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ON INTEGRAL LATTICES HAVING AN ODD MINIMUM

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Abstract. The kissing number of integral lattices of odd minimum is studied, with special emphasis on the case of minimum 3.

§1. Introduction

In this paper, we consider the problem of the kissing number for integral lattices having an odd minimum m, with special emphasis on the case where $m = 3$. Thus we try to give some precisions about the set of possible values for the number s of pairs of minimal vectors on such a lattice, and in particular for its maximum value $s_{\max}$ for a given minimum m and dimension n. Our results include

• A list of upper and lower bounds for $m = 3$ and $1 \le n \le 24$.

• The value of $s_{\max}$ for all odd m in the range $1 \le n \le 7$ and, moreover, for $m = 3$ when $n = 8, 9, 16, 22, 23$.

• The complete list of possible s for $m = 3$ and $n \le 8$.

We may and most of the time shall restrict ourselves to *well-rounded lattices*, i.e., lattices possessing n independent minimal vectors, and even to lattices that are generated by their minimal vectors.

Our results are based on the use of various techniques arising from the theories of spherical designs and root systems, Watson's study [W2] of lattices that do not possess any hexagonal cross-section having the same minimum, classification according to the index of well-rounded sublattices, and estimations for the minimum of the dual lattice; we have also used calculations with the PARI package. However, the reader may check that the proofs of the bounds for $s_{\max}$ displayed in Table 1.1 below, except in dimension 9, do not involve heavy computations; more generally, hand-computational techniques suffice to obtain (up to dimension 8) the classification of lattices of minimum 3 having a relatively large kissing number.

Here are the most important results which we prove in this paper.

Theorem 1.1. *Lower and upper bounds for the values of s on an integral lattice of minimum* 3 *are as displayed in Table* 1.1.

Moreover, *in dimensions* 3–9, 16, 22, *and* 23, *for which the upper and lower bounds in the table coincide, the lattices on which these bounds are attained are unique.*

The lower bounds are proved in §2 and the upper bounds in §3, except for dimensions 8 and 9, for which we need techniques to be developed later in §§8–10. In §3, we also prove the uniqueness assertions for $n = 16$, 22, and 23. A proof of those for low dimensions is given in §4, where we introduce root systems associated with suitable sublattices. (More subtle constructions of root systems will occur in §9.)

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TABLE 1.1. Lower and upper bounds for $s_3(n)$, $n \leq 24$. .

n	1	2	3	4	5	6	7	8
$s_3(n) \geq$	1	2	4	6	10	16	28	30
$s_3(n) \leq$	1	2	4	6	10	16	28	30
n	9	10	11	12	13	14	15	16
$s_3(n) \geq$	34	40	52	68	88	112	160	256
$s_3(n) \leq$	34	63	81	103	129	162	203	256
n	17	18	19	20	21	22	23	24
$s_3(n) \geq$	288	352	448	640	896	1408	2300	2301
$s_3(n) \leq$	322	411	531	703	965	1408	2300	4991

§5, which we shall not use in the sequel, is devoted to lattices having any odd minimum $m \geq 3$ in the range $2 \leq n \leq 7$. To state our result precisely, we first introduce some notation. We say that a lattice Λ is *primitive* if it is integral and the scalar products on Λ generate $\mathbb{Z}$. Given integers $m \geq 1$ and $n \geq 2$, we denote by $s_m(n)$ the maximum value of s on primitive n-dimensional lattices of minimum m. (This makes sense for all m and all $n \geq 2$, but $\mathbb{Z}$ is the only primitive 1-dimensional lattice.)

Theorem 1.2. *Let $m \geq 5$ be odd. We have $s_m(n) = s_3(n)$ for $2 \leq n \leq 6$ and $s_m(7) = 27 < s_3(7) = 28$.*

Define the *index* of a well-rounded lattice Λ of dimension n to be the largest possible value of the index $[\Lambda : L]$ where L is generated by n independent minimal vectors in Λ.

In §6, essentially, we consider lattices of index 1 and dimension $n \leq 6$. We describe the *minimal classes* (in the sense of [M, Chapter 9]) for lattices of odd minimum in dimensions $n \leq 5$; general results under the less restrictive condition that the lattices do not possess hexagonal sections having the same minimum are displayed in Appendix 1. We obtain a complete description of the possible values of s for all odd m in dimensions $n \leq 5$, as well as the analogous 6-dimensional results for $m = 3$ and $m = 5$. (Probably, the result we obtain for $m = 5$ is valid for all $m \geq 5$ odd.)

In §7, we calculate the maximum value of the index up to dimension 8 under various restrictions, notably under the hypothesis that Λ be similar to an integral lattice having an odd minimum. These results are then applied to lattices of dimension $n \leq 6$. We also consider integral lattices of minimum 3 in dimensions up to 8 more thoroughly; in particular, we characterize 7-dimensional integral lattices of index 3 (and minimum 3).

The remainder of the paper essentially deals with integral lattices of minimum 3.

In §8, we prove new upper bounds for the minimum of the dual lattice of an integral lattice of minimum $m = 3$ and dimension $n \leq 9$. Our bounds in dimensions 8 and 9 (1 and $\frac{4}{3}$) are most certainly far from being sharp.

§9, in which we utilize the results of §8, is devoted to the construction of root systems that we can attach (under some severe restrictions) to integral lattices of minimum 3. These constructions are used to bound the difference $s(\Lambda) - s(\Lambda_0)$ from above, where Λ_0 denotes a suitably chosen hyperplane cross-section. We also employ these constructions to prove the uniqueness assertions of Theorem 1.1 for $3 \leq n \leq 8$.

More precise results for dimensions 7 and 8 are proved in §10, in which, in particular, we establish the list of all possible kissing numbers. §11 is devoted to dimension 9, and four appendices complete this paper.

The results we prove for dimension 9 give us an opportunity for making two comments.

(1) The bound $s \le 34$ that we shall prove in §11 significantly improves on the bound $s \le 49$ of [V], indeed valid for any system of norm 3 vectors having mutual scalar products ± 1. However, under this weaker hypothesis, a system with $s = 48$ was found by Neumaier [Neu]; see the final remark in §3.

(2) In the entire paper, we have focused on large values of s. Lattices having a small s, say, slightly larger than n, cannot be expected to admit a reasonable classification. Indeed, here we meet a problem in graph theory: consider the set Γ of graphs of valence 3 with n vertices; with such a graph, we associate its adjacency matrix A and then the matrix $B = 3I_n - A$; this matrix is positive, semidefinite, and determines a relative lattice, which is integral of minimum $m \le 3$ and has a small kissing number. To classify graphs of a given valency is considered to be a highly complicated problem. So, the same conclusion should be true for lattices having a low kissing number.

§2. Constructions of lattices

In this section, we explain various constructions of lattices of minimum 3, which will in particular provide examples of lattices on which the lower bounds of Table 1.1 are attained.

Given a lattice Λ and $a > 0$, we denote by ${}^a\Lambda$ the lattice Λ equipped with the scalar product $a(x \cdot y)$.

Let m be a positive integer. In dimension 1, for every m there exists a unique lattice of minimum m, namely ${}^m\mathbb{Z}$; for $n = 2$, there are $\lfloor \frac{m+1}{2} \rfloor$ well-rounded lattices, determined by a basis (e_1, e_2) such that $e_1 \cdot e_1 = e_2 \cdot e_2 = m$ and $e_1 \cdot e_2 = i$, $0 \le i \le \lfloor \frac{m-1}{2} \rfloor$, hence two lattices if $m = 3$.

In the remainder of this section, we restrict ourselves to the case where $m = 3$.

Our first construction is an adaptation of various lamination procedures to the problem of the kissing number. It produces lattices denoted by W_n, or maybe W_n^a, W_n^b, etc., when they are not unique. We set $W_1 = {}^3\mathbb{Z}$, take the lattice with $e_1 \cdot e_2 = 1$ for the role of W_2, and construct the further W_n by induction: starting with a lattice W_n of minimum 3 that has a basis $(e_1, \dots, e_n)$ of minimal vectors, we consider (up to isometry) $(n+1)$-dimensional lattices of minimum 3 which have a basis $(e_1, \dots, e_n, e_{n+1})$ extending the previous one and for which s is as large as possible. These are the lattices W_{n+1}.

Theorem 2.1. *The lattices W_n are unique in the range $1 \le n \le 16$, except for $n = 10$ and $n = 11$, where there are two of them. For $n \le 14$, $s(W_n)$ is the lower bound for s displayed in Table 1.1; we have $s(W_{15}) = 140$ and $s(W_{16}) = 156$.*

[In dimensions 10 and 11, we distinguish W_n^a and W_n^b by the properties of their duals: the annihilator of Λ^*/Λ is 4 for $\Lambda = W_{10}^a$ and 16 for $\Lambda = W_{10}^b$ (in both cases, $s(\Lambda^*) = 4$); we have $s({W_{11}^a}^*) = 19$ and $s({W_{11}^b}^*) = 27$, (in both cases, the annihilator of Λ^*/Λ is 4). We have the inclusions $W_{10}^a \subset W_{11}^a$, $W_{10}^a \subset W_{11}^b$ and $W_{10}^b \subset W_{11}^a$.]

Sketch of the proof. We have used a computer to list all possible extensions of a given W_n with $n \le 15$. In all cases, we found lattices with determinant equal to the determinant found by Plesken and Pohst in [Pl-P] for their *arithmetical laminations* for minimum 3. In this way, we could identify the W_n with some of the Plesken–Pohst lattices. □

The notation W_n has been chosen after Watson, who discovered these lattices in [W2] up to dimension 7. We have $W_7 \simeq {}^2\mathbb{E}_7^*$, but W_7 is better understood as the pull-back in $\mathbb{Z}^7$ of the binary $[7,4,3]$ Hamming code. The lattices W_6, W_5, ... are then obtained as antilaminations of W_7 (see the definition below).

Now, we briefly explain explicit constructions for dimensions up to 12.

Given a lattice $\Lambda \subset \mathbb{Z}^n$, we easily construct lattices of dimension $n+1$ and $n+2$ by the following trick: adjoin to Λ the vectors $(0^{n-1},1,1,1) \in \mathbb{Z}^{n+2}$ and then $(0^{n-2},1,0,0,1,1) \in \mathbb{Z}^{n+3}$, and consider the sections of these lattices first by the hyperplanes $x_{n+1}+x_{n+2}=0$ in $\mathbb{R}^{n+2}$, then by the hyperplane $x_{n+1}+x_{n+2}+x_{n+3}=0$ in $\mathbb{R}^{n+3}$. We obtain lattices in dimension $n+1$ with $s=s(\Lambda)+2$, then in dimension $n+2$ with $s=s(\Lambda)+6$. Taking $\Lambda=W_7$, we obtain W_8 and W_9 in this way. Identifying two weight 3 words in the concatenation of two copies of the Hamming code, we obtain a $[12,8,3]$ code possessing 17 weight 3 words. Its pull-back in $\mathbb{Z}^{12}$ has $s=17\cdot 4=68$; it is the lattice W_{12}. Its dual has two orbits of minimal vectors, with corresponding hyperplane sections W_{11}^a and W_{11}^b; repeating the antilamination procedure yields all the W_n with $n \le 10$.

Let Λ be a lattice, with minimal m. By the *antilaminations of* Λ, we mean the descending chain of successive hyperplane sections of Λ of the lattices having the highest Hermite invariant. The antilaminations of O_{23} (the only 23-dimensional unimodular lattice of minimum 3) are described in [Bt-M], from which we can extract the following theorem.

Theorem 2.2. *There is a unique "antilaminated" lattice in* O_{23} *for each of the dimensions from* $n=23$ *to* $n=14$, *denoted by* O_n. *The values of* $s(\mathrm{O}_n)$ *are the lower bounds for* s *that are displayed in Table* 1.1. *We have* $s(\mathrm{O}_n) > s(W_n)$ *for* $n=15,16$ *and* $s(\mathrm{O}_{14}) = s(W_{14})$, *but* O_{14} *and* W_{14} *are not isometric.*

Proof of the lower bounds in Theorem 1.1. Consider the lattices W_n for $1 \le n \le 14$, O_n for $14 \le n \le 23$, and $\mathrm{O}_{23} \perp W_1$ for $n=24$. □

We now indicate a way of obtaining integral lattices of minimum 3 from even integral lattices of minimum 4, for which numerous examples are known.

Proposition 2.3. *An integral lattice* Λ *of minimum* 3 *with* $s>n$ *possesses norm* 4 *vectors.*

Proof. Let $S(\Lambda)=\{\pm e_1,\dots,\pm e_s\}$. If the e_i are pairwise orthogonal, then $s \le n$. Otherwise, there exist i,j such that $e_i\cdot e_j=\pm 1$. Then $N(e_i \mp e_j)=4$.

Theorem 2.4. *An integral lattice of minimum* 3 *contains vectors of norm* 4 *if and only if there exists an even lattice* L *of minimum* 4 *and a vector* $e \in L$ *that satisfies the following three conditions*:

(1) $N(e)=12$;

(2) e *is not congruent modulo* 2 *to a shorter vector of* L;

(3) $e \in 2L^*$.

[Note that, because of the identity $N(x+2z)-N(x)=4(x.z+N(z))$, $y \equiv x \mod 2L \Rightarrow N(y) \equiv N(x) \mod 4$, so that it suffices to consider vectors of norm 4 or 8 to test condition (2).]

Proof. If Λ exists, let $L=\Lambda_{\text{even}}$. Then $\min L=4$. Let $f\in S(\Lambda)$, and set $e=2f$. Then $N(e)=12$, and $e\in 2L^*$, because all scalar products $e\cdot x=2f\cdot x$ are even on L, thus for all $x\in\Lambda$. Moreover, if $e'\equiv e \mod 2\Lambda$, then $\frac{e'}{2}$ is a nonzero vector of Λ, which implies $N(e')\ge 4\min\Lambda \ge 12$.

Conversely, given L of norm 4, suppose a vector $e\in L$ satisfies the three conditions of the theorem. Set $f=\frac{e}{2}$ and $\Lambda=\langle L,f\rangle$. Then $\Lambda = L\cup(f+L)$. Thanks to condition (3), all scalar products $f\cdot x$, $x\in L$, are integral. Moreover, given $x,y\in L$, we have the identity

$$(f+x)\cdot(f+y)=N(f)+f\cdot(x+y)+x\cdot y,$$

which shows that Λ is integral. Finally, for $x \in \Lambda$, we have either $x \in L$, hence $x = 0$, or $N(x) \geq 4$, or $x = f + y, y \in L$, hence $2x \equiv e \mod 2L$, whence $N(2x) \geq 12$. □

For $14 \leq n \leq 23$, this construction transforms the laminated lattices Λ_n into O_n and, similarly, we obtain W_{11}^b, W_{12}, and W_{13} from $\Lambda_{11}^{\max}$, $\Lambda_{12}^{\mathrm{mid}}$, and $\Lambda_{13}^{\max}$. Also, two important examples show up in dimension 10. The lattices K'_{10} [M, Chapter 8, Definition 5.8] and Q_{10} (a 4-modular lattice discovered by Souvignier) yield integral lattices of minimum 3, which have the same value of s ($s = 40$) as W_{10}^a and W_{10}^b.

[Curiously, K'_{10} and Q_{10} play an important rôle in Nebe and Venkov's classification of 10-dimensional strongly eutactic lattices; compare [N-V, Theorem 3.6].]

Now, for further use, we apply the above theorem to root lattices (rescaled to minimum 4). We restrict ourselves to the cases where the resulting lattices Λ of minimum 3 are well rounded. The results, the proofs of which are left to the reader, are summarized in the table below, which contains an integer n, an n-dimensional root lattice L, the kissing number of the only lattice Λ of minimum 3 that comes from ${}^2L$, and the name of Λ if any.

TABLE 2.5. Application of root lattices. .

n	3	4	5	6	7	7	8	9	$n \geq 5$
L	$\mathbb{A}_1^{\perp 3}$	$\mathbb{A}_3 \perp \mathbb{A}_1$	$\mathbb{A}_5$	$\mathbb{D}_6$	$\mathbb{E}_7$	$\mathbb{E}_6 \perp \mathbb{A}_1$	$\mathbb{E}_7 \perp \mathbb{A}_1$	$\mathbb{E}_8 \perp \mathbb{A}_1$	$\mathbb{D}_{n-1} \perp \mathbb{A}_1$
s	4	6	10	16	28	10	12	16	$2(n-1)$
Λ	W_3	W_4	W_5	W_6	W_7				

[The lattice constructed from $\mathbb{D}_{n-1} \perp \mathbb{A}_1$ (indeed, also for $n = 3$ or 4) possesses a basis $(e_1, \dots, e_n)$ of minimal vectors such that the other minimal vectors (up to sign) are $e_1 + e_2 + e_3, \dots, e_1 + e_2 + e_n$.]

