-\psi'' + v(x)\psi = E\psi, \quad x \geq 0, \ \ \psi(0) = 0, \tag{1.1}$$

assuming that the potential v is a real-valued smooth function decaying sufficiently fast as $x \to \infty$ (e.g., a Schwartz class function). Also, we assume that $E > 0$. The standard solution $\psi(x,k)$ of the Cauchy problem $\psi(0) = 0, \psi'(0) = 1$, can be expressed explicitly in terms of the Jost solution $f(x,k)$ distinguished by its asymptotic behavior as $x \to \infty$, $f(x,k)e^{-ikx} \to 1, k^2 = E$:

$$\psi(x,k) = \frac{1}{2ik}[\overline{M}f - M\bar{f}], \quad M(k) = f(0,k). \tag{1.2}$$

The properties of the functions f and M are well known (see, e.g., [1]). The function $R = -\frac{\overline{M}}{M}$, defined for $k \in \mathbb{R}$, $|R| = 1$, is called the *reflection coefficient*.

If $v(x) = p(\epsilon x)$, where $\epsilon > 0$ is a small parameter, then all these objects become functions of ϵ, and we may ask about the asymptotic behavior as $\epsilon \to 0$ of, say, the solution f, via which M, R, and ψ can be expressed. Such asymptotics is said to be *quasiclassical*. The possibility itself of constructing this asymptotics, as well as its form, depend crucially on the location of the curve $k^2 + p(\xi) = E, k \in \mathbb{C}, \xi \in \mathbb{R}$, on the phase plane (k,ξ), i.e., on the graph of the potential p. The example of

$$p(\xi) = \frac{A}{[1 + (\xi - \alpha)^2]^r} \tag{1.3}$$

(with various choices of the parameters) typifies the situation.

As a substantial generalization of the quasiclassical case, we choose the equation with a potential of the form $v(x) = p(x, \epsilon x)$, where the function $p(x,\xi)$ of two variables is assumed to be periodic in x with period a. In order to stay within the framework of the boundary value problem introduced at the very beginning, it is natural to assume that the function $p(x,\xi)$ decays rapidly (in the power scale) as $\xi \to \infty$. Again, the question arises about the asymptotic behavior of solutions, e.g., of the solution f, as $\epsilon \to 0$.

In this case, first of all, we consider the equation

$$-\psi'' + p(x,\xi)\psi = E\psi, \tag{1.4}$$

viewing $\xi \geq 0$ as an external parameter. We recall that for an equation with purely periodic potential there is a well-defined *band function* $\mathcal{E}(k,\xi)$, which depends on the

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quasimomentum k (and, in our case, on ξ) and links E and k for the so-called Bloch solutions $\chi(x,k,\xi)$:

$$\chi(x,k,\xi) = e^{ikx}\varphi(x,k,\xi), \quad \mathcal{E}(k,\xi) = E. \tag{1.5}$$

Here φ is an a-periodic function of x.

Again, the possibility itself of finding the asymptotics of solutions of the equation with potential $v(x) = p(x,\epsilon x)$, as well as the explicit form of asymptotic formulas, depend on the location of the curve $\mathcal{E}(k,\xi) = E$ on the phase plane.

No general and explicit method (except, to some extent, for the case of finite-gap potentials) exists for describing this curve. Only for large ξ, when the potential v becomes small, can a simple perturbation theory be developed. Such a theory is well known (see, e.g., [2]) for the equations of the form

$$-\psi'' + \delta p_0(x)\psi = E\psi, \tag{1.6}$$

but it extends easily to the case of the general dependence of the potential on the parameter (on ξ in our case). Here we are not going to dwell on elementary details of that theory (it does not give any information about the general form of the curve $\mathcal{E}(k,\xi) = E$ anyway). Instead, we formulate some assumptions concerning the location of this curve, which, in particular, are in natural agreement with the perturbation theory mentioned above.

These assumptions represent a wide class of situations and are quite natural under our conditions on the potential $p(x,\xi)$. However, justification of these assumptions would have required a separate and substantial publication.

1.2. Assumptions. To describe the postulated structure of the curve $\mathcal{E}(k,\xi) = E$, we note that the function $\mathcal{E}(k,\xi)$ can be assumed invariant under the transformations

$$k \to k + a^*n, \quad n \in \mathbb{Z}, \;\; a^* = \frac{2\pi}{a}, \;\; k \to -k. \tag{1.7}$$

For experts, we mention that the periodicity property of $\mathcal{E}(k,\xi)$ depends on how this function is treated. We can view $\mathcal{E}$ as an (in general, infinite) collection of independent branches that correspond, for quasiperiodic boundary conditions with quasimomentum k, to the system $\mathcal{E}_n(k,\xi)$ of eigenvalues of equation (1.4) on the interval $[a,b]$. The separated real branches corresponding to different eigenvalues close up via complex branches, on which the quasimomentum can be assumed to run over a complex curve, so that $\mathcal{E}(k,\xi)$ remains real. On the other hand, $\mathcal{E}(k,\xi)$ may be defined as an analytic function on a suitable two-sheeted Riemann surface, and then its periodicity emerges only after analytic continuation along some specific cycles on that surface.

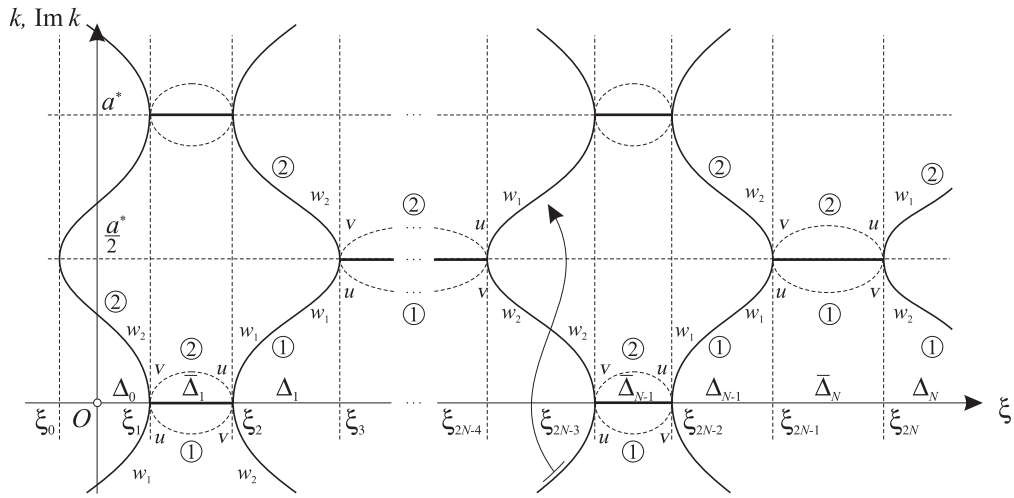
Because of the invariance mentioned above, it suffices to describe only one branch that covers the ξ-axis. The other branches are obtained from that branch via transformations (1.7).

So, our assumptions are as follows (see the figure).

1) For $\xi \neq \xi_1 < \xi_2 < \cdots < \xi_{2N} < \infty$, the curve $\mathcal{E}(k,\xi) = E$ consists of continuous isolated branches possessing one-to-one projections to the ξ-axis, $k = \kappa(\xi)$.

2) Outside of the intervals $[\xi_1,\xi_2],\ldots,[\xi_{2N-1},\xi_{2N}]$ the branches are real and the corresponding functions are monotone; over these intervals the branches are complex, and the real parts of the variable k remain constant, while the imaginary parts of κ have a unique nondegenerate extremum point inside each interval.

3) Over infinitesimally small neighborhoods of the points ξ_l (by analogy with the quasiclassical case, these points are called the *turning points*), the joining of the branches is modeled on the curves $(k - k_0 - a^*n)^2 = c(\xi - \xi_l)$, k_0, where k_0 can take only the



special values 0 or $a^*/2$. In a full neighborhood, up to the adjacent turning points, we can put $k = k_0 + a^*n \pm r$, and then the model takes the form $r^2 = c(\xi - \xi_l)$.

