

**BERNSTEIN-TYPE INEQUALITIES FOR THE DERIVATIVES
OF RATIONAL FUNCTIONS IN L_p -SPACES, $0 < p < 1$,
ON LAVRENT'EV CURVES**

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ABSTRACT. Let S be a simple or a closed Lavrent'ev curve on the complex plane, let $0 < p < 1$ with $1/p \notin \mathbb{N}$, and let $s \in \mathbb{N}$. It is shown that for an arbitrary rational function r of degree n such that $|r|^p$ is integrable on S the following inequality is fulfilled:

$$\left(\int_S |r^{(s)}(z)|^\sigma |dz| \right)^{1/\sigma} \leq cn^s \left(\int_S |r(z)|^p |dz| \right)^{1/p},$$

where $1/\sigma = s + 1/p$, and $c > 0$ depends only on S, p , and s .

Earlier (in 1995) this result was obtained by the author and Stahl for the segment and the circle. The inequality is used to deduce an inverse rational approximation theorem in the Smirnov class E_p . Other rational approximation problems in L_p and E_p are also treated.

§1. INTRODUCTION

Let S be a rectifiable Jordan curve (either simple or closed) on the complex plane \mathbb{C} . Then S is called an *Ahlfors curve* if there exists $\kappa \geq 2$ such that for all $z \in \mathbb{C}$ and $t > 0$ we have

$$(1.1) \quad |\{\zeta \in S : |\zeta - z| \leq t\}| \leq \kappa t.$$

Here and below $|I|$ stands for the linear Lebesgue measure of a set $I \subset S$. A simple curve S is called a *Lavrent'ev curve* if there exists $\theta \geq 1$ such that for every $\zeta_1, \zeta_2 \in S$ the arc $I(\zeta_1, \zeta_2) \subset S$ with endpoints ζ_1 and ζ_2 satisfies the inequality

$$(1.2) \quad |I(\zeta_1, \zeta_2)| \leq \theta |\zeta_1 - \zeta_2|.$$

A closed curve S is called a *Lavrent'ev curve* if for every $\zeta_1, \zeta_2 \in S$ inequality (1.2) is fulfilled for the shortest of the two arcs on S with endpoints ζ_1, ζ_2 .

It is easily seen that a Lavrent'ev curve is an Ahlfors curve; the converse is not true. If S is a Lavrent'ev curve, then the smallest constants κ and θ in (1.1) and (1.2) are related by the inequality $\kappa \leq 2\pi\theta$.

For a rectifiable Jordan curve S and for $0 < p \leq \infty$, let $L_p = L_p(S)$ denote the Lebesgue space of complex-valued functions on S . Specifically, $f \in L_p(S)$ if f is measurable and the following quasinorm (a norm for $1 \leq p \leq \infty$ and a p -norm for $0 < p < 1$) is finite:

$$\|f\|_p = \|f\|_{L_p(S)} = \left(\int_S |f(\zeta)|^p |d\zeta| \right)^{1/p}, \quad 0 < p < \infty;$$
$$\|f\|_\infty = \|f\|_{L_\infty(S)} = \operatorname{ess\,sup}_{\zeta \in S} |f(\zeta)|, \quad p = \infty.$$

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We denote by \mathcal{R}_n , $n = 0, 1, 2, \dots$, the set of all rational functions of degree at most n ($\deg r \leq n$).

The following theorem is the main result of the paper.

Theorem 1.1. *Suppose S is a Lavrent'ev curve (simple or closed). Let $0 < p < 1$ with $1/p \notin \mathbb{N}$, let $s \in \mathbb{N}$, and let $1/\sigma = s + 1/p$. Then for every $r \in \mathcal{R}_n \cap L_p(S)$ we have*

$$(1.3) \quad \|r^{(s)}\|_\sigma \leq cn^s \|r\|_p,$$

where $c > 0$ depends only on p , s , and the number θ in (1.2).

Earlier, Theorem 1 was obtained by the author and Stahl for the segment and the circle.

For $p \in (1, \infty]$, inequality (1.3) is also true (see [15, 4, 19] for the details) under the only assumption that S is an Ahlfors curve. This latter condition is in fact necessary for the validity of (1.3) for an arbitrary $p \in (0, +\infty]$, $1/p \notin \mathbb{N}$. If $1/p \in \mathbb{N}$, then (1.3) fails for all curves S . See Subsection 2A below for more details. If $0 < p < 1$ and $1/p \notin \mathbb{N}$, the following question remains open: What is the maximal class of curves S for which (1.3) is fulfilled? We note also that the question about the validity of (1.3) was raised by Sevast'yanov [22].

The standard Bernstein method allows us to deduce the next Theorem 1.2 from Theorem 1.1. Theorem 1.2 is an inverse theorem for rational approximation in the Smirnov space $E_p = E_p(G)$. We recall the definition of this space.

A domain $G \subset \mathbb{C}$ will be called a Jordan, Ahlfors, Lavrent'ev, etc., domain if its boundary ∂G is a respective curve. Let G be a bounded simply connected Jordan domain with rectifiable boundary, and let $0 < p \leq \infty$. A function f analytic in G belongs to E_p if $\sup_{n \in \mathbb{N}} \|f\|_{L_p(\partial G_n)} < \infty$ for at least one sequence $\{G_n\}_{n=1}^\infty$ of simply connected domains (with rectifiable boundaries ∂G_n) such that $\overline{G}_n \subset \overline{G}_{n+1} \subset G$ for all n and $\bigcup_{n=1}^\infty G_n = G$. This definition was suggested by Keldysh and Lavrent'ev. Smirnov's original definition (equivalent to the above definition) will be given in §3 (see also [10, 20]).

A function $f \in E_p$ has a nontangential boundary value $f(\zeta)$ at almost every $\zeta \in \partial G$ (with respect to linear Lebesgue measure). It is known that $f(\zeta) \in L_p(\partial G)$. By definition, the quasinorm $\|f\|_{E_p}$ is $\|f\|_{L_p(\partial G)}$.

For $f \in E_p$, we introduce the *best approximation* by rational functions of degree at most n :

$$R_n(f)_p = \inf\{\|f - r\|_{E_p} : r \in \mathcal{R}_n \cap E_p\}.$$

Theorem 1.2. *Suppose G is a simply connected bounded Lavrent'ev domain. Let $0 < p < 1$ with $1/p \notin \mathbb{N}$, let $s \in \mathbb{N}$, and let $1/\sigma = s + 1/p$. If $f \in E_p$ and*

$$\sum_{n=1}^{\infty} \frac{1}{n} (n^s R_n(f)_p)^\sigma < \infty,$$

then $f^{(s)} \in E_\sigma$.

The corresponding direct theorem was obtained by the author in [19] in the following form. *Suppose G is a bounded simply connected Lavrent'ev domain. Let $0 < p < \infty$, let $s \in \mathbb{N}$, and let $1/\sigma = s + 1/p$. If f is analytic in G and $f^{(s)} \in E_\sigma$, then $f \in E_p$ and*

$$R_n(f)_p = o\left(\frac{1}{n^s}\right) \quad \text{as } n \rightarrow \infty,$$

$$R_n(f)_p \leq \frac{c}{n^s} \|f\|_{E_\sigma} \quad \text{as } n = s, s+1, \dots$$

Here c is a positive constant independent of n and f .

In [19] the direct theorem was obtained also for $p = \infty$, i.e., for uniform approximations. However, in this case G must be an Alper or Radon domain.

In the paper [17] by the author and Stahl, where a segment or a circle plays the role of S , Theorem 1.1 was verified in two steps. First, it was shown that it suffices to obtain (1.3) for rational functions with poles on S only. Then (1.3) was proved for such rational functions. In the general case the arguments follow the same pattern but, unlike [17], now the first step is the most difficult. Among other things, we shall employ Carleson's lemma [10] used in his original proof of the corona theorem. At the second step, the general Hardy–Littlewood embedding theorem from [19] is applied. Note also that at the first step all constructions are done for Ahlfors curves, whereas at the second step we are forced to restrict ourselves to Lavrent'ev curves.

At the end of the paper, other applications of the methods used here will be given. In particular, a new proof of a result by Aleksandrov [2] will be presented.

§2. INEQUALITIES FOR CERTAIN INTEGRALS ALONG AHLFORS CURVES

2A. Estimates of simplest singular integrals. Various positive constants will be denoted by c, c_1, c_2, \dots . The parameters on which these constants may depend will be either clear from the context or indicated explicitly. We agree to distinguish between these constants in the notation only within a current statement or proof. Throughout, κ and θ denote the constants in the Ahlfors condition (1.1) and Lavrent'ev condition (1.2), respectively.

Let K and S be subsets of \mathbb{C} . We denote by $\rho(K, S)$ the distance between K and S , i.e., $\rho(K, S) = \inf\{|\xi - z| : \xi \in K, z \in S\}$. In particular, $\rho(\xi, S)$ is the distance from a point ξ to the set S .

Lemma 2.1 (see [3, 5]). *Suppose S is a rectifiable Jordan curve and $1 \geq \alpha > 0$. The following conditions are equivalent:*

- (i) S is an Ahlfors curve;
- (ii) for every $\xi \in \mathbb{C} \setminus S$ we have

$$(2.1) \quad \int_S \frac{|dz|}{|z - \xi|^{\alpha+1}} \leq \frac{c(S, \alpha)}{\rho(\xi, S)^\alpha}.$$

In [3, 5] this lemma was proved for $\alpha = 1$. In the general case the proof is similar.

Considering the simplest rational functions of the form $r(z) = (z - \xi)^{-1}$, $\xi \in \mathbb{C} \setminus S$, and using Lemma 2.1 together with the remark in the Introduction about the validity of (1.3) for $1 < p \leq \infty$, it is easy to show that the set of Ahlfors curves is the maximal class of curves for which (1.3) is fulfilled for $p = \infty$. In the case of $s = 1$ this was observed by Dolzhenko and Danchenko [5]. The same condition is necessary for $0 < p < \infty$, $1/p \notin \mathbb{N}$. To prove this, we need Lemma 2.2.