Now, we briefly describe two constructions based on weight 6 codes.

For the first construction, we consider the lattice $\mathbb{A}_1^{\perp n}$ endowed with its canonical basis $(\varepsilon_1, \dots, \varepsilon_n)$, its "doubly even sublattice" L_n (the scaled copy of $\mathbb{D}_n$ defined by the congruence $\sum_i x_i \equiv 0 \mod 2$), and a nonzero, self-dual code C (i.e., such that $C \subset C^\perp$) of length n and weight 6. Let Λ_C be the lattice generated by L_n and the vectors $\frac{1}{2}(a_1\varepsilon_1 + \cdots + a_n\varepsilon_n)$, where $(a_1, \dots, a_n)$ reduce modulo 2 to a word of C. Then Λ_C is an integral lattice of minimum 3, and $s(\Lambda_C) = 16t$, where t denotes the number of weight 6 words in C. For instance, if $n = 6$ and C is the one-word code, then Λ_C is isometric to W_6. These codes were studied by Bachoc and Gaborit in [B-G]. From [B-G], we can extract examples in dimensions 6, 10, 14, 15, 16, 19, 20, and 21 for which $s(\Lambda_C)$ meets the lower bound displayed in Table 1.1.

For the second construction, we start with the lattice $\mathbb{Z} \perp \mathbb{A}_1^{\perp(n-1)}$, and consider a self-dual code C of length $n-1$ and weight 6. This time, we attach to C the lattice Λ'_C generated by L_{n-1}, the vectors $\frac{1}{2}(a_1\varepsilon_1 + \cdots + a_{n-1}\varepsilon_{n-1})$ as above, and the $2(n-1)$ vectors $\varepsilon + \tau$ for τ a root of $\mathbb{A}_1^{\perp n-1}$. Then, again, Λ'_C is an integral lattice of minimum 3, with $s = 16t + 2(n-1)$. For instance, if $n = 7$ and C is the one-word code, Λ'_C is isometric to W_7. From [B-G], we can extract examples in dimensions 7, 11, 13, and 17 for which $s(\Lambda_C)$ meets the lower bound in Table 1.1.

§3. Upper bounds from spherical designs theory

First, we state a theorem due to the second author:

Theorem 3.1. *Let $S \subset \mathbb{R}^n$ be a finite symmetric set whose elements are $2s$ norm 3 vectors with pairwise scalar products $0, \pm 1, \pm 3$. Then we have the following upper bounds for s:*

(1) $s \le \dfrac{8n}{9-n}$ *for* $n \le 8$.

(2) $s \le \dfrac{8n(n+2)}{25-n}$ *for* $n \le 24$.

Proof. See [V, Theorem 7.13]. □

The proof in [V] of the two above assertions consists in comparing two inequalities. In both cases, one of them is a Cauchy–Schwarz inequality $[f, f] \ge 0$ for the scalar product on the set of degree 2 harmonic polynomials [V, Section 1], an inequality which properly belongs to the theory of spherical 2-designs. In case (2), the second inequality is obtained in the same way, by using degree 4 polynomials this time; in case (1), one simply writes that the number of pairs of orthogonal vectors is bounded from below by zero. This implies the following statement.

Theorem 3.2. *Equality occurs in* 3.1 (1) *if and only if S is a spherical* 3*-design and S does not contain any pair of orthogonal vectors*; *equality occurs in* 3.1 (2) *if and only if S is a spherical* 5*-design.*

[We refer to [V, Section 3] for the definition of a spherical design.]

The integral lattices of minimum 3 whose sets of minimal vectors constitute a spherical 5-design were classified in [V, Theorem 7.4]. In our notation, this result reads as follows.

Theorem 3.3. *The set of minimal vectors of an integral lattice Λ of minimum* 3 *is a spherical* 5*-design if and only if Λ is isometric to* W_1, W_7, O_{16}, O_{22}, *or* O_{23}.

Using Theorem 3.1 and taking into account Theorem 3.3, which tells us when the inequality in 3.1 (2) is strict, we justify the upper bounds given in Theorem 1.1 except for $n = 8$ and $n = 9$; we also obtain the uniqueness assertions for $n = 7$, 16, 22 and 23.

To complete the proof of Theorem 1.1, it only remains to prove the upper bounds stated for dimensions 8 and 9 and the uniqueness assertions for dimensions 3–8. A quick proof of the uniqueness assertions for $n = 3$, 5, 6, 7 will be given in the next section; easy proofs for $n = 3$ and 4 can be read at the beginning of §6.

Remark 3.4. The set of $2s = 96$ vectors in $\mathbb{Z}^9$ with mutual scalar products $\pm$ 1 found by Neumaier in [Neu], and referred to at the end of the introduction, is constructed in the following way: start with the affine plane P over $\mathbb{F}_3$, and identify its nine elements with the canonical basis $\mathcal{B} = (\varepsilon_1, \dots, \varepsilon_9)$ of $\mathbb{R}^9$; a line in P contains three points corresponding to three basis vectors $\varepsilon_i, \varepsilon_j, \varepsilon_k$, and the latter give rise to the 8 vectors $\varepsilon_i \pm \varepsilon_j \pm \varepsilon_k \in \mathbb{Z}^9$. Thus, the 12 affine lines in P determine $48 \times 2 = 96$ vectors in $\mathbb{Z}^9$.

§4. A first construction of root systems

We still denote by Λ an n-dimensional integral lattice of minimum 3.

Proposition 4.1. *Let T be a nonempty set of minimal vectors in Λ. Suppose T is stable under the map $x \mapsto -x$ and does not contain any pair of orthogonal vectors. Set $r = \operatorname{rk} T$ and $t = \frac{1}{2}|T|$. For $e \in T$, put*

$$T_e^+ = \{x \in T \mid e \cdot x = +1\}.$$

(*T is the disjoint union of T_e^+ and $T_e^- = -T_e^+$.*)

(1) *If all scalar products $x \cdot y$ for $x, y \in T_e^+$, $y \neq x$, are equal to $+1$, then $t = r$, and the vectors $e - x$, $x \in T_e^+$, constitute a basis for a copy of the root lattice $\mathbb{A}_{r-1}$ scaled to minimum* 4.

(2) *Otherwise, the lattice L generated by the vectors $e - x$, $x \in T_e^+$, and $x + y$, $x, y \in T_e^+$, $x \cdot y = -1$, is a copy of a rank r root lattice scaled to minimum* 4.

Proof. For $x, x' \in T_e^+$, we have $N(x \pm x') = 6 \pm x \cdot x'$, which shows that, for $x \cdot y = -1$, the vectors $\frac{1}{\sqrt{2}}(e - x)$ and $\frac{1}{\sqrt{2}}(x + y)$ have norm 2 and integral mutual scalar products. Hence they generate a root lattice.

Under hypothesis (1), let $e_1 = e, e_2, \dots, e_t$ be the vectors of T_e^+. The Gram matrix of the $t - 1$ vectors $\frac{1}{\sqrt{2}}(e_1 - e_i)$ has entries 2 on the diagonal and 1 outside. Thus, it is a Gram matrix for $\mathbb{A}_{t-1}$. In particular, it has rank $t - 1$, which shows that

$$\mathrm{rk}(e_1, \dots, e_t) = \mathrm{rk}(e_1, e_1 - e_2, \dots, e_i - e_t) = t,$$

hence $t = r$.

Under hypothesis (2), let $e_1, \dots, e_r$ be independent vectors in T_e^+. The vectors $e_1 - e_k$, together with one vector $e_i + e_j$, constitute a rank r system. Choosing i, j with $e_i \cdot e_j = -1$ proves the second assertion. □

Corollary 4.2. *If Λ is well rounded, if $S(\Lambda)$ contains no pair of orthogonal vectors, and if $s(\Lambda) > n$, then Λ can be constructed by the procedure of Theorem* 2.4; *as lattice L, the copy of a rank n root lattice scaled to minimum* 4 *is used.*

Proof. Since $s > n$, hypothesis (2) of Proposition 4.1 is satisfied. Clearly, we have $\Lambda = \langle L, e_1 \rangle$, and this construction of Λ by L is precisely that of Theorem 2.4. □

Proof of the uniqueness assertions for $n = 3, \dots, 7$. By Theorems 3.1 and 3.2, the scalar product of two minimal vectors in a lattice of the dimensions indicated above at which s attains its maximum are nonzero. Hence we may apply the above corollary, and the conclusion follows from the results displayed in Table 2.5. □

§5. A theorem of Watson

In this section, we use a theorem of Watson to prove Theorem 1.2. We consider *general* lattices in the Euclidean space $\mathbb{R}^n$ (our lattices are no longer assumed to satisfy integrality conditions). To state Watson's theorem referred to above, we recall the notion of a *minimal class* (see [M, Chapter 9] for an extensive study): these are the classes for the equivalence relation

$$\Lambda \mathcal{R} \Lambda' \iff \exists u \in \mathrm{GL}_n(\mathbb{R}) \text{ such that } u(\Lambda) = \Lambda' \text{ and } u(S(\Lambda)) = S(\Lambda').$$

Clearly, a class is a union of similarity classes of lattices. On the set of minimal classes, we define an ordering relation by

$$\mathrm{cl}(\Lambda) \prec \mathrm{cl}(\Lambda') \iff \text{there exists } u \in \mathrm{GL}_n(\mathbb{R}) \text{ such that } u(\Lambda) = \Lambda' \text{ and } u(S(\Lambda)) \subset S(\Lambda').$$

The *perfection rank of a lattice* Λ is the rank r in the space $\mathrm{Sym}_n(\mathbb{R})$ of symmetric endomorphisms of the set of orthogonal projections onto the minimal vectors of Λ; we have $1 \le r \le \frac{n(n+1)}{2}$; the difference $r' = \frac{n(n+1)}{2} - r$ is the *perfection corank of* Λ. These are invariants of the minimal class of Λ. The similarity classes of lattices belonging to a class of corank r' are represented by the set of their Gram matrices scaled to a given (arbitrary) minimum; this is parametrized by the relative interior of a convex set in an affine space of dimension r' whose extremal points are perfect classes (for which $r' = 0$). A class with $r' = 0$ is the similarity class of a perfect lattice. A class with $r' = 1$ is represented by the interior of a *Voronoĭ path* $t \in (0,1) \mapsto M(t)$ connecting the perfect matrices $M(0)$ and $M(1)$.

Theorem 5.1 (Watson)**.** *Let* Λ *be a lattice of dimension* $n \le 7$ *that does not possess hexagonal sections having the same minimum and whose kissing number is maximal among all such lattices. Then the minimal class of* Λ *is that of* W_n*. In particular, for a lattice of dimension* $n = 1, 2, 3, 4, 5, 6,$ *or* 7 *possessing no hexagonal section with the same minimum, we have* $s \le 1, 2, 4, 6, 10, 16,$ *or* 28, *respectively.*

Proof. See [W2]. □

Using Watson's theorem, we immediately recover the values of $s_n(3)$ for $n \le 7$ that we gave in Theorem 1.1, because two minimal vectors in an integral lattice having an odd minimum cannot generate a hexagonal lattice. Now, we proceed to the proof of Theorem 1.2, which concerns integral lattices having an odd minimum $m \ge 5$:

Theorem 5.2. *Let* $m \ge 5$ *odd. We have* $s_m(n) = s_3(n)$ *for* $2 \le n \le 6$ *and* $s_m(7) = 27 < s_3(7) = 28$*.*

Proof. The upper bounds for s in the range 1–6 follow immediately from Theorem 4.1. To prove the sharper bound $s \le 27$ for $n = 7$, we observe that W_7 is perfect. This shows that a lattice with $s = 28$ having an odd minimum is similar to W_7, hence isometric to $\sqrt{a}W_7$ for some integer $a \ge 1$. But such a lattice is not primitive unless $a = 1$.

Now, we prove that the upper bound 27 is attained for all $m \ge 5$. To this end, we consider a Voronoĭ path $\mathcal{C}$ connecting W_7 to a neighboring perfect lattice. For W_7, we have $s = r$. Hence the Voronoĭ paths having W_7 as an endpoint are in one-to-one correspondence with the complementary sets of *one* pair $\pm x$ of minimal vectors [M, Section 7.5]. Since $\mathrm{Aut}(W_7) \simeq W(\mathbb{E}_7)$ acts transitively on $S(W_7^*) \sim S(\mathbb{E}_7)$, all these Voronoĭ paths are isometric to one of them, and connect scaled copies of $\mathbb{E}_7^*$ and $\mathbb{E}_7$. An explicit one-parameter family of matrices can be found in Jaquet's thesis [J]. Scaled to minimum 6, this family is given by the matrices

$$M(t) = \begin{pmatrix} 6 & t+3 & t+3 & t+3 & t+3 & t+3 & 2t \\ t+3 & 2t+6 & t+3 & t+3 & t+3 & t+3 & 0 \\ t+3 & t+3 & 2t+6 & t+3 & t+3 & t+3 & t+3 \\ t+3 & t+3 & t+3 & 2t+6 & t+3 & t+3 & t+3 \\ t+3 & t+3 & t+3 & t+3 & 2t+6 & t+3 & t+3 \\ t+3 & t+3 & t+3 & t+3 & t+3 & 2t+6 & t+3 \\ 2t & 0 & t+3 & t+3 & t+3 & t+3 & 6 \end{pmatrix},$$

which connect the perfect lattices ${}^3\,\mathbb{E}_7$ and ${}^2W_7$. Set $t = \frac{3p}{m}$ where p and m are integers with $0 < 3p < m$. The entries of $\frac{m}{6}M(t)$ are m (its minimum), $p+m$, $\frac{p+m}{2}$, p, and 0. Choosing for p an odd integer strictly smaller than $\frac{m}{3}$, e.g., $p = 1$, we obtain an integral lattice of minimum m that belongs to $\mathcal{C}$, hence has $s = 27$. Denote by $L_7(m)$ any such lattice.

By orthogonality to one of the 63 directions of minimal vectors in $\mathbb{E}_7$, we obtain 63 hyperplane cross-sections of W_7 with $s = 16$, all having the same configuration of minimal vectors (i.e., the minimal vectors have the same system of components in a suitably chosen basis for W_7). Removing one pair of vectors from $S(W_7)$ preserves the existence of such configurations (indeed, there remains 27 of them). This proves that $L_7(m)$ contains hyperplane sections $L_6(m)$ with $s = 16$.

Since the configuration of $S(L_6(m))$ is that of $S(W_6)$, $L_6(m)$ contains a hyperplane section $L_5(m)$ with $s = 10$, which itself contains a hyperplane section with $s = 6$, etc. This completes the proof of Theorem 1.2. □

It is easy to verify that the lattices $L_7(5)$ and $L_7(7)$ are unique in the class $\mathcal{C}$. However, this does not prove the uniqueness of these lattices *as integral lattices with minimum* 5 *or* 7, for Watson's theorem does not tell us that $\mathcal{C}$ is the only 7-dimensional, well-rounded minimal class with $s = 27$ that contains integral lattices of odd minimum. Nevertheless, we believe that this is actually the case.

[The lattice $L_7(5)$ is 9-modular. This was first constructed by Conway and Sloane in [C-S2], and later interpreted by Bergé and the first author (unpublished) as the fixed point (up to similarity) of an involution on $\mathcal{C}$ that exchanges a lattice and its dual.]

§6. Low-dimensional lattices of index 1

Let Λ be an n-dimensional, well-rounded lattice. This means that Λ contains systems of n independent minimal vectors. These systems generate sublattices of finite index in Λ. The largest possible value for this index is called the *index of* Λ, and is denoted by $\imath(\Lambda) = \imath$. The index solely depends on the minimal class of Λ, and is bounded from above by $\gamma_n^{n/2}$; see §7 below.

In this section, we consider lattices of index 1 and dimension $n \le 6$. In particular, we give an explicit description of all well-rounded minimal classes of index 1 that contain an integral lattice having an odd minimum for dimensions $n \le 5$. This is done by using conditions (C1) and (C2) in the lemma below; a complete minimal classification of 5-dimensional lattices having no hexagonal section with the same minimum is displayed in Appendix 1.

We begin with an easy lemma.

Lemma 6.1. *Let* Λ *be an integral lattice having an odd minimum. Then* Λ *satisfies the following two conditions*:

(C1) *For any* $\mathbb{Z}$*-linear relation* $a_1e_1 + \cdots + a_re_r$ *among minimal vectors* $e_1, \dots, e_r$, *the sum* $a_1 + \cdots + a_r$ *is even.*

(C2) *If* $e_1, \dots, e_r$ *are* independent *minimal vectors and if* Λ *contains* $e = \frac{e_1+\cdots+e_r}{2}$, *then* r *is even and* ≥ 6.

Proof. If $r = 4$ in condition (C2), Λ contains a 4-dimensional centered cubic lattice, which itself contains hexagonal sections. The other assertions result immediately from norm calculations. □

In what follows, we restrict ourselves to minimal classes whose elements satisfy conditions (C1) and (C2).

