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APPROXIMATION OF SUBHARMONIC FUNCTIONS

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Abstract. In certain classes of subharmonic functions u on $\mathbb{C}$ distinguished in terms of lower bounds for the Riesz measure of u, a sharp estimate is obtained for the rate of approximation by functions of the form $\log|f(z)|$, where f is an entire function. The results complement and generalize those recently obtained by Lyubarskiĭ and Malinnikova.

§1. Introduction

We use the standard notions of subharmonic function theory (see [1]). We put $D_z(t) = \{\zeta \in \mathbb{C} : |\zeta - z| < t\}$, $z \in \mathbb{C}$, $t > 0$. For a function u subharmonic in $\mathbb{C}$, we write $B(r,u) = \max\{u(z) : |z| = r\}$, $r > 0$, and define the order $\rho[u]$ by the relation $\rho[u] = \limsup_{r\to+\infty} \log B(r,u)/\log r$. Also, let μ_u denote Riesz measure associated with the subharmonic function u, and let m be plane Lebesgue measure. The symbol C with indices stands for some positive constants. If $u = \log|f|$, where f is an entire function with zeros $\{a_k\}$, then $\mu_u = \sum_k n_k \delta(z - a_k)$, where n_k is the multiplicity of the zero a_k, and $\delta(z - a_k)$ is the Dirac function concentrated at a_k. However, the class of functions subharmonic in $\mathbb{C}$ is broader than that of functions of the form $\log|f|$, where f is an entire function. Since it is often easier to construct a subharmonic function rather than an entire one with desired asymptotic properties, a natural problem arises of approximating subharmonic functions by the logarithms of the moduli of entire functions. Apparently, Azarin [2] was the first to investigate this problem in the general form in the class of functions subharmonic in the plane and having finite order of growth. The results cited below have numerous applications in function theory and potential theory (see, e.g., [3]–[6]).

In 1985 Yulmukhametov [7] obtained the following remarkable result. *For any function u subharmonic in $\mathbb{C}$ and of order $\rho \in (0,+\infty)$, and for any $\alpha > \rho$, there exists an entire function f and a set $E_\alpha \subset \mathbb{C}$ such that*

$$\big|u(z) - \log|f(z)|\big| \le C_\alpha \log|z|, \quad z \to \infty, \ z \notin E_\alpha, \tag{1.1}$$

and E_α can be covered by a family of disks $D_{z_j}(t_j)$, $j \in \mathbb{N}$, satisfying the estimate $\sum_{|z_j|>R} t_j = O(R^{\rho-\alpha})$, $R \to +\infty$. In the special case where the subharmonic function u is homogeneous, $\log|z|$ in (1.1) can be replaced by $O(1)$ (see [3, 8]). Passage to an integral metric allows us to drop an exceptional set in the case of approximation of subharmonic functions of finite order. Let $\|\cdot\|_q$ be the norm in the space $L^q(0,2\pi)$, and let

$$Q(r,u) = \begin{cases} O(\log r),\ r \to +\infty, & \rho[u] < +\infty, \\ O(\log r + \log n(r,u)),\ r \to +\infty, & \\ \qquad r \notin E,\ \operatorname{meas} E < +\infty, & \rho[u] = +\infty. \end{cases}$$

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In [9] Gol′dberg and Girnyk proved that for an arbitrary subharmonic function u there exists an entire function f such that

$$\left\|u(re^{i\theta}) - \log|f(re^{i\theta})|\right\|_q = Q(r,u), \quad r>0,\ q>0.$$

From the recent result of Lyubarskiĭ and Malinnikova [10] it follows that integration of the approximation rate over the plane measure makes it possible to lift the assumption that u is of finite order and leads to sharp estimates.

Theorem A ([10])**.** *Let $u(z)$ be a subharmonic function in $\mathbb{C}$. Then for each $q > 1/2$ there exists $R_0 > 0$ and an entire function f such that*

$$\frac{1}{\pi R^2}\int_{|z|<R}\bigl|u(z)-\log|f(z)|\bigr|\,dm(z) < q\log R, \quad R>R_0. \tag{1.2}$$

An example constructed in [10] shows that we cannot take $q < 1/2$ in (1.2). In connection with Theorem A, the following Sodin's question is known, which is a refinement of Question 1 in [11, p. 315].