Lemma 2.2. *Suppose S is a rectifiable Jordan curve and $1 \geq \alpha > 0$, $\beta > \alpha$. If there exists $c = c(S, \alpha, \beta) > 0$ such that*

$$(2.2) \quad \left(\int_S \frac{|dz|}{|z - \xi|^{\alpha+1}} \right)^{1/\alpha} \leq c \left(\int_S \frac{|dz|}{|z - \xi|^{\beta+1}} \right)^{1/\beta}$$

for every $\xi \in \mathbb{C} \setminus S$, then S is an Ahlfors curve.

Proof. Since $\beta > \alpha > 0$, we have $|z - \xi|^{\beta+1} \geq \rho(\xi, S)^{\beta-\alpha} |z - \xi|^{\alpha+1}$ for $z \in S$ and $\xi \in \mathbb{C} \setminus S$. Therefore, (2.2) implies

$$\left(\int_S \frac{|dz|}{|z - \xi|^{\alpha+1}} \right)^{1/\alpha} \leq \frac{c}{\rho(\xi, S)^{1-\frac{\alpha}{\beta}}} \left(\int_S \frac{|dz|}{|z - \xi|^{\alpha+1}} \right)^{1/\beta}.$$

After easy transformations, this yields (2.1) with $c_1 = c(S, \alpha, \beta)^{\frac{\beta}{\beta-\alpha}}$. By Lemma 2.1, S is an Ahlfors curve. □

Suppose that (1.3) is fulfilled for some fixed p and s , $p \in (0, \infty)$, $s \in \mathbb{N}$. Putting $n_1 = n_1(p) = [1/p] + 1$, we substitute the rational function $r_1(z) = (z - \xi)^{-n_1}$, $\xi \in \mathbb{C}$, in (1.3). Lemma 2.2 shows that S is an Ahlfors curve. Assuming, in addition, that $1/p \in \mathbb{N}$, we substitute the rational function $r_2(z) = (z - \xi)^{-1/p}$, $\xi \in \mathbb{C} \setminus S$, in (1.3). After easy transformations, we obtain

$$\int_S \frac{|dz|}{|z - \xi|} \leq c_2(S) \quad \text{for } \xi \in \mathbb{C} \setminus S.$$

The left-hand side of this inequality grows unboundedly as $\rho(\xi, S) \rightarrow 0$, a contradiction. Thus, for $1/p \in \mathbb{N}$ inequality (1.3) fails for all curves S .

In the next statement (Lemma 2.3) we give more explicit expressions for the constant $c(S, \alpha)$ in (2.1) and estimate other integrals of similar nature. These integrals also occurred in [2, 6, 7].

Lemma 2.3. *If S is an Ahlfors curve, then*

$$(2.3) \quad \int_{S_\varepsilon(\xi)} \frac{|dz|}{|z - \xi|^{\alpha+1}} \leq \frac{\kappa(\alpha + 1)}{\alpha\varepsilon^\alpha} \quad \text{for } \alpha > 0, \varepsilon > 0, \xi \in \mathbb{C}, \text{ and} \\ S_\varepsilon(\xi) = \{z \in S : |z - \xi| \geq \varepsilon\};$$

$$(2.4) \quad \int_S \frac{|dz|}{|z - \xi|^{\alpha+1}} \leq \frac{\kappa(\alpha + 1)}{\alpha\rho(\xi, z)^\alpha} \quad \text{for } \alpha > 0 \text{ and } \xi \in \mathbb{C} \setminus S;$$

$$(2.5) \quad \int_S \frac{|dz|}{|z - \xi|^p} \leq \frac{\kappa^p}{1 - p} |S|^{1-p} \quad \text{for } p \in (0, 1) \text{ and } \xi \in \mathbb{C}.$$

Proof. We use the following identity for the Lebesgue integral. If $f \in L_p(S)$, $0 < p < \infty$, and $m(f, t) = |\{z \in S : |f(z)| > t\}|$, $t \geq 0$, is the distribution function for $|f|$, then

$$(2.6) \quad \int_S |f(z)|^p |dz| = p \int_0^{+\infty} t^{p-1} m(f, t) dt.$$

To prove (2.3), take $f(z) = (z - \xi)^{-1}$ for $z \in S_\varepsilon(\xi)$ and $f(z) = 0$ for $z \in S \setminus S_\varepsilon(\xi)$. Then $m(f, t) = |\{z \in S_\varepsilon(\xi) : |z - \xi|^{-1} > t\}| = |\{z \in S : \varepsilon \leq |z - \xi| < 1/t\}|$ for $t > 0$. Consequently, $m(f, t) = 0$ for $t > 1/\varepsilon$, and, by (1.1), we have $m(f, t) \leq \kappa/t$ for $t \in (0, 1/\varepsilon]$. Taking (2.6) into account, we obtain (2.3):

$$\int_{S_\varepsilon(\xi)} \frac{|dz|}{|z - \xi|^{\alpha+1}} \leq \kappa(\alpha + 1) \int_0^{1/\varepsilon} t^{\alpha-1} dt = \frac{\kappa(\alpha + 1)}{\alpha\varepsilon^\alpha}.$$

Putting $\varepsilon = \rho(\xi, S)$ in (2.3), we arrive at (2.4).

In order to establish (2.5), we take $f(z) = (z - \xi)^{-1}$, $\xi \in \mathbb{C}$. Then $m(f, t) = |\{z \in S : |z - \xi|^{-1} > t\}| = |\{z \in S : |z - \xi| < 1/t\}| \leq \min\{\frac{\kappa}{t}, |S|\}$, $t > 0$. Here we have used the obvious inequality $m(f, t) \leq |S|$ and condition (1.1). Therefore,

$$\int_S \frac{|dz|}{|z - \xi|^p} \leq p \int_0^\infty t^{p-1} \min\left\{\frac{\kappa}{t}, |S|\right\} dt. \quad \square$$

2B. Integral estimates for algebraic polynomials. A measurable function f on S is said to belong to the Lebesgue space $L_0 = L_0(S)$ if $-\infty \leq \int_S \ln |f(z)| |dz| < +\infty$. For such a function, we introduce the geometric mean

$$\|f\|_0 = \|f\|_{L_0(S)} = \exp \left(\frac{1}{|S|} \int_S \ln |f(z)| |dz| \right).$$

By the Jensen inequality, it is easily seen that if $f \in L_p(S)$ with $0 < p \leq \infty$, then $f \in L_0(S)$ and

$$(2.7) \quad \|f\|_0 \leq |S|^{-1/p} \|f\|_p.$$

We denote by \mathcal{P}_m ($m = 0, 1, 2, \dots$) the set of algebraic polynomials of degree at most m .

Lemma 2.4. *If S is an Ahlfors curve and $f \in \mathcal{P}_m$, then $\|f'\|_0 \leq 4\kappa m^2 |S|^{-1} \|f\|_0$.*

Proof. There is no loss of generality in assuming that $\deg f = m$ and $m \geq 1$. Let z_1, z_2, \dots, z_m be the zeros of f (taken with multiplicities). Then

$$\frac{\|f'\|_0}{\|f\|_0} = \left\| \frac{f'}{f} \right\|_0 = \left\| \sum_{k=1}^m \frac{1}{z - z_k} \right\|_0.$$

We denote the sum in the third term by $g = g(z)$. We must prove that

$$(2.8) \quad \|g\|_0 \leq 4\kappa m^2 |S|^{-1}.$$

Fixing some $p \in (0, 1)$, we apply (2.7); this yields

$$\|g\|_0^p \leq \frac{1}{|S|} \|g\|_p^p \leq \frac{1}{|S|} \sum_{k=1}^m \int_S \frac{|dz|}{|z - z_k|^p}.$$

We continue the estimate by using (2.5). As a result, we obtain

$$\|g\|_0^p \leq \frac{\kappa^p}{1-p} \cdot \frac{m}{|S|^p}.$$

Now, to verify (2.8), it suffices to put $p = 1/2$. □

Lemma 2.5. *Suppose S is an Ahlfors curve and $f \in \mathcal{P}_m$. Then for every $k = 0, 1, \dots, m$ we have*

$$\|f^{(k)}\|_\infty \leq \frac{3(4\kappa)^m (m!)^2}{|S|^k} \|f\|_0.$$

Proof. For every $k = 0, 1, \dots, m$, let ξ_k denote a point on S satisfying $\min\{|f^{(k)}(\xi)| : \xi \in S\} = |f^{(k)}(\xi_k)|$. Next, put $a_k = f^{(k)}(\xi_k)$. We have arrived at the so-called Birkhoff interpolation problem: to recover the polynomial f by the data (ξ_k, a_k) . This problem has a unique solution, which can be found by iteration:

$$f^{(m)}(\eta_m) = a_m \quad (\eta_m \in S)$$

and

$$f^{(k)}(\eta_k) = \int_{\xi_k}^{\eta_k} f^{(k+1)}(\eta_{k+1}) d\eta_{k+1} + a_k \quad (\eta_k \in S)$$

for $k = m - 1, m - 2, \dots, 0$. It easily follows that

$$\|f^{(k)}\|_\infty \leq \sum_{j=k}^m |a_j| \cdot |S|^{j-k} \quad \text{for } k = 0, 1, 2, \dots, m.$$

Next, by Lemma 2.4 and the choice of ξ_k and a_k , we obtain

$$|a_j| \leq \|f^{(j)}\|_0 \leq \left(\frac{m!}{(m-j)!}\right)^2 \cdot (4\kappa)^j \frac{\|f\|_0}{|S|^j}.$$

Therefore,

$$\|f^{(k)}\|_\infty \leq \frac{\|f\|_0}{|S|^k} \sum_{j=k}^m \left(\frac{m!}{(m-j)!}\right)^2 (4\kappa)^j \leq 3(4\kappa)^m (m!)^2 \frac{\|f\|_0}{|S|^k}. \quad \square$$

2C. Relationship between rational and piecewise-polynomial functions. Let S be a simple rectifiable oriented Jordan curve, and let I be an arc of it. We denote by χ_I the characteristic function of I , i.e., $\chi_I(\xi) = 1$ for $\xi \in I$ and $\chi_I(\xi) = 0$ for $\xi \in S \setminus I$. Suppose S is split into n ($n \geq 1$) pairwise disjoint arcs I_1, I_2, \dots, I_n (they are indexed in accordance with the fixed direction on S). Next, let h_1, h_2, \dots, h_n be some polynomials of degree at most $l - 1$ ($l \in \mathbb{N}$). The function $g(\xi) = \sum_{k=1}^n \chi_{I_k}(\xi) h_k(\xi)$, $\xi \in S$, is called a *piecewise-polynomial function* of degree at most $l - 1$ and with at most n pieces (in symbols, $g \in \mathcal{P}_n^l(S)$).