A complete classification of minimal classes up to dimension 4 (first made by Štogrin) can be found in [M, Sections 9.3 and 9.4]. Minimal classes in dimension $n \le 4$ satisfying conditions (C1) and (C2) are as follows. (We describe the minimal vectors in a basis $(e_1, \dots, e_n)$ of minimal vectors; the notation a_s, b_s, etc. is that of [M].)

$\underline{n = 2}, \underline{s = 2}$ (a_2).
$\underline{n = 3}, \underline{s = 3}$ (a_3); $\underline{s = 4}$: a_3, $e_4 = e_1 + e_2 + e_3$ (a_4).
$\underline{n = 4}, \underline{s = 4}$ (a_4); $\underline{s = 5}$: a_4, $e_5 = e_1 + e_2 + e_3$ (b_5);
$\underline{s = 6}$: b_5, $e_6 = e_1 + e_2 + e_4$ (a_6).

Using this list of classes, we easily verify that all these classes contain integral lattices of minimum m for all $m \ge 3$ odd, and that such a lattice is unique if $n = 3$, $s = 4$ and $m = 3$ or 5, and if $n = 4$, $s = 6$ and $m = 3$; here are the Gram matrices for the three lattices above (for $n = 4$ and $m = 5$, there are three lattices):

$$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 5 & -2 & -2 \\ -2 & 5 & -1 \\ -2 & -1 & 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & 1 \\ -1 & -1 & 1 & 3 \end{pmatrix}.$$

A general 5-dimensional classification is known (see [Bt]). However, the result is very complicated, and the data published is not quite suitable for extracting the list of classes we are interested in. Following the method of [M], we have classified directly the minimal classes that have no hexagonal section with the same minimum. We have done this starting with the classes with $s = 5$, and increasing inductively the kissing number. We give below a detailed description of what concerns the classes satisfying conditions (C1) and (C2), denoted by λ_s or λ'_s.

$\underline{n = 5}$, $\underline{s = 5}$: $S = \{\pm e_1, \dots, \pm e_5\}$ (λ_5).
$\underline{s = 6}$: λ_5, and $e_6 = e_1 + e_2 + e_3$ or $e_1 + e_2 + e_3 + e_4 + e_5$ (λ_6, λ'_6).
$s \geq 7$: $\mathcal{C} \succ \lambda_6$.
$\underline{s = 7}$: λ_6, and $e_7 = e_1 + e_2 + e_4$ or $e_1 + e_4 + e_5$ (λ_7, λ'_7).
$s \geq 8$: $\mathcal{C} \succ \lambda_7$.
$\underline{s = 8}$: λ_7, and $e_8 = e_3 - e_4 + e_5$ or $e_1 + e_2 + e_5$ (λ_8, λ'_8);
$s \geq 8$ and $e_8 = e_1 + e_2 + e_5 \Longrightarrow s = 8$.
$\underline{s = 9}$: λ_8, and $e_9 = e_1 + e_3 + e_5$ (λ_9).
$\underline{s = 10}$: λ_9, and $e_{10} = e_2 + e_4 - e_5$ (λ_{10}).

[A more symmetric definition for λ_8, λ_9, λ_{10} could be obtained by extending λ_7 successively with $e'_8 = e_1 + e_3 + e_5$, $e'_9 = e_1 + e_4 + e_5$ and $e'_{10} = e_1 + e_2 + e_3 + e_4 + e_5$.]

Sketch of the proof. We shall not justify the details. Since the index is one, there exist bases of minimal vectors. As in [M], we use the fact that the *characteristic determinants* (the determinants whose entries are components of systems of $r \leq n$ minimal vectors on r basis vectors) are 0 or ± 1, which implies that

(1) all components are 0 or ± 1;
(2) there do not exist systems of two vectors with two components $(1, 1)$ and $(1, -1)$;
(3) there do not exist systems of three vectors with three components $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$.

We also prove that if $s \geq 7$ (respectively, $s \geq 8$), we can find a base change that puts in evidence a hyperplane section with $s \geq 5$ (respectively, $s \geq 6$), so that we can use the results we know for dimension 4. □

[The class λ'_8 deserves a special remark: it belongs to a series with $s = 2(n-1)$, defined for all $n \geq 3$, with minimal vectors $e_1, \dots, e_n$ (which constitute a basis) and $e_1 + e_2 + e_i$, $i = 3, \dots, n$. One may take $e_i \cdot e_i = m$, $e_1 \cdot e_2 = -1$, $e_1 \cdot e_i = e_2 \cdot e_i = -\frac{m-1}{2}$ for $3 \leq i \leq n$, and $e_i \cdot e_j = \frac{m-1}{2}$ for $3 \leq i < j \leq n$; the decomposition

$$N(x) = \frac{m-1}{2}(x_1 + x_2 - x_3 - \cdots - x_n)^2 + \frac{m+1}{2}(x_1 - x_2)^2 + \frac{m+1}{2}(x_3^2 + \cdots + x_n^2)$$

of the norm form easily shows the existence of lattices that have the above properties. Moreover, such a lattice is unique if $m = 3$, as one easily sees by using the uniqueness of hyperplane sections with $s = 6$, and indeed coincides with the lattice associated in Table 2.5 with the root lattice $\mathbb{D}_{n-1} \perp \mathbb{A}_1$.]

Theorem 6.2. *All classes λ_s, λ'_s contain integral lattices of any odd minimum $m \geq 3$, except λ_9 if $m = 3$. Lattices of minimum 3 in each of the classes λ'_7, λ_8, λ'_8, λ_{10} are* *unique, whereas* λ_7 *contains three lattices. Similarly,* λ_{10} *contains a unique lattice of minimum* 5.

Proof. The construction of lattices having an odd minimum m has been done using the techniques given above in the case of λ_8'. Examples show up in all classes except in λ_9. We shall consider in detail only lattices with $m = 3$ and $s \geq 7$, the only cases we shall need later. (Note, however, that the existence of a lattice in λ_{10} with any minimum $m \geq 3$ was proved in §4.)

If $\mathcal{C} \succ \lambda_7$, since $e_1 + e_2 + e_3$ and $e_1 + e_2 + e_4$ are minimal, we have $e_i \cdot e_j = -1$ for $1 \leq i < j \leq 4$, except $e_3 \cdot e_4 = 1$ (for the last condition, note that $N(e_1 + e_2 + e_3 + e_4) = 2 + 2e_3 \cdot e_4 \geq 4$). Using the conditions $e_i \cdot e_5 \in \{0, \pm 1\}$, $N(e_i + e_j + e_5) > 3$, $N(e_3 + e_4 - e_5) > 3$, and the invariance of $\mathcal{C}_7$ under the elements $(1,2)$, $(3,4)$ of S_5, we quickly find the three isometry classes of lattices belonging to $\mathcal{C}_7$.

For a lattice Λ in a class containing λ_8, by the condition $N(e_3 - e_4 + e_5) = 3$, we have $e_4 \cdot e_5 = -e_3 \cdot e_5 = 1$. Using the lower bounds $N(e_1 + e_2 + e_3 + e_5) \geq 4$ and $N(e_1 + e_2 + e_4 - e_5) \geq 4$, we obtain $e_1 \cdot e_5 + e_2 \cdot e_5 = 0$. Exchanging e_1 and e_2, we may, moreover, assume that $e_1 \cdot e_5 \leq e_2 \cdot e_5$, and we are left with the two possibilities: $e_1 \cdot e_5 = e_2 \cdot e_5 = 0$, and then $\Lambda \in \lambda_8$, and $e_1 \cdot e_5 = -1$, $e_2 \cdot e_5 = +1$, and then $\Lambda \in \lambda_{10}$.

The easier cases of λ_7 and λ_8 are dealt with in the same way. □

Now, we prove partial classification results for 6-dimensional lattices *of index* 1, which will show that the large values of s exist only for lattices having a larger index, a situation to be treated in the next section.

Lemma 6.3. *If* $s(\Lambda) \geq 7$ (*respectively,* $s(\Lambda) \geq 9$), *then* Λ *has a hyperplane section* Λ_0 *with* $\min \Lambda_0 = \min \Lambda$ *and* $s(\Lambda_0) \geq 6$ (*respectively,* $s(\Lambda_0) \geq 7$).

Proof. Since $\imath(\Lambda) = 1$, any system $e_1, \ldots, e_6$ of six independent minimal vectors of Λ constitutes a basis for Λ. Since the other minimal vectors have three or five nonzero components on $e_1, \ldots, e_6$, it is clear that if $s \geq 7$, we can find five basis vectors that generate a sublattice with $s = 7$.

Suppose now that $s \geq 9$. If two minimal vectors have five components, say, $e = e_1 + \cdots + e_4 + e_5$ and $e' = e_1 + \cdots + e_4 + e_6$, then $e' = e - e_5 + e_6$ and we have a vector with three components on the basis $(e, e_2, \ldots, e_6)$. A similar argument shows that if three minimal vectors have five components, we can get rid of two of them. So we assume that $S(\Lambda)$ contains two vectors e_7 and e_8 with three components. If they have a common component, we are done. Otherwise, we may assume that $e_7 = e_1 + e_2 + e_3$ and $e_8 = e_4 + e_5 + e_6$. Then an extra vector with three (respectively, five) components shares two (respectively, three) components with e_7 or e_8, which completes the proof of the lemma. □

Remark 6.4. By a detailed analysis of classes having a large kissing number, we can show that, when $s \geq 11$, Λ contains a hyperplane section Λ_0 with $\min \Lambda_0 = \min \Lambda$ and $s(\Lambda_0) \geq 8$.

By the lemma above, for the study of the kissing number of integral 6-dimensional lattices with an odd minimum, we can restrict ourselves to lattices having a hyperplane section of type λ_7, λ_7', λ_8, λ_8', λ_9, or λ_{10}. Using PARI-GP, and some more programs due to Batut, we have made a computer search for lattices of minimum 3 containing as a hyperplane section one of the seven lattices quoted in Theorem 6.2, and we have tested for isometry the lattices we found. We state the result as a theorem.

Theorem 6.5. *There are two* (*respectively, five, respectively, ten*) *well-rounded* 6*-dimensional lattices with* $s = 11$ (*respectively,* $s = 10$, *respectively,* $s = 9$).

[Remarks. (1) The two lattices with $s = 11$ belong to the same minimal class; both contain a hyperplane section isometric to W_5. (2) One of the five lattices with $s = 10$ is the strongly eutactic 5-modular lattice $\bigwedge^2 \mathbb{A}_4$.]

Using Remark 6.4, we can show that for a 6-dimensional lattice of index 1 with an odd minimum, we have $s \le 12$ (and also we can give a hand-computational proof for the bound $s \le 11$ when $m = 3$). We shall not give the nevertheless complicated details of the proof, and content ourselves with the following statement, which is easily verified by using a computer.

Theorem 6.6. *On the set of integral, well-rounded* 6*-dimensional lattices of index* 1 *and minimum* $m = 3$ (*respectively,* $m = 5$)*, the values taken by* s *constitute the interval* $[6, 11]$ (*respectively,* $[6, 12]$).

[Note that, since s runs through the entire interval $[5, 10]$ for $n = 5$ and all $m \ge 5$ odd, the set of values of s certainly contains the interval $[6, 11]$ for $n = 6$ and all $m \ge 5$ odd. Probably, the set of values taken by s on lattices of index 1 is the interval $[6, 12]$ for all $m \ge 5$ odd.]

§7. The index of a sublattice

In the preceding section, we proved upper bounds for the kissing number of well-rounded 6-dimensional lattices of index 1. Now, we consider lattices having a larger index, and first study upper bounds of the index under various conditions for low-dimensional lattices. We shall give detailed proofs for $n \le 7$, leaving some verifications for $n = 8$ to the reader. Then we prove some precise results for $n = 6$, $\imath = 2$ or 3, and $n = 7$, $\imath = 3$.

In the following table, the first two lines give the exact upper bound for the index, first among all lattices, then among those not similar to a root lattice. The third line (NHS) gives this bound on the set of lattices having *no hexagonal section* with the same minimum, and the last one (OddMin), on the set of lattices that are integral when scaled to some *odd minimum.*

Theorem 7.1. *Let* Λ *be a well-rounded lattice of dimension* $n \le 8$*. Then for the index of* Λ *we have the following sharp upper bounds*:

n	≤ 3	4	5	6	7	8
$\imath_{\max}$	1	2	2	4	8	16
$\imath'_{\max}$	1	1	2	3	4	8
NHS	1	1	2	3	3	6
OddMin	1	1	1	3	3	5

Proof. The proof heavily depends on results of [M1], in particular Table 11.1.

• *Calculation of* $\imath_{\max}$ *and* $\imath'_{\max}$. It was proved in [M1] that the easy bound $\imath_{\max} \le \gamma_n^{n/2}$ is attained on suitably chosen root lattices, and that the better bound $\imath \le \imath'_{\max}$ occurs except for the case where Λ is similar to one of the lattices $\mathbb{D}_4$, $\mathbb{D}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$. (These results were indeed proved by Watson in [W1], except for the slightly weaker result $\imath'_{\max} \le 9$ if $n = 8$.)

From now on we only consider lattices satisfying condition (NHS).

• *Dimension* $n \le 5$. There is nothing to prove for $n \le 4$. If $n = 5$, consideration of the centred cubic lattice shows that index 2 may occur, and condition (C1) shows that integral lattices having an odd minimum have index 1.

• *Dimension* $n = 6$. It suffices to exhibit an integral lattice of index 3 having an odd minimum. A more precise result will be proved below (Proposition 7.6).

• 2-*elementary quotients*. Suppose that Λ contains a sublattice L generated by vectors $e_1, \dots, e_n \in S$ and such that Λ/L is 2-elementary of dimension $r \geq 2$ over $\mathbb{F}_2$. Then Table 11.1 of [M1] shows that $n \geq 8$, and that if $n = 8$, then $r = 2$ (and condition (C2) is not fulfilled). Hence, for such a lattice, the index is smaller than the upper bound given in Theorem 2.1 under condition (OddMin). Moreover, a look at the three types listed in [M1] shows that extensions to index 8 with a quotient of type $(4, 2)$ and no hexagonal sections are impossible. Since cyclic quotients of order 8 or 7 do not exist, we have proved the bound $\imath \leq 6$ for 8-dimensional lattices.

• *Dimension* $n = 7$. Table 11.1 of [M1] shows that lattices of index 4 and cyclic type satisfying condition (NHS) are generated by minimal vectors $e_1, \dots, e_7$ together with $e = \frac{e_1+e_2+e_3+e_4+e_5+2e_6+2e_7}{4}$. The example given there, for which $s = 7$, obviously satisfies condition (NHS), but condition (C2) is not fulfilled, since $2e \equiv \frac{e_1+e_2+e_3+e_4+e_5}{2}$ mod 4Λ. The existence of integral lattices of index 3 having an odd minimum follows from Theorem 1.2.

• *Dimension* $n = 8$. The bound $\imath \leq 6$ has been proved. Table 11.1 of [M1] lists six types of index 6. A close look at the conditions given there shows that only one type (denoted by $(1, 1, 1, 2, 2, 2, 3, 3)$ in [M1]) may occur under condition (NHS), but the existence of components $1, 1, 1, 3, 3$ shows that this type cannot be realized by an integral lattice of odd minimum; we leave the details to the reader, as well as the proof that an example of index 5 satisfying (OddMin) can be constructed by using the type $(1, 1, 1, 1, 1, 2, 2, 2)$. □

To study lattices of index 2 or 3, we first recall an identity of Watson.

Lemma 7.2 (Watson). *Let $e_1, \dots, e_r \in E$, let $a_1, \dots, a_r \in \mathbb{R}$, let $d > 0$, and let $e = \frac{a_1e_1+\cdots+a_re_r}{d}$. Denote by $\operatorname{sgn}(x)$ the sign of the real number x (0 if $x = 0$). Then*

$$\left(\left(\sum_{i=1}^{n}|a_i|\right) - 2d\right)N(e) = \sum_{i=1}^{n}|a_i|\big(N(e - \operatorname{sgn}(a_i)e_i) - N(e_i)\big).$$

From this identity, we easily deduce (see [M1, Section 2] for the details) the following statement.

Proposition 7.3 (Watson). *Let Λ be a lattice, let $e_1, \dots, e_r$ be minimal vectors in Λ, and let $a_1, \dots, a_r$ and $d \geq 2$ be integers such that $e = \frac{a_1e_1+\cdots+a_re_r}{d}$ belongs to Λ. Then we have*

$$\sum_{i=1}^{n}|a_i| \geq 2d,$$

and equality occurs if and only if $e - \operatorname{sgn}(a_i)e_i$ is minimal for every index i with $a_i \neq 0$. Moreover, if equality occurs and if $a_i = \frac{d}{2}$ for some i, then e is also minimal.

