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ABSOLUTE CONTINUITY OF THE "EVEN" PERIODIC SCHRÖDINGER OPERATOR WITH NONSMOOTH COEFFICIENTS

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§1. Introduction

The question about the absolute continuity of the spectrum of the Schrödinger operator is actively studied (see [2, 3, 6, 8, 13, 10]). The operator in question can be described in detail as follows. For any set $\Xi \subset \mathbb{R}^d$, let $h_\Xi[u,v]$ denote the integral

$$h_\Xi[u,v] = \int_\Xi \left(\langle g(i\nabla + A)u, (i\nabla + A)v\rangle + Vu\overline{v}\right)\,dx, \quad u,v \in H^1(\Xi), \tag{1}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{C}^d$. In $L_2(\mathbb{R}^d)$, consider the bilinear form $h_{\mathbb{R}^d}$ defined on the Sobolev space $\operatorname{Dom} h_{\mathbb{R}^d} = H^1(\mathbb{R}^d)$. We assume that the coefficients satisfy the following conditions.

Condition 1. *The functions g, A, and V are real-valued and 2π-periodic in all variables.*

Condition 2. a) *The matrix-valued function g (metric) is positive and bounded,*

$$c_0 \mathbb{1} \le g(x) \le c_1 \mathbb{1}, \quad 0 < c_0 \le c_1 < \infty.$$

b) *The magnetic potential A and the electric potential V belong to the following classes:*

$$A \in L_{q,\mathrm{loc}},\ V \in L_{q/2,\mathrm{loc}}, \quad \text{where } q = d \text{ if } d \ge 3, \quad \text{and} \quad q > 2 \text{ if } d = 2.$$

Under Conditions 1 and 2, $h_{\mathbb{R}^d}$ is a semibounded closed form. The selfadjoint operator H corresponding to $h_{\mathbb{R}^d}$ will be called the *Schrödinger operator*.

At present, the absolute continuity of the spectrum of H is known under the following assumptions. For $d = 2$, it suffices to assume that $\det g \in W^1_{q,\mathrm{loc}}$, $q > 2$ (see [13]). For $d \ge 3$, absolute continuity was proved if $g(x) = a(x)\mathbb{1}$, where a is a scalar function, $a \in C^1$, $A \in H^s_{\mathrm{loc}}$, $s > (3d-2)/2$, and $V \in L_{p,\mathrm{loc}}$, $p = \max\{d/2, d-2\}$ (see [9]).

In [12], Friedlander considered the situation under an additional condition of "evenness".

Condition 3. *The operator H is invariant under the inversion $x_1 \mapsto -x_1$. In terms of its coefficients, this means that*

$$\begin{gathered} g_{11}(-x_1,x') = g_{11}(x_1,x'), \qquad g_{\alpha\beta}(-x_1,x') = g_{\alpha\beta}(x_1,x'), \\ A_\alpha(-x_1,x') = A_\alpha(x_1,x'), \qquad V(-x_1,x') = V(x_1,x'), \\ g_{1\alpha}(-x_1,x') = -g_{1\alpha}(x_1,x'), \quad A_1(-x_1,x') = -A_1(x_1,x'), \\ \alpha,\beta = 2,\dots,d. \end{gathered}$$

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Friedlander showed that the absolute continuity of H follows from Conditions 1, 2a), and 3 if g, A, and V are infinitely smooth. Our goal in this paper is to generalize his result to the case of nonsmooth coefficients. For this, we only need to require that H satisfy the *unique continuation property.*

Condition 4. *For any* $\Xi \subset \mathbb{R}^d$, *if* $u \in H^1(\Xi)$, $h_\Xi[u,v] = 0$ *for all* $v \in \overset{\circ}{H}{}^1(\Xi)$, *and* $u\,|_B = 0$, *where* $B \subset \Xi$ *is a ball, then* $u = 0$ *in* Ξ.

§2. Formulation of the result

Let T^{d-1} be a $(d-1)$-dimensional torus, $\Omega = (-\pi,\pi) \times \mathrm{T}^{d-1}$ a toroidal cylinder, $\Gamma_\pm = \{\pm\pi\} \times \mathrm{T}^{d-1}$ the left and right edges of Ω, $\partial\Omega = \Gamma_- \cup \Gamma_+$. For the set Ω, we omit the index in the form (1), $h[u,v] := h_\Omega[u,v]$.