Let n and l be natural numbers with $n \geq l + 1$. We denote by $\mathcal{R}_n^l(S)$ the set of functions $r \in \mathcal{R}_n$ such that all poles of r lie on S and are of multiplicity at most l , and $r(z) = O(z^{-l-1})$ as $z \rightarrow \infty$.

For a function $g \in L_1(S)$, we put

$$(Cg)(z) = \frac{1}{2\pi i} \int_S \frac{g(\xi) d\xi}{\xi - z}, \quad z \in \mathbb{C} \setminus S$$

(the *Cauchy-type integral*).

Lemma 2.6. *For every $r \in \mathcal{R}_n^l(S)$ there is a unique function g in $\mathcal{P}_{n+1}^l(S)$ with*

$$(2.9) \quad r(z) = (Cg)^{(l)}(z), \quad z \in \mathbb{C} \setminus S.$$

On the other hand, if $g \in \mathcal{P}_n^l(S)$, then the function $r(z)$ given by (2.9) belongs to $\mathcal{R}_{l(n+1)}^l(S)$.

For $S = \mathbb{R}$ this lemma was proved in [17]. In the present case the proof is similar.

Lemma 2.7. *Suppose S is a simple Ahlfors curve, $1 < q < \infty$, $l \in \mathbb{N}$, and $1/\lambda = 1/q + l + 1$. Then for every $g \in \mathcal{P}_n^l(S)$ we have*

$$(2.10) \quad \|(Cg)^{(l+1)}\|_{L_\lambda(S)} \leq c(\kappa, q, l) n^{l+1} \|g\|_{L_q(S)}.$$

Remark. By Lemma 2.6, $(Cg)^{(l+1)}(z)$ is a rational function for $z \in \mathbb{C} \setminus S$; for $z \in S$ the symbol $(Cg)^{(l+1)}(z)$ means an analytic extension of this function.

Proof of Lemma 2.7. Let $g = \sum_{k=1}^n \chi_{I_k} h_k$, where $I_k \subset S$ and $h_k \in \mathcal{P}_{l-1}$ (see above). Then

$$(Cg)^{(l+1)}(z) = \sum_{k=1}^n (C\chi_{I_k} h_k)^{(l+1)}(z) \quad \text{for } z \in \mathbb{C}.$$

Suppose that (2.10) is fulfilled for $n = 1$. Then the above identity implies

$$\|(Cg)^{(l+1)}\|_{L_\lambda(S)}^\lambda \leq \sum_{k=1}^n \|(C\chi_{I_k} h_k)^{(l+1)}\|_{L_\lambda(S)}^\lambda \leq c^\lambda \sum_{k=1}^n \|h_k\|_{L_q(I_k)}^\lambda.$$

We estimate the sum on the right by using the Hölder inequality:

$$\sum_{k=1}^n \|h_k\|_{L_q(I_k)}^\lambda \leq \left(\sum_{k=1}^n 1\right)^{1-\frac{\lambda}{q}} \left(\sum_{k=1}^n \|h_k\|_{L_q(I_k)}^q\right)^{\lambda/q} \leq n^{\lambda(l+1)} \|g\|_{L_q(S)}^\lambda.$$

Thus, we have proved that if (2.10) is fulfilled for $n = 1$, then it is fulfilled for all $n \geq 1$.

Now, we prove (2.10) for $n = 1$. Suppose an arc I lies on S , $h \in \mathcal{P}_{l-1}$, and $g = \chi_I h$. We denote by ξ_0 the point of the arc I that splits it into two parts of equal length. Next, we split S into two disjoint sets J and T : $J = \{\xi \in S : |\xi - \xi_0| \leq |I|\}$, $T = S \setminus J$.

We have

$$(2.11) \quad (Cg)^{(l+1)}(z) = \frac{(l+1)!}{2\pi i} \int_I \frac{h(\xi) d\xi}{(\xi - z)^{l+2}}, \quad z \in \mathbb{C} \setminus I.$$

If $\xi \in I$ and $z \in T$, then $|\xi - z| = |(\xi_0 - z) - (\xi_0 - \xi)| \geq |\xi_0 - z| - |\xi_0 - \xi| \geq |\xi_0 - z| - \frac{1}{2}|I| \geq \frac{1}{2}|\xi_0 - z|$. Hence, (2.11) implies the inequality

$$\left| (Cg)^{(l+1)}(z) \right| \leq \frac{2^{l+1}(l+1)!}{\pi |\xi_0 - z|^{l+2}} \|h\|_{L_1(I)} \quad \text{for } z \in T.$$

Next, we apply (2.3) and the inequality $\|h\|_{L_1(I)} \leq |I|^{1-\frac{1}{q}} \|h\|_{L_q(I)}$. As a result, we obtain

$$(2.12) \quad \int_T \left| (Cg)^{(l+1)}(z) \right|^\lambda |dz| \leq c_1 \|h\|_{L_q(I)}^\lambda, \quad c_1 = c_1(\kappa, q, l).$$

To estimate the integral of the same function over J , we evaluate $(Cg)^{(l+1)}(z)$ by integration by parts. Namely, if ξ_1 and ξ_2 are the endpoints of I , then (2.11) implies the identity

$$(Cg)^{(l+1)}(z) = \sum_{j=0}^{l-1} \frac{(-1)^j (l-j)!}{2\pi i} \left(\frac{h^{(j)}(\xi_2)}{(\xi_2 - z)^{l+1-j}} - \frac{h^{(j)}(\xi_1)}{(\xi_1 - z)^{l+1-j}} \right) \quad \text{for } z \neq \xi_1, \xi_2.$$

Let $\Lambda_{kj}(z)$ ($k = 1, 2$; $j = 0, 1, 2, \dots, l-1$) denote the summands on the right in the latter identity. We have

$$\int_J |\Lambda_{kj}(z)|^\lambda |dz| = \left[\frac{(l-j)! |h^{(j)}(\xi_k)|}{2\pi} \right]^\lambda \int_J \frac{|dz|}{|\xi_k - z|^{\lambda(l+1-j)}}.$$

Next, we estimate the factor outside the integral by Lemma 2.5, and we estimate the integral with the help of (2.5). This leads to

$$(2.13) \quad \int_J |(Cg)^{(l+1)}(z)|^\lambda |dz| \leq c_2 \|h\|_{L_q(I)}^\lambda, \quad c_2 = c_2(\kappa, q, l).$$

Adding (2.12) and (2.13), we obtain (2.10) for $n = 1$. □

§3. APPLICATION OF THE EMBEDDING THEOREM AND CARLESON LEMMA

3A. The Hardy space and Blaschke products. A function f analytic in the disk $D = \{w : |w| < 1\}$ belongs to the Hardy space H_p , $0 < p \leq \infty$, if the following quasinorm (a norm for $1 \leq p \leq \infty$ and a p -norm for $0 < p < 1$) is finite: $\|f\|_{H_p} = \sup_{r \in (0,1)} \|f(\cdot r)\|_{L_p(\partial D)}$. It is well known (see [10, 11]) that any function f in H_p has nontangential boundary values $f(\xi)$ at almost all points $\xi \in \partial D$. Moreover, it turns out that $\|f\|_{H_p} = \|f\|_{L_p(\partial D)}$.

We shall also consider the Hardy space H_p^- consisting of functions analytic in $\overline{\mathbb{C}} \setminus \overline{D}$. Specifically, $g \in H_p^-$ if $g(\infty) = 0$ and the function $f(w) := g(1/w)$, $w \in D$, is in H_p . In this case $\|g\|_{H_p^-} = \|g\|_{L_p(\partial D)} = \|f\|_{H_p}$.

We introduce the family $\mathcal{L}' = \{Q'\}$ of simple polar dyadic squares Q' :

$$\begin{aligned} & \{w : |w| \geq 2\}; \\ & \left\{ r e^{i\theta} : 1 + \frac{1}{2^{k-2}} \leq r \leq 1 + \frac{1}{2^{k-3}}, \frac{\pi j}{2^{k-1}} \leq \theta \leq \frac{\pi(j+1)}{2^{k-1}} \right\}, \end{aligned}$$

where $k = 3, 4, 5, \dots$, and $j = 0, 1, 2, \dots, 2^k - 1$. Note that the domain $\{w : |w| \geq 2\}$ is called a square for convenience: its boundary is the circle $|w| = 2$. The boundary of any other square consists of four sides: two arcs of concentric circles centered at the origin and two radial segments between these circles. Any two squares in \mathcal{L}' may intersect at most in their boundaries, and the union of all these squares is the domain $\overline{\mathbb{C}} \setminus \overline{D}$.