In practice, if L is a sublattice of Λ having a basis $(e_1, \dots, e_n)$ of minimal vectors and if Λ/L is cyclic of order d, we may write $\Lambda = \langle L, e\rangle$ with $e = \frac{a_1e_1+\cdots+a_re_r}{d}$, where r and the a_i are integers such that $r \leq n$ and $1 \leq a_1 \leq \cdots \leq a_r \leq \frac{d}{2}$.

Remark 7.4. If $f = \frac{b_1e_1+\cdots+b_re_r}{d}$ is minimal, then for any index i the lattice generated by f and the e_j with $j \neq i$ has index $|b_i|$ in Λ. Hence the b_i are bounded by the index bound for dimension r.

Example 7.5. If $n = 6$ and Λ has index 3, then, over a suitably chosen sublattice L having a basis of minimal vectors, Λ is generated by the vector $e = \frac{e_1+e_2+e_3+e_4+e_5+e_6}{3}$. The vectors in $e + L$ are of the form $f = \frac{b_1e_1+\cdots+b_6e_6}{3}$ with $b_i \equiv 1 \mod 3$. If such a vector is minimal, then the b_i are equal to 1 or -2. Hence the minimal vectors in $\Lambda\smallsetminus L$ (up to sign) are of the form $e - e_{i_1} - \cdots - e_{i_k}$.

Proposition 7.6. *Well-rounded, 6-dimensional lattices of index 3 satisfying condition* NHS *constitute a single minimal class, whose elements have $s = 12$. This class contains integral lattices having an odd minimum m for all $m \geq 7$ (unique if $m = 7$), but none for $m \leq 5$.*

Proof. As above, write $\Lambda = \langle L, e\rangle$. We know that the vectors e_i and $e'_i = e - e_i$ are minimal, and we must show that under condition (NHS) there are no other minimal vectors in Λ.

First, if $f = a_1e_1 + \cdots + a_ke_k$, say, belongs to $S(L)$, with nonzero a_i, a sublattice of index $3a_i$ in Λ shows up. Thus, we have $a_i \in \{\pm 1\}$, and $k \geq 3$. But if, say, $f = e_1 + e_2 - e_3 \in S(L)$, then we can write $e = e_3 + \frac{f - e_3 + e_4 + e_5 + e_6}{3}$, in contradiction with Proposition 7.3.

Next, suppose that $f = e - e_1 - \cdots - e_k$, say, is an extra minimal vector. Then $k \geq 3$ (because $e - e_1 \in S$), and we have $\frac{f + e_{k+1} + \cdots + e_6}{2} \in \Lambda$, which implies $7 - k \geq 5$, i.e., $k \leq 2$, again a contradiction. This proves that $s = 12$, and that the minimal class of Λ is well defined.

Lemma 7.2 applied to the minimal vector $e - e_i$ yields

$$2\big(N(e) - m\big) + \sum_{j \neq i} \big(N(e - e_i - e_j) - m\big) = N(e - e_i).$$

Since the six terms on the left-hand side are not zero, this identity implies $m \geq 7$ (and even $m \geq 14$ if m is even). If $m = 7$, we must have $N(e) = 8$, $e \cdot e_i = \frac{1}{2}N(e) = 4$, and $e_i \cdot e_j = 1$ for $j \neq i$. These conditions determine a unique lattice Λ of minimum $m = 7$. Thus, the lower bound $m \geq 7$ is optimal. Examples for odd $m \geq 7$ are obtained by taking for instance $e_i \cdot e_j = \frac{m-5}{2}$ if $i + j = 7$ and $e_i \cdot e_j = 1$ otherwise. □

An n-dimensional, well-rounded lattice Λ of index 3 is of the form $\Lambda = L \cup \pm(e + L)$ where L has a basis $(e_1, \dots, e_n)$ of minimal vectors and $e = \frac{e_1 + \cdots + e_r}{3}$ for some integer r with $6 \leq r \leq n$. For $n = 7$, we have $r = 6$ or 7. If Λ is integral with an odd minimum m, we have seen that the first case may occur if and only if $m \geq 7$. Now, we consider the case where $n = r = 7$. We are going to prove a characterization of integral lattices of index 3 and minimum 3. However, before stating the result, we write down some identities valid for any lattice as above with $r = n$. The first two are Watson's identity (7.2) relative to e and $e - e_i$:

$$(*) \qquad \sum_{i=1}^{n} \big(N(e - e_i) - m\big) = N(e);$$

$$(**) \qquad 2\big(N(e) - m\big) + \sum_{j \neq i} \big(N(e - e_i - e_j) - m\big) = (n - 5)N(e - e_i) \quad \text{for all } i.$$

Taking the sum over i in $(**)$ and using $(*)$, we obtain the new identity

$$(**') \qquad \sum_{1 \leq i < j \leq n} \big(N(e - e_i - e_j) - m\big) = \frac{n(n-3)}{2}m - \frac{n+5}{2}N(e),$$

which yields the upper bound $N(e) \leq \frac{n(n-3)}{n+5}m$ for the norm of e, where equality occurs if and only if the $\frac{n(n-1)}{2}$ vectors $e - e_i - e_j$ are minimal.

Lemma 7.7. *If Λ is integral and if both n and m are odd, then $N(e) \geq n$.*

Proof. Since $3e = \sum_i e_i$, we have $N(e) \equiv nm \equiv 1 \mod 2$. This implies that $N(e - e_i)$ is even for all i, whence $N(e - e_i) \geq m + 1$ and $N(e) \geq n$ by $(*)$. $\square$

Theorem 7.8. *An integral* 7*-dimensional lattice of minimum* 3 *and index* 3 *is isometric to* $W_7 \simeq \sqrt{2}\,\mathbb{E}_7^*$.

Proof. Since $m = 3 < 7$, Λ is of the above form. We have $N(e) \leq \frac{n(n-3)}{n+5} m = 7$ and $N(e) \geq n = 7$ by the previous lemma, hence $N(e) = 7$; then $N(e - e_i) = 4$ for all i by $(*)$, which implies $e \cdot e_i = 3$, whence $N(e - e_i - e_j) = 1 + 2e_i \cdot e_j$, and finally $e_i \cdot e_j = 1$, because $N(e - e_i - e_j) = 3$ by $(**')$. This shows that the Gram matrix of the basis $(e_1, \dots, e_6, e)$ of Λ is uniquely determined, hence, up to isometry, there exists at most one possible lattice, which indeed exists, since we know W_7. $\square$

We display the Gram matrices first for $(e_1, \dots, e_7)$, and then for the basis $(e - e_6 - e_7, e_2, \dots, e_7)$ of Λ:

$$A_{28} = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}; \quad C_{28} = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 3 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}.$$

Lattices of minimum 5 and index 3 also are of the above form, and the upper bound that follows from $(**')$ shows that $N(e)$ takes one of the values 7, 9, or 11. The corresponding lattices have not been classified.

Now, we come back to dimension 6 and consider the problem of classifying integral lattices of index 2 and minimum 3.

Theorem 7.9. *Let* Λ *be an integral* 6*-dimensional lattice of minimum* 3 *and index* 2*. Then* Λ *is isometric to one of the three lattices* W_6 (*the Watson lattice*), W_6', W_6'', *with* $s = 16,\ 12,\ 10$, *respectively.*

Proof. Since index 2 does not occur in lower dimensions, Λ is of the form $\Lambda = L \cup (e + L)$, where L possesses a basis $(e_1, \dots, e_6)$ of minimal vectors and

$$e = \frac{e_1 + e_2 + e_3 + e_4 + e_5 + e_6}{2}.$$

Since we may change any signs in e (because, say, $e - e_1 = \frac{-e_1 + e_2 + \cdots + e_6}{2}$), we may assume that e is the shortest of the vectors $\frac{e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6}{2}$.

In our setting, Watson's identity now reads

$$(\dagger) \qquad \sum_i \big(N(e - e_i) - m\big) = 2N(e).$$

Using $(\dagger)$ and the inequality $N(e - e_i) \geq N(e)$, we obtain $6\big(N(e) - 3\big) \leq 2N(e)$, i.e., $N(e) \leq \frac{9}{2}$, hence $N(e) = 3$ or 4. If $N(e) = 4$, then $N(e - e_i) \equiv 1 \mod 2$, thus $N(e - e_i) \geq 5$, and $(\dagger)$ implies $2N(e) \geq 12$, a contradiction. Hence $N(e) = 3$, and, again by $(\dagger)$, for all i we have $N(e - e_i) = 4$, i.e., $e \cdot e_i = 1$, which also reads

$$\sum_{j \neq i} e_i \cdot e_j = -1 \quad \text{for all } i.$$

Since $|e_i \cdot e_j| \leq 1$, the numbers of indices $j \neq i$ such that $e_i \cdot e_j = -1, 0, 1$ are equal to $(3, 0, 2)$, $(2, 2, 1)$, or $(1, 4, 0)$. Denoting by t_1, t_2, t_3 the respective numbers of the systems above, we have $s(\Lambda) = 7 + \frac{1}{2}(3t_1 + 2t_2 + t_1)$. In this way, we recover the bound $s \le 16$, and the fact that this is attained if and only if $t_3 = 6$.

Also, we have $N(e - e_i - e_j) = 5 + 2e_i \cdot e_j$, so that $e - e_i - e_j$ is minimal if and only if $e_i \cdot e_j = -1$, and the condition $N(e - e_i - e_j - e_k) \ge 4$ (which is fulfilled because this vector has an even norm) amounts to $e_i \cdot e_j + e_i \cdot e_k + e_j \cdot e_k \ge -1$.

Suppose first that a system $(3, 0, 2)$ exists for some i. Then we may assume that $e_1 \cdot e_2 = e_1 \cdot e_3 = e_1 \cdot e_4 = -1$ and $e_1 \cdot e_5 = e_1 \cdot e_6 = +1$. This implies successively $e_2 \cdot e_3 = e_2 \cdot e_4 = +1$, then $e_2 \cdot e_5 = e_2 \cdot e_6 = -1$, and similarly $e_3 \cdot e_4 = +1$ and $e_3 \cdot e_5 = e_3 \cdot e_6 = -1$, and finally $e_4 \cdot e_5 = e_4 \cdot e_6 = -1$ and $e_5 \cdot e_6 = +1$. This proves that, for all i, we have a system $(3, 0, 2)$, that the corresponding lattice is unique up to isometry (hence isometric to W_6), and that we have $s < 16$ otherwise.

Suppose now that no system $(3, 0, 2)$ exists, but that there is a system $(2, 2, 0)$. We may then assume that $e_1 \cdot e_2 = e_1 \cdot e_3 = -1$, $e_1 \cdot e_4 = e_1 \cdot e_5 = 0$, and $e_1 \cdot e_6 = +1$. Arguments like those we used in the first case show that we have a unique choice for the other products $e_i \cdot e_j$, and that for $i = 1, 2, 3, 6$ (respectively, $i = 4, 5$), we have systems of type $(2, 0, 1)$ (respectively, $(1, 4, 0)$). This time, we have $s = 12$, again attained on a unique lattice (up to isometry), which we denote by W'_6, but we have $s = 10$ if all systems are of type $(1, 4, 0)$. The case where $s = 10$ can be dealt with by similar arguments that we leave to the reader. □

Combining Theorems 6.5 and 7.9, we indeed obtain the following theorem, which proves the results stated in the Introduction for dimension 6.

Theorem 7.10. *The set of values taken by* s *on* 6*-dimensional, well-rounded lattices of minimum* 3 *is* $[6, 12] \cup \{16\}$.

A computer search easily produces lattices of minimum 5 and index 2 with s running through the interval $[12, 16]$. This shows the following.

Theorem 7.11. *The set of values taken by* s *on* 6*-dimensional, well-rounded lattices of minimum* 5 *is the interval* $[6, 16]$.

Probably, the same result is valid for all $m \ge 7$ odd.

Remark 7.12. We have found only one lattice of minimum 5 for each of the values $13, 14, 15, 16$ of s. Probably, these four lattices are unique. In contrast, we have found many examples with a lower value of s, including one with index 2 and $s = 6$.

A lattice Λ of dimension 7 and index $\imath = 2$ having an odd minimum extends a 6-dimensional lattice with the same property. This proves that when its minimum is 3, if $\imath \ge 2$, then Λ has a hyperplane section isometric to one of the three lattices W_6, W'_6, or W''_6 (if $\imath = 3$, then $\Lambda \simeq W_7 \supset W_6$). However, new possibilities exist in dimension 8. We give a partial result below.

Proposition 7.13. *Let* Λ *be an* 8*-dimensional lattice of minimum* 3 *and index* $\imath \ge 3$. *Then either* Λ *has a section isometric to* W_6, W'_6 *or* W''_6, *or* $s(\Lambda) \le 15$.

Proof. For the proof, we assume that Λ has no 6-dimensional section of minimum 3 and isometric to W_6, W'_6, or W''_6. Hence all sublattices of Λ of minimum 3 and dimension at most 7 have index 1.

(a) First, we prove that the index 4 is impossible. Indeed, otherwise we could write $\Lambda = \langle e_1, \dots, e_8, e \rangle$ with minimal e_i and $e = \frac{e_1 + \cdots + e_8}{4}$. By Watson's identity, the vectors $e - e_i$ are minimal. Applied to $e - e_i$, this identity takes the form $\big(N(e) - 3\big) + \sum_{j \ne 1} \big(N(e - e_1 - e_j) - 3\big) = 2N(e - e_1) = 6$, a contradiction, because the left-hand side is made of eight odd terms.

(b) Next, we observe that in the case of index 3 or 5, writing Λ as above, for the lattice L generated by the e_i we have $s(L) = 8$. For instance, if $e = \frac{e_1+\cdots+e_8}{3}$ and if, say, $e' = e_1 + e_2 - e_3$ were minimal, we could write $e - e' = \frac{e'-e_3+e_4+\cdots+e_8}{3}$, and an index 3 would have shown up in dimension 7; the same kind of argument works in all cases of index 3 or 5.

(c) Now we prove that $s \le 8$ if $\imath = 5$. Write $e = \frac{e_1+\cdots+e_r+2e_{r+1}+\cdots+2e_8}{5}$. We have $\Lambda = L \cup \pm(e+L) \cup \pm(2e+L)$, and replacing e by $2e$ amounts to interchanging r and $8-r$; we also have $2 \le r \le 6$. The minimal vectors of $e+L$ are of the form $e-e_i-e_j-\cdots$. If $i \le r$, an index 4 occurs by (a); by Watson's identity, this always occurs if $r = 6$. We may thus assume that $3 \le r \le 5$. If some $e - e_i - e_j - \cdots$ is minimal, then we can construct a lattice of dimension ≤ 7 and index 3. If the minimal vectors are among e or the $e - e_i$, then there are at least two coefficients 2 in the denominator of e, and we may use them this time to construct a lattice of index 2 and dimension 7. Hence we have $S(\Lambda) \subset S(L)$, whence $s(\Lambda) \le s(L) = 8$.

(d) Finally, let $e = \frac{e_1+\cdots+e_8}{3}$. Then e has even norm. Hence the minimal vectors of $\Lambda \smallsetminus L$ are of the form $e - e_i - e_j - \cdots$ with $1, 3, \ldots$ indices $i, j, \ldots$. If, say, some vector $e' = e - e_1 - e_2 - e_3 - \cdots$ is minimal, then $e' + e_1 + e_2 + e_3 = \frac{e'+e_4+\cdots+e_8}{2}$, and an index 2 shows up in dimension 6. This shows that the minimal vectors in $\Lambda \smallsetminus L$ lie among the $e - e_i$. Since $\sum_i (N(e-e_i) - 3) = 2N(e) > 0$, there are at most 7 such vectors, whence the (actually, not optimal) bound $s \le 15$. □

Remark 7.14. The previous proposition shows that an 8-dimensional lattice Λ of index $\imath \ge 2$ with no section W_6, W_6', W_6'' and $s \ge 16$, if any, may be written in the form $\Lambda = \langle e_1, \ldots, e_8, e\rangle$ with e_i and $e = \frac{e_1+\cdots+e_8}{2}$ minimal. Then its other minimal vectors, up to sign, are of one of the forms $e - e_i - e_j$ or $e - e_i - e_j - e_k - e_\ell$.

§8. The minimum of the dual lattice

In the remainder of the paper, we only consider integral lattices Λ of minimum 3 (unless otherwise explicitly stated). To construct cross-sections with the same minimum and comparatively large values of s, we shall need to bound the minimum of the dual lattice Λ^* of Λ. Some "critical" values (especially $\frac{2}{3}$, 1, $\frac{4}{3}$) will show up.