4) On the interval (ξ_{2N}, ∞) we have two branches along which k tends to the values $\pm\sqrt{E}$ rapidly (in the power scale). It is assumed that, under an appropriate choice of k_{2N}, which can only take the values mentioned in 3), the quantity $\sqrt{E}$ lies strictly inside the interval $(k_{2N}, k_{2N} + 1/2a^*)$.

Though these assumptions exhaust the system of conditions under which we can construct the asymptotics of the solutions of the equation

$$-\psi'' + p(x, \epsilon x)\psi = E\psi, \tag{1.8}$$

it is clear that other versions are possible. Now we describe the version to be dealt with; three other versions can be treated at the expense of minor modifications. So, we assume the following: 1) on the branch with the asymptotics $k \to \sqrt{E}$ as $\xi \to \infty$ at the turning point, k takes the value k_{2N}; 2) $\xi_1 > 0$, so that a real branch adjoins the point $\xi = 0$.

1.3. The layout of the paper. To a large extent, we lean upon a series of earlier papers where we studied the asymptotic behavior of solutions of the equation

$$-\psi'' + p(x, \epsilon x)\psi = 0 \tag{1.9}$$

and applied those asymptotics to the spectral problem on the axis for the equation

$$-\psi'' + (p(x) - \epsilon x)\psi = E\psi. \tag{1.10}$$

We mention the papers [3, 6], which are close to the present one. References to other publications can also be found in [3, 6]. The first ideas appeared in [3, 7] and were developed technically in [6].

In the next section we list some results obtained in the papers mentioned above. However, the presentation adopted here may be regarded as new, and, we hope, is more transparent. On the intervals that are bounded by two turning points or contain one turning point strictly inside, we introduce solutions of equation (1.8) that admit a relatively simple asymptotic description as $\epsilon \to 0$. Next, we consider matrices that link pairs of similar linearly independent solutions on different but adjacent intervals, and we deduce the asymptotics of these matrices from that of the solutions.

In the final §3 we link the solutions with specific asymptotic behavior as $\epsilon \to 0$ and those with specific behavior as $\xi \to \infty$, i.e., the solutions f and $\bar{f}$. Uniting these results with the results of §2, we come to a possibility to find the asymptotics as $\epsilon \to 0$ of the

solutions f and $\bar{f}$ everywhere for $\xi \geq 0$. After that, we can describe the asymptotic behavior of other spectral characteristics, in particular, of the reflection coefficient.

§2. Preliminaries

In this section we collect several dissimilar results to be applied in the sequel for the deduction of an asymptotic formula for the solution f. Mainly, these results are taken from earlier publications, but we present them in a form allowing independent reading.

2.1. Formal solutions. Consider an interval (a, b) on the ξ-axis. If this interval lies at a positive distance from the turning points, then on it (over it) we can define an infinite collection of separated smooth branches $k = \kappa(\xi)$ of the isoenergy curve $\mathcal{E}(k, \xi) = E$, with E fixed. These branches fall into two classes; within one class the branches are transformed to one another by the period shift $k \to k + a^* n$, $n \in \mathbb{Z}$. Two branches of different classes, e.g., $k = \kappa(\xi)$ and $k = -\kappa(\xi)$, cannot be transformed to each other by such a shift.

Each branch as above gives rise to a formal solution of (1.8) that possesses an asymptotic character as $\epsilon \to 0$, i.e., to an exponential formal asymptotic solution of the form

$$\mathbf{f} = e^{\frac{i}{\epsilon}\int^{\xi} k\, d\xi} \sum_{n \geq 0} \epsilon^n f_n(x, \xi), \quad f_0 \neq 0, \tag{2.1}$$

where integration is along the branch chosen, and the functions f_n are a-periodic in x. This solution is determined up to an arbitrary factor belonging to the field $\mathcal{F}$ the nonzero elements of which are formal series

$$e^{\frac{i}{\epsilon}\theta} \sum_{n \geq 0} \epsilon^n c_n, \quad c_0 \neq 0, \tag{2.2}$$

where θ and c_n are complex constants. Two formal solutions corresponding to branches of one and the same class are linearly dependent. On the contrary, if solutions correspond to branches of different classes, then they are linearly independent over the field $\mathcal{F}$.

If a branch is real, the exponential factors have constant moduli. If a branch is complex, these factors are monotone increasing or decreasing depending on the sign of $\operatorname{Im} k$.

On the interval (a, b), equation (1.8) has an exact (true) solution for which the given formal solution serves as the asymptotic expansion as $\epsilon \to 0$ if $\epsilon^{-2/3}|\xi - \xi_0| \to \infty$, where ξ_0 is any one of the turning points that bound the interval. Of course, this true solution is nonunique. For real branches, uniform norms like the C^r-norm can be used in the expansion mentioned above, while in the case of a complex branch the norm must include a natural additional weight that depends exponentially on ϵ. The form of this weight is dictated by the modulus of the exponential factor of the formal solution in question.

It is not hard to obtain an explicit expression for the leading term of the formal solution (see [3] or [5]). However, the case to be treated here is somewhat more general than that in [3, 5]: there it was assumed that $p(x, \xi) = v(x) + q(\xi)$. We do not repeat the elementary calculations leading to a convenient answer; instead, here we present the result itself in the form that remains valid in the general case. We have

$$f_0 = A(\xi)\varphi(x, k, \xi), \tag{2.3}$$

where φ is the periodic part of the Bloch solution $\chi = e^{ikx}\varphi$, and A solves the equation

$$A' = -\frac{N}{D} A, \tag{2.4}$$

where

$$D = 2\int_0^a \varphi_2 \left(\frac{1}{i}\frac{\partial}{\partial x} + k\right)\varphi_1\, dx \tag{2.5}$$

and

$$N = \int_0^a \varphi_2 \left[\left(\frac{1}{i}\frac{\partial}{\partial x} + k\right)\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \xi}\left(\frac{1}{i}\frac{\partial}{\partial x} + k\right)\right]\varphi_1\, dx. \tag{2.6}$$

In these formulas, $\varphi_1 = \varphi(x, k, \xi)$, $\varphi_2 = \varphi(x, -k, \xi)$, and $\frac{\partial}{\partial \xi}$ is the total derivative under the constraint $\mathcal{E}(k, \xi) = E$.

The denominator D can also be written as follows:

$$D = ia\,\mathrm{wr}\{\chi_1, \chi_2\} = \mathcal{E}_k \int_0^a \chi_1\chi_2\, dx, \tag{2.7}$$

where $\chi_1 = e^{ikx}\varphi_1$ and $\chi_2 = e^{-ikx}\varphi_2$ are Bloch solutions, $\mathrm{wr}\{\chi_1, \chi_2\} = \chi_1\chi_2' - \chi_1'\chi_2$ is the standard Wronskian of the two solutions, and $\mathcal{E}_k$ is the partial derivative for fixed ξ. Since the Wronskian can vanish only at turning points, the equation for A is consistent. However, we cannot say the same about the separate factors in the representation (2.7) for D. In particular, inside the interval covered by a complex branch, there is a point where the first factor turns into infinity; then the second factor is zero, so that the solutions will be orthogonal on the period.

The expression for A can be written in the form

$$A = c_0 D^{-1/2} e^{\int \omega}, \tag{2.8}$$

where c_0 is an arbitrary constant and

$$\omega = \frac{\int_0^a \left[\frac{\partial}{\partial \xi}\varphi_2 \left(\frac{1}{i}\frac{\partial}{\partial x} + k\right)\varphi_1 - \left(-\frac{1}{i}\frac{\partial}{\partial x} + k\right)\varphi_2 \frac{\partial}{\partial \xi}\varphi_1\right] dx}{D}\, d\xi. \tag{2.9}$$

Again, the derivatives of φ_1, φ_2 with respect to ξ, as well as the differential $d\xi$, must be total. The form ω can be reshaped to

$$\omega = \frac{\int_0^a p_\xi(\varphi_1\varphi_{2\xi} - \varphi_2\varphi_{1\xi})\, dx}{2\int_0^a p_\xi\varphi_1\varphi_2\, dx}\, d\xi. \tag{2.10}$$

Though the denominator in (2.10) can vanish at some points, ω itself remains consistent. Note that in the case of a real branch the functions φ_1 and φ_2 are complex conjugate to each other; therefore, ω will be purely imaginary.