Question. Given a subharmonic function u on $\mathbb{C}$, does there exist an entire function f and a constant $\alpha \in [0,1)$ such that

$$\int_{|z|<R}\bigl|u(z)-\log|f(z)|-\alpha\log|z|\bigr|\,dm(z) = O(R^2), \quad R>R_0? \tag{1.3}$$

Remark 1. Question 1 in [11, p. 315] corresponds to the case of $\alpha = 0$. The example in [10] mentioned above implies the negative answer to this question.

On the other hand, restrictions on Riesz measure that bound it from below make it possible to refine estimate (1.2).

Theorem B ([10])**.** *Let $u(z)$ be a subharmonic function in $\mathbb{C}$. If, for some $R_0 > 0$ and $q > 1$, we have*

$$\mu_u(\{z : R < |z| \le qR\}) > 1, \quad R > R_0, \tag{1.4}$$

then there exists an entire function f satisfying

$$\sup_{R>0} R^{-2}\int_{|z|<R}\bigl|u(z)-\log|f(z)|\bigr|\,dm(z) < \infty.$$

Moreover, for every $\varepsilon > 0$ there exists a set $E_\varepsilon \subset \mathbb{C}$ such that

$$\limsup_{R\to\infty} m(\{z\in E : |z| < R\})R^{-2} < \varepsilon$$

and

$$u(z) - \log|f(z)| = O(1), \quad z \notin E_\varepsilon,\ z\to\infty. \tag{1.5}$$

A gap is seen between the statements of Theorems A and B. The following question arises: how can the estimate of the left-hand side of (1.2) be improved if (1.4) fails? The answer to this question is given by Theorem 1.

Let Φ be the class of slowly varying functions $\psi\colon [1,+\infty) \to (1,+\infty)$ (in particular, $\psi(2r) \sim \psi(r)$ as $r\to+\infty$).

Theorem 1. *Let u be a function subharmonic in $\mathbb{C}$, and let $\mu = \mu_u$. If for some $\psi \in \Phi$ there exists a constant R_1 such that*

$$\mu(\{z : R < |z| \le R\psi(R)\}) > 1 \quad \textit{for all } R > R_1, \tag{1.6}$$

then there exists an entire function f satisfying the condition

$$\int_{|z|<R}\bigl|u(z)-\log|f(z)|\bigr|\,dm(z) = O(R^2\log\psi(R)) \quad (R\ge R_1). \tag{1.7}$$

Corollary 1. *Under the hypothesis of Theorem* 1, *for every* $\varepsilon > 0$ *there exist* $K(\varepsilon) > 0$ *and a set* $E_\varepsilon \subset \mathbb{C}$ *such that*

$$\limsup_{r\to\infty} \frac{m(E \cap \overline{D}_r)}{r^2} < \varepsilon \tag{1.8}$$

and

$$\bigl|u(z) - \log|f(z)|\bigr| \le K_\varepsilon \log\psi(|z|), \quad z \notin E_\varepsilon. \tag{1.9}$$

The following example and Theorem 2 show that estimate (1.7) is sharp in the class of subharmonic functions satisfying (1.6).

For $\varphi \in \Phi$, put

$$u(z) = u_\varphi(z) = \frac{1}{2}\sum_{k=1}^{+\infty} \log\Bigl|1 - \frac{z}{r_k}\Bigr|, \tag{1.10}$$

where $r_0 = 2$, $r_{k+1} = r_k\varphi(r_k)$, $k \in \mathbb{N} \cup \{0\}$. Then μ_u satisfies condition (1.6) with $\psi(x) = \varphi^2(x)$.

Theorem 2. *Let* $\psi \in \Phi$ *be such that* $\psi(r) \to +\infty$ $(r \to +\infty)$. *There exists no entire function* f *with*

$$\int_{|z|<R} \bigl|u_\psi(z) - \log|f(z)|\bigr|\, dm(z) = o(R^2 \log\psi(R)), \quad R \to \infty.$$

From the next Theorem 2′ it follows that the answer to Sodin's question formulated above is in the negative. It was Girnyk who drew the author's attention to that question, as well as to the fact that the negative answer follows from the above example.

Theorem 2′. *Let* σ *be an arbitrary positive continuous function defined on* $[1, +\infty)$, *and let* $\sigma(t) \to 0$ $(t \to \infty)$. *No entire function* f *and no constant* $\alpha \in [0, 1)$ *can satisfy*

$$\int_{|z|<R} \bigl|u_\psi(z) - \log|f(z)| - \alpha\log|z|\bigr|\, dm(z) = o\Bigl(R^2 \int_1^R \frac{\sigma(t)}{t}\, dt\Bigr), \quad R \to \infty.$$

Remark 2. The growth of $\int_1^R \frac{\sigma(t)}{t}\, dt$ as $R \to +\infty$ is restricted only by the condition $\int_1^R \frac{\sigma(t)}{t}\, dt = o(\log R)$.