Recall that the Hermite invariant of a lattice L is $\gamma(L) = \frac{\min L}{\det(L)}$ and that the Hermite constant for dimension n is $\gamma_n = \sup_{\dim L = n} \gamma(L)$. What we really need is their dual versions (the geometric means of their values for a lattice and its dual) $\gamma'(L)$ and γ_n': indeed, we have

$$\gamma'(L)^2 = \gamma(L)\gamma(L^*) = \min L \, \min L^*,$$

hence

$$(*) \qquad \min \Lambda^* = \frac{\gamma'(\Lambda)^2}{3} \le \frac{{\gamma_n'}^2}{3}.$$

However, our knowledge on γ_n' is very poor, and most of the time, we shall have to content ourselves with the trivial bound $\gamma_n' \le \gamma_n$. The values of γ_n and the corresponding critical lattices are known up to $n = 8$. Beyond dimension 8, upper bounds for γ_n, coming from estimations of Rogers, can be obtained from Table 1.2 of [C-S]; in [C-S], the *center density* δ is used; we have $\gamma_n = 4\delta_n^{2/n}$. (Recently, Roger's bounds were improved by Cohn and Elkies in [C-E]; the new bounds can be used to shorten the proof of Theorem 8.2.)

We are going to prove three theorems for $n \le 7$, $n = 8$, and $n = 9$ respectively. Recall that Λ stands for an integral lattice of minimum 3.

Theorem 8.1. *For $n \le 7$, we have $\min \Lambda^* \le 1$, and equality occurs if and only if $n = 7$ and $\Lambda \simeq W_7$.*

Proof. For $n \leq 6$, the result follows from $(*)$, by using the crude estimate $\gamma_n < \sqrt{3}$. Now let $n = 7$. Set $d = \det(\Lambda)$. The even part of Λ has determinant $4d$ and minimum $m \geq 4$. We have $\gamma(\Lambda_{\text{even}}) \geq \frac{4}{(4d)^{1/7}}$ and $\gamma(\Lambda_{\text{even}}) \leq \gamma_7 = \gamma(\mathbb{E}_7) = 2^{6/7}$, hence $d \geq 2^6$. In the other direction, we have $\det(\Lambda^*) = \frac{1}{d}$. Thus, if $\min \Lambda^* \geq 1$, then $\gamma(\Lambda^*) \geq d^{1/7}$, and $\gamma(\Lambda^*) \leq \gamma_7 = 2^{6/7}$ implies $d \leq 2^6$. Hence if $\min \Lambda^* \geq 1$, then $\min \Lambda^* = 1$, $\gamma(\Lambda^*) = \gamma(\mathbb{E}_7)$, and since $\mathbb{E}_7$ is a unique (up to scaling) critical 7-dimensional lattice, Λ^* is isometric to $\frac{1}{\sqrt{2}}\,\mathbb{E}_7$, and Λ to $\sqrt{2}\,\mathbb{E}_7^* \simeq W_7$. Conversely, W_7^* obviously has minimum 1. □

The same kind of methods can be used to handle the case of dimension 9, but the result we obtain is probably far from being optimal.

Theorem 8.2. *For* $n = 9$, *we have* $\min \Lambda^* < \frac{4}{3}$.

Proof. We use Roger's bound $\gamma_9 < 2.1411672$ (Table 1.2 of [C-S] gives $\delta_9 \leq 0.06007$).

Let $d = \det(\Lambda)$. As above, we have $\det(\Lambda_{\text{even}}) = 4d$ and $\min \Lambda_{\text{even}} \geq 4$, which implies $\gamma(\Lambda_{\text{even}}) \geq \frac{4}{(4d)^{1/9}}$, hence $d \geq 4^8 \gamma_9^{-9} > 69.28\ldots$; similarly, if $\min \Lambda^* \geq \frac{4}{3}$, we have $\gamma(\Lambda^*) \geq \frac{4}{3} \cdot d^{1/9}$, hence $d \leq (\frac{3}{4})^9 \gamma_9 < 71.02\ldots$. Thus, we have just proved that d must be equal to 70 or 71. Since these integers are square free, d is the annihilator of Λ^*/Λ, which shows that $\min \Lambda^* = \frac{p}{q}$ with $q = d$ and $(p, q) = 1$. We then have $p \leq \lfloor \gamma_9 d^{1/9} q \rfloor$, i.e., $p \leq 93$ if $d = 70$ and $p \leq 94$ if $d = 71$. But $\min \Lambda^* = \frac{p}{q} \geq \frac{4}{3}$ now reads $p \geq 93.33\ldots$ if $d = 70$ and $p \geq 94.66\ldots$ if $d = 71$, which contradicts the bounds above. □

To handle the case of dimension 8, we shall use different techniques, namely the possibility of embedding any integral lattice into a unimodular one.

Theorem 8.3 (Conway and Sloane)**.** *An* n*-dimensional integral lattice can be embedded in an* (*odd*) *unimodular lattice of dimension* $n' \leq n + 3$.

Proof. This is Corollary 8 of [C-S1]. □

First, we prove a general lemma.

Lemma 8.4. *Let* L *be an odd* n*-dimensional sublattice of a unimodular lattice* M *of the form* $M_0 \perp \mathbb{Z}^k$, *where* M_0 *is even. Then* $\min L^* \leq 1$. *Moreover, if equality occurs, then either* $M_0 \neq 0$ *or* L *can be embedded in* $\mathbb{Z}^n$.

Proof. We may assume that k is minimal. Denote by $(\varepsilon_1, \ldots, \varepsilon_k)$ the canonical basis for $\mathbb{Z}^k$. Since L is odd, k is nonzero. Since k is minimal, the ε_i do not lie in $L^\perp$ (for, otherwise, L could be embedded in $M_0 \perp \mathbb{Z}^{k-1}$). Hence the projections ε_i' of the ε_i to the span of L are nonzero. For all $x \in L$, we have $x \cdot \varepsilon_i' = x \cdot \varepsilon_i \in \mathbb{Z}$. Hence the vectors ε_i' are nonzero vectors in L^*. Thus, we have

$$\min L^* \leq \min_i N(\varepsilon_i') \leq \min_i N(\varepsilon_i) = 1.$$

Moreover, if $\min L^* = 1$, all ε_i' have norm 1, hence coincide with the ε_i. Since the ε_i are independent, we then have $\dim L \geq k$, whence $\dim L = k$ if $M_0 = 0$. □

[Actually, the strict inequality $\min L^* < 1$ is fulfilled for any nonzero sublattice of $\mathbb{Z}^k$ (even or odd) that is not isometric to a sublattice of $\mathbb{Z}^n$.]

Now we return to the case of a lattice Λ with $\min \Lambda = 3$ and $\dim \Lambda = 8$. Again, the results we are going to prove are probably far from being optimal. The lemma below is merely a first step towards the proof of Theorem 8.6:

Lemma 8.5. *For* $n = 8$, *we have* $\min \Lambda^* \leq 1$.

Proof. More generally, we consider an integral lattice Λ of minimum 3 and dimension $n \le 8$. By Conway and Sloane's Theorem 8.3, Λ can be embedded in an (odd) unimodular lattice M of dimension $n' \le n+3$. We still assume that n' is the smallest possible dimension. Since $\dim M \le 11$, M is isomorphic to a lattice $\mathbb{Z}^k$, $k \le 11$, or $\mathbb{E}_8 \perp \mathbb{Z}^k$, $k \le 3$. (This is a theorem of Kneser; see [C-S, Table 16.7].) The lemma now follows immediately from Lemma 8.4. □

With respect to the bounds for s to be proved in the next section, the value 1 for $\min \Lambda^*$ is critical: the results we shall obtain under the assumption $\min \Lambda^* \le 1$ are better when the strict inequality occurs. The following partial result will suffice.

Theorem 8.6. *An* 8*-dimensional lattice* Λ *that is generated by its minimal vectors satisfies the strict inequality* $\min \Lambda^* < 1$.

Proof. First, we prove the theorem in the case where Λ can be embedded in some $\mathbb{Z}^k$. By Lemma 8.4, it suffices to consider the case where Λ is a sublattice of $\mathbb{Z}^8$. Then we denote by h the index $[\mathbb{Z}^8 : \Lambda]$. Suppose that $\min \Lambda^* = 1$, and put $d = \det(\Lambda)$. We have $\gamma(\Lambda^*) = d^{1/8} \le \gamma_8 = 2$, hence $d \le 2^8$, and in fact $d < 2^8$, because Λ is not similar to $\mathbb{E}_8$. Calculating $\gamma(\Lambda_{\text{even}})$, we obtain the bound $4(4d)^{-1/8} < 2$, i.e., $d > 2^6$. Hence we have $8 < h < 16$.

Let $(\varepsilon_1, \dots, \varepsilon_8)$ be the canonical basis for $\mathbb{Z}^8$, set $H = \mathbb{Z}^8/\Lambda$ and $h = |H|$, and let $\varphi : \mathbb{Z}^8 \to H$ be the canonical surjection. Then the 16 vectors $\varphi(\pm\varepsilon_i)$ are nonzero and distinct except perhaps for a pair $\pm\varepsilon_i$ (for, otherwise, Λ would contain vectors of norm 1 or 2 (namely, some ε_i or some $\varepsilon_i \pm \varepsilon_j$), and we would have $h > 16$, a contradiction).

In case H contains a pair $\pm\varepsilon_i$, h is even, hence equal to 10, 12, or 14. Since $[\Lambda^* : \mathbb{Z}^8] = [\mathbb{Z}^8 : \Lambda] = h$, in Λ^* there exists a vector of the form $e = \frac{a_1\varepsilon_1 + \cdots + a_8\varepsilon_8}{p}$ for $p = 3$, 5, or 7. Reducing e modulo $\mathbb{Z}^8$, we may assume that $|a_i| \le \frac{p-1}{2}$. Now, we consider the three possible values for p.

$\underline{p=3}$. We have $|a_i| \le 1$, hence $N(e) \le \frac{8}{9} < 1$.

$\underline{p=5}$. Replacing e by $2e$ if need be, we may assume that there are at most $\lfloor \frac{8}{2} \rfloor = 4$ coefficients a_i equal to ± 2. Hence $N(e) \le \frac{4\cdot 2^2 + 4\cdot 1^2}{25} = \frac{4}{5} < 1$.

$\underline{p=7}$. Replacing e by $2e$ or $3e$ if need be, we may assume that there are at most $\lfloor \frac{8}{3} \rfloor = 2$ coefficients a_i equal to ± 3. Hence $N(e) \le \frac{2\cdot 3^2 + 6\cdot 2^2}{49} = \frac{6}{7} < 1$.

This completes the proof of Theorem 8.6 for lattices contained in some $\mathbb{Z}^k$.

We are now left with the three cases where $\Lambda \subset \mathbb{Z}^k \perp \mathbb{E}_8$, $k = 1$, 2, or 3. We still denote by (ε_i) the canonical basis for $\mathbb{Z}^k$. The minimal vectors of Λ are of the form either $\varepsilon_i + \tau$, where τ is a root of $\mathbb{E}_8$, or also $\pm(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3)$ if $k = 3$. We denote by $\mathcal{R}_i$ the set of roots $\tau \in \mathbb{E}_8$ such that $\varepsilon_i + \tau$ is minimal in Λ.

Let L be the orthogonal projection of Λ onto the span of $\mathbb{E}_8$. This is a lattice of dimension $8-k$ and generated by the roots $\tau \in \mathbb{E}_8$ such that $\varepsilon_i + \tau \in S(\Lambda)$ for some index i. We have the inclusion $\Lambda \subset \mathbb{Z}^k \perp L$. If L has a component of type $\mathbb{A}_\ell$, then $\min \Lambda^* < 1$, and there is nothing to prove. Otherwise, L is isometric either to $\mathbb{D}_{8-k} \subset \mathbb{Z}^{8-k}$ (and L is again a sublattice of $\mathbb{Z}^8$ for which we have already proved the bound $\min \Lambda^* < 1$), or to $\mathbb{E}_7$ (respectively, $\mathbb{E}_6$), and then $k = 1$ (respectively, 2).

First consider the case where $\Lambda \subset \mathbb{Z} \perp \mathbb{E}_7$. The set $\mathcal{R}$ must then span $\mathbb{R}\,\mathbb{E}_7$ and be of affine rank 7, hence must be equal to $\widetilde{\boldsymbol{E}}_7$. A Gram matrix for Λ is of the form $J_8 + M$ where M is a Gram matrix for $\widetilde{\boldsymbol{E}}_7$ and J_8 denotes the all ones 8×8 matrix. A direct calculation shows that $\min \Lambda^* = \frac{43}{81} = 0.530\ldots < 1$.

Finally, we consider the more difficult case where $\Lambda \subset \mathbb{Z}^2 \perp \mathbb{E}_6$. We assume that $\min \Lambda^* = 1$.

Let $H = (\mathbb{Z}^2 \perp \mathbb{E}_6)/\Lambda$ and let $h = |H|$. We first prove that H is elementary Abelian of order 9. We have

$$\gamma(\Lambda^*) = \det(\Lambda^*)^{-1/6} = \det(\Lambda)^{1/6} = (3h^2)^{1/6} \le \gamma_8 = 2,$$

whence $h^2 \le \sqrt{\frac{256}{3}}$, i.e., $h \le 9$. For each $a \in H$, denote by $\mathcal{R}_a$ the set of roots of $\mathbb{E}_6$ which map to a. If r_1, r_2 are distinct roots in $\mathcal{R}_a$, then $r_1 - r_2$ is a nonzero element of Λ, which implies $N(r_1 - r_2) \ge 3$, hence $r_1 \cdot r_2 \le 0$. Thus $\mathcal{R}_a$ is an affine root system. If $2a = 0$, since $r_1 + r_2$ also belongs to Λ, we have $r_1 \cdot r_2 = 0$, hence $\mathcal{R}_a$ is of type $k\mathbb{A}_1$ with $k \le 4$ ($\mathbb{E}_6$ has deficiency 2). Thus, we have $|\mathcal{R}_a| \le 8$ in this case. Conversely, if $\mathcal{R}_a$ contains a component $\boldsymbol{A}_1$, then $2a = 0$, because $r_1 + (-r_1) = 0$. Consequently, if $2a \ne 0$, $\mathcal{R}_a$ has no component $\boldsymbol{A}_1$, which implies $|\mathcal{R}_a| \le 9$, with equality if and only if $\mathcal{R}_a$ is of type $3\widetilde{\boldsymbol{A}}_2$, the only largest possible system of rank 6. Since $\mathcal{R}_0 = \varnothing$, we have $\sum_{a\in H}|\mathcal{R}_a| \le 9(h-1) = 72$, and since $\mathbb{E}_6$ contains exactly 72 roots, we must have $h = 9$, and indeed H must be elementary, since $\mathcal{R}_a$ is of type $3\widetilde{\boldsymbol{A}}_2$. So $\boldsymbol{E}_6$ is the union of four systems $\{a, -a\}$ of type $3\boldsymbol{A}_2$ graded by $H/\{1, -1\}$.

Choose a system $3\boldsymbol{A}_2$ as above in some class $a \in H$, and denote by r_i, $1 \le i \le 9$, its elements, ordered in such a way that $r_i + r_{i+1} + r_{i+2} = 0$ for $i = 1, 4, 7$. Let r be any root in $\boldsymbol{E}_6$. Then $r \cdot r_1 + r \cdot r_2 + r \cdot r_3 = 0$, so that if $r \ne r_1, r_2, r_3$, the set $(r \cdot r_1, r \cdot r_2, r \cdot r_3)$ is $(0, 0, 0)$ or a permutation of $(1, 0, -1)$. Clearly, the first case occurs for any $r \in \{r_4, \dots, r_9\}$ and the second one for any $r \ne \pm r_1, \dots, \pm r_9$ (write r as a combination of $r_1, r_2, r_4, r_5, r_7, r_8$ with coefficients $a_i \in \mathbb{Z}$ not all zero, hence, say, $a_1 \ne 0$; then $r \cdot r_1 = 2a_1 - a_2$ must be ± 1). Since $\mathrm{Aut}(\mathbb{E}_6)$ acts transitively on its sublattices of type $\mathbb{A}_2^{\perp 3}$, we may assume (up to isometry), first, that L contains the lattice L_7 with basis $(\varepsilon_1 + r_i)$, $i = 1, 2, 4, 5, 7, 8, 3$, whose minimal vectors are the 9 pairs $\pm(\varepsilon_1 + r_i)$, $i = 1, \dots, 9$, and then that Λ is obtained by extending L_7 with a vector $\varepsilon_2 + r$. Its minimal vectors are the 18 pairs $\pm(\varepsilon_1 + r_i)$, $\pm(\varepsilon_2 + r'_i)$, where $\{r'_i\}$ is a system of type $3\boldsymbol{A}_2$ containing r. Using automorphisms of $\mathbb{E}_6$, we can perform all circular permutations inside the three systems $(-1, 0, 1)$ occurring as scalar products $r \cdot r_i$, and we can realize the transpositions by using convenient sign changes of r, (r_4, r_5, r_6) and (r_7, r_8, r_9). This shows that the lattice Λ is unique up to isometry and that

$$A = \begin{pmatrix} 3 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 3 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 & -1 & 3 \end{pmatrix}$$

is a Gram matrix for Λ. Using this matrix, we have verified that $\min \Lambda^* = \frac{7}{9} < 1$. This completes the proof of Theorem 8.6. □

[Remarks. (1) For a convenient ordering of the r_i, the vector $\frac{-\varepsilon_1 + r_1 + r_4 + r_7}{3}$ belongs to Λ^*; this proves "by hand" the bound $\min \Lambda^* \le \frac{7}{9}$.