Sometimes, we shall assume that, at least near the turning points,

$$\int_0^a \chi_1\chi_2\, dx = a. \tag{2.11}$$

Then the factor $D^{-1/2}, D = \mathcal{E}_k$, will lose smoothness at one of the points of the interval covered by the complex branch under consideration, but simultaneously the form ω will acquire a pole at that point. Taken together, these singularities cancel. Since the elementary representation for A that we discuss will only be used off the turning points, often it will be convenient to write $|\mathcal{E}_k|^{-1/2}$ instead of $D^{-1/2} = \mathcal{E}_k^{-1/2}$. Under these agreements,

$$A(\xi) = c_0|\mathcal{E}_k|^{-1/2} e^{\int \omega}. \tag{2.12}$$

However, this formula requires caution when applied near a point where $\mathcal{E}_k = 0$, because $\mathcal{E}_k$ changes the sign if such a point is crossed.

We shall dwell on this when we describe f_0; an explicit calculation of the next order in the asymptotics is already a bulky deal.

2.2. Turning points. In accordance with the terminology introduced in [6], the turning points we deal with are parabolic turning points. Also in [6], it was shown that in finite neighborhoods of a parabolic turning point ξ_0 the asymptotic behavior of the solutions of (1.8) can be described by formal asymptotic solutions of the form

$$\psi = e^{\frac{i}{\epsilon}(k_0+a^*n)(\xi-\xi_0)}[t_1(x,k,\xi)z_1 + t_2(x,k,\xi)z_2]. \tag{2.13}$$

This solution corresponds to the pair of branches

$$k = k_0 + a^*n \pm r, \quad r(\xi_0) = 0, \tag{2.14}$$

intersecting at ξ_0. Here t_1 and t_2 are series as in (2.1), and $z = (z_1, z_2)^t$ is an arbitrary solution of the equation

$$i\epsilon\frac{dz}{d\xi} = M(\xi)z, \quad M(t) = \zeta'(\xi)\begin{pmatrix} 0 & 1 \\ -\zeta(\xi) & 0 \end{pmatrix}, \tag{2.15}$$

where ζ denotes a unique solution of $r^2(\xi) = -{\zeta'}^2\zeta$ smooth at the point ξ_0. Qualitatively, near ξ_0 the function r^2 is similar to a linear function with nonzero derivative; ζ behaves in the same way, but differs from r^2 by the sign. The series t_1 and t_2 are independent of the choice of z. The formal solution is determined uniquely up to a factor belonging to the field $\mathcal{F}$ the nonzero elements of which are formal series $c = \sum_{n\geq 0}\epsilon^n c_n, c_0 \neq 0$, with complex constants c_n. Like in the case of the exponential type solutions (2.1), the leading terms of the above series can easily be computed (see [6]). Not presenting the corresponding formulas, here we dwell on the transformations of the asymptotic solutions (2.13) under crossing a neighborhood of a turning point ξ_0. The results were described and used in several publications (see, e.g., [5, 6]), but largely we refer to the paper [6], where these transformations were represented in the form most suitable for us.

Observe that the change of variables

$$\zeta(\xi) = \epsilon^{2/3}\eta, \quad w = E(\epsilon)z, \;\; E(\epsilon) = \begin{pmatrix} \epsilon^{1/6} & 0 \\ 0 & \epsilon^{-1/6} \end{pmatrix} \tag{2.16}$$

reshapes (2.15) to the form

$${w_1}'' = \eta w_1, \quad w_2 = i{w_1}', \tag{2.17}$$

i.e., to the classical Airy equation.

We recall some properties of the solutions of the Airy equation $w'' = \eta w$. Its solutions admit integral representations. In particular, there exist solutions w_1 and w_2 distinguished uniquely by their asymptotic behavior as $\eta \to -\infty$:

$$w_1 \sim |\eta|^{-1/4}e^{i2/3|\eta|^{3/2}}, \quad w_2 \sim |\eta|^{-1/4}e^{-i2/3|\eta|^{3/2}}. \tag{2.18}$$

Also, there is a solution v determined uniquely by its asymptotics as $\eta \to +\infty$:

$$v \sim \frac{1}{\sqrt{2}}|\eta|^{-1/4}e^{-2/3|\eta|^{3/2}}. \tag{2.19}$$

We have

$$v = \frac{1}{\sqrt{2}i}\left(e^{i\frac{\pi}{4}}w_1 - e^{-i\frac{\pi}{4}}w_2\right). \tag{2.20}$$

As $\eta \to +\infty$, the solution

$$u = \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}w_1 + e^{-i\frac{\pi}{4}}w_2\right) \tag{2.21}$$

admits the asymptotics

$$u \sim \sqrt{2}|\eta|^{-1/4}e^{2/3|\eta|^{3/2}}. \tag{2.22}$$

In the vector form, the relationship among the solutions mentioned above looks like this:

$$\begin{pmatrix} v \\ u \end{pmatrix} = S \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} \end{pmatrix}. \tag{2.23}$$

These solutions are associated with the branches of the curve

$$k^2 + z = 0, \tag{2.24}$$

which arises naturally when we study the asymptotic properties of solutions of the Airy equation for large z. If η is real, then w_1 and w_2 are complex conjugate to each other, and u and v are real.

2.3. Transformation of solutions. With any formal solution, we can associate a true solution of the initial equation in such a way that the formal solution coincides with the asymptotic expansion of the true one. Essentially, this is done in the same way as in the case of exponential solutions, but some additional details must be taken into account.

Using in (2.13) any of the four solutions (2.18)–(2.21) of the Airy equation, we obtain four different formal solutions.

Now we discuss the question as to how the formal solutions described by Airy functions are transformed to exponential formal solutions as we recede from a turning point. Far from a turning point, as $\epsilon^{-2/3}|\xi - \xi_0| \to \infty$, these exponential solutions can be used for the asymptotic description of the true solutions mentioned above.

Two possibilities may occur for the mutual location of the real and the complex branches of the isoenergy curve near ξ_0 (see the figure). Also, in the figure we marked the correspondence between the branches of the curve and the solutions that are defined near ξ_0 and turn into exponential solutions associated with relevant branches after asymptotic degeneration when we recede from ξ_0.

It is easily seen that

$$\begin{aligned} \mathbf{w}_1^{(\xi_0)} &\sim e^{\frac{i}{\epsilon}(k_0+a^*n)(\xi-\xi_0)}|\eta|^{-1/4} e^{i2/3|\eta|^{3/2}} \left(t_{10}(x,k,\xi) + \cdots\right) \\ &= \alpha e^{\frac{i}{\epsilon}\int_{\xi_0}^{\xi} k\, d\xi} \left(|\mathcal{E}_k|^{-1/2} e^{\int_{\xi_0}^{\xi} \omega} \varphi(x,k,\xi) + \cdots\right), \end{aligned} \tag{2.25}$$

where t_{10} is the leading term of the coefficient t_1, α is some complex constant, and the variables k and ξ are assumed to lie on the corresponding branch of the isoenergy curve (this assumption also refers to the integrals).