Remark 3. The author does not know whether it is possible to refine estimate (1.8) of the exceptional set for (1.9). In [12], sharp estimates of the exceptional set outside of which (1.9) is true were obtained for a class of subharmonic functions subject to some additional restriction on Riesz measure.

§2. Proof of Theorem 1

2.1. Partition of measures. It has turned out that, in order to have a "good" approximation of a subharmonic function by the logarithm of the modulus of an entire function, we need a "good" approximation of the corresponding Riesz measure by a discrete measure. The Riesz measure defined on the Borel sets in the plane is subject to the only requirement that it should be finite on the compact sets. The following theorem on partition of measures is the principal step in the proofs of Theorems 1, 2, and 2′.

Theorem C. *Let* μ *be a measure in* $\mathbb{R}^2$ *with compact support*, *and let* $\operatorname{supp}\mu \subset \Pi$ *and* $\mu(\Pi) \in \mathbb{N}$, *where* Π *is a square. Suppose*, *moreover*, *that for any line* L *parallel to a side of the square* Π, *there is at most one point* $p \in L$ *such that*

$$0 < \mu(\{p\}) < 1 \text{ and } \mu(L \setminus \{p\}) = 0. \tag{2.1}$$

Then there exists a system of rectangles $\Pi_k \subset \Pi$ *with sides parallel to the sides of* Π, *and a system of measures* μ_k *with the following properties*:

1) $\operatorname{supp}\mu_k \subset \Pi_k$;
2) $\mu_k(\Pi_k) = 1$, $\sum_k \mu_k = \mu$;
3) *the interiors of the convex hulls of the supports of* μ_k *are pairwise disjoint*;
4) *the ratio of the side lengths of rectangles* Π_k *lies in the interval* $[1/3, 3]$;
5) *each point of the plane belongs to the interiors of at most* 4 *rectangles* Π_k.

Theorem C was proved by Yulmukhametov (see [7, Theorem 1]) for absolutely continuous measures (i.e., for ν such that $m(E) = 0 \implies \nu(E) = 0$). In this case condition (2.1) is fulfilled automatically. In [4, Theorem 2.1], Drasin showed that Yulmukhametov's proof works if the condition of continuity is replaced by condition (2.1). In its turn, condition (2.1) can be met by rotating the initial square (see [4]). Though in [10] it was noted that Theorem C remains valid even without condition (2.1), the author does not know any proof of this fact. Moreover, in the proof of Theorem 2.1 in [4] (this is a version of Theorem C), condition (2.1) was used substantially. In this connection, we mention the paper [13] by Grishin and Makarenko, where a two-dimensional version of Theorem C was proved under the condition that the measure loads no line parallel to a coordinate axis.

Remark 4. In the proof of Theorem C given in [4], the rectangles Π_k are obtained by splitting the given rectangles, starting with Π, into smaller rectangles in the following way. The length of the smaller side of an initial rectangle coincides with that of a side of the resulting rectangle, and the length of the other side of the resulting rectangle is between one third and two thirds of the length of the other side of the initial rectangle. Thus, the following form of Theorem C, which will be used in the sequel, is true.

Theorem 3. *Let* μ *be a measure in* $\mathbb{R}^2$ *with compact support*, *and let* $\operatorname{supp}\mu \subset \Pi$ *and* $\mu(\Pi) \in 2\mathbb{N}$, *where* Π *is a rectangle with the ratio* $b_0/a_0 = l_0 \in [1, +\infty)$ *of the side lengths* a_0, b_0 $(a_0 \le b_0)$. *If*, *moreover*, *condition* (2.1) *is fulfilled*, *then there exists a system of rectangles* $\Pi_k \subset \Pi$ *with sides parallel to the sides of* Π, *and a system of measures* μ_k *with the following properties*:

1) $\operatorname{supp}\mu_k \subset \Pi_k$;
2) $\mu_k(\Pi_k) = 2$, $\sum_k \mu_k = \mu$;
3) *the interiors of the convex hulls of the supports of* μ_k *are pairwise disjoint*;
4) *the ratio* b_k/a_k *of the lengths* a_k, b_k $(a_k \le b_k)$ *of the sides of* Π_k *lies in the interval* $[1, l_0]$, *and*, *moreover*, *if* $l_k > 3$, *then* $a_k = a_0$;
5) *each point of the plane belongs to the interiors of at most four rectangles* Π_k.