(2) The lattice Λ above is a section of codimension 2 of the short Coxeter-Todd lattice. We have verified that it cannot be embedded into a strictly larger integral, 8-dimensional lattice of minimum 3.]

As for dimension 9, the methods used to prove Theorem 8.6 do not apply, because of the existence of the odd unimodular lattice $\mathbb{D}_{12}^+$, which would only yield the weak bound $\min \Lambda^* \le 2$.

§9. Bounds relative to a hyperplane section

We still consider an integral lattice of minimum 3, whose dimension is denoted by n. Hyperplane cross-sections of Λ are in one-to-one correspondence with pairs $\pm e$ of primitive vectors in its dual Λ^* (we associate with e the section of Λ by $e^\perp$). Under various assumptions about the norm of such a vector, we shall prove lower bounds for the kissing number of $e^\perp$, and use these bounds to prove the assertions of the introduction relative to dimensions $n \le 8$. Most of the proofs make use of root systems. However, our first results are based on the following well-known lemma, for which we shall nevertheless give a proof.

Lemma 9.1. *Let V be a Euclidean space, and let $T \subset V$ be a set of vectors with strictly negative mutual scalar products. Then $|T| \le 1 + \dim V$.*

Proof. If the vectors in T are independent, then $|T| \le \dim V$. Otherwise, there exists a nontrivial dependence relation, which can be written in the form

$$\sum_{x \in T_1} \lambda_x x = \sum_{y \in T_2} \lambda_y y,$$

where $T = T_1 \cup T_2$ is a partition of T and $\lambda_x > 0$ (respectively, $\lambda_x \le 0$) if $x \in T_1$ (respectively, $x \in T_2$). Since it is a norm, the scalar product of the two sides of the identity above is nonnegative. Hence $\sum_{x \in T_1, y \in T_2} \lambda_x \lambda_y (x \cdot y) \ge 0$, which implies that all λ_y, $y \in T_2$, are zero, or otherwise stated, that any linear relation on T has nonnegative (or nonpositive) coefficients. It is then easy to see that this implies that such a relation is unique up to proportionality. □

[An example of $n+1$ vectors as above in an n-dimensional lattice is provided by a suitably chosen half-system of minimal vectors in $\mathbb{A}_n^*$.]

We shall apply the lemma to systems of vectors associated with a suitably chosen vector $e \in \Lambda^*$. We say that a nonzero vector $e \in \Lambda^*$ is *reduced* if its norm is minimal in the coset $e + \Lambda$ modulo Λ. This amounts to the condition

$$\forall x \in \Lambda, \quad |e \cdot x| \le \frac{N(x)}{2};$$

Λ^* contains reduced vectors if and only if Λ is not unimodular.

Let $e \in \Lambda^*$ be reduced. Then we have $|e \cdot x| \le 1$ for all $x \in S = S(\Lambda)$. Let

$$S_i = \{x \in S \mid x \cdot e = i\}.$$

Then $S = S_0 \cup S_1 \cup S_{-1}$ is a partition of S (with $S_{-1} = -S_1$), and S_0 is the set of norm 3 vectors in the lattice $\Lambda_0 = \Lambda \cap (\mathbb{R}e)^\perp$, of dimension $\le n-1$. Set $s_1 = |S_1|$.

We first state and prove a lemma concerning norm 1 vectors in Λ^*.

Lemma 9.2. *Suppose that S_1 is nonempty. Then, for a reduced vector $e \in \Lambda^*$, the following conditions are equivalent*:

(1) $2e$ *is a sum of two vectors in* S_1;

(2) $N(e) = 1$ *and* $2e \in \Lambda$.

Proof. Under hypothesis (1), we have $2N(e) = e \cdot x + e \cdot y = 2$, and of course $2e \in \Lambda$. Conversely, under hypothesis (2), given any $x \in S_1$, we have $e \cdot (2e - x) = 1$ and $N(2e - x) = 3$, hence $2e = x + (2e - x)$. This proves that (1) is satisfied (and moreover, that $x \mapsto 2e - x$ is an involution on S_1). □

Lemma 9.3. *Let $e \in S(\Lambda^*)$, set $\alpha = N(e)$ and $\beta = 1/\alpha$, and let x, y be distinct vectors in S_1.*

(1) *If $2e = x + y$, then $\alpha = 1$ and $x \cdot y = -1$.*

(2) *If* $\alpha < \frac{4}{3}$ *and either* $\alpha \neq 1$ *or* $\alpha = 1$ *and* e *is not a sum* $x_1 + x_2$, $x_i \in S_1$, *then* $x \cdot y = 0$ *or* 1.

(3) *If* $\alpha < \frac{2}{3}$, *then* $x \cdot y = 1$.

Proof. If $2e = x + y$, then $\alpha = 1$ by the lemma above, and $N(x) + N(y) + 2x \cdot y = N(x + y) = 4$, hence $x \cdot y = -1$.

Under the hypotheses of (2) or (3), $2e - x - y$ is nonzero. Since e is minimal in Λ^*, this implies $N(2e - x - y) = 4\alpha - 2 + 2x \cdot y \geq \alpha$, i.e.,

$$x \cdot y \geq 1 - \frac{3}{2}\alpha.$$

If $\alpha < \frac{4}{3}$ (respectively, $\alpha < \frac{2}{3}$), we then have $x \cdot y > -1$ (respectively, $x \cdot y > 0$), which completes the proof of the lemma. □

Let $\overline{S}_1$ be the image of S_1 under the translation $x \mapsto \bar{x} = x - \beta e$. Note that $\overline{x}$ is the orthogonal projection of x to $(\mathbb{R}e)^\perp$. In particular, $\overline{S}_1$ is contained in $(\mathbb{R}e)^\perp$.

Lemma 9.4. *Let* $e \in \Lambda^*$ *with* $N(e) < 1$, *and let* $\Lambda_0 = \Lambda \cap (\mathbb{R}e)^\perp$. *Then*

(1) $s_1 \leq n$.

(2) *If* $s_1 = n$, S_1 *generates a sublattice* Λ_1 *of finite index in* Λ.

(3) *If, moreover,* $s > n$, *then the index* $[\Lambda : \Lambda_1]$ *is even, and* Λ *contains a vector of the form* $f = \frac{e_1 + \cdots + e_p - e_{p+1} - \cdots - e_{2p}}{2}$ *for some* p *with* $6 \leq 2p \leq n$.

Proof. (1) For $x, y \in S_1$ we have

$$\bar{x} \cdot \bar{y} = (x - \beta e) \cdot (y - \beta e) = x \cdot y - \beta,$$

which shows that distinct vectors in S_1 have a strictly negative scalar product, hence $s_1 \leq \dim(\mathbb{R}e)^\perp + 1 = n$ by Lemma 9.1.

(2) Let $\sum_i \lambda_i e_i$ be a nontrivial dependence relation among the e_i. Among the projections of the e_i onto $(\mathbb{R}e)^\perp$, we have the relation $\sum_i \lambda_i \bar{e}_i$. The proof of Lemma 9.1 shows that this is unique up to proportionality, and that changing the signs if need be, we may assume that it has nonnegative coefficients. Then $\sum_i \lambda_i = \sum_i \lambda_i (e_i \cdot e) = 0$, which implies that all λ_i are zero, a contradiction.

(3) The lattice $\Lambda_1 \cap (\mathbb{R}e)^\perp$ is generated by the differences $e_i - e_j$, hence is contained in the even part of Λ_0. Since $s > n$, Λ_0 contains a norm 3 vector f_0. Let a be the smallest integer such that $af_0 \in \Lambda_1$. We have $af_0 \in \Lambda_1 \cap (\mathbb{R}e)^\perp \subset \Lambda_{\text{even}}$. Since $f_0 \in \Lambda \smallsetminus \Lambda_{\text{even}}$, a is even. Since $e \cdot e_i = 1$ for all i, $\Lambda \cap (\mathbb{R}e)^\perp$ contains a vector of the form $f = \frac{e_1 + \cdots + e_p - e_{p+1} - \cdots - e_{2p}}{2}$ for some p with $2p \leq n$, and we have $2p \geq 6$ because the integral lattices of dimensions $n \leq 5$ having an odd minimum must have index 1. This completes the proof of the lemma. □

Corollary 9.5. *Let* Λ *be a lattice with* $\min \Lambda = 3$ *and* $\min \Lambda^* < 1$. *Then* Λ *has a hyperplane section* Λ_0 *such that* $s(\Lambda) \leq S(\Lambda_0) + n$.

Proof. Simply apply the above proposition to a vector $e \in \Lambda^*$ with norm $N(e) < 1$. □

Corollary 9.6. *If* $n \leq 8$ *and* Λ *is not isometric to* W_7, *then* Λ *has a hyperplane section* Λ_0 *such that* $s(\Lambda) \leq S(\Lambda_0) + n$.

Proof. Apply the above corollary together with Theorems 8.1 and 8.6. □

By the results of §7, if $n \leq 5$, every system of n independent minimal vectors in Λ generates Λ. Lemma 9.4 then shows that the above corollary is not optimal in these dimensions. It is for $n = 6$, since $s(W_6) = 16$ and $s(W_5) = 10$, and it allows us to recover the bound $s \leq 16$ for $n = 6$ from the bound $s \leq 10$ for $n = 5$. Note that in dimension 6, if Λ is not isometric to W_6, it has a hyperplane section Λ_0 such that $s(\Lambda) - s(\Lambda_0) \leq 5$:

this is clear if $s(\Lambda) \leq 10$, and the three lattices with $11 \leq s < 16$ all have a section W_5, with $s = 10$. The classification results to be proved in the next section will show that in dimension 7, every lattice Λ with $\min \Lambda^* < 1$ has a section Λ_0 with $s(\Lambda) - s(\Lambda_0) \leq 6$.

Applied to dimension 7, the above results imply interesting restrictions on the possible values for s. However, these results will be improved significantly in the next section, so that we state them as a mere proposition.

Proposition 9.7. *Let Λ be an integral, 7-dimensional lattice of minimum 3. Then either $s = 28$ and $\Lambda \simeq W_7$, or Λ is isometric to a (uniquely determined) lattice W_7' with $s = 18$, or $s(\Lambda) \leq 17$.*

Proof. Let Λ_0 be a hyperplane section of Λ on which $s_0 = s(\Lambda_0)$ attains its maximum. We know by Proposition 7.6 and Theorems 6.5 and 7.9 that either $s_0 \leq 10$, or Λ_0 is one of the lattices W_6 or W_6', and then $s_0 = 16$ or 12, or Λ_0 is one of two lattices with $s = 11$. In particular, we have $s_0 \leq 16$, which immediately implies $s \leq 23$. Classifying the extensions of W_6 (which could fairly easily be done by hand, thanks to the rich structure of $S(W_6)$), we see that, besides W_7, there are exactly three extensions of W_6, namely, a lattice W_7' of determinant 128 with $s = 18$ and two lattices with $s = 17$ (a lattice W_7'' of determinant 144 and $W_6 \perp W_1$). This shows that either Λ is isometric to W_7, or $s(\Lambda) \leq s(W_6') + 7 = 19$. To complete the proof of the proposition, we have used a computer search to classify the extensions of the remaining three lattices, and again only W_7', W_7'', and $W_6 \perp W_1$ showed up. □

Using the above proposition together with the bound $s - s_0 \leq 8$, we can now prove the following fact.

Proposition 9.8. *Let Λ be an integral, 8-dimensional lattice of minimum 3. Then either $s = 30$ and $\Lambda \simeq W_8$, or $s = 29$ and $\Lambda \simeq W_7 \perp W_1$, or $s \leq 25$.*

Proof. First, we classify the extensions of W_7 (which can fairly easily be done by hand, thanks to the rich structure related to the Hamming code of $S(W_7)$). This way we find W_8 and $W_7 \perp W_1$. If Λ does not extend W_7, we have $s_0 \leq 18$, whence $s \leq 26$, and even $s \leq 25$ if Λ does not extend W_7'. In this last case, we have checked that extensions of W_7' have $s \leq 20$ except for W_8 and a uniquely determined lattice W_8' of determinant 192 with $s = 22$. □

Now we return to the case of an arbitrary dimension n. The lemma that we state and prove below will be used in §11 to bound the kissing number for 9-dimensional, integral lattices of dimension 9.

Lemma 9.9. *With the notation of Lemma 9.3, suppose that $N(e) \leq 1$, i.e., $\beta \geq 1$. Let $t = (\beta(\beta-1))^{(1/2)}$, and let $\overline{\overline{S}}_1$ be the image of S_1 under the translation*

$$x \mapsto \overline{\overline{x}} = te + \bar{x} = (t - \beta)e + x.$$

(We have $\overline{\overline{x}} = \bar{x}$ if and only if $N(e) = 1$.) If $N(e) < 1$, or if $N(e) = 1$ and e is not *a sum $x + y$, $x, y \in S_1$, then $\overline{\overline{S}}_1$ is the set of* simple *roots (of norm 2) in an* affine *root system of rank $\leq n$. We have $s_1 \leq n + 2$.*

Proof. We have

$$\overline{\overline{x}} \cdot \overline{\overline{y}} = ((t-\beta)e + x) \cdot ((t-\beta)e + y) = \frac{1}{\beta}(t-\beta)^2 + 2(t-\beta) + x \cdot y = x \cdot y - 1.$$

This shows that $\overline{\overline{x}} \cdot \overline{\overline{y}} = -2, -1, 0$ for $x \cdot y = -1-, 0, 1$, that all vectors in $\overline{\overline{S}}_1$ have norm 2, and that $\overline{\overline{x}} \cdot \overline{\overline{y}} = -2$ is equivalent to $x \cdot y = -1 \iff N(x+y) = 4$ on the one hand, and to $x+y=-2(t-\beta)e$ on the other hand. Comparing the norms, we obtain

$$x\cdot y=-1 \iff (t-\beta)^2\beta=1 \iff \beta=1,$$

and if $\beta=1$, then $2e=x+y$. Consequently, vectors in $\overline{\overline{S}}_1$ have mutual scalar products 0 or -1. Thus the elements of $\overline{\overline{S}}_1$ can be viewed as the vertices of an affine Dynkin diagram defined on an affine space of dimension $n-1$.

Let S' be an irreducible component of the Dynkin diagram, and let $x\in S'$. The structure of the Dynkin diagram (see [C-S, Table 4.1]) shows that the relation $\overline{\overline{x}}\cdot\overline{\overline{y}}=0$ is fulfilled on $S_1\smallsetminus S'$ and for at least $|S'|-3$ elements in S', hence for at least s_1-3 vectors in $\overline{\overline{S}}_1$. Now, for $y,z\in S_1$ such that $\overline{\overline{x}}\cdot\overline{\overline{y}}=\overline{\overline{x}}\cdot\overline{\overline{z}}=0$, we have $x\cdot y=x\cdot z=1$, hence $(x-y)\cdot(x-z)=2$ and $N(x-y)=4$. This shows that the set $\{x-y\mid\overline{\overline{x}}\cdot\overline{\overline{y}}=0\}$ is a scaled copy of a root system of type $\mathbb{A}_r$ contained in $\mathbb{R}e^{\perp}$, with $r\geq s_1-3$. Thus, we have $s_1-3\leq n-1$.

This completes the proof of the proposition. □

Remark 9.10. Except for the case where S' determines an affine system of type $\widetilde{\mathbb{A}}_r$, $r\geq 2$, x can be chosen in such a way that $|S'|-2$ or $|S'|-1$ elements in S' are orthogonal to $\overline{\overline{x}}$. Hence if the bound $s_1=n+2$ is sharp, all components of the root system $\overline{\overline{S}}_1$ must be of type $\widetilde{\boldsymbol{A}}_r$, $r\geq 2$. Examples with $s_1=n+2$ are known for $n=10$, with systems $4\widetilde{\boldsymbol{A}}_2$ and $3\widetilde{\boldsymbol{A}}_3$, obtained on the lattices with even parts K'_{10} and Q_{10}, respectively, that we mentioned in §§2 and 8.