It is convenient to introduce a specific normalization of the formal and the true solutions under consideration. We describe this normalization. Note that the normalized solutions will depend not only on ξ_0, but also on the choice of a level $k_0 + na^*$, i.e., in fact on ξ_0 and n. If we vary n, i.e., shift the branches (in k) by the period, then the normalized solutions will acquire an additional numerical factor. We assume that the constant α in (2.25) is equal to 1; then the periodicity of the Bloch solutions relative to k allows us to put $\mathbf{w}_{1,2}^{(\xi_0,n)} = e^{-\frac{i}{\varepsilon}a^*\xi_0}\mathbf{w}_{1,2}^{(\xi_0,n-1)}$. On the solutions $\mathbf{w}_1^{(\xi_0,n)}$ and $\mathbf{w}_2^{(\xi_0,n)}$ we can impose the condition $\bar{\mathbf{w}}_1^{(\xi_0,n)} = e^{-\frac{i}{\varepsilon}\xi_0 2(k_0+a^*n)}\mathbf{w}_2^{(\xi_0,-n)}$. Moreover, there is a good reason to subject the Bloch solutions to the normalization condition $\int_0^a \chi_1\chi_2\, dx = a$ and to fix the following Wronskian: $\mathrm{wr}\{\mathbf{w}_1^{(\xi_0,n)}, \mathbf{w}_2^{(\xi_0,-n)}\} = -i$. Under these assumptions, the solution $\mathbf{w}_1^{(\xi_0)}$ is still determined only up to a unimodular factor belonging to $\mathcal{F}_0$, but the leading element of this factor is fixed: it is equal to 1.

We collect the properties of the solutions described above in the following proposition.

Proposition 2.1. *Near each parabolic turning point ξ_0 separated away from the other turning points, we can construct four true solutions $w_1^{(\xi_0)}, w_2^{(\xi_0)}, v^{(\xi_0)}$, and $u^{(\xi_0)}$ of equation* (1.8)*. Asymptotically, these solutions are described by the corresponding formal*

solutions $\mathbf{w}_1^{(\xi_0)}$, $\mathbf{w}_2^{(\xi_0)}$, $\mathbf{v}^{(\xi_0)}$, $\mathbf{u}^{(\xi_0)}$, *and the norms employed correspond to the increase or decrease of the exponential factors that arise in the asymptotics of formal solutions when we recede from* ξ_0. *The true solutions can be constructed in such a way that, together with the formal solutions, they satisfy the relation*

$$\begin{pmatrix} v^{(\xi_0)} \\ u^{(\xi_0)} \end{pmatrix} = S \begin{pmatrix} w_1^{(\xi_0)} \\ w_2^{(\xi_0)} \end{pmatrix}, \tag{2.26}$$

where S *is the matrix* (2.23). *In the process of receding from* ξ_0, *the formal solutions* $\mathbf{w}_1^{(\xi_0)}, \dots$ *are rebuilt asymptotically into exponential formal solutions in accordance with a formula of type* (2.25).

2.4. Extension of solutions through an interval covered by a real branch. Let a and b ($> a$) be the turning points that bound an interval $\Delta = (a, b)$ covered by real branches (see the figure). Also, in the figure we indicated the branches that correspond to the solutions constructed in the preceding subsection. The indices linking the solutions with turning points are not written, but this relationship is clear from the figure. It is also clear that two solutions (true as well as formal) that correspond to one and the same turning point but to different systems of branches shifted by the period a^* relative to k, can be assumed to differ by a simple exponential factor.

Obviously, the true solutions in neighborhoods of two turning points are linked to each other via some matrix $Q_\Delta(E, \epsilon)$:

$$\begin{pmatrix} w_1^{(b)} \\ w_2^{(b)} \end{pmatrix} = Q_\Delta \begin{pmatrix} w_1^{(a)} \\ w_2^{(a)} \end{pmatrix}, \quad Q_\Delta = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}. \tag{2.27}$$

To be more precise, we should write $w_{1,2}^{(a,n)}$ in place of $w_{1,2}^a$ and $w_{1,2}^{(b,n')}$ in place of $w_{1,2}^b$. The index n' will be chosen so that the points a, n and b, n' be linked by the branch associated with the common solution w_1. These agreements are tacitly assumed throughout this subsection.

Our goal now is to describe the asymptotic behavior of the entries of the above matrix. Recall that the asymptotics of true solutions is given by formal solutions like $\mathbf{w}_j^{(l)}$, $j = 1, 2$, $l = a, b$:

$$w_j^{(l)} \sim \mathbf{w}_j^{(l)}(1 + O(\epsilon^\infty)). \tag{2.28}$$

The entries of Q can be expressed in terms of the Wronskians of true solutions:

$$\operatorname{wr}\{w_1^{(b)}, w_2^{(a)}\} = q_{11} \operatorname{wr}\{w_1^{(a)}, w_2^{(a)}\}, \qquad \operatorname{wr}\{w_1^{(b)}, w_1^{(a)}\} = q_{12} \operatorname{wr}\{w_2^{(a)}, w_1^{(a)}\}, \tag{2.29}$$

$$\operatorname{wr}\{w_2^{(b)}, w_2^{(a)}\} = q_{21} \operatorname{wr}\{w_1^{(a)}, w_2^{(a)}\}, \qquad \operatorname{wr}\{w_2^{(b)}, w_1^{(a)}\} = q_{22} \operatorname{wr}\{w_2^{(a)}, w_1^{(a)}\}. \tag{2.30}$$

In place of the true solutions involved in the above Wronskians, we plug in the corresponding formal solutions; as a result, we obtain asymptotic representations for the entries of the matrix Q. The explicit formulas for $\mathbf{w}_j^{(l)}$ imply that, asymptotically, all entries of Q are bounded. The solutions $\mathbf{w}_j^{(a)}$ and the solutions $\mathbf{w}_j^{(b)}$, $j = 1, 2$, are linearly dependent because they correspond to the same branches of the isoenergy curve. Therefore, for the nondiagonal entries of Q we have

$$q_{12} = O(\epsilon^\infty), \quad q_{21} = O(\epsilon^\infty). \tag{2.31}$$

As to the diagonal entries, their leading orders can be computed explicitly by direct comparison of the leading orders in the asymptotic formulas for the solutions $w_j^{(l)}$. Indeed, the above estimates for the nondiagonal entries show that

$$w_1^{(b)} = q_{11} w_1^{(a)} + O(\epsilon^\infty), \quad w_2^{(b)} = q_{22} w_2^{(a)} + O(\epsilon^\infty). \tag{2.32}$$

Note that $k_{1,2} = k_0 + a^*n \pm r_a, r_a(a) = 0$ or $k_{1,2} = k_0 + a^*n' \pm r_b, r_b(b) = 0$. For definiteness, we choose the second possibility; in what follows we shall use the representation $k_{1,2} = k_0 + a^*n \pm r, r(b) = 0, r \geq 0$. The symbols "1" and "2" will mark the branches that correspond to the signs $-$ and $+$, respectively.

We recall the leading orders of the asymptotic formulas:

$$w_1^{(a)} \sim e^{\frac{i}{\epsilon}\int_a^\xi k_1\,d\xi}\big[|\mathcal{E}_k|^{-1/2}e^{\int_a^\xi \omega_1}\varphi(x,k_1,\xi) + \cdots\big] \tag{2.33}$$

and

$$w_1^{(b)} \sim e^{\frac{i}{\epsilon}\int_b^\xi k_1\,d\xi}\big[|\mathcal{E}_k|^{-1/2}e^{\int_b^\xi \omega_1}\varphi(x,k_1,\xi) + \cdots\big]. \tag{2.34}$$

The indices "1" of k and ω in these expressions mean that the points k and ξ lie on the branch "1". The factors φ involved in both solutions are also associated with one and the same branch. Therefore,

$$q_{11} = e^{-\frac{i}{\epsilon}\int_a^b k_1\,d\xi}\big(e^{-\int_a^b \omega_1} + \cdots\big). \tag{2.35}$$

On the other hand, w_2 is associated with another branch, namely with that linking the points a, n and $b, n'-1$. Therefore, the formula for q_{22} must be modified somewhat:

$$q_{22} = e^{-\frac{i}{\epsilon}a^*a}e^{-\frac{i}{\epsilon}\int_a^b k_2\,d\xi}\big(e^{-\int_a^b \omega_2} + \cdots\big). \tag{2.36}$$

Remark 2.2. Since the forms ω depend on k in a periodic way, the integrals of these forms do not change if we shift branches periodically. For real branches, up to oscillating exponential factors independent of k, the functions $\varphi(x,k_1,\xi)$ and $\varphi(x,k_2,\xi)$ are complex conjugate to each other, and the explicit formulas for ω show that ω_1 and ω_2 are purely imaginary and mutually complex conjugate. It can be checked that the diagonal entries of the matrix Q are of modulus 1 up to terms of the form $O(\epsilon^\infty)$. This follows from the fact that the moduli of the Wronskians of two solutions w_1, w_2 coincide at the boundary points of an interval covered by a real branch. The details can be found in [5, 6].