Theorem 2.1. *Suppose the coefficients* g, A, V *of the form* (1) *satisfy Conditions* 1–4. *Then the set of points* $\zeta \in \mathbb{C}$ *satisfying the following properties is finite*:

1) $\operatorname{Im}\zeta \neq 0$, $|\zeta| \neq 1$;
2) *there exists* $u_\zeta \in H^1(\Omega) \setminus \{0\}$ *such that* $u_\zeta\,|_{\Gamma_+} = \zeta u_\zeta\,|_{\Gamma_-}$ *and*

$$h[u_\zeta, v] = 0 \quad \text{for all } v \in H^1(\Omega) \text{ with } \overline{\zeta} v|_{\Gamma_+} = v|_{\Gamma_-}.$$

This theorem implies by standard methods (see [3, 7]) that the number $\lambda = 0$ cannot be an eigenvalue of the operator H.

The unique continuation property for the solutions of elliptic equations is well investigated.

Theorem 2.2 (see [14])**.** *Let* $d = 2$, *and let* $g, V \in L_\infty$, $A = 0$. *Then* H *has the unique continuation property.*

Theorem 2.3 (see [5])**.** *Suppose* $d \geq 3$, $g \in \mathrm{Lip}$, $A \in L_{q,\mathrm{loc}}$, $q > d$, *and* $V \in L_{d/2,\mathrm{loc}}$. *Then* H *has the unique continuation property.*

Under these assumptions about the coefficients, we shall see that $\lambda = 0$ is not an eigenvalue of H. Since the conditions on the potential V are invariant with respect to the shift by a constant, the point spectrum of H is empty. So, the following statement is true.

Theorem 2.4. *Suppose that the coefficients of the operator* H *satisfy Conditions* 1, 2a), *and* 3, *and that*

a) $V \in L_\infty$, $A = 0$ *if* $d = 2$;
b) $g \in \mathrm{Lip}$, $A \in L_{q,\mathrm{loc}}, q > d$, $V \in L_{d/2,\mathrm{loc}}$ *if* $d \geq 3$.

Then the spectrum of H *corresponding to the form* (1) *with* $\Xi = \mathbb{R}^d$ *is absolutely continuous.*

Remark 2.1. From the unique continuation theorem (see [5]) it follows that in the two-dimensional case with $g \in \mathrm{Lip}$ and $A \in L_{q,\mathrm{loc}}$, $V \in L_{q/2,\mathrm{loc}}$, $q > 2$, the spectrum of the "even" operator H is absolutely continuous. But this result has already been known without the assumption of "evenness" (see [13]).

Remark 2.2. The assumptions in b) are close to optimal. If $d \geq 3$, then there exists a Schrödinger operator H with $g \in \bigcap_{\alpha<1} C^\alpha$, $A = 0$, and $V = 0$ that does not possess the unique continuation property and has an eigenvalue of infinite multiplicity (see [11]).

Remark 2.3. In the same way, a similar statement for a waveguide can be proved. Let U be a bounded domain in $\mathbb{R}^k$, $k < d$, with Lipschitz boundary. Suppose that g, A, V satisfy the assumptions of Theorem 2.4 in $\Xi = \mathbb{R}^{d-k} \times U$ (g, A, V are periodic in the first $(d-k)$ variables). Then the spectrum of the operator H corresponding to the form

(1) and defined on $\operatorname{Dom} h_\Xi = \mathring{H}^1(\Xi)$ (the Dirichlet problem) or $\operatorname{Dom} h_\Xi = H^1(\Xi)$ (the Neumann problem) is absolutely continuous.

The remaining part of the paper is devoted to the proof of Theorem 2.1. Mostly, we use the method of [12], where the Dirichlet-to-Neumann operators play a crucial role. We do not use the pseudodifferential technique, working with Dirichlet–Neumann forms (defined on the trace space $H^{1/2}(\Gamma_\pm)$) instead of operators; this approach permits us to handle nonsmooth coefficients.

In §3, the solvability of various boundary value problems in the cell Ω is discussed. In §4, we prove an auxiliary estimate. In §5 we define the Dirichlet–Neumann forms and investigate their properties. The proof of Theorem 2.1 will be completed in §6.