Let w_1, w_2, \dots, w_m be some points of $\overline{\mathbb{C}} \setminus \overline{D}$ (a point may be repeated several times, in accordance with its multiplicity). Put

$$b_m(w) = \prod_{k=1}^m \frac{w - w_k^*}{1 - \overline{w_k^*}w}, \quad w_k^* = 1/\overline{w_k};$$

this is the *Blaschke product* with zeros at $w_1^*, w_2^*, \dots, w_m^*$ (consequently, with poles at w_1, w_2, \dots, w_m).

Lemma 3.1 (the author [16] and Dyn'kin [8, 9]). *For every $a > 1$ the number of squares Q' in \mathcal{L}' that meet the set $\{w : |b_m(w)| \geq a\}$ does not exceed $c(a)m$.*

This lemma was obtained in [8, 9, 16] for $a = 2$. Therefore, it is also true for $a \geq 2$. But if $a \in (1, 2)$, we arrive at the required statement by replacing $b_m(z)$ with $b_m(z)^k$, where k is the smallest positive integer satisfying $a^k \geq 2$. As a result, we obtain the assertion for every $a > 0$ with $c(a) = ca/(a - 1)$, where $c > 0$ is a universal constant.

Lemma 3.1 was applied efficiently in [8, 14, 16] in order to prove certain Bernstein-type inequalities for derivatives of rational functions. However, the proof of Theorem 1.1 will also require the following Carleson lemma.

Lemma 3.2 (Carleson [10, pp. 207–219]). *There exist absolute constants $1 < a_1 < 4$ and $a_2 \geq 1$ such that some squares Q' of the family \mathcal{L}' form a set Π' that is a union of at most m pairwise disjoint simply connected domains satisfying the following conditions:*

- (i) $\{w_k\}_{k=1}^m \subset \Pi'$, and each of the above simply connected domains contains at least one point w_k ;
- (ii) $a_1 \leq |b_m(w)| \leq 4$ for all $w \in \partial\Pi'$;
- (iii) for all $\theta \in [0, 2\pi]$ and $h \in (0, 1]$ we have $|\{re^{i(\theta+t)} \in \partial\Pi' : 0 \leq t \leq h, 1 < r \leq 1+h\}| \leq a_2h$, i.e., arc length on $\partial\Pi'$ is a Carleson measure.

We emphasize that the Carleson lemma is stated here in the form convenient for our present purposes. Therefore, when consulting [10], one should address the corresponding lemma together with its proof. A somewhat different version of this lemma (see [11]) was employed by Dyn'kin [9].

Consider the set Π' the existence of which was stated in Lemma 3.2. By Lemma 3.1, the boundary $\partial\Pi'$ is formed by at most $4c(a_1)m$ sides of the squares Q' that constitute Π' . Recall also that Π' is split into at most m simply connected closed domains. Therefore, $\partial\Pi'$ can be split into ν ($\nu \leq 2m$) arcs $\Gamma'_1, \Gamma'_2, \dots, \Gamma'_\nu$ consisting of at most $[4c(a_1)] + 1$ sides of squares Q' . Indeed, let $\partial\Pi'$ consist of r ($1 \leq r \leq m$) closed Jordan curves T'_1, T'_2, \dots, T'_r . Each of these curves consists of links (the latter are certain sides of the squares Q'). Let m'_j be the number of links of T'_j ($j = 1, \dots, r$). Then $m'_1 + m'_2 + \dots + m'_r \leq 4c(a_1)m$. The curve T'_s ($1 \leq s \leq r$) can be split into $[m'_s/([4c(a_1)] + 1)] + 1$ arcs so that each arc is a union of at most $[4c(a_1)] + 1$ links. For the total number ν of these arcs we have

$$\nu := \sum_{s=1}^r \left\{ \left[\frac{m_s}{[4c(a_1)] + 1} \right] + 1 \right\} \leq r + \frac{1}{4c(a_1)} \sum_{s=1}^r m_s \leq r + m \leq 2m.$$

Lemma 3.3. *For every function g in H_p^- with $0 < p < \infty$ and every point $\eta'_j \in \Gamma'_j$ ($j = 1, 2, \dots, \nu$) the inequality*

$$(3.1) \quad \sum_{j=1}^{\nu} |g(\eta'_j)|^p |\Gamma'_j| \leq c \|g\|_{H_p^-}^p$$

is true with some absolute constant c .

Proof. We introduce a discrete measure by putting

$$\mu(K) := \sum_{\eta'_j \in K} |\Gamma'_j|$$

for $K \subset \mathbb{C} \setminus \overline{D}$. By Lemma 3.2(iii), μ is a Carleson measure in $\mathbb{C} \setminus \overline{D}$. Hence, (3.1) is fulfilled by the Carleson embedding theorem [10, 11]. \square

3B. The Smirnov spaces E_p and E_p^- . Let G be a bounded simply connected Jordan domain with rectifiable boundary ∂G . We fix a point $z_0 \in G$ and denote by $z = \varphi_+(w)$ the Riemann function that maps the unit disk conformally onto G and satisfies $\varphi_+(0) = z_0$, $\varphi'_+(0) > 0$. Let $w = \psi_+(z)$ be the inverse mapping. In accordance with Smirnov's definition (see [10, 20]), a function f analytic in G belongs to the space $E_p = E_p(G)$, $0 < p \leq \infty$, if the function $g(w) := f[\varphi_+(w)] \sqrt[p]{\varphi'_+(w)}$ belongs to H_p (if $p = \infty$, we agree that $\sqrt[p]{\varphi'_+(w)} = 1$). An arbitrary function $f \in E_p$ has nontangential boundary values $f(\xi)$ at almost every $\xi \in \partial G$. Moreover, $\|f\|_{E_p} := \|f\|_{L_p(\partial G)} = \|g\|_{H_p}$. This definition of E_p is equivalent to the definition by Keldysh and Lavrent'ev (see the Introduction). The space $E_p^- = E_p^-(\Omega)$ in the domain $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ is defined similarly. Namely, let $z = \varphi_-(w)$ be the Riemann function that maps conformally the domain $|w| > 1$ onto Ω and satisfies $\varphi_-(\infty) = \infty$, $\varphi'_-(\infty) > 0$. Denote by $w = \psi_-(z)$ the inverse mapping. Then a function f analytic in Ω belongs to $E_p^- = E_p^-(\Omega)$ if $g(w) := f[\varphi_-(w)] \sqrt[p]{\varphi'_-(w)} \in H_p^-$. In this case we also have $\|f\|_{E_p^-} := \|f\|_{L_p(\partial\Omega)} = \|g\|_{H_p^-}$.

Let S be a simple rectifiable Jordan curve. The boundary $\partial\Omega$ of the domain $\Omega := \overline{\mathbb{C}} \setminus S$ consists of two copies of S passed in opposite directions. By analogy, we can introduce the space E_p^- also in this situation.

The main constructions will be carried out in domains Ω of the form $\overline{\mathbb{C}} \setminus \overline{G}$ or $\overline{\mathbb{C}} \setminus S$. In what follows, we denote by K the continuum \overline{G} or S ; in place of $z = \varphi_-(w)$ and $w = \psi_-(z)$ we write $z = \varphi(w)$ and $w = \psi(z)$, respectively.

We list some properties of φ and ψ . Namely, $\varphi' \in H_1^- + \varphi'(\infty)$ and $\psi' \in E_1^- + \psi'(\infty)$. Therefore, φ and ψ extend to continuous functions in the closed domains $\overline{\mathbb{C}} \setminus D$ and $\overline{\Omega}$, respectively. Moreover, the functions $\varphi|_{\partial D}$ and $\psi|_{\partial\Omega}$ are absolutely continuous. The quantity $\gamma(K) := \varphi'(\infty)$ coincides with the *analytic capacity of the continuum K* . The following inequalities are true (see [12]):

$$(3.2) \quad \rho(z, K) \leq \gamma(K) |\psi(z)|, \quad z \in \Omega;$$

$$(3.3) \quad \gamma(K) \leq \text{diam}(K) \leq 4\gamma(K),$$

where $\text{diam}(K) = \max\{|z_1 - z_2| : z_1, z_2 \in K\}$ is the diameter of K .

The following double inequality is a consequence (see [6]) of the Koebe theorem:

$$(3.4) \quad \frac{1}{4} \frac{\rho(\varphi(w), K)}{|w| - 1} \leq |\varphi'(w)| \leq 4 \frac{\rho(\varphi(w), K)}{|w| - 1} \quad \text{for } |w| > 1.$$

Let z_k ($k = 1, 2, \dots, m$) be some points in Ω , and let $w_k = \psi(z_k)$. Starting with these points, we construct sets Π' and Γ'_j ($j = 1, 2, \dots, \nu$) as in Subsection 3A. Next, we transfer Π' and Γ'_j into Ω by putting $\Pi = \varphi(\Pi')$, $\Gamma_j = \varphi(\Gamma'_j)$.

Applying the distortion theorem for univalent functions (see, e.g., [21]), we easily see that there is an absolute constant $\delta \in (0, 1)$ such that

$$(3.5) \quad \delta \leq \left| \frac{\varphi'(w_1)}{\varphi'(w_2)} \right| \leq \frac{1}{\delta}$$

for every $w_1, w_2 \in \Gamma'_j$ ($j = 1, 2, \dots, \nu$).

Lemma 3.4. *If Γ' is an arbitrary curve among $\Gamma'_1, \Gamma'_2, \dots, \Gamma'_\nu$, and $\Gamma = \varphi(\Gamma')$, then*

$$(3.6) \quad \rho(\Gamma, K) \leq 2 \operatorname{diam}(K),$$

$$(3.7) \quad \delta |\varphi'(w)| |\Gamma'| \leq |\Gamma| \leq \frac{1}{\delta} |\varphi'(w)| |\Gamma'| \quad \text{for } w \in \Gamma',$$

$$(3.8) \quad c_1 \rho(\Gamma, K) \leq |\Gamma| \leq c_2 \rho(\Gamma, K),$$

where δ is the number occurring in (3.5), and c_1, c_2 are positive absolute constants.