§10. Dimensions 7 and 8

In this section, in which we keep the notation of the previous section, we shall sharpen the results of Theorems 9.7 and 9.8. To this end, we shall prove new bounds for the difference $s(\Lambda)-s(\Lambda_0)$ on the basis of properties of the index. As above, Λ stands for an integral, n-dimensional lattice ($n=7$ or 8) of minimum 3 and $s_0=s(\Lambda_0)$, where Λ_0 is a hyperplane section of Λ whose kissing number is maximal. Recall that 6-dimensional lattices of minimum 3 have index $\imath=1$ to within the three exceptions of W_6,W'_6,W''_6 for which $\imath=2$ (Theorem 6.5), that 7-dimensional lattices of minimum 3 have index 1 or 2, except W_7 for which $\imath=3$, and that lattices with $\imath\geq 2$ are those extending one of the lattices W_6,W'_6,W''_6. (In dimension 8, it is easily verified that $\imath(W_8)=4$ and $\imath(W_7\perp W_1)=3$; a consequence of the results we are going to prove is that $\imath=1$ or 2 otherwise.)

Unless otherwise stated, all lattices are integral and have minimum 3.

Theorem 10.1. *Let Λ be an integral, 7-dimensional lattice of minimum 3. Then $s(\Lambda)\leq$ 14 to within the following five exceptions: W_7 ($s=28$), W'_7 ($s=18$), W''_7 and $W_6\perp W_1$ ($s=17$), and one lattice with $s=16$. Moreover, there are four (respectively, nine, respectively, 27) lattices with $s=14$ (respectively, $s=13$, respectively, $s=12$).*

Since lattices of the form $\Lambda\perp W_1$ with $\dim\Lambda=6$ cover the interval $[7,13]$, we have the following statement.

Corollary 10.2. *The set of values taken by s on the set of well-rounded, 7-dimensional lattices is $[7,14]\cup[16,18]\cup\{28\}$.*

To prove Theorem 10.1, we shall take the index of Λ into account, making use of three lemmas.

Lemma 10.3. *Besides the five lattices with $s\geq 16$ listed in Theorem 10.1, there are seventeen 7-dimensional lattices of index 2, namely, four with $s=14$, six with $s=13$, three with $s=12$, and four with $s=11$.*

Proof. Just list the extensions of W_6,W'_6,W''_6 and test them for isometry. □

Lemma 10.4. *The* 7*-dimensional lattices of index* 1 *all have* $s \leq 13$*. There are three such lattices with* $s = 13$ *and* 24 *with* $s = 12$*.*

Proof. We know by Theorem 6.5 that there are altogether seventeen 6-dimensional lattices of index 1 with $s \geq 9$ (and in fact, $s = 9$, 10, or 11). Listing the extensions of these lattices, we find 24 lattices with $s = 12$, three lattices with $s = 13$, and none with $s \geq 14$. The problem now is to show that no new lattice arises among the extensions of well-rounded, 6-dimensional lattices with $s = 6$, 7 or 8. This is a consequence of the following lemma, in which, more generally, we consider integral lattices of any odd minimum. □

Lemma 10.5. *Let* L *be an integral, well-rounded,* 7*-dimensional lattice having an odd minimum* m*. If* $s(L) \geq 12$*, then* L *possesses a hyperplane section of minimum* m *with* $s \geq 9$*.*

Proof. It involves many details, so that we only sketch it. We classify minimal classes of index 1, and it is a relatively simple matter to describe $1 + 3 + 6 + 7 = 17$ classes with $s \leq 10$ that do not extend a 6-dimensional class with $s = 9$. The method is the one we sketched in §6 to deal with dimension 5, together with the following improvement: we have shown that the equivalence class of a *Bacher matrix* (defined in [Bt]) characterizes the corresponding minimal class; the proof is merely a modification of the proof of a proposition due to Bergé (Proposition 2.9 in [Bt]). We prove that all these classes can be expressed by using uniquely vectors with 3 or 5 components on a convenient basis $\mathcal{B} = (e_1, \dots, e_7)$ of minimal vectors (except for one obvious class with $s = 7$). We distinguish two classes with $s = 9$, namely (a_9) and (b_9), defined by

(a_9): $e_8 = e_1 + e_2 + e_3$, $e_9 = e_1 + e_2 + e_4$ and (b_9): $e_8 = e_1 + e_2 + e_3$, $e_9 = e_1 + e_4 + e_5$,

which we now use to list the seven classes with $s = 10$ that do not extend a 6-dimensional class with $s = 9$:

(a_{10}): $(a_9), e_5 + e_6 + e_7$; (b_{10}): $(b_9), e_1 + e_6 + e_7$; (c_{10}): $(b_9), e_2 + e_6 + e_7$;

(d_{10}): $(a_9), e_1 + e_2 + e_5 + e_6 + e_7$; (e_{10}): $(a_9), e_1 + e_3 + e_5 + e_6 + e_7$;

(f_{10}): $(b_9), e_1 + e_2 + e_4 + e_6 + e_7$; (g_{10}): $(b_9), e_2 + e_3 - e_4 + e_6 + e_7$.

Now we consider extensions of these seven minimal classes to a class with $s = 11$. It is easily verified that adding a vector with seven nonzero components to one of these classes amounts to adding a vector with three or five nonzero components to some other one (maybe the same). Then we consider in detail the first three classes and prove that adding a vector with five components yield classes that can all be expressed with four vectors with only three components. It is easy to check that a minimal class defined on $\mathcal{B}$ by four extra minimal vectors having only three nonzero components always possesses a hyperplane section containing 9 pairs of minimal vectors. This proves Lemma 10.5 for the classes with $s \geq 11$ that extend (a_{10}), (b_{10}) or (c_{10}).

Since extending one of the four classes (d_{10})–(g_{10}) by a vector having three nonzero components on $\mathcal{B}$ amounts to extending one of the classes (a_{10}), (b_{10}), (c_{10}), we are left with extensions of (d_{10})–(g_{10}) by a vector with five components.

In the case of (d_{10}) or (e_{10}), any extension without a convenient hyperplane section must involve e_5, e_6, and e_7. It turns out that a characteristic determinant equal to ± 2 always shows up. In each of the cases (f_{10}), (g_{10}), we find an essentially unique extension, both giving rise to the same class (a_{11}) with $s = 11$, and such that they cannot be extended to a class of index 1 with $s \geq 12$. □

[Explicitly, $(a_{11}) = (b_9), e_1 + e_2 + e_4 + e_6 + e_7, e_2 + e_3 + e_5 + e_6 + e_7$.]

Proof of Theorem 10.1. Clear using the three previous lemmas. □

Now we consider 8-dimensional lattices.

Theorem 10.6. *Let Λ be an integral, 8-dimensional lattice of minimum 3. Then $s(\Lambda) \leq 20$ to within the following three exceptions: W_8 $(s = 30)$, $W_7 \perp W_1$ $(s = 29)$, and W_8' $(s = 22)$. Moreover, there are seven lattices with $s = 20$.*

Proof. We know by Theorem 8.6 and Lemma 9.4 (1) that Λ possesses a hyperplane section Λ_0 of minimum 3 with $s(\Lambda_0) \geq s(\Lambda) - 8$. Since we have a complete list of small 7-dimensional lattices Λ_0 with $s(\Lambda_0) \geq 12$, we can list all 8-dimensional lattices Λ with $s \geq 20$. □

Corollary 10.7. *The set of values taken by s on the set of well-rounded, 8-dimensional lattices is $[8, 20] \cup \{22\} \cup \{29, 30\}$.*

Proof. Using the transformation $\Lambda \mapsto \Lambda \perp W_1$ together with the results of Corollary 10.2, we see that the set of values for s contains the union $[8, 15] \cup [17, 19]$. By Theorem 10.6, it suffices to exhibit a lattice with $s = 16$. An easy example is provided by the pull-back of the binary code generated by the four vectors

$$(1, 1, 1, 0, 0, 0, 0, 0), (0, 0, 0, 1, 1, 1, 0, 0), (1, 0, 0, 0, 0, 1, 1, 0), (0, 1, 0, 0, 1, 0, 0, 1),$$

a 4-modular lattice of determinant 256. □

We could prove a more precise result, the proof of which we only sketch.

Theorem 10.8. *There are exactly five integral, 8-dimensional lattices of minimum 3 with $s = 19$.*

To prove this theorem, since we know all Λ_0 with $s(\Lambda_0) \geq 12$, it suffices to prove the existence of hyperplane sections Λ_0 such that $s(\Lambda) - s(\Lambda_0) \leq 7$. We can produce two kinds of proofs for this result. The first one consists in the close study of the possible root systems that we constructed in §9. The second one relies on considerations about the index. Using Proposition 7.13 for lattices of index $\imath \geq 3$ and Theorem 10.1 for lattices that extend a 7-dimensional lattice with $s \geq 12$, we are left with lattices of index 2 of the form $\Lambda = \langle e_1, \ldots, e_8, f \rangle$, where $e_1, \ldots, e_8$ are minimal vectors and $f = \frac{e_1+e_2+e_3+e_4-e_5-e_6-e_7-e_8}{2}$. (See Remark 7.14; f is orthogonal to a minimal vector in Λ^*.) For these particular lattices, we can prove the existence of a hyperplane section Λ_0 such that $s(\Lambda) - s(\Lambda_0) \leq 5$.

§11. Bounds for dimension 9

Let Λ be a 9-dimensional lattice of minimum 3 and let $e \in S(\Lambda^*)$ be reduced. As above, let $S_1 = \{x \in S(\Lambda) = S \mid e \cdot x = 1\}$, $s_1 = |S_1|$, $\alpha = N(e)$, and $\beta = \frac{1}{\alpha}$.

Lemma 11.1. *If $N(e) \neq 1$, or if $N(e) = 1$ and $2e \notin \Lambda$, then $s_1 \leq 13$.*

[Note that if $N(e) < 1$ (respectively, $N(e) = 1$), we have $s_1 \leq 9$ (respectively, $s_1 \leq 11$) by Corollary 9.5 (respectively, by Lemma 9.9). The proof will also show the following refinement: if $\beta > 0.902$, then $s_1 \leq 12$.]

Proof. Let $\overline{S}_1$ and $\overline{\overline{S}}_1$ be the images of S_1 under the translations $x \mapsto \bar{x} = x - \beta e$ and $x \mapsto \bar{\bar{x}} = \bar{x} + \zeta e$, where ζ is the (positive) square root of $\beta(\beta - \frac{3}{4})$. (This definition makes sense because we have proved the upper bound $N(\Lambda^*) < \frac{4}{3}$.) Note that $\overline{S}_1$ is contained in the set of norm 3 vectors in the $(n-1)$-dimensional lattice $\Lambda \cap e^\perp$, hence $\overline{\overline{S}}_1$ is contained in an affine hyperplane of $\mathbb{R}^9$.

Let $x, y \in S_1$. We have $x \cdot y \geq 0$ by Lemma 9.3. An easy calculation then shows the formulas

$$\bar{x} \cdot \bar{y} = x \cdot y - \beta \geq -\beta$$

and

$$\bar{\bar{x}} \cdot \bar{\bar{y}} = \bar{x} \cdot \bar{y} + \zeta^2 \alpha = \bar{x} \cdot \bar{y} + \beta - \frac{3}{4} = x \cdot y - \frac{3}{4} \geq -\frac{3}{4}.$$

This proves that $\bar{\bar{x}} \cdot \bar{\bar{x}} = 3 - \frac{3}{4} = \frac{9}{4}$ and that $\bar{\bar{x}} \cdot \bar{\bar{y}} = \frac{1}{4}$ or $-\frac{3}{4}$ if $\bar{\bar{y}} \neq \bar{\bar{x}}$.

To get rid of the denominators, we now replace $\overline{\overline{S}}_1$ by its scaled copy

$$Z = \{2z \mid z \in \overline{\overline{S}}_1\};$$

the above formulas now read $z \cdot z = 9$ and $z_1 \cdot z_2 = 1$ or -3 if $z_2 \neq z_1$.

Denote by A, B, C the number of pairs $(z_1, z_2) \in Z \times Z$ such that $z_1 \cdot z_2 = 9, 1, -3$, respectively. We have

$$A = |S_1| \quad \text{and} \quad A^2 = A + B + C.$$

We shall use this to express $|S_1| = A$ in terms of the parameter C only.

Set $Z = \{z_1, \ldots, z_A\}$, and let $z = z_1 + \cdots + z_A$. We have

$$N(z) = \sum_{i,j} z_i \cdot z_j = 9A + B - 3C = A^2 + 8A - 4C \geq 0,$$

hence $C \leq \frac{1}{4}A^2 + 2A$.

To obtain a lower bound for C, consider the harmonic polynomial $F_z(t) = \sum_i P_2^{(t)}(z_i)$. (The notation is that of [V, §3].) We have

$$[F_z, F_z] = \sum_i \big((z_i \cdot z_j)^2 - 0\big) = 9^2 A + B + 9C - 9A^2 = 8(-A^2 + 10A + C) \geq 0.$$

Hence $C \geq A^2 - 10A$.

Comparing these two bounds for C, we get

$$A^2 - 10A \leq \frac{1}{4}A^2 + 2A \iff A \leq 16,$$

which is not yet optimal.

In the proof above, we used the weak inequality $v \cdot v \geq 0$ valid for any vector v in any Euclidean vector space E; this inequality was applied successively with $E = \mathbb{R}^9$ and $E = \mathrm{Harm}_2(\mathbb{R}^9)$. To go further, on the one hand, we shall make a better choice for v, and on the other hand, we shall use the Cauchy–Schwarz inequality in the form

$$v \cdot v \geq \frac{(v \cdot u)^2}{u \cdot u}$$

with a suitably chosen vector u.

First, we consider the vector $t = z - \xi e$ where we choose ξ in order that t be orthogonal to e. We have $t \cdot e = (\sum_i z_i - \xi e) \cdot e = 2A\zeta\alpha - \xi\alpha$. Hence we take $\xi = 2A\alpha$. Then

$$\begin{aligned} t \cdot t &= \Big(\sum_i z_i - \xi e\Big) \cdot \Big(\sum_j z_j - \xi\Big) = \sum_{i,j} z_i \cdot z_j - \xi(e \cdot z) \\ &= A^2 + 8A - 4C - 4(\beta - 3/4)A^2 = 4(1 - \beta)A^2 + 8A - 4C, \end{aligned}$$

which implies the new bound $C \leq (1-\beta)A^2 + 2A$, better than the previous one, because we have replaced $\frac{1}{4}$ by $1 - \beta < \frac{1}{4}$ in front of the A^2 term.

To improve on the lower bound for C, together with F_z we consider another harmonic polynomial, namely $F_e(t) = P_2^{(e)}(t) = (e \cdot t)^2 - \frac{\alpha}{9}(t \cdot t)$. We have

$$[F_z, F_e] = \sum_i \big((z_i \cdot e)^2 - \alpha\big) = 4\zeta^2\alpha^2 A - \alpha A = 4(\beta - 1)\alpha A$$

and $[F_e, F_e] = (e \cdot e)^2 - \frac{1}{9}\alpha^2 = \frac{8}{9}\alpha^2$.

Using the inequality $[F_z,F_z]\ge [F_z,F_e]^2/[F_e,F_e]$, we obtain

$$8(-A^2+10A+C)\ge \frac{16(\beta-1)^2\alpha^2A^2}{(8/9)\alpha^2}=18(\beta-1)^2A^2,$$

i.e., $C\ge \bigl(1+\frac{9}{4}(\beta-1)^2\bigr)A^2-10A$, better than the previous lower bound $C\ge A^2-10A$.

Comparing the last upper and lower bounds for C, we get

$$\Bigl(1+\frac{9}{4}(1-\beta)^2\Bigr)A-10A\le (1-\beta)A^2+2A \iff A\le \frac{12}{(\frac{9}{4}\beta^2-\frac{7}{2}\beta+\frac{9}{4})}.$$

The maximum of the right-hand side is attained for $\beta=\frac{7}{9}$, and is then equal to $\frac{27}{2}<14$. This proves the desired bound $A\le 13$ and the refinement $A<13$ for $\beta>(91+4\sqrt{13})/117=0.90104\ldots$. □

Lemma 11.2. *If $N(\Lambda^*)=1$, then there exists $e\in S(\Lambda^*)$ with $2e\notin\Lambda$.*

Proof. The argument resembles that of Theorem 8.6, and involves many details. Since the proof of Theorem 11.4 only needs a weaker result which could be obtained by the methods of Appendix 4, we only sketch the argument.

Since e belongs to Λ^* and $N(e)$ is integral, the lattice $M=\langle\Lambda,e\rangle$ is integral. By Theorem 8.3, M can be embedded in a unimodular lattice M' of dimension $n'\le 12$. Since M contains a vector of norm 1, M' is not isometric to $\mathbb{D}_{12}^+$. Hence, we have $M'\simeq\mathbb{Z}^k$ ($k\le 12$, and in fact $k=9$ if we assume that k is minimal) or $M'\simeq\mathbb{E}_8\perp\mathbb{Z}^k$ ($k\le 4$). As usual, we denote by (ε_i) the canonical basis for $\mathbb{Z}^k$.