As was noted in [3], the idea of partition into rectangles of mass 2 is due to Grishin. In order to apply Theorem 3, we need the following lemma (see also [4, Lemma 2.4]).

Lemma 1. *Let* ν *be a locally finite measure in* $\mathbb{C}$. *Then in any neighborhood of the origin there exists a point* z' *with the following properties*:

a) *on each line* L_α *through* z', *there is at most one point* ζ_α *such that* $\nu(\{\zeta_\alpha\}) > 0$, *and*, *moreover*, $\nu(L_\alpha \setminus \{\zeta_\alpha\}) = 0$;
b) *on each circle* C_ρ *with center* z', *there exists at most one point* ζ_ρ *such that* $\nu(\{\zeta_\rho\}) > 0$, *and*, *moreover*, $\nu(C_\rho \setminus \{\zeta_\rho\}) = 0$.

We give a simple example for illustration. Let $\nu(z) = \sum_{n\in\mathbb{N}} \delta(z - n)$. Then $\nu(\mathbb{R}) = +\infty$. As z' we can take any point of the disk $\{z : |z| < 1\}$ with nonzero imaginary part.

Proof of Lemma 1. We put $B_n = \{z : 2^n < |z| \le 2^{n+1}\}$, $n \in \mathbb{N}$, and $B_0 = \{z : |z| \le 2\}$, $\nu_n = \nu\big|_{B_n}$. Since ν is a locally finite measure, $\nu_n(\mathbb{C}) = \nu_n(B_n) < +\infty$. There is an at most countable set ζ_{nk} of points such that $\nu_n(\{\zeta_{nk}\}) > 0$. Therefore, the set $E_1 = \bigcup_n \bigcup_k \{\zeta_{nk}\}$ is at most countable. Given a pair of points in E_1, we consider the straight line through these points and the middle perpendicular to the segment connecting these points. All these lines cover some set $A \subset \mathbb{C}$ with $m(A) = 0$. Let $z_1 \in \mathbb{C} \setminus A$. By construction, an arbitrary straight line through the point z_1 contains at most one point with positive mass. The same is true for an arbitrary circle centered at z_1. We define $\nu'_n = \nu_n - \sum_{\nu_n(\{\zeta\})>0} \nu_n(\{\zeta\})\delta_\zeta$, $n \in \mathbb{Z}_+$, and $\delta_\zeta(z) = \delta(z-\zeta)$. Then, for any $z \in \mathbb{C}$, we have $\nu'_n(\{z\}) = 0$, $n \in \mathbb{Z}_+$. Since the intersection of two different circles (straight lines) is either the empty set, or a point, or two points, and $\nu_n(\mathbb{C}) < +\infty$, the countable additivity of ν_n shows that there exists at most countable set of circles and straight lines with positive ν_n-measure. The union F_n of all these straight lines and all centers of these circles has zero area. Now, if $z' \in \mathbb{C} \setminus (A \cup \bigcup_{n\in\mathbb{Z}_+} F_n)$, then for any $n \in \mathbb{Z}_+$ the measure ν_n of any circle with center z' as well as that of a straight line through z' is equal to zero. Consequently, their ν-measure is also equal to zero. Finally, by the countable additivity of the plane measure, we have $m(A \cup \bigcup_{n\in\mathbb{Z}_+} F_n) = m(A) + \sum_n m(F_n) = 0$. Thus, any point $z' \in (\mathbb{C} \setminus (A \cup \bigcup_{n\in\mathbb{Z}_+} F_n)) \cap U$, where U is an arbitrary neighborhood of the origin, possesses the required properties. □