Proof. If $w \in \Gamma'$, then $1 < |w| \leq 2$, and (3.6) is a consequence of (3.2) and (3.3): $\rho(\varphi(w), K) \leq |w| \gamma(K) \leq 2 \operatorname{diam}(K)$.

To obtain (3.7), it suffices to note that $|\Gamma| = \int_{\Gamma'} |\varphi'(w)| |dw|$ and to use estimates (3.5). Inequalities (3.8) are consequences of (3.4) and (3.7). \square

Lemma 3.5. *If $f \in E_p^-, 0 < p < \infty$, then for arbitrary points $\eta_j \in \Gamma_j$ ($j = 1, 2, \dots, \nu$) we have*

$$\sum_{j=1}^{\nu} |f(\eta_j)|^p |\Gamma_j| \leq c \|f\|_{E_p^-}^p,$$

where $c > 0$ is an absolute constant.

Proof. By Smirnov's definition of E_p^- , the function $g(w) := f[\varphi(w)] \sqrt[p]{\varphi'(w)}$ belongs to H_p^- and $\|f\|_{E_p^-} = \|g\|_{H_p^-}$. We use Lemma 3.3, where we take $\eta'_j = \psi(\eta_j)$. Since $|g(\eta'_j)|^p = |f(\eta'_j)|^p \cdot |\varphi'(\eta'_j)|$, inequality (3.1) takes the form

$$\sum_{j=1}^{\nu} |f(\eta_j)|^p |\varphi'(\eta'_j)| |\Gamma'_j| \leq c \|f\|_{E_p^-}^p.$$

It remains to apply the second inequality in (3.7). \square

3C. Withdrawal of poles of a meromorphic function from a domain to the boundary. Here we employ the constructions and notation described in Subsection 3B. For $u \in L_1(\Gamma)$, denote by $(C_\Gamma u)(z)$ the Cauchy-type integral (see Subsection 2C) along Γ .

Lemma 3.6. *Suppose $\partial\Omega$ is an Ahlfors curve and Γ is an arbitrary arc among the arcs Γ_j ($j = 1, 2, \dots, \nu$) constructed above. Let $u \in L_\infty(\Gamma)$, and let $0 < p < 1$; we put $l = [1/p]$ and $\lambda = p/(p+1)$. Then there exists a rational function $(N_\Gamma u)(z)$ of degree at most l , with poles on $\partial\Omega$, and such that*

$$(3.9) \quad \int_{\partial\Omega} |(C_\Gamma u - N_\Gamma u)(z)|^p |dz| \leq c_1 \|u\|_{L_\infty(\Gamma)}^p |\Gamma|;$$

$$(3.10) \quad \int_{\partial\Omega} |(C_\Gamma u - N_\Gamma u)'(z)|^\lambda |dz| \leq c_2 \|u\|_{L_\infty(\Gamma)}^\lambda |\Gamma|^\frac{\lambda}{p},$$

where c_1 and c_2 depend only on p and κ .

Proof. We use the identity

$$(3.11) \quad \frac{1}{\eta - z} = \frac{1}{\xi_1 - z} + \frac{\xi_1 - \eta}{(\xi_1 - z)(\xi_2 - z)} + \frac{(\xi_1 - \eta)(\xi_2 - \eta)}{(\xi_1 - z)(\xi_2 - z)(\xi_3 - z)} + \cdots \\ + \frac{(\xi_1 - \eta)(\xi_2 - \eta) \cdots (\xi_{l-1} - \eta)}{(\xi_1 - z)(\xi_2 - z) \cdots (\xi_{l-1} - z)} + \frac{(\xi_1 - \eta)(\xi_2 - \eta) \cdots (\xi_l - \eta)}{(\xi_1 - z)(\xi_2 - z) \cdots (\xi_l - z)(\eta - z)}.$$

Choose $\xi_1, \xi_2, \dots, \xi_l \in \partial\Omega$ in such a way that

$$(3.12) \quad c_3|\Gamma| \leq \rho(\xi_k, \Gamma) \leq c_4|\Gamma| \quad \text{for } k = 1, 2, \dots, l,$$

$$(3.13) \quad c_5|\Gamma| \leq |\xi_k - \xi_s| \leq c_6|\Gamma| \quad \text{for } l \geq 2 \text{ and } k \neq s.$$

Here the constants c_j ($j = 3, 4, 5, 6$) depend only on l (in other words, on p) and κ . The existence of such points ξ_k follows from (3.6) and (3.8).

We denote the sum of the first l terms on the right in (3.11) by $v(z, \eta)$ and put

$$(N_\Gamma u)(z) = \frac{1}{2\pi i} \int_\Gamma u(\eta)v(z, \eta) d\eta.$$

Clearly, $N_\Gamma u$ is a rational function of the required form. Next, we put

$$\mu(z) = (\xi_1 - z)(\xi_2 - z) \cdots (\xi_l - z).$$

By (3.11), we have

$$(3.14) \quad (C_\Gamma u - N_\Gamma u)(z) = \frac{1}{2\pi i \mu(z)} \int_\Gamma \frac{u(\eta)\mu(\eta)}{\eta - z} d\eta, \quad z \in \partial\Omega.$$

Now, we prove (3.9). We split $\partial\Omega$ into two disjoint sets I and T : $I = \{z \in \partial\Omega : |z - \xi_1| \leq c_7|\Gamma|\}$ and $T = (\partial\Omega) \setminus I$. Here $c_7 = 1$ if $l = 1$ and $c_7 = lc_6$ if $l \geq 2$, where c_6 is the constant occurring in (3.13). For brevity, put $a = \|u\|_{L^\infty(\Gamma)}$. From (3.8) and (3.12)–(3.14) we deduce the existence of a constant $c_8 = c_8(p, \kappa)$ such that

$$|(C_\Gamma u - N_\Gamma u)(z)| \leq \frac{c_8 a |\Gamma|^{l+1}}{|z - \xi_1|^{l+1}} \quad \text{for } z \in T.$$

Applying (2.3), we obtain the required estimate for the contribution to (3.9) given by the integral over T :

$$(3.15) \quad \int_T |C_\Gamma u - N_\Gamma u|^p \leq c_9 a^p |\Gamma|, \quad c_9 = c_9(p, \kappa).$$

To obtain a similar estimate for the contribution of the integral over I , we observe that

$$\frac{1}{\mu(z)} = \sum_{k=1}^l \frac{1}{\mu'(\xi_k)(\xi_k - z)}, \quad z \in \mathbb{C}.$$

Taking (3.12) and (3.13) into account, we deduce the estimate

$$\left| \frac{\mu(\eta)}{\mu(z)} \right| \leq c_{10}(p, \kappa) |\Gamma| \sum_{k=1}^l \frac{1}{|\xi_k - z|} \quad \text{for } \eta \in \Gamma \text{ and } z \in I.$$

Combining this inequality with (3.8) and (3.14), we see that

$$|(C_\Gamma u - N_\Gamma u)(z)| \leq c_{11} a |\Gamma| \sum_{k=1}^l \frac{1}{|\xi_k - z|} \quad \text{for } z \in I$$

with some constant $c_{11} = c_{11}(p, \kappa)$. Applying (2.5), we obtain the required estimate for the contribution of the integral over I to (3.9):

$$(3.16) \quad \int_I |C_\Gamma u - N_\Gamma u|^p \leq c_{12} a^p |\Gamma|, \quad c_{12} = c_{12}(p, \kappa).$$

To finish the proof of (3.9), it remains to add (3.15) and (3.16).

Finally, (3.10) is proved along the same lines as (3.9). □

Lemma 3.7. *Let K be the continuum described above. Suppose that $\partial\Omega$ is an Ahlfors curve, r is a rational function of degree $m \geq 1$ all poles of which lie in Ω , $h \in E_p^-$, $0 < p < 1$, $u = r + h$, and $\lambda = \frac{p}{p+1}$. Then there exists a rational function r_0 of degree at most $2m/p$ all poles of which lie on $\partial\Omega$ and*

$$(3.17) \quad \|r - r_0\|_p \leq c_1 \|u\|_p,$$

$$(3.18) \quad \|r' - r_0'\|_\lambda \leq c_2 m \|u\|_p.$$

Here the constants c_1 and c_2 depend on p and κ only, and the quasinorms are evaluated on $\partial\Omega$.

Proof. Starting with the poles z_1, z_2, \dots, z_m of r , we construct the sets Π , $\partial\Pi$, and Γ_j ($j = 1, 2, \dots, \nu$) as in Subsection 3B. By the residue calculus, it is easily seen that, under a proper orientation of $\partial\Pi$, we have

$$(3.19) \quad r(z) = \frac{1}{2\pi i} \int_{\partial\Pi} \frac{u(\eta) d\eta}{\eta - z}, \quad z \in \partial\Omega.$$

We show also that

$$(3.20) \quad \sum_{j=1}^\nu \|u\|_{L^\infty(\Gamma_j)}^p |\Gamma_j| \leq c_3 \|u\|_p^p$$

for some $c_3 = c_3(p)$. Indeed, the function $f(\eta) := u(\eta)/\psi(\eta)b_m[\psi(\eta)]$ (see the notation after formula (3.4)) belongs to E_p^- and $|f(\eta)| = |u(\eta)|$ a.e. on $\partial\Omega$. Therefore, by Lemma 3.5, we see that for every $\eta_j \in \Gamma_j$ the following inequality is true:

$$(3.21) \quad \sum_{j=1}^\nu |f(\eta_j)|^p |\Gamma_j| \leq c_4 \|u\|_p^p, \quad c_4 = c_4(p).$$

However, by Carleson's lemma 3.2(ii), for $\eta \in \partial\Pi$ we have $|\psi(\eta)b_m[\psi(\eta)]| \leq 8$. Consequently, (3.20) is implied by (3.21). Identity (3.19) can be written in the form

$$(3.22) \quad r(z) = \sum_{j=1}^\nu (C_{\Gamma_j} u)(z), \quad z \in \partial\Omega.$$

Let $(N_{\Gamma_j} u)(z)$ be the rational function whose existence is guaranteed by Lemma 3.6. We show that

$$(3.23) \quad r_0(z) := \sum_{j=1}^\nu (N_{\Gamma_j} u)(z)$$

is the required rational function.