In the next proposition we summarize the results obtained in this subsection.

Proposition 2.3. *On an interval covered by a real branch of the isoenergy curve, the true solutions $w_j^{(l)}$ corresponding to the turning points that bound the interval in question are linked by formula* (2.27). *Asymptotically as $\epsilon \to 0$, the entries of the matrix Q occurring in* (2.27) *are described by* (2.31), (2.35), *and* (2.36).

2.5. Extensions of solutions through an interval covered by a complex branch. Let a and b ($> a$) be the turning points that bound an interval $\Delta = (a,b)$ covered by complex branches (see the figure). These branches are of the form $k = k_0 + a^*n \pm i\tau$.

Let $Z_\Delta(E,\epsilon)$ denote the matrix that links two solutions constructed near two turning points:

$$\begin{pmatrix} v^{(b)} \\ u^{(b)} \end{pmatrix} = Z_\Delta \begin{pmatrix} v_1^{(a)} \\ u_2^{(a)} \end{pmatrix}, \quad Z_\Delta = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}. \tag{2.37}$$

Now the index n accompanying the points a and b is the same.

Again, our goal is to describe the asymptotic behavior of the entries of Z_Δ as $\epsilon \to 0$. Recall that the asymptotics of true solutions is given by formal solutions like $\mathbf{v}^{(l)}$, $\mathbf{u}^{(l)}$, $l = a, b$:

$$v^{(l)}, \quad u^{(l)} \sim \mathbf{v}^{(l)}, \quad \mathbf{u}^{(l)}(1 + O(\epsilon^\infty)). \tag{2.38}$$

Again, the entries in question can be expressed in terms of the Wronskians of true solutions:

$$\operatorname{wr}\{v^{(b)}, u^{(a)}\} = z_{11} \operatorname{wr}\{v^{(a)}, u^{(a)}\}, \qquad \operatorname{wr}\{v^{(b)}, v^{(a)}\} = z_{12} \operatorname{wr}\{u^{(a)}, v^{(a)}\}, \tag{2.39}$$

$$\operatorname{wr}\{u^{(b)}, u^{(a)}\} = z_{21} \operatorname{wr}\{v^{(a)}, u^{(a)}\}, \qquad \operatorname{wr}\{u^{(b)}, v^{(a)}\} = z_{22} \operatorname{wr}\{u^{(a)}, v^{(a)}\}. \tag{2.40}$$

Substituting the corresponding formal solutions for the true solutions in the above Wronskians and computing these Wronskians at appropriate points, we can obtain asymptotic representations for the entries of Z, in terms of elements of the field $\mathcal{F}$. It is easy to check that the Wronskians on the right in (2.39), (2.10) have finite order. The explicit formulas for $\mathbf{v}^{(l)}$ and $\mathbf{u}^{(l)}$ show that, asymptotically, three of four entries of Z are exponentially small as $\epsilon \to 0$. Namely, all elements except for z_{21} involve the exponentially small factor $e^{-\frac{1}{\epsilon}\int_a^b |\operatorname{Im} k|\, d\xi}$. Moreover, it is easily seen that the solutions $v^{(a)}$ and $u^{(b)}$, as well as $u^{(a)}$ and $v^{(b)}$, are linearly dependent, because they are associated with one and the same branch of the isoenergy curve. Thus, the diagonal entries admit the estimates

$$z_{11}, z_{22} = O(\epsilon^\infty) e^{-\frac{1}{\epsilon}\int_a^b |\operatorname{Im} k|\, d\xi}. \tag{2.41}$$

On the contrary, the element z_{21} contains an exponentially large factor. The elements z_{12} and z_{21} can be calculated explicitly in the leading order by using the relations

$$v^{(b)} = O(\epsilon^\infty) e^{-\frac{1}{\epsilon}\int_a^b |\operatorname{Im} k|\, d\xi} + z_{12} u^{(a)}, \quad u^{(b)} = z_{21} v^{(a)} + O(\epsilon^\infty). \tag{2.42}$$

Recalling that the true solutions correspond asymptotically to formal solutions, we calculate the leading term of the asymptotics of z_{12}, (2.37):

$$\begin{aligned} \mathbf{v}^{(b)} &\sim \frac{1}{\sqrt{2}} e^{\frac{i}{\epsilon}\int_b^\xi k_1\, d\xi} \big(|\mathcal{E}(k_1,\xi)|^{-1/2} e^{\int_b^\xi \omega_1} \varphi(x,k_1,\xi) + \cdots\big) \\ &\sim z_{12}\mathbf{u}^{(a)} \sim z_{12}\sqrt{2} e^{\frac{i}{\epsilon}\int_a^\xi k_1\, d\xi} \big(|\mathcal{E}(k_1,\xi)|^{-1/2} e^{\int_a^\xi \omega_1} \varphi(x,k_1,\xi) + \cdots\big). \end{aligned} \tag{2.43}$$

However, these formulas do not give the leading term in question directly. The trouble is that the products $|\mathcal{E}(k_1,\xi)|^{-1/2} e^{\int_a^\xi \omega_1}$ are formed by factors that have singularities at the extremum point of $\operatorname{Im} k(\xi)$. These extrema are nondegenerate. When the factors are multiplied, the singularities cancel. This fact and the nature of the singularity of the local factor (which can be established by direct inspection) imply that the form ω has a simple pole at its singular point. We agree to compute z_{12} by comparison of both sides of (2.43) at the singular point. The values of both sides can be obtained as the limits upon approaching this point from the left and from the right. The local contributions will cancel in the limit if we approach the singular point in a symmetric way. This means that *the resulting integral of the form* ω must be understood *in the sense of the Cauchy principal value*. Under this agreement, we have

$$z_{12} \sim \frac{1}{2} e^{-\frac{i}{\epsilon}\int_a^b k_1\, d\xi} e^{-\int_a^b \omega_1} (1 + \cdots). \tag{2.44}$$

The leading term (as $\varepsilon \to 0$) of the exponentially growing coefficient z_{21} is obtained similarly:

$$\begin{aligned} \mathbf{u}^{(b)} &\sim \sqrt{2} e^{\frac{i}{\epsilon}\int_b^\xi k_2\, d\xi} \big(|\mathcal{E}(k_2,\xi)|^{-1/2} e^{\int_b^\xi \omega_2} \varphi(x,k_2,\xi) + \cdots\big) \\ &\sim z_{21}\mathbf{v}^{(a)} \sim z_{21} \frac{1}{\sqrt{2}} e^{\frac{i}{\epsilon}\int_a^\xi k_2\, d\xi} \big(|\mathcal{E}(k_2,\xi)|^{-1/2} e^{\int_a^\xi \omega_2} \varphi(x,k_2,\xi) + \cdots\big), \end{aligned} \tag{2.45}$$

which implies that

$$z_{21} \sim 2 e^{-\frac{i}{\epsilon}\int_a^b k_2\, d\xi} e^{-\int_a^b \omega_2} (1 + \cdots). \tag{2.46}$$

This time, the arguments similar to those in the preceding subsection show that ω_1 and ω_2 are real.