Taking the above lemma into account, we may assume that properties a) and b) in Lemma 1 are fulfilled at the origin. We follow the method of proof used in [10], assuming that $\psi(x) \nearrow +\infty$ as $x \to +\infty$, because otherwise Theorem 1 is equivalent to Theorem B. Without loss of generality, we can assume that $u(z)$ is harmonic in a neighborhood of $z = 0$. Otherwise, we choose an arbitrary $a > 0$ such that $n(a) \le N \le n(a+0)$ for some $N \in \mathbb{N}$ and introduce the measure ν that is equal to μ in $D_0(a)$ and contains the part $\mu\big|_{\{z:|z|=a\}}$ so that $\nu(\overline{D_0(a)}) = N$. Then $\mu - \nu \equiv 0$ in $D_0(a)$ and, in place of u, we consider the function $\tilde{u}(z) = u(z) - \int_{\mathbb{C}} \log|z-\zeta|\,d\nu(\zeta)$. The quantity $\int_{\mathbb{C}} \log|z-\zeta|\,d\nu(\zeta) - N\log|z|$ is bounded as $z \to +\infty$. Therefore, without loss of generality we assume that $R_0 = \sup\{r > 0 : \operatorname{supp}\mu \cap D_0(r) = \varnothing\} > 1$. Put $\Psi_1(R) = R\psi(R)$, $\Psi_n(R) = \Psi_1(\Psi_{n-1}(R))$ for $n \in \mathbb{N}$, $\Psi_0(R) \equiv R$, $R > 1$. By induction, we define measures $\mu_k^{(j)}$, $j \in \{1,2,3\}$, and a sequence (R_k), $k \in \mathbb{N}$. Suppose that the $\mu_l^{(j)}$ have already been defined for $l < k$, and the R_l are defined for $l \le k$. Let

$$Q_k = \{\zeta \in \mathbb{C} : R_k \le |\zeta| \le \Psi_1(R_k)\}, \quad \mu_k^- = \Big(\mu - \sum_{j=1}^{k-1}(\mu_j^{(1)} + \mu_j^{(2)} + \mu_j^{(3)})\Big)\Big|_{Q_k}.$$

If $\mu_k^-(Q_k) < 2$, we put $\mu_k^{(1)}(Q_k) \equiv 0$, $\mu_k^{(2)} = \mu_k^-$, $\mu_k^{(3)} \equiv 0$. If $\mu_k^-(Q_k) \ge 2$, we write $\mu_k^- = \mu_k^{(1)} + \mu_k^{(2)}$ so that $\mu_k^{(1)}(Q_k) = 2[\mu_k^-(Q_k)/2]$, where $[a]$ denotes the integral part of a. If $\mu_k^{(2)}(Q_k) \ge 1$, we put $\mu_k^{(3)}(Q_k) = 0$ and $R_{k+1} = \Psi_1(R_k)$. Otherwise, we define

$$R_{k+1} = \inf\big\{R > R_k : \mu(\{\Psi_1(R_k) < |z| \le R\}) \ge 1\big\},$$

and by $\mu_k^{(3)}$ we mean the sum of the restriction of μ to $\{\zeta : \Psi_1(R_k) < |\zeta| < R_{k+1}\}$ and $\tilde{\mu}$, where $\tilde{\mu}$ is a part of the restriction $\mu\big|_{\{\zeta:|\zeta|=R_{k+1}\}}$ such that $\mu_k^{(3)}(\mathbb{C}) = 1$. By construction, for all $k \in \mathbb{N}$ we have the following:

1) $\operatorname{supp}\mu_k^{(1)} \subset Q_k$, $\mu_k^{(1)}(Q_k) \in 2\mathbb{Z}_+$;
2) $\Psi_1(R_k) \le R_{k+1} \le \Psi_2(R_k)$;
3) $\operatorname{supp}(\mu_k^{(2)} + \mu_k^{(3)}) \subset \{\zeta : R_k \le |\zeta| \le R_{k+1}\}$;
4) $1 \le (\mu_k^{(2)} + \mu_k^{(3)})\big(\{\zeta : R_k \le |\zeta| \le R_{k+1}\}\big) \le 2$.

Let $\mu^{(1)} = \sum_{j=1}^{+\infty} \mu_j^{(1)}$, $\mu^{(2)} = \sum_{j=1}^{+\infty}(\mu_j^{(2)} + \mu_j^{(3)})$. From 3) and 4) it follows that $\mu^{(2)}(\overline{D_0(R)}) = O(\log R)$, $R \to +\infty$. Therefore, $u_2(z) = \int_{\mathbb{C}} \log\bigl|1 - \frac{z}{\zeta}\bigr|\, d\mu^{(2)}(\zeta)$ is a subharmonic function in $\mathbb{C}$. Let $u_1(z) = u(z) - u_2(z)$. Then $\mu_{u_1} = \mu^{(1)}$. We shall approximate u_1 and u_2 separately. It suffices to prove that

$$I_n \stackrel{\text{def}}{=} \int_{2^n \le |z| \le 2^{n+1}} \bigl|u(z) - \log|f(z)|\bigr|\, dm(z) = O(4^n \log \psi(2^n)), \quad n \to +\infty. \tag{2.2}$$

Indeed, let $R \in [2^n, 2^{n+1})$. Then from (2.2) it follows that

$$\begin{aligned}\frac{\int_{|z|<R} |u(z) - \log|