Indeed, the poles of r_0 lie on $\partial\Omega$, and $\deg r_0 \leq \sum_{j=1}^{\nu} \deg N_{\Gamma_j} \leq \nu l \leq 2m/p$. We prove (3.17):

$$\begin{aligned} \|r - r_0\|_p^p &\leq \left\| \sum_{j=1}^{\nu} (C_{\Gamma_j} u - N_{\Gamma_j} u) \right\|_p^p \\ &\leq \sum_{j=1}^{\nu} \|C_{\Gamma_j} u - N_{\Gamma_j} u\|_p^p \leq c_5 \sum_{j=1}^{\nu} \|u\|_{L^\infty(\Gamma_j)}^p |\Gamma_j| \leq c_6 \|u\|_p^p. \end{aligned}$$

(We have used (3.9), (3.20), (3.22), and (3.23).) A similar argument with (3.10) in place of (3.9) yields (3.18):

$$\|r' - r'_0\|_\lambda^\lambda \leq c_7 \sum_{j=1}^{\nu} \|u\|_{L^\infty(\Gamma_j)}^\lambda |\Gamma_j|^{\frac{\lambda}{p}} \leq c_8 m^\lambda \|u\|_p^\lambda.$$

(We have used the Hölder inequality at the last step.) □

Remarks. (i) Suppose G is an Ahlfors domain, $h \in E_p(G)$, $0 < p < 1$, r is a rational function of degree $m \geq 1$ with poles in G , and $u = r + h$. An analog of Lemma 3.7 is true in this case. Specifically, it is possible to construct a rational function r_0 of degree at most $2m/p$, with poles on ∂G only, and such that r_0 satisfies (3.17) and (3.18).

(ii) If G is a Lavrent'ev domain, $h \in E_1(\Omega)$, $\Omega = \overline{\mathbb{C}} \setminus G$, r is a rational function of degree $m \geq 1$ all poles of which lie in G , and $u = r + h$, then there exists a rational function r_0 of degree at most $2m$, with all poles in G , and satisfying (3.17), (3.18) with $p = 1$.

We indicate the changes to be made in the proof of Lemma 3.6 in order to treat the setting (ii). After these changes, an analog of Lemma 3.7 is proved much as before.

Here we need to use some results about quasiconformal maps (the necessary references can be found in [19]). Since ∂G is a Lavrent'ev curve, it is a quasicircle; consequently, there exists a quasiconformal involution $*$ of the plane $\overline{\mathbb{C}}$ relative to ∂G . In our case $l = 1$, and we must specify the choice of ξ_1 . We put $\xi_1 = \eta_1^*$, where η_1 is a fixed point of Γ . The remaining arguments are similar to the case where $1/2 < p < 1$.

Naturally, Remark (ii) has an analog in the setting described in Remark (i).

§4. APPLICATION OF A GENERAL HARDY-LITTLEWOOD EMBEDDING THEOREM

4A. Piecewise polynomial approximation in E_q^- . We split ∂D into $m \geq 2$ arcs with endpoints at $w_k = e^{it_k}$, $t_1 < t_2 < \dots < t_m < t_{m+1} := t_1 + 2\pi$. Next, we introduce the domains

$$\Delta'_k = \{\rho e^{it} : t_k < t < t_{k+1}, 1 < \rho < 1 + t_{k+1} - t_k\}, \quad k = 1, 2, \dots, m.$$

Lemma 4.1. *Suppose $g \in H_p^-$, $0 < p < \infty$, and $m \geq 2$. Then the points $\{w_k\}_{k=1}^m$ can be chosen in such a way that*

$$(4.1) \quad \int_{\partial\Delta'_k} |g(w)|^p |dw| \leq \frac{c}{m} \|g\|_{H_p^-}^p \quad \text{for } k = 1, 2, \dots, m,$$

where $c > 0$ is an absolute constant. Moreover, any two points among $\{w_k\}_{k=1}^m$ can be fixed arbitrarily.

Proof. For every $w \in \partial D$, we denote by $\Gamma(w)$ the set of points η satisfying at least one of the following two conditions: $|\eta| > 2$; η is inside the right angle formed by the rays $\{tw \pm i(1-t)w : t > 1\}$. We introduce the *Hardy-Littlewood maximal function*

$$(Mg)(w) = \sup\{|g(\eta)| : \eta \in \Gamma(w)\}, \quad w \in \partial D.$$

It is well known [10, 11] that $Mg \in L_p(\partial D)$ and

$$(4.2) \quad \int_0^{2\pi} (Mg)^p(e^{it}) dt \leq c_1 \|g\|_{H_p^-}^p.$$

We show that for every $t_1 < t_2 < \dots < t_m < t_{m+1} := t_1 + 2\pi$ we have

$$(4.3) \quad \int_{\partial\Delta'_k} |g(w)|^p |dw| \leq 10 \int_{t_k}^{t_{k+1}} (Mg)^p(e^{it}) dt, \quad k = 1, 2, \dots, m.$$

For instance, let $k = 1$. The curve $\partial\Delta'_1$ consists of four parts: two arcs $T_1 = \{e^{it} : t_1 < t < t_2\}$ and $T_2 = \{(1 + t_2 - t_1)e^{it} : t_1 < t < t_2\}$, and two radial segments $I_s = \{\rho e^{it_s} : 1 < \rho < 1 + t_2 - t_1\}$, where s is 1 or 2. The definition of Mg readily yields

$$\int_{T_1 \cup T_2} |g(w)|^p |dw| \leq (2 + t_2 - t_1) \int_{t_1}^{t_2} (Mg)^p(e^{it}) dt.$$

We estimate the contributions of integration over I_1 and I_2 to the right-hand side of (4.3). For definiteness, consider the segment I_1 . It is easily seen that $\rho e^{it_1} \in \Gamma[\exp(i(t_1 + \frac{t_2 - t_1}{2}))]$ for $\rho > 1$, and therefore, $|g(\rho e^{it_1})| \leq (Mg)[\exp(i(t_1 + \frac{t_2 - t_1}{2}))]$. This readily implies the inequality

$$\int_{I_1} |g(w)|^p |dw| \leq 2 \int_{t_1}^{(t_1+t_2)/2} (Mg)^p(e^{it}) dt.$$

Similarly,

$$\int_{I_2} |g(w)|^p |dw| \leq 2 \int_{(t_1+t_2)/2}^{t_2} (Mg)^p(e^{it}) dt.$$

In order to obtain (4.3), it remains to add the estimates for the contributions of integration over $T_1 \cup T_2$, I_1 , and I_2 .

We show how to choose the points $w_k = e^{it_k}$ so as to ensure (4.1). Let $w' = e^{it'}$, $w'' = e^{it''}$ ($t' < t'' < t' + 2\pi$) be two arbitrary points of ∂D . If $m = 2$, put $t_1 = t'$, $t_2 = t''$. In this case, (4.2) and (4.3) imply (4.1) with $c = 20c_1$.

Now, let $m \geq 3$. Putting $t_1 = t'$, we define t_2, t_3, \dots, t_{m-1} by induction in such a way that

$$\int_{t_k}^{t_{k+1}} (Mg)^p(e^{it}) dt = \frac{1}{m-1} \int_0^{2\pi} (Mg)^p(e^{it}) dt, \quad k = 1, 2, \dots, m-1,$$

and put $t_m := t_1 + 2\pi$. If t'' is not among t_2, t_3, \dots, t_{m-1} , we add t'' to the collection $\{t_k\}_{k=1}^m$ and renumber the resulting collection. But if t'' is among t_2, t_3, \dots, t_{m-1} , we may take, e.g., $(t_1 + t_2)/2$ as an additional point. In any case, (4.2) and (4.3) again imply (4.1) with $c = 20c_1$. □

Let S be a simple rectifiable Jordan curve, and let $\Omega = \overline{\mathbb{C}} \setminus S$. In this case $\partial\Omega$ consists of two copies S_1 and S_2 of S passed in opposite directions. We agree that Ω remains to the right when we go around $\partial\Omega$. Suppose z_1, z_2, \dots, z_m are pairwise distinct points of $\partial\Omega$ enumerated in accordance with the direction fixed on $\partial\Omega$. Putting $w_k = \varphi(z_k)$, $k = 1, 2, \dots, m$, we construct a domain Δ'_k starting with these points, and denote by Δ_k the image of Δ'_k under the mapping $z = \varphi(w)$. The change of variables $z = \varphi(w)$ allows us to readily deduce the next Lemma 4.2 from Lemma 4.1.

Lemma 4.2. *Suppose $f \in E_p^-(\Omega)$, $0 < p < \infty$, and $m \geq 2$. Then there exist points $\{z_k\}_{k=1}^m$ on $\partial\Omega$ satisfying the following conditions:*

- (i) *the endpoints of the arc S belong to the collection $\{z_k\}_{k=1}^m$;*
- (ii) *$\int_{\partial\Delta_k} |f(z)|^p |dz| \leq \frac{c}{m} \|f\|_{E_p^-}^p$ for $k = 1, 2, \dots, m$.*

The next Lemma 4.3 is well known for the disk, in which case it is called the Hardy–Littlewood embedding theorem [1, 10].