The above results are summarized as follows.

Proposition 2.4. *On the interval covered by a complex branch of the isoenergy curve, the true solutions $v^{(l)}$ and $u^{(l)}$ corresponding to the turning points ξ_l that bound the interval are linked by formula* (2.37). *Asymptotically as $\epsilon \to 0$, the entries of the matrix Z occurring in* (2.37) *are described by* (2.41), (2.44), *and* (2.46).

§3. Asymptotic behavior of the functions f, M, and R

3.1. Asymptotics of the solutions f for $\xi > \xi_{2N}$. First, we construct the asymptotics as $\epsilon \to 0$ of the Jost solution $f(x,k)$ that is characterized by its asymptotic behavior as $x \to \infty$ (see Subsection 1.1). Here we consider the domain $\xi > \xi_{2N}$ remote from ξ_{2N}: $\epsilon^{-2/3}|\xi - \xi_{2N}| \to \infty$.

As usual, we start with the formula

$$\begin{pmatrix} f \\ \bar{f} \end{pmatrix} = A_N \begin{pmatrix} w_1^{(2N)} \\ w_2^{(2N)} \end{pmatrix}, \quad A_N = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{3.1}$$

We are interested in the asymptotics of the entries of the matrix A_N. The asymptotic behavior of the solutions $w_1^{(2N)}, w_2^{(2N)}$ is described by the formal solutions $\mathbf{w}_1^{(2N)}, \mathbf{w}_2^{(2N)}$ corresponding to the branches "2" and "1" in the figure. The branch "2" is distinguished by the property that $k \to \sqrt{E} > 0$ as $\xi \to \infty$: $k_2 = k_{2N} + r$.

The relationship between the solution corresponding to a certain branch and the solution having a certain behavior as $\xi \to \infty$ was studied in [4, 5]. The restrictions imposed in [4, 5] on the behavior of the potential at infinity were different; accordingly, the behavior of the solutions and branches was also different. Nevertheless, the arguments suggested there show that the asymptotics of the solution f in our case is described by the formal solution associated with the branch "2". In other words,

$$a_{12} = O(\epsilon^\infty), \quad a_{21} = O(\epsilon^\infty). \tag{3.2}$$

In order to find the leading terms for the diagonal entries of A_N, it suffices to study the behavior as $\xi \to \infty$ of the formal solutions $\mathbf{w}_1^{(2N)}, \mathbf{w}_2^{(2N)}$ (in the leading order in ε). This follows from the relations

$$f = a_{11} w_1^{(2N)} + O(\epsilon^\infty), \quad \bar{f} = a_{22} w_2^{(2N)} + O(\epsilon^\infty). \tag{3.3}$$

Here and in what follows we use the abbreviation $w_j^{(2N)} = w_j^{(\xi_{2N})}$.

Thus, asymptotically in ϵ and up to a factor belonging to $\mathcal{F}$, the true solution f is described by the formal solution

$$\begin{aligned} \mathbf{w}_1^{(2N)} &= e^{\frac{i}{\epsilon}\int_{\xi_{2N}}^{\xi} k_2\, d\xi} \big(|\mathcal{E}_k|^{-1/2} e^{\int_{\xi_{2N}}^{\xi} \omega_2} \varphi(x, k_2, \xi) + \cdots\big) \\ &= e^{i\frac{\sqrt{E}}{\epsilon}(\xi - \xi_{2N})} e^{\frac{i}{\epsilon}\int_{\xi_{2N}}^{\xi} (k_2 - \sqrt{E})\, d\xi} \big(|\mathcal{E}_k|^{-1/2} e^{\int_{\xi_{2N}}^{\xi} \omega_2} \varphi(x, k_2, \xi) + \cdots\big). \end{aligned} \tag{3.4}$$

Observe that as $\xi \to \infty$ we have $\mathcal{E}_k \sim 2\sqrt{E}, \varphi \sim 1, \omega \to 0$ (sufficiently fast in the power scale), and $k_2 - \sqrt{E} \to 0$. Therefore,

$$\mathbf{w}_1^{(2N)} \sim e^{i\sqrt{E}x} e^{-i\frac{\sqrt{E}}{\epsilon}\xi_{2N}} e^{\frac{i}{\epsilon}\int_{\xi_{2N}}^{\infty} (k_2 - \sqrt{E})\, d\xi} \big((2\sqrt{E})^{-1/2} e^{\int_{\xi_{2N}}^{\infty} \omega_2} + \cdots\big) \tag{3.5}$$

as $\xi \to \infty$. It follows that

$$a_{11} \sim e^{i\frac{\sqrt{E}}{\epsilon}\xi_{2N}} e^{-\frac{i}{\epsilon}\int_{\xi_{2N}}^{\infty} (k_2 - \sqrt{E})\, d\xi} \big((2\sqrt{E})^{1/2} e^{-\int_{\xi_{2N}}^{\infty} \omega_2} + \cdots\big). \tag{3.6}$$

Recalling the formula $w_2^{(\xi_{2N},k_{2N})} = e^{-2\frac{i}{\varepsilon}\xi_{2N}k_{2N}} w_2^{(\xi_{2N},-k_{2N})}$ and the relationship between w_1 and w_2 on the interval covered by a real branch, we arrive at the following identity:

$$a_{22} = e^{2\frac{i}{\varepsilon}k_{2N}\xi_{2N}}\bar{a}_{11}. \tag{3.7}$$

We collect the above results.

Proposition 3.1. *The solutions $w_j^{(2N)}, j = 1, 2,$ and $f, \bar{f}$ are linked by formula* (3.1). *Asymptotically as $\epsilon \to 0$, the entries of the matrix A_N are described by* (3.2), (3.6), *and* (3.7).

3.2. Asymptotics of the matrix that links solutions on different intervals. Now we can unite the results of the constructions above in order to find the asymptotic behavior of the solution f everywhere on the semiaxis. After that, we establish the asymptotics of the solution ψ of the boundary value problem and the asymptotics of its spectral characteristics.

We consider the solutions f and $\bar{f}$ inside an interval $\Delta_l = (\xi_{2l}, \xi_{2l+1})$. This interval is covered by real branches of the isoenergy curve. On Δ_l, as the canonical solutions with simple asymptotic behavior we can take $w_1^{(2l+1)}, w_2^{(2l+1)}$. The solutions f and $\bar{f}$ can be expressed in terms of $w_1^{(2l+1)}, w_2^{(2l+1)}$ as follows:

$$\begin{pmatrix} f \\ \bar{f} \end{pmatrix} = T_l \begin{pmatrix} w_1^{(2l+1)} \\ w_2^{(2l+1)} \end{pmatrix}, \tag{3.8}$$

where

$$T_l = A_N S^{-1} Z_N S Q_{N-1} S^{-1} \cdots Z_{l+1} S. \tag{3.9}$$

Here Z_l and Q_l denote the matrices corresponding to the intervals $\bar{\Delta}_l = (\xi_{2l-1}, \xi_{2l})$ and Δ_l . In particular, on Δ_0 we have

$$\begin{pmatrix} f \\ \bar{f} \end{pmatrix} = T_0 \begin{pmatrix} w_1^{(1)} \\ w_2^{(1)} \end{pmatrix}. \tag{3.10}$$

Since we know the asymptotics of all matrices that form T_l, we obtain explicit asymptotic formulas for f and $\bar{f}$ on any interval of type Δ_l. Similar formulas can be written in the case where the interval under consideration is covered by a complex branch of the isoenergy curve.