If $\Lambda\subset\mathbb{Z}^9$, all norm 1 vectors in $\mathbb{Z}^9$ must belong to Λ^* (see the proof of Theorem 8.6). Hence we have $2\mathbb{Z}^9\subset\Lambda\subset\mathbb{Z}^9$, which shows that Λ is the pull back of a code of length 9 and weight 3 (together with a vector $\varepsilon_i+\varepsilon_j+\varepsilon_k$, Λ contains the eight vectors $\varepsilon_i\pm\varepsilon_j\pm\varepsilon_k$). It can be verified that such codes do not exist.

Suppose now that $\Lambda\subset\mathbb{E}_8\perp\mathbb{Z}^k$. As in the proof of Theorem 8.6, consider the projection L of Λ to the span of $\mathbb{E}_8$. This is a lattice of dimension $8-k$, generated by the roots $\tau\in\mathbb{E}_8$ such that $\varepsilon_i+\tau\in S(\Lambda)$ for some index i.

We have the inclusion $\Lambda\subset\mathbb{Z}^k\perp L$. If L has a component of type $\mathbb{A}_\ell$, then $\min\Lambda^*<1$. If $L\simeq\mathbb{D}_{8-k}$, then we may embed L into some $\mathbb{Z}^\ell$. Hence, it suffices to consider the cases where $L\subset\mathbb{E}_7\perp\mathbb{Z}^2$ or $L\subset\mathbb{E}_6\perp\mathbb{Z}^3$. These cases are proved to be impossible by arguments like those we used in the proof of Theorem 8.6. □

Theorem 11.3. *Let Λ be a 9-dimensional integral lattice of minimum 3. Then either Λ is isometric to W_9, with $s=34$, or $s(\Lambda)\le 32$.*

Proof. By Theorem 10.6, a hyperplane section of Λ is isometric to W_8, $W_7\perp W_1$, W_8', or to one of seven lattices with $s=20$, or has $s\le 19$. By a direct computer computation, we have checked that the extensions of these ten lattices either are isometric to W_9, or have $s\le 32$. Lemmas 11.1 and 11.2 then prove that extensions of the other 7-dimensional lattices have $s\le 19+13=32$. □

[As a matter of fact, extensions of W_8 and of $W_7\perp W_1$ are easily dealt with by hand, but this time, such calculations do not suffice to prove the bound $s\le 34$.]

The same method, involving Theorem 10.8 (which we have not proved in detail) instead of Theorem 10.6, immediately proves the following more precise result.

Theorem 11.4. *Let Λ be a 9-dimensional integral lattice of minimum 3. Then Λ is either isometric to W_9, with $s=34$, or to one of the three lattices with $s=32$, or has $s\le 31$.*

We conjecture that on the set of 9-dimensional, integral lattices of minimum 3 that are generated by their minimal vectors, the bound $N(\Lambda^*) < \frac{4}{3}$ can be improved to $N(\Lambda^*) \leq 1$ (which would then be optimal). Moreover, it might well be that a hyperplane section Λ_0 of minimum 3 with $s(\Lambda) - s(\Lambda_0) \leq n-1$ exists in dimension 9 as in dimension 8. We could then be able to list all lattices with $s \leq 27$. We state the result as a conjecture.

Conjecture 11.5. *There are* 11 *integral, well-rounded,* 9*-dimensional lattices of minimum* 3 *with* $s \geq 27$*, one with* $s = 34$*, two with* $s = 31$ *and* 30*, and three with* $s = 32$ *and* 28.

We have found well-rounded lattices for small values of s in the range 9–26. Thus, a consequence of the conjecture above is that the set of possible values for s on the set of integral, well-rounded, 9-dimensional lattices should be

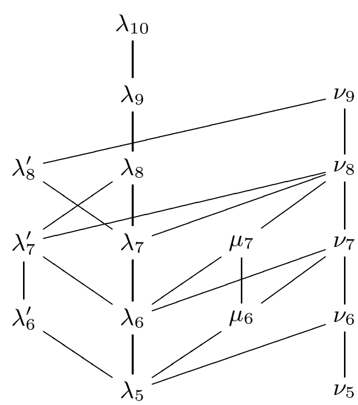
$$[9, 26] \cup \{28\} \cup [30, 32] \cup \{34\}.$$

Things are less clear for dimension 10, where the largest known value for s, namely, 40, is attained on four lattices, including two "non-Watson" lattices. Experimental data suggest that for $n = 10$, we have $s = 40$ or $s \leq 38$.

We end this section with a few words about integral lattices of minimum 5. We know that up to $n = 6$, the set of possible values for s on a well-rounded lattice is the entire interval $[n, s_5(n)]$ and that $s_5(n) = s_3(n)$. (This probably is also true for all $m \geq 7$ odd.) When $n = 7$, there is probably a gap: a computer search only found lattices with $s \in [7, 21] \cup \{27\}$. Maybe, one could use methods *à la Watson* to prove the inequality $s \leq 21$ on lattices having index $\imath \leq 2$, and then prove that there exists a unique lattice of index 3.

Probably, all 8-dimensional lattices have $s \leq 30$. However, the bounds we proved or conjectured for lattices of minimum 3 in dimensions 9 or 10 do not extend to larger minima. Indeed, for $r \equiv 3 \mod 4$, a unique sublattice of $\mathbb{A}_r^*$ containing $\mathbb{A}_r$ to index $\frac{r+1}{2}$ is proportional to an integral lattice of minimum $m = \frac{r-1}{2} \equiv 1 \mod 2$, denoted by Cox_r. We have $\mathrm{Cox}_7 \simeq W_7$. For $r \geq 11$, the lattice Cox_r has $s = \frac{r(r+1)}{2}$, and possesses sections with $s = \frac{n(n-1)}{2} + 1$ if $n = r-1$ and $s = \frac{n(n-1)}{2}$ if $n \leq r-2$. Hence, for all $m \geq 5$ odd, lattices with $s = 36 > 34$ (respectively, $s \geq 45 > 40$) exist in dimension $n = 9$ (respectively, 10), and we can even have $s = 46$ for $m = 5$ and $n = 10$. Note that, for $m = 5$ (taking $r = 11$), we obtain the lattices with $n = 6$ and $s = 15$ of Theorem 7.11 and with $n = 5$ and $s = 10$ of Theorem 6.2. Bergé [Be] proved that the values of s on sections of Cox_r given above are the largest possible for all $n \geq 4$.

Appendix 1: 5- and 6-dimensional lattices without a hexagonal section having the same minimum



λ_i, λ'_i: classes of index 1 satisfying condition (C1);

μ_i: other classes of index 1; ν_i: classes of index 2.

Inclusion graphs for NHS minimal classes in dimension 5.

Minimum minimorum. Below, for each class as above, we list the smallest possible minimum for an integral lattice belonging to this class.
$m=1$: λ_5; $m=3$: λ_i, $i\neq 1,9$; λ'_i; $m=4$: ν_5, μ_6; $m=5$: λ_9;
$m=6$: μ_7; $m=10$: ν_6, ν_7, ν_8; $m=16$: ν_9.
All classes λ_i, λ'_i contain integral lattices of any odd minimum $m\geq 5$.

Correspondence with Batut's classification. λ_i: a_5, b_6, e_7, m_8, p_9, w_{10}; λ'_i: d_6, g_7, j_8; μ_i: c_6, f_7; ν_i: b_5, e_6, j_7, p_8, w_9.

[Incidentally, the fifth column of the row w_9 in [Bt], which reproduces that of the row u_9, is a slip. One should read $[36,88,2]$.]

Dimension 6. We know that for an integral, well-rounded lattice having an odd minimum, of index $\imath=1$ (respectively, 2, respectively, 3), we have $6\leq s\leq 12$ (respectively, $6\leq s\leq 16$, respectively, $s=12$). We have verified more generally that minimal classes having no hexagonal sections and such that $s\geq 14$ are contained in the class determined by W_6. In particular, this way we recover Watson's theorem for dimension 6. As to classes of index 1 containing an integral lattice, for $s=7$ (respectively, 8, 9, 10) there are 2 (respectively, 4, 4, 5) classes.

Appendix 2: Improved bounds in the range 2–24

The second statement in Theorem 3.1 can be made more precise. The proof, which involves a complicated calculation, resembles that of Lemma 11.1. We state the result without giving its proof.

Theorem. *Let Λ be an integral lattice of minimum 3 and dimension n in the range 2–24. Suppose that Λ is not unimodular. Let $e\in S(\Lambda^*)$, $\alpha=N(e)$,*

$$s_1=|\{x\in S(\Lambda)\mid e\cdot x=1\}|,$$

and $\lambda=\frac{s_1}{s}$. Then we have

$$s(\Lambda)\leq\frac{8n(n+2)}{25-n}-A_1-A_2,$$

where

$$A_1=\frac{n^2(n+2)(50-n)}{9(n-1)(n+4)(25-n)}\times\frac{(\lambda-3\alpha/n)^2 s}{\alpha^2}$$

and

$$A_2=\frac{n}{9(n-1)(n+4)(25-n)}\times\frac{\Big(3^3\alpha^2-\big(18\alpha-(n+4)\big)(n+2)\lambda\Big)^2 s}{\alpha^4}.\qquad\square$$

The restriction that Λ should not be unimodular is necessary: indeed, the proof makes use of the existence of a nonzero reduced vector e, a property which fails for unimodular lattices.

We know that, on nonunimodular lattices, equality occurs in Theorem 3.1 if and only if $n=7$, 16 or 22, and in each case, we know the value of s. An easy calculation shows that $A=B=0\implies\alpha=\frac{n+2}{9}$ and $\frac{s_1}{s}=\frac{n+2}{3}$. For $n=7$, 16, 22 this gives the following values for s and $s_0=s-s_1$:
$n=7$: $s=28$, $s_0=16$;
$n=16$: $s=256$, $s_0=190$;
$n=22$: $s=1408$, $s_0=896$.

We recognize the values of s_0, namely those of $s(\Lambda_0)$ for $\Lambda_0=W_6$, O_{15}, and O_{22} respectively. It should be noticed that our results imply that a lattice of dimension $n_0\in\{6,15,21\}$, if not isometric to one of the lattices Λ_0 listed above, cannot be embedded in a Λ with $\dim\Lambda=n_0+1$ and $s(\Lambda)$ maximal. The existence of such a lattice has been proved to be impossible if $n_0=6$, and is very unlikely if $n_0=15$ or 21.

Appendix 3: Large values of $\min \Lambda^*$

We display the largest known values of $\min \Lambda^*$ for integral lattices of minimum 3 *that are generated by their minimal vectors* in the range $1 \le n \le 24$. We have verified that these bounds are optimal for $1 \le n \le 7$; thanks to the results of §7, it then suffices to assume that Λ is well rounded.

For $n = 1, 2, 3, 7$, the bounds are even optimal on the set of *all* lattices of minimum 3: for $n = 1, 2$, because Λ and Λ^* are similar; for $n = 3$, because γ'_3 is attained on $\mathbb{A}_3^* \sim W_3$ — cf. [M, Theorem 6.3.4]; and for $n = 7$, by Theorem 8.1. The case of $n = 5$ would follow from a proof of the conjectural value of γ'_5.

The lattice $\Lambda \subset \mathbb{Z}^2 \perp \mathbb{E}_6$ constructed in the proof of Theorem 8.6 attains the largest known value (in fact, $\frac{7}{9}$) for $n = 8$. However, for $n = 8$ there exists a well-rounded lattice (of index 2) for which $\min \Lambda^*$ outdoes the value given in the table ($\frac{4}{5} = 0.8 > \frac{7}{9} = 0.77\dots$). Note that the results of §7 imply that well-rounded lattices that are not generated by their minimal vectors do not exist in dimensions $n \le 7$.

Next, we find the value 1 for $9 \le n \le 13$, and then larger values (2 is attained at O_{16}, 3 at O_{23} and O_{24}); for $n = 10$ examples are provided by the lattices related to K'_{10} or to Q_{10} that were constructed in §2; for $n = 9$ and $11 \le n \le 23$, one may consider convenient sublattices of O_{23} or Plesken–Pohst lattices (denoted by plp_* with suitable subscripts $*$), for which Gram matrices can be found in [Bt-M].

Large values of $\min \Lambda^*$.

n	1	2	3	4	5	6	7	8
$\min \Lambda^*$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	1	$\frac{7}{9}$
n	9	10	11	12	13	14	15	16
$\min \Lambda^*$	1	1	1	1	1	$\frac{4}{3}$	$\frac{3}{2}$	2
n	17	18	19	20	21	22	23	24
$\min \Lambda^*$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	2	2	$\frac{8}{3}$	3	3

Below we list the lattices that are known to us to meet the bound for dimensions which were not considered above.

<u>$n = 9$</u>: O_{9a3}, O_{9b2}; <u>$n = 11$</u>: W_{11a}, W_{11b}, O_{11};
<u>$n = 12$</u>: W_{12}, O_{12a}, O_{12b}; <u>$n = 13$</u>: W_{13}, O_{13a}, O_{13b}, plp_{13a};
<u>$n = 14, 15, 16$</u>: O_n; <u>$n = 17$</u>: plp_{17c};
<u>$n = 18$</u>: plp_{18c}, plp_{18d}; <u>$n = 19$</u>: O_{19}, plp_{19d};
<u>$n = 20$ to 24</u>: O_n.

Appendix 4: On lattices with $\min \Lambda^* = 1$

Now, we consider bounds for S_1 for lattices Λ with $\min \Lambda^* = 1$ for which all norm 1 vectors in Λ^* satisfy the conditions of Lemma 9.3. We assume that these lattices are generated by their minimal vectors.

Proposition. *Let $e \in \Lambda^*$. If $N(e) = 1$ and $2e \in \Lambda$, then $\overline{S}_1$ is a root system of type $\kappa \boldsymbol{A}_1$ for some $\kappa \in [1, n-1]$. (Otherwise stated, S_1 consists of $\kappa \le n - 1$ pairs $\pm z$ of norm 2 vectors.)*

Proof. Let $2e = x + y$, $x, y \in S_1$. For any $z \ne x$, y in S_1, set $t = x + y - z$. Then t is an element of S such that $t \cdot e = 1$, hence an element of S_1, and we have $t \cdot z = 2e \cdot z - z \cdot z = -1$, which shows that $\bar{t} = -\bar{z}$. Moreover, we have $x \cdot z + y \cdot z = 2e \cdot z = 2$, hence $x \cdot z = y \cdot z = 1$ because both terms are bounded from above by 1, whence $\bar{x}\cdot\bar{z}=\bar{y}\cdot\bar{z}=0$. Since $N(\bar{z})=N(z)-1=2$ for every z, $\overline{S}_1$ is a root system of type $\kappa\boldsymbol{A}_1$ for some $\kappa\le\dim\mathbb{R}e^\perp=n-1$, and κ is not zero because Λ is well rounded. □

Denote by r_i $(i=1,\dots,\kappa)$ the elements of $\overline{S}_1$ (the roots of the system $\kappa\boldsymbol{A}_1$ above), let P be the lattice they generate, let V be the span of P, and let P_0 be the lattice of index 2 in P (a scaled copy of $\mathbb{D}_\kappa$) generated by the $r_i\pm r_j$. We have $P_0=\Lambda\cap P=\Lambda_{\text{even}}\cap P$. Up to permutation and change of signs of the r_i, a norm 3 vector $f\in\Lambda\cap V$ is of the form $f=\frac{r_1+\cdots+r_6}{2}$: indeed, $2f$ (of norm 12) is a sum of the form $2r_1+r_2+r_3$ or $r_1+\cdots+r_6$, and the first case must be excluded, since then $(e+r_1)\cdot f=2$. Together with f, $\Lambda\cap V$ contains all vectors that differ from f by an even number of sign changes, and the supports of such vectors f constitute a self-dual code of length κ and weight 6. When $\kappa=n-1$, we recover the construction we gave at the end of §2.

We saw in the previous sections that for $n\le 9$, $N(e)=1$ and $e\in S_1+S_1$ occurs only for $\Lambda\simeq W_7$. We can show that this is also impossible for $n=10$. The discussion above immediately proves the weaker result $\kappa\le n-2$ for $n\le 10$ to within the only exception of W_7, which corresponds to the fact that every root in $\mathbb{E}_7$ is a sum $x+y$ with $x,y\in\frac{1}{2}S(\mathbb{E}_7^*)$.

It is relatively easy to handle the cases where $\kappa=n-3$, for which Λ contains the lattice Λ' generated by the vectors $e\pm r_i$ together with some norm 3 vector $e'\in(\mathbb{Q}e)^\perp\smallsetminus V$. Using this device, we can obtain a short proof for $n=9$ of the bound $\kappa\le n-3=6$, which can be used in place of Lemma 11.2 to prove Theorem 11.4.

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