§3. Boundary-value problems in the cell

Consider the kernel of the Dirichlet problem

$$\ker_D := \{w \in \mathring{H}^1(\Omega) : h[w,v] = 0, v \in \mathring{H}^1(\Omega)\}.$$

It is easily seen that $\dim \ker_D < \infty$.

Let $B : H^{1/2}(\mathrm{T}^{d-1}) \to H^1(\Omega)$ be a bounded linear extension operator such that $(B\varphi)\,|_{\Gamma_-} = \varphi$, $(B\varphi)\,|_{\Gamma_+} = 0$. By definition, we put

$$\mathrm{Dom} := \{\varphi \in H^{1/2}(\mathrm{T}^{d-1}) : h[B\varphi, w] = 0, w \in \ker_D\}.$$

Remark 3.1. 1) The definition of Dom does not depend on the choice of the operator B. 2) The codimension of Dom in $H^{1/2}$ is finite.

Lemma 3.1. *The problem*

$$\begin{cases} h[u,v] = 0 \quad \text{for all } v \in \mathring{H}^1(\Omega), \\ u\,|_{\Gamma_-} = \varphi, \quad u\,|_{\Gamma_+} = 0, \end{cases} \tag{2}$$

has a solution in $H^1(\Omega)$ if and only if $\varphi \in$ Dom. If $\varphi \in$ Dom, then there exists a unique solution u of (2) *such that $u \perp \ker_D$.*

Proof. Obviously, if (2) is solvable, then $\varphi \in$ Dom. Conversely, let $\varphi \in$ Dom. The Fredholm theory shows that there exists $u_0 \in \mathring{H}^1(\Omega)$ such that

$$h[u_0, v] = h[B\varphi, v] \quad \text{for all } v \in \mathring{H}^1(\Omega).$$

Then $u = B\varphi - u_0$ is a solution of (2). The final statement of the lemma is obvious. □

Definition 1. By the Poisson operator $\mathcal{P}$ we shall mean the operator $\mathcal{P} : \mathrm{Dom} \to H^1(\Omega)$ that takes $\varphi \in$ Dom to the solution u of (2) orthogonal to $\ker_D$.

Remark 3.2. The *two-sided* estimate

$$c_2\|\varphi\|_{H^{1/2}} \le \|\mathcal{P}\varphi\|_{H^1} \le c_3\|\varphi\|_{H^{1/2}} \tag{3}$$

is true.

Also, we introduce the inversion operator

$$(Ju)(x_1, x') := u(-x_1, x')$$

and the operators $\tilde{B} := JB$, $\tilde{\mathcal{P}} := J\mathcal{P}$. By Condition 3, we have

$$h[Ju, Jv] = h[u,v], \quad u, v \in H^1(\Omega),$$

and $J \ker_D = \ker_D$.

Lemma 3.2. *Let $\zeta \neq \pm 1$. The problem*

$$\begin{cases} h[u,v] = 0 \quad \text{for all } v \in \mathring{H}^1(\Omega), \\ u\,|_{\Gamma_-} = \varphi, \quad u\,|_{\Gamma_+} = \zeta\varphi, \end{cases} \tag{4}$$

has a solution in H^1 if and only if $\varphi \in \mathrm{Dom}$. In this case

$$u = \mathcal{P}\varphi + \zeta\tilde{\mathcal{P}}\varphi + w, \quad \text{where } w \in \ker_D. \tag{5}$$

Proof. Let u be a solution of (4). Then $(1-\zeta^2)^{-1}(u - \zeta J u)$ is a solution of (2), whence $\varphi \in \mathrm{Dom}$. Conversely, let $\varphi \in \mathrm{Dom}$. Then the function $u = \mathcal{P}\varphi + \zeta\tilde{\mathcal{P}}\varphi$ is a solution of (4). The final statement of the lemma is obvious. □

§4. The space $R(\mathcal{P})$

The image of the Poisson operator can be described explicitly:

$$R(\mathcal{P}) = \{u \in H^1(\Omega) : u\,|_{\Gamma_+} = 0,\ h[u,v] = 0\ \forall v \in \mathring{H}^1(\Omega),\ u \perp \ker_D\}.$$

Thus, $R(\mathcal{P})$ is a closed subspace of $H^1(\Omega)$, and $R(\mathcal{P}) \cap \mathring{H}^1(\Omega) = \{0\}$.