Naturally, the formulas for T_l admit substantial simplification. In the above products of matrices there is a "block" $F_l = SQ_lS^{-1}Z_l$. In terms of F_l, formula (3.9) can be rewritten as follows:

$$T_l = A_N S^{-1} Z_N F_{N-1} F_{N-2} \cdots F_{l+1} S. \tag{3.11}$$

We simplify F_l. Recall the formulas for Z_l and Q_l. The branches corresponding to these operators are, respectively, of the form

$$k_{1,2} = k_l \pm i\tau, \quad k_{1,2} = k_l \pm r, \quad \text{and} \quad k_l = k_0 + na^*. \tag{3.12}$$

Therefore, F_l reduces to

$$F_l = e^{-\frac{i}{\varepsilon}k_l\delta_l} e^{-\frac{i}{2\varepsilon}a^*\xi_{2l}} F_l^{(0)}, \tag{3.13}$$

where δ_l is the sum of the lengths of the intervals Δ_l and $\bar{\Delta}_l$, respectively. Our main interest is in the leading order of the matrix $F_l^{(0)}$ as $\epsilon \to 0$. It can be characterized explicitly: in the formulas obtained above for the entries of the matrices Z_l and Q_l that determine F_l (see Propositions 2.3 and 2.4), we can replace $k_{1,2}$ by the corresponding

last components in (3.12). Let $Z_l^{(0)}$, $Q_l^{(0)}$, and $F_l^{(0)}$ denote the matrices obtained after this replacement. In particular,

$$Q_l^{(0)} \sim \begin{pmatrix} e^{\frac{i}{2\varepsilon}a^*\xi_{2l}} e^{\frac{i}{\epsilon}\int_{\xi_{2l}}^{\xi_{2l+1}}(r\,d\xi+i\epsilon\omega_1+\cdots)} & O(\epsilon^\infty) \\ O(\epsilon^\infty) & e^{-\frac{i}{2\varepsilon}a^*\xi_{2l}} e^{\frac{i}{\epsilon}\int_{\xi_{2l}}^{\xi_{2l+1}}(-r\,d\xi+i\epsilon\omega_2+\cdots)} \end{pmatrix}. \tag{3.14}$$

We put

$$[q_l^{(0)}]_{11} \sim e^{i\Psi}, \quad \Psi_l = \Phi_l + \epsilon\cdots, \quad \Psi_l = \frac{1}{\varepsilon}\left(\frac{1}{2}a^*\xi_{2l} + \int_{\xi_{2l}}^{\xi_{2l+1}}(r\,d\xi + i\varepsilon\omega_1)\right). \tag{3.15}$$

Here and in what follows, by $\epsilon\cdots$ we denote certain asymptotic power expansions which, in principle, are determined by some recurrence procedures, but at the moment allow no explicit formulas for the terms of order higher than those written out. As has already been mentioned, the expansion of Ψ is purely real; in particular, on an interval covered by a real branch, ω_l is purely imaginary. The form ω_1 is associated with the points ξ_{2l+1}.

So,

$$Q_l^{(0)} \sim \begin{pmatrix} e^{i\Psi_l} & O(\epsilon^\infty) \\ O(\epsilon^\infty) & e^{-i\Psi_l} \end{pmatrix}. \tag{3.16}$$

Transforming the matrix $Q_l^{(0)}$ with the help of S, we obtain

$$SQ_l^{(0)}S^{-1} = U_l + O(\epsilon^\infty), \quad U_l = \begin{pmatrix} \cos\Psi_l & -\sin\Psi_l \\ \sin\Psi_l & \cos\Psi_l \end{pmatrix}. \tag{3.17}$$

Recall that in the matrix $Z_l^{(0)}$ two nondiagonal entries are dominating, but one of them is exponentially large as $\epsilon \to 0$, and the other is exponentially small:

$$Z_l^{(0)} = \begin{pmatrix} 0 & 0 \\ [z_l^{(0)}]_{21} & 0 \end{pmatrix} + O(e^{-\frac{A}{\epsilon}}), \quad A > 0. \tag{3.18}$$

We introduce new notation:

$$[z_l^{(0)}]_{21} \sim e^{\Gamma_l}, \quad \Gamma_l = \Lambda_l + \epsilon\cdots, \quad \Lambda_l = \frac{1}{\epsilon}\int_{\xi_{2l-1}}^{\xi_{2l}}(\tau - \epsilon\omega_l)\,d\xi, \ \ \tau > 0. \tag{3.19}$$

Here, in the case of a complex branch, ω is real, together with the entire series Γ_l.

If we keep only the exponentially growing terms in all factors $Z_l^{(0)}$, then we easily calculate the product $F_{N-1}^{(0)}\cdots F_1^{(0)} = U_{N-1}Z_{N-1}^{(0)}\cdots U_1Z_1^{(0)}$:

$$F_{N-1}^{(0)}\cdots F_1^{(0)} = e^{\Gamma_{N-1}}\cdots e^{\Gamma_1}\left[\sin\Psi_{N-2}\cdots\sin\Psi_1\begin{pmatrix} \sin\Psi_{N-1} & 0 \\ -\cos\Psi_{N-1} & 0 \end{pmatrix} + O(\epsilon^\infty)\right]. \tag{3.20}$$

However, this answer needs a comment, because from it we could have concluded that, asymptotically, the solution f may have roots for E real. This contradicts the well-known fact that for real E the Jost solution f has no roots. In order to see that there are no such roots also for small ϵ (to be more precise, that they are shifted to the complex plane and represent resonances), we must take into account the contributions made by the exponentially small entries of the matrices Z_l. The study of resonances goes beyond the framework of this paper: this issue requires entering into additional details that we did not plan to touch on here. The conclusion we can draw from this discussion is that, though the solution f cannot have roots for real E, the values of f on certain rich systems of points with mutual distances of order ϵ can be exponentially small (as $\epsilon \to 0$).

Thus, we have

$$
\begin{aligned}
T_0 \sim & A_N S^{-1} Z_N F_{N-1} \cdots F_1 S \\
&= e^{\frac{i}{2\varepsilon}(\xi_0+\xi_2+\cdots+\xi_{2N-2})} B_N e^{\Gamma_{N-1}} \cdots e^{\Gamma_1} A_N S^{-1} Z_N^{(0)} \\
&\quad \times \Big[\sin\Psi_{N-2} \cdots \sin\Psi_1 \begin{pmatrix} \sin\Psi_{N-1} & 0 \\ -\sin\Psi_{N-1} & 0 \end{pmatrix} + O(\epsilon^\infty) \Big] S,
\end{aligned} \tag{3.21}
$$

where

$$
B_N = e^{-\frac{i}{2\varepsilon} a^*(\xi_2+\cdots+\xi_{2N-2})} e^{-\frac{i}{\varepsilon} k_{2N}(\xi_{2N}-\xi_{2N-1})} e^{-\frac{i}{\varepsilon}(k_{N-1}\delta_{N-1}+\cdots+k_1\delta_1)}. \tag{3.22}
$$

Proposition 3.2. *The asymptotics of the matrix T_0 that transforms the solutions $w_1^{(2l+1)}$ and $w_2^{(2l+1)}$ to the solutions f and $\bar f$ on the interval $(0,\xi_1)$ (see (3.8)) is given by formula (3.21).*

3.3. Asymptotics of the solution f and the reflection coefficient. Simple and natural calculations lead now to the following asymptotic formula for f on the interval $\Delta_0 = (0,\xi_1)$:

$$
\begin{aligned}
\begin{pmatrix} f \\ \bar f \end{pmatrix} \sim & \frac{1}{2} B_N e^{\Gamma_N} \cdots e^{\Gamma_1} \Big[\sin\Psi_{N-1} \cdots \sin\Psi_1 \begin{pmatrix} -ia_{11} & a_{11} \\ a_{22} & ia_{22} \end{pmatrix} + O(\epsilon^\infty) \Big] \\
& \times e^{\frac{i}{\epsilon} k_1(\xi-\xi_1)} |\mathcal{E}_k|^{-1/2} \begin{pmatrix} e^{\frac{i}{\epsilon}\int_{\xi_1}^{\xi}(-r-i\epsilon\omega_1)} \phi(x,k_1,\xi,\epsilon) \\ e^{\frac{i}{\epsilon}\int_{\xi_1}^{\xi}(r-i\epsilon\omega_2)} \phi(x,k_2,\xi,\epsilon) \end{pmatrix}.
\end{aligned} \tag{3.23}
$$

Here

$$
\phi(x,k_1,\xi,\epsilon) = \varphi(x,k_1,\xi) + \epsilon \cdots . \tag{3.24}
$$

It has already been mentioned that ω_1 and ω_2 are purely imaginary on $(0,\xi_1)$.