Lemma 4.3 (the author [19]). *Suppose Δ is a Lavrent'ev domain, $q \in (0, \infty]$, $l \in \mathbb{N}$, and $1/p = l + 1/q$. If f is analytic in Δ and $f^{(l)} \in E_p$, then $f \in E_q$ and there exists a polynomial h of degree at most $l - 1$ such that*

$$\|f - h\|_{E_q} \leq c \|f^{(l)}\|_{E_p}, \quad c = c(\theta, q, l).$$

Here θ is the constant in the Lavrent'ev condition (1.2).

In Lemma 4.4 we shall use the piecewise polynomial functions $\mathcal{P}_m^l(S_j)$ introduced in Subsection 2C. Recall also that by S_j ($j = 1, 2$) we denote two shores of the boundary of $\Omega = \overline{\mathbb{C}} \setminus S$. For $\zeta \in S$, let $f_j(\zeta)$ denote the nontangential boundary value of $f(z)$, $z \in \Omega$, evaluated as z tends to S from the side of S_j .

Lemma 4.4. *Suppose S is a simple Lavrent'ev curve, $q \in (0, \infty]$, $l \in \mathbb{N}$, and $1/p = l + 1/q$. If f is analytic in Ω , $f(\infty) = 0$, and $f^{(l)} \in E_p^-$, then*

- (i) $f \in E_q^-$ and $\|f\|_{E_q} \leq c_1 \|f^{(l)}\|_{E_p^-}$;
- (ii) for every $m \geq 2$ there exist $g_j \in \mathcal{P}_{m_j}^{(l)}(S_j)$ ($j = 1, 2$; $m_1 + m_2 \leq m$) such that

$$\|f_j - g_j\|_{L_q(S_j)} \leq \frac{c_2}{m^l} \|f^{(l)}\|_{E_q^-},$$

where c_1 and c_2 depend only on θ , q , and l .

Proof. We apply Lemma 4.2 with $f^{(l)}$ in place of f . Consider the domains Δ_k , $k = 1, 2, \dots, m$, arising in that lemma. It turns out that all of them are Lavrent'ev domains. Moreover [13, Proposition 1.5], there exists $\tilde{\theta} \geq 1$ depending only on θ in condition (1.2) and such that for every $\zeta', \zeta'' \in \partial\Delta_k$ we have $|I(\zeta', \zeta')| \leq \tilde{\theta} |\zeta' - \zeta''|$, where $I(\zeta', \zeta'')$ is the shortest among the two arcs of $\partial\Delta_k$ that join ζ' and ζ'' . It should be emphasized that $\tilde{\theta}$ depends neither on m , nor on the choice of the points z_k . Therefore, $f|_{\Delta_k} \in E_q(\Delta_k)$ by Lemma 4.3, and for every $k = 1, 2, \dots, m$ there exists a polynomial $h_k \in \mathcal{P}_{l-1}$ satisfying

$$(4.4) \quad \|f - h_k\|_{E_q(\Delta_k)} \leq \frac{c_3}{m^{l+1/q}} \|f^{(l)}\|_{E_p^-(\Omega)},$$

where $c_3 = c_3(\theta, q, l)$.

The parts of the boundaries of all domains Δ_k adjacent to S_j will be denoted by I_{js} ($s = 1, 2, \dots, m_j$); let h_{js} be the corresponding polynomials. We show that the function

$$g_j := \sum_{s=1}^{m_j} \chi_{I_{js}} h_{js}$$

has the required properties. Indeed, for $0 < q < \infty$ formula (4.4) implies

$$\|f_j - g_j\|_{L_q(S_j)}^q = \sum_{s=1}^{m_j} \|f_j - h_{js}\|_{L_q(I_{js})}^q \leq \frac{c_3^q}{m^{ql+1}} m_j \|f^{(l)}\|_{E_p^-}^q \leq \frac{c_3^q}{m^{ql}} \|f^{(l)}\|_{E_p^-}^q.$$

This yields (ii) for $0 < q < \infty$. If $q = \infty$, it suffices to modify the latter calculation in an obvious way.

We turn to statement (i). The arguments given above for $m = 2$ show that $f \in E_q^-(\Omega)$. It remains to verify the inequality in (i). By (ii) with $m = 2$, it suffices to show that $\|g_j\|_{L_q(S_j)} \leq c_4 \|f^{(l)}\|_{E_p^-}$. This is easy if we observe (see [19]) that g_j is a Taylor polynomial of f . The details are left to the reader. \square

4B. Proof of Theorem 1.1. We need also Lemmas 4.5–4.7 below. Recall (see Subsection 2C) that $(Cg)(z)$, $z \in \Omega$, stands for the Cauchy-type integral over S . We agree that S in this integral is oriented as S_1 .

Lemma 4.5. *Suppose S is a simple Lavrent'ev curve, $1 < q < \infty$, $l \in \mathbb{N}$, $1/p = l + 1/q$, and $g \in L_1(S)$. If $(Cg)^{(l)} \in E_p^-$, then*

- (i) $g \in L_q(S)$ and $\|g\|_{L_q(S)} \leq c_1 \|(Cg)^{(l)}\|_{E_p^-}$;
- (ii) for every $m \geq 1$ there exists a function $u \in \mathcal{P}_m^l(S)$ such that

$$\|g - u\|_{L_q(S)} \leq \frac{c_2}{m^l} \|(Cg)^{(l)}\|_{E_p^-}.$$

Here the constants c_1 and c_2 depend only on θ, q , and l .

Proof. We use Lemma 4.4 with $f = Cg$. Then (see [3, 7, 13]) $g = f_2 - f_1$ a.e. on S . Thus, (i) and (ii) for $m = 1$ are implied by Lemma 4.4(i). If $m \geq 2$, we use Lemma 4.4(ii). Clearly, the function $u = g_2 - g_1$ satisfies condition 4.5(ii). \square

Lemma 4.6. *Suppose S is an Ahlfors curve, $f \in L_p(S)$, $0 < p < 1$, $k \in \mathbb{N}$. Then there exists a monic algebraic polynomial a of degree k , with zeros only on S , and such that*

$$\int_S \left| \frac{f(z)}{a(z)} \right|^p |dz| \leq \frac{\kappa^{kp}}{|S|^{kp(1-p)^k}} \int_S |f(z)|^p |dz|.$$

Proof. Let $\eta = (\eta_1, \eta_2, \dots, \eta_k) \in S^k$, let $a(z, \eta) = (z - \eta_1)(z - \eta_2) \cdots (z - \eta_k)$, and let $|d\eta| = |d\eta_1| \cdots |d\eta_k|$. By the Fubini theorem and (2.5), we have

$$\int_{S^k} |d\eta| \int_S \left| \frac{f(z)}{a(z, \eta)} \right|^p |dz| \leq |S|^{k(1-p)} \cdot \frac{\kappa^{pk}}{(1-p)^k} \int_S |f(z)|^p |dz|.$$

Arguing by contradiction, from this inequality we deduce that for some $\eta_0 \in S^k$ the polynomial $a(z) := a(z, \eta_0)$ satisfies the claim of the lemma. \square

Lemma 4.7. *Suppose S is a simple Lavrent'ev curve, $0 < p < 1$, $1/p \notin \mathbb{N}$, and $1/\lambda = 1 + 1/p$. If $r \in \mathcal{R}_n \cap L_p(S)$ and all poles of r lie on S , then*

$$(4.5) \quad \|r'\|_\lambda \leq c_1 n \|r\|_p, \quad c_1 = c_1(\theta, p).$$

Proof. Suppose first that $r(z) = O(z^{-l-1})$ as $z \rightarrow \infty$, where $l = [1/p]$. In other words, $r \in \mathcal{R}_n^l(S)$. By Lemma 2.6, there exists a function $g \in \mathcal{P}_{n+1}^l(S)$ such that $r(z) = (Cg)^{(l)}(z)$ for $z \in \mathbb{C} \setminus S$. Next, by Lemma 4.5, for every $j = 0, 1, \dots, s := [\log_2(n + 1)]$ there exists a function $u_j \in \mathcal{P}_{2^j}^l(S)$ satisfying

$$(4.6) \quad \|g - u_j\|_q \leq \frac{c_2}{2^{lj}} \|r\|_p, \quad 1/q = 1/p - l.$$

Moreover, $u_0 \equiv 0$. Almost everywhere on S we have $g = (u_1 - u_0) + (u_2 - u_1) + \cdots + (u_s - u_{s-1}) + (g - u_s)$. For short, we put $v_1 = u_1 - u_0, v_2 = u_2 - u_1, \dots, v_s = u_s - u_{s-1}$, and $v_{s+1} = g - u_s$. Clearly, $v_j \in \mathcal{P}_{2^j}^l$ and

$$(4.7) \quad \|v_j\|_q \leq \frac{c_3}{2^{lj}} \|r\|_p, \quad j = 1, 2, \dots, s + 1,$$

by (4.6).