Lemma 4.1. *Let W be a closed subspace of $H^1(\Omega)$ with $W \cap \mathring{H}^1(\Omega) = \{0\}$. Let $\psi \in L_r(\Omega)$, where $r > 1$ if $d = 2$ and $r = d/2$ if $d \geq 3$. Then for any $\varepsilon > 0$ there exists $c(\varepsilon)$ such that*

$$\int_\Omega \left|\psi u^2\right| dx \leq \varepsilon \int_\Omega |\nabla u|^2\, dx + c(\varepsilon) \int_{\partial\Omega} |u|^2\, dS, \quad u \in W.$$

Remark 4.1. Of course, this statement is valid for any bounded domain with Lipschitz boundary.

Proof. Suppose that for some $\varepsilon > 0$ there exists a sequence $\{u_n\} \subset W$ such that

$$\int_\Omega \left|\psi u_n^2\right| dx = 1 \geq \varepsilon \|\nabla u_n\|^2_{L_2(\Omega)} + n \|u_n\,|_{\partial\Omega}\|^2_{L_2(\partial\Omega)}.$$

Without loss of generality we may assume that $u_n \xrightarrow[H^1]{w} u$. It is well known that the embedding

$$H^1(\Omega) \subset L_2(\Omega, |\psi| dx)$$

is compact (for $d = 2$ this is a consequence of the fact that the embedding $H^1 \subset L_{2r/(r-1)}$ is compact; for $d \geq 3$ we refer the reader to [1]). Thus, $\int_\Omega \left|\psi u^2\right| dx = 1$. On the other hand, since $\|u_n\,|_{\partial\Omega}\|_{L_2} \to 0$ and the embedding $H^1(\Omega) \subset L_2(\partial\Omega)$ is also compact, we have $u\,|_{\partial\Omega} = 0$. Therefore,

$$u \in \mathring{H}^1(\Omega) \cap W \implies u = 0,$$

a contradiction. □

Corollary 1. *Suppose g, A, and V satisfy Condition 2. Then for any $\varepsilon > 0$ there exists $C(\varepsilon)$ such that*

$$\int_\Omega \left(\langle gA, A\rangle + |V|\right)|u|^2\, dx + 2\int_\Omega |\langle gAu, \nabla u\rangle|\, dx$$
$$\leq \varepsilon \int_\Omega |\nabla u|^2\, dx + C(\varepsilon) \int_{\Gamma_-} |u|^2\, dS, \quad u \in R(\mathcal{P}).$$

§5. Dirichlet–Neumann forms

In $L_2(\mathrm{T}^{d-1})$, consider the subspace $\mathcal{H} := \overline{\mathrm{Dom}}^{L_2}$ (if $\ker_D = \{0\}$, then $\mathcal{H}$ coincides with L_2).

Definition 2. The forms n_0, n_1 in the Hilbert space $\mathcal{H}$ that are defined on the domain

$$\mathrm{Dom}\, n_0 = \mathrm{Dom}\, n_1 = \mathrm{Dom}$$

by the formulas

$$n_0[\varphi,\psi] = h[\mathcal{P}\varphi, \mathcal{P}\psi], \quad n_1[\varphi,\psi] = h[\mathcal{P}\varphi, \tilde{\mathcal{P}}\psi],$$

will be called the Dirichlet–Neumann forms.

It is easily seen that n_0 is Hermitian. The form n_1 is also Hermitian, which follows from Condition 3:

$$n_1[\varphi,\psi] = h[\mathcal{P}\varphi, \tilde{\mathcal{P}}\psi] = h[\tilde{\mathcal{P}}\varphi, \mathcal{P}\psi] = \overline{n_1[\psi,\varphi]}.$$

Lemma 5.1. *The form n_0 is semibounded and closed.*

Proof. Let $\varphi \in \mathrm{Dom}$, $\mathcal{P}\varphi = u$. By the corollary to Lemma 4.1, we get

$$\begin{aligned} n_0[\varphi,\varphi] &= \int_\Omega \left(\langle g(i\nabla + A)u, (i\nabla + A)u\rangle + V|u|^2 \right) dx \\ &\geq \frac{c_0}{2} \int_\Omega |\nabla u|^2\, dx - c_4 \int_{\Gamma_-} |\varphi|^2\, dS. \end{aligned} \tag{6}$$

Thus, the form n_0 is semibounded. Combining (6) and (3), we obtain

$$c_5 \|\varphi\|^2_{H^{1/2}} \leq n_0[\varphi,\varphi] + c_4\|\varphi\|^2_{L_2} \leq c_6 \|\varphi\|^2_{H^{1/2}},$$

so that n_0 is closed. □

To prove that $\ker n_1 = \{0\}$, we need the following facts.