We write the asymptotics of f:

$$
\begin{aligned}
f \sim & \frac{e^{-i\frac{\pi}{4}}}{2} B_N e^{\frac{i}{\epsilon} k_1(\xi-\xi_1)} e^{\Gamma_N} \cdots e^{\Gamma_1} a_{11} |\mathcal{E}_k|^{-1/2} \\
& \times \Big[\sin\Psi_{N-1} \cdots \sin\Psi_1 \big(e^{-i\frac{\pi}{4}} e^{\frac{i}{\epsilon}\int_{\xi_1}^{\xi}(-r-i\epsilon\omega_1)} \phi(x,k_1,\xi,\epsilon) \\
& \qquad + e^{i\frac{\pi}{4}} e^{\frac{i}{\epsilon}\int_{\xi_1}^{\xi}(r-i\epsilon\omega_2)} \phi(x,k_2,\xi,\epsilon) + O(\epsilon^\infty) \big) \Big].
\end{aligned} \tag{3.25}
$$

Up to the factor of $(2\sqrt{E})^{1/2}$, the element a_{11} is described asymptotically by an oscillating exponential (the argument is the same as in the case of the matrices Q_l):

$$
a_{11} \sim (2\sqrt{E})^{1/2} e^{i\Omega}, \quad \Omega = \Theta + \epsilon \cdots , \quad \Theta = \frac{\sqrt{E}}{\epsilon}\xi_{2N} - \frac{1}{\epsilon}\int_{\xi_{2N}}^{\infty} (k_2 - \sqrt{E}\, d\xi - i\epsilon\omega_2). \tag{3.26}
$$

Keeping only the leading orders and dropping the terms of higher order, i.e., the terms decaying as $\epsilon \to 0$, we obtain

$$
\begin{aligned}
f \sim & \frac{e^{-i\frac{\pi}{4}}}{2} (2\sqrt{E})^{1/2} B'_N e^{\frac{i}{\epsilon} k_1(\xi-\xi_1)} e^{\Gamma_N} \cdots e^{\Gamma_1} \\
& \times \sin\Phi_{N-1} \cdots \sin\Phi_1 (e^{-i\frac{\pi}{4}} e^{\frac{i}{\epsilon}\int_{\xi_1}^{\xi}(-r-i\epsilon\omega_1)} \varphi(x,k_1,\xi) + e^{i\frac{\pi}{4}} e^{\frac{i}{\epsilon}\int_{\xi_1}^{\xi}(r-i\epsilon\omega_2)} \varphi(x,k_2,\xi)).
\end{aligned} \tag{3.27}
$$

Here

$$
B'_N = e^{\frac{i}{\epsilon}[\sqrt{E}\xi_{2N} - \int_{\xi_{2N}}^{\infty}\{(k_2-\sqrt{E})\, d\xi - i\epsilon\omega_2\}]} B_N. \tag{3.28}
$$

Proposition 3.3. *The asymptotic behavior of the solutions f and $\bar f$ on the interval $(0,\xi_1)$ is given in the leading order by formula (3.27), and the error estimates are described by (3.25).*

We turn to the asymptotic behavior of the functions M and R. In the case of M, it suffices to use the formulas for f, putting $x = \xi = 0$ in them. We do not write the corresponding formula explicitly. As to the function R, in the leading order without error control, the asymptotic formula for R has the form

$$\begin{aligned} R &= -\frac{\overline{M}}{M} \\ &\sim -i{B'}_N^{-2} e^{2\frac{i}{\epsilon}k_1\xi_1} \\ &\quad \times \frac{e^{i\frac{\pi}{4}} e^{-\frac{i}{\epsilon}\int_{\xi_1}^0 (-r-i\epsilon\omega_1)}\varphi^*(0,k_1,0) + e^{-i\frac{\pi}{4}} e^{-\frac{i}{\epsilon}\int_{\xi_1}^0 (r-i\epsilon\omega_2)}\varphi^*(0,k_2,0)}{e^{-i\frac{\pi}{4}} e^{\frac{i}{\epsilon}\int_{\xi_1}^0 (-r-i\epsilon\omega_1)}\varphi(0,k_1,0) + e^{i\frac{\pi}{4}} e^{\frac{i}{\epsilon}\int_{\xi_1}^0 (r-i\epsilon\omega_2)}\varphi(0,k_2,0)}. \end{aligned} \tag{3.29}$$

These formulas become bulkier if we include the error control:

$$\begin{aligned} R \sim &- i{B'}_N^{-2} e^{2\frac{i}{\epsilon}k_1\xi_1} \\ &\times \Big(\sin\Psi_{N-1}\cdots\sin\Psi_1 (e^{i\frac{\pi}{4}} e^{-\frac{i}{\epsilon}\int_{\xi_1}^0 (-r-i\epsilon\omega_1)}\phi^*(0,k_1,0,\epsilon) \\ &\qquad + e^{-i\frac{\pi}{4}} e^{-\frac{i}{\epsilon}\int_{\xi_1}^0 (r-i\epsilon\omega_2)}\phi^*(0,k_2,0,\epsilon) + O(\epsilon^\infty)) \Big) \\ &\times \Big(\sin\Psi_{N-1}\cdots\sin\Psi_1 (e^{-i\frac{\pi}{4}} e^{\frac{i}{\epsilon}\int_{\xi_1}^0 (-r-i\epsilon\omega_1)}\phi(0,k_1,0,\epsilon) \\ &\qquad + e^{i\frac{\pi}{4}} e^{\frac{i}{\epsilon}\int_{\xi_1}^0 (r-i\epsilon\omega_2)}\phi(0,k_2,0,\epsilon) + O(\epsilon^\infty)) \Big)^{-1}. \end{aligned} \tag{3.30}$$

Theorem 3.4. *The asymptotics of the reflection coefficient R as $\epsilon \to 0$ is given by formula* (3.30)*. The leading terms of this asymptotics are described by* (3.29)*.*

Of course, given in the above short form, this theorem needs some comment. If the numerator (or, equivalently, the denominator) vanishes for some E, then the expression suggested above makes no sense. At such a point, the limit of the fraction remains unimodular and gives the asymptotics correctly, but the leading term of this limit can no longer be described by our explicit formula. To obtain an explicit formula at the roots in question, we must take into account the exponentially small terms of the matrices Z_l, i.e., we must show that, for real E, there are no such roots in fact. This is related to analysis of complex resonances, which can be exponentially close to the real axis. Another point deserving attention is that, as $\epsilon \to 0$, the roots of different sines may become closer (typically, such roots lie at a distance of the order of ϵ from one another). This causes additional troubles when we analyze the behavior of R near the roots. The only way to overcome this difficulty is to consider only sequences of ϵ along which the roots do not approach. For a different behavior of $p(x,\xi)$ as $\xi \to \infty$, both issues mentioned above were considered in the papers [4, 5], where also references to close works can be found. To acquaint themselves with the ideas touched upon in this paragraph, the readers can consult the same papers, but, of course, a complete analysis in the case under study in this paper requires special investigation.

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Department of Physics, St. Petersburg State University, Ulyanovskaya 1, Petrodvorets, St. Petersburg 198504, Russia

E-mail address: buslaev@mph.phys.spbu.ru

St. Petersburg State Electrotechnical University, Ul. Prof. Popova 5, St. Petersburg 197376, Russia

Department of Mathematics, University Paris 13, Av. J-B Clement, Villetaneuse 93430, France

E-mail address: grigis@math.univ-paris13.fr

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