Now we apply Lemma 2.7. In our case, $1/q + l + 1 = 1/p + 1 = 1/\lambda$. Therefore, (2.10) and (4.7) imply

$$\|(Cv_j)^{(l+1)}\|_\lambda \leq c_4 2^j \|r\|_p, \quad j = 1, 2, \dots, s + 1.$$

This enables us to verify (4.5) in the case in question:

$$\|r'\|_\lambda^\lambda = \|(Cg)^{(l+1)}\|_\lambda^\lambda \leq \sum_{j=1}^{s+1} \|(Cv_j)^{(l+1)}\|_\lambda^\lambda \leq \sum_{j=1}^{s+1} (c_4 2^j \|r\|_p)^\lambda \leq c_5^\lambda n^\lambda \|r\|_\lambda^\lambda.$$

The general case reduces to the above with the help of Lemma 4.6. In that lemma, we take $k = l + 1$, $f(z) = r(z)$ and define the corresponding polynomial $a(z)$. Then the function $r_1(z) := r(z)/a(z)$ belongs to $\mathcal{R}_{n+l+1}^l(S)$ and

$$(4.8) \quad \|r_1\|_p \leq c_6 |S|^{-l-1} \|r\|_p.$$

By the case already proved, we can write

$$(4.9) \quad \|r'_1\|_\lambda \leq c_7 |S|^{-l-1} (n + l + 1) \|r_1\|_p.$$

Since $r = ar_1$, we have $r' = a'r_1 + ar'_1$, whence $\|r'\|_\lambda^\lambda \leq \|a'r_1\|_\lambda^\lambda + \|ar'_1\|_\lambda^\lambda$. We estimate the summands separately. For this we observe that $\|a\|_\infty \leq |S|^{l+1}$ and $\|a'\|_1 \leq (l + 1)|S|^{l+1}$. Here we have used the fact that a is a monic polynomial with zeros on S . Taking these inequalities into account, we obtain

$$(4.10) \quad \|a'r_1\|_\lambda \leq \|a'\|_1 \|r_1\|_p \leq c_6 (l + 1) \|r\|_p,$$

$$(4.11) \quad \|ar'_1\|_\lambda \leq c_7 (n + l + 1) \|r\|_p.$$

Here we have used (4.8) and the Hölder inequality to verify (4.10). Inequality (4.11) is a consequence of (4.9).

Collecting the estimates, we obtain (4.5) in the general case. □

Proof of Theorem 1.1. It suffices to prove (1.3) for $s = 1$ (then the general case follows by iteration). Also, we may assume that S is a simple curve. (If S is closed, we split it into two simple curves of equal length and apply the inequality already proved to each of the halves.) Since (1.3) is obvious for $n = 0$, we assume that $n \geq 1$.

Let $u \in \mathcal{R}_n \cap L_p(S)$. We must prove that, under the above assumptions about S, n , and p , we have

$$(4.12) \quad \|u'\|_\lambda \leq c_1 n \|u\|_p, \quad 1/\lambda = 1 + 1/p, \quad c_1 = c_1(\theta, p).$$

If all poles of u lie on S , then (4.12) is true by Lemma 4.7. Suppose now that u has $m \geq 1$ poles in $\Omega = \overline{\mathbb{C}} \setminus S$. Then u has a unique representation in the form $u = h + r_1$, where h and r_1 are rational functions with poles only on S and Ω , respectively, and $h(\infty) = 0$.

By Lemma 3.7, there exists a rational function r_0 of degree at most $2m/p$ and with poles only on S that satisfies (3.17) and (3.18). By (3.17), we have $\|h + r_0\|_p^p = \|(h + r_1) - (r_1 - r_0)\|_p^p = \|u - (r_1 - r_0)\|_p^p \leq \|u\|_p^p + \|r_1 - r_0\|_p^p \leq (1 + c_1^p) \|u\|_p^p$. Thus, $\|h + r_0\|_p \leq c_3 \|u\|_p$. Furthermore, $h + r_0$ is a rational function with poles only on S and of degree at most $(n - m) + 2m/p \leq 2n/p$. Therefore, by Lemma 4.7, we have

$$(4.13) \quad \|h' + r'_0\|_\lambda \leq c_4 n \|h + r_0\|_p.$$

From (3.18) and (4.13) we deduce (4.12):

$$\begin{aligned} \|u'\|_\lambda^\lambda &= \|h' + r'_1\|_\lambda^\lambda \\ &= \|(h' + r'_0) + (r'_1 - r'_0)\|_\lambda^\lambda \leq \|h' + r'_0\|_\lambda^\lambda + \|r'_1 - r'_0\|_\lambda^\lambda \leq c_5^\lambda n^\lambda \|u\|_p^\lambda. \end{aligned} \quad \square$$

4C. Other applications of Lemmas 3.5 and 3.7. It was mentioned in the Introduction that for $1 < p \leq \infty$ inequality (1.3) had been obtained earlier (and, moreover, for the Ahlfors curves). Here we give yet another proof of this result. As a matter of fact, we verify a more general statement (which can also be found in [15, 4]).

Theorem 4.1 (the author [15]; Danchenko [4]). *Let K be a continuum of the form indicated above, and let $\Omega = \overline{\mathbb{C}} \setminus K$. Suppose $\partial\Omega$ is an Ahlfors curve. If $h \in E_p^-$ ($1 < p \leq \infty$) and r are rational functions of degree m and with poles only in Ω , $u = h + r$, $s \in \mathbb{N}$,*

and $1/\sigma = s + 1/p$, then

$$(4.14) \quad \|r^{(s)}\|_\sigma \leq cm^s \|u\|_p,$$

where $c = c(\kappa, p, s)$, and the quasinorms are evaluated on $\partial\Omega$.

Proof. First, consider the case where $1 < p < \infty$. We employ relations (3.19) and (3.20) already used in the proof of Lemma 3.7. From (3.19) we deduce that

$$r^{(s)}(z) = \frac{s!}{2\pi i} \int_{\partial\Pi} \frac{u(\eta) d\eta}{(\eta - z)^{s+1}}, \quad z \in \partial\Omega.$$

We find $\eta_j \in \Gamma_j$ such that $|u(\eta_j)| = \max\{|u(\eta)| : \eta \in \Gamma_j\}$. Together with (3.8), this identity implies

$$|r^{(s)}(z)| \leq c_1 \sum_{j=1}^\nu \frac{a_j |\Gamma_j|}{|\eta_j - z|^{s+1}}, \quad z \in \partial\Omega.$$

Here we have put $a_j = |u(\eta_j)|$ for brevity. Now, we use (2.4):

$$(4.15) \quad \int_{\partial\Omega} |r^{(s)}(z)|^\sigma |dz| \leq c_1^\sigma \sum_{j=1}^\nu a_j^\sigma |\Gamma_j|^\sigma \int_{\partial\Omega} \frac{|dz|}{|\eta_j - z|^{\sigma(s+1)}} \leq c_2^\sigma \sum_{j=1}^\nu a_j^\sigma |\Gamma_j|^{\sigma/p}.$$

Applying the Hölder inequality and (3.20) to the last sum, we obtain

$$\sum_{j=1}^\nu a_j^\sigma |\Gamma_j|^{\sigma/p} \leq \left(\sum_{j=1}^\nu 1\right)^{1-\frac{\sigma}{p}} \left(\sum_{j=1}^\nu |a_j|^p |\Gamma_j|\right)^{\sigma/p} \leq c_3^\sigma m^{\sigma s} \|u\|_{L_p(\partial\Omega)}^\sigma.$$

This proves (4.14) for $1 < p < \infty$.

In the case of $p = \infty$, the last calculation is superfluous, and the result follows directly from (4.16). \square

In the Introduction we defined the quantity $R_n(f)_p$, i.e., the best approximation of a function $f \in E_p(G)$. For $f \in L_p(\partial G)$ we introduce the quantity $R_n(f, L_p) := \inf\{\|f - r\|_p : r \in \mathcal{R}_n \cap L_p(\partial G)\}$, i.e., the best L_p -approximation on ∂G . In the case of L_p -approximation, the poles of approximants may lie in $\mathbb{C} \setminus G$ and in G . Therefore, $R_n(f, L_p) \leq R_n(f)_p$ for $f \in E_p(G)$. David's theorem (see [3, 7]) implies that if G is an Ahlfors domain and $1 < p < \infty$, then the reverse inequality is true: $R_n(f)_p \leq cR_n(f, L_p)$, $c = c(\kappa, p)$. In the next Theorems 4.2 and 4.3 this inequality is extended to $p \in (0, 1]$.

Theorem 4.2. *If G is an Ahlfors domain and $f \in E_p(G)$, $0 < p < 1$, then for every $n \geq 0$ we have*

$$R_{2n/p}(f)_p \leq cR_n(f, L_p), \quad c = c(\kappa, p).$$

Proof. An analog of Lemma 3.7 for G (see Remark (i) after the proof of that lemma) allows us to expel the possible poles of the approximants from G to ∂G . The required inequality is a consequence of (3.17). \square

Similar arguments and Remark (ii) yield the following theorem.

Theorem 4.3. *If G is a Lavrent'ev domain and $f \in E_1(G)$, then*

$$R_{2n}(f)_1 \leq cR_n(f, L_1), \quad c = c(\theta),$$

for $n \geq 0$.

We note that, for the disk, Theorems 4.2 and 4.3 were obtained in [18] by less involved means.

Suppose G is a Jordan domain with rectifiable boundary ∂G , $\Omega = \mathbb{C} \setminus \overline{G}$, and $0 < p < 1$. We denote by $E_p \cap E_p^-$ (see [1, 2]) the set of functions $f \in L_p(\partial G)$ satisfying the following

condition: there exist $g \in E_p(G)$ and $\omega \in E_p^-(\Omega)$ such that $g = \omega = f$ a.e. on ∂G . For $f \in L_p(\partial G)$, $0 < p < 1$, we denote by $R_n^*(f)_p$ the best L_p -approximation of f by rational functions r of degree at most n and such that all poles of r lie on ∂G and $r(\infty) = 0$.

The following statement is proved by analogy with Theorem 4.2.

Theorem 4.4. *If G is an Ahlfors domain, $0 < p < 1$, and $f \in E_p \cap E_p^-$, then*

$$R_{2n/p}^*(f)_p \leq cR_n(f, L_p), \quad c = c(\kappa, p),$$

for every $n \geq 0$.

Corollary (Aleksandrov [2]). *If G is an Ahlfors domain and $0 < p < 1$, then $E_p \cap E_p^-$ coincides with the L_p -closure on ∂G of the linear combinations of Cauchy kernels $\{1/(z - \eta)\}_{\eta \in \partial G}$.*

Counterparts of Theorem 4.4 and its corollary hold for simple Ahlfors curves.

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