Lemma 5.2 (see, e.g., [4])**.** *Let $l, l_1, \dots, l_n$ be bounded linear functionals on a Hilbert space. If $\bigcap_{k=1}^n \ker l_k \subset \ker l$, then l is a linear combination of $l_1, \dots, l_n$, i.e., $l = \sum_{k=1}^n \alpha_k l_k$.*

Lemma 5.3. *Suppose the coefficients of the form h satisfy Condition 4. If the function $w \in H^1(\Omega)$ is such that*

1) $w|_{\Gamma_+} = 0$,
2) $h[v,w] = 0$ *for any* $v \in H^1(\Omega)$ *with* $v\,|_{\Gamma_-} = 0$,

then $w = 0$.

Proof. Consider the cylinder $\Xi = (-\pi, 4) \times \mathrm{T}^{d-1}$ and the function $\hat{w}(x) \in H^1(\Xi)$ such that $\hat{w}(x) = w(x)$ if $x_1 \leq \pi$ and $\hat{w}(x) = 0$ if $x_1 > \pi$. Then $\hat{w} \in H^1(\Xi)$ and

$$h_\Xi[\hat{w}, v] = h[w, v\,|_\Omega] = 0, \quad v \in \mathring{H}^1(\Xi).$$

Now, the identity $w = 0$ follows from Condition 4. □

Theorem 5.1. *The kernel of the form n_1 is trivial*, $\ker n_1 = \{0\}$.

Proof. Let $\varphi \in \ker n_1$, $\mathcal{P}\varphi = u$. Then

$$h[v,u] = 0 \quad \text{for all } v \in R(\tilde{\mathcal{P}}).$$

Let $\{u_k\}$ be a basis in $\ker_D$. Consider the linear functionals

$$l(\psi) = h[\tilde{B}\psi, u], \quad l_k(\psi) = h[\tilde{B}\psi, u_k]$$

on $H^{1/2}(\mathrm{T}^{d-1})$. By definition, $\bigcap_k \ker l_k = \mathrm{Dom}$. If $\psi \in \mathrm{Dom}$, then $h[\tilde{B}\psi, u] = h[\tilde{\mathcal{P}}\psi, u] = 0$, whence $\mathrm{Dom} \subset \ker l$. By Lemma 5.2, there exists $u_0 \in \ker_D$ such that

$$h[\tilde{B}\psi, u - u_0] = 0 \quad \text{for all } \psi \in H^{1/2}(\mathrm{T}^{d-1}).$$

If $w = u - u_0$, then $h[v, w] = 0$, $v \in \mathring{H}^1(\Omega)$. Consequently,

$$h[v, w] = 0 \quad \text{for all } v \in H^1(\Omega) : v\,|_{\Gamma_-} = 0.$$

Lemma 5.2 yields $w = 0$, so that $u = u_0 \in \mathring{H}^1(\Omega) \iff u = 0$. □

§6. Proof of Theorem 2.1

We shall use the following abstract statement.

Lemma 6.1. *Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{F}$ be a dense linear set in $\mathcal{H}$, and let n_0, n_1 be Hermitian bilinear forms such that* $\mathrm{Dom}\, n_0 = \mathrm{Dom}\, n_1 = \mathcal{F}$. *Assume that n_0 is semibounded from below and closed,* $\ker n_1 = \{0\}$. *If the embedding of $\mathcal{F}$ with norm $\|f\|^2_{\mathcal{F}} = n_0[f, f] + \gamma\|f\|^2_{\mathcal{H}}$ in $\mathcal{H}$ is compact, then the set of nonreal z such that* $\ker(n_0 + zn_1) \neq \{0\}$ *is finite.*

Proof. Without loss of generality we can assume that $\mathrm{Im}\, z > 0$. Consider the set

$$M = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0, \ker(n_0 + zn_1) \neq \{0\}\},$$

and let

$$\mathcal{M} = \sum_{z \in M} \ker(n_0 + zn_1)$$

be the linear span of such kernels. For $z_1, z_2 \in M$ (including the case where $z_1 = z_2$) and $\varphi_k \in \ker(n_0 + z_k n_1)$, $k = 1, 2$, we have

$$n_0[\varphi_1, \varphi_2] + z_1 n_1[\varphi_1, \varphi_2] = 0, \qquad n_0[\varphi_1, \varphi_2] + \overline{z}_2 n_1[\varphi_1, \varphi_2] = 0.$$

Therefore, $n_0[\varphi_1, \varphi_2] = n_1[\varphi_1, \varphi_2] = 0$, whence $n_0[\varphi, \psi] = n_1[\varphi, \psi] = 0$, $\varphi, \psi \in \mathcal{M}$. Combining this and the compactness of the embedding $\mathcal{F} \subset \mathcal{H}$, we see that $\dim \mathcal{M} < \infty$.

Next, there exist operators $N_0, N_1 : \mathcal{M} \to \mathcal{F}$ such that

$$(N_k\varphi, \psi)_{\mathcal{F}} = n_k[\varphi, \psi], \quad \varphi \in \mathcal{M}, \ \psi \in \mathcal{F}.$$

Obviously, $\ker N_1 = \{0\}$. If $\varphi \in \ker(n_0 + zn_1)$, then $N_0\varphi = -zN_1\varphi$, whence $R(N_0) = R(N_1)$, and we can define the operator $N_1^{-1}N_0$ acting in $\mathcal{M}$. Then the cardinality of M does not exceed the dimension of $\mathcal{M}$, because $M \subset \sigma(-N_1^{-1}N_0)$. □

Proof of Theorem 2.1. Suppose that $\zeta \in \mathbb{C}$, $\mathrm{Im}\, \zeta \neq 0$, $|\zeta| \neq 1$ and there exists $u_\zeta \in H^1(\Omega)$, $u_\zeta \neq 0$, such that $u_\zeta\,|_{\Gamma_+} = \zeta u_\zeta\,|_{\Gamma_-}$ and

$$h[u_\zeta, v] = 0 \quad \text{for all } v \in H^1(\Omega) \text{ with } \overline{\zeta} v\,|_{\Gamma_+} = v\,|_{\Gamma_-}.$$

By Lemma 3.2 (formula (5)), we have $u_\zeta = \mathcal{P}\varphi + \zeta\tilde{\mathcal{P}}\varphi + w$, where $w \in \ker_D$, $\varphi = u_\zeta\,|_{\Gamma_-}$.

First, suppose $\varphi = 0$. Then $u_\zeta \in \ker_D$. If $f \in H^1(\Omega)$ and $f\,|_{\Gamma_-} = 0$, then

$$h[\zeta J u_\zeta + u_\zeta, f] = h[u_\zeta, \overline{\zeta} Jf + f] = 0.$$

By Lemma 5.3, $Ju_\zeta = -\zeta^{-1}u_\zeta$. Since the spectrum of J consists of two numbers 1 and -1, we have $u_\zeta = 0$, a contradiction.

Now, let $\varphi \neq 0$. Then, for any $\psi \in \mathrm{Dom}$, choosing $v = \tilde{\mathcal{P}}\psi + \overline{\zeta}\mathcal{P}\psi$, we get

$$0 = h[u_\zeta, v] = h[\mathcal{P}\varphi + \zeta\tilde{\mathcal{P}}\varphi + w, \tilde{\mathcal{P}}\psi + \overline{\zeta}\mathcal{P}\psi] = (1 + \zeta^2)n_1[\varphi, \psi] + 2\zeta n_0[\varphi, \psi]$$
$$\implies n_0[\varphi, \psi] + zn_1[\varphi, \psi] = 0,$$

where $z = (1+\zeta^2)/(2\zeta)$. Recall that $z \notin \mathbb{R}$ by assumption. The compactness of the embedding $\mathcal{F} = \mathrm{Dom} \subset \mathcal{H}$ follows from the compactness of the embedding $H^{1/2}(\mathrm{T}^{d-1}) \subset L_2(\mathrm{T}^{d-1})$, which allows us to apply Lemma 6.1. So, the number of such ζ is finite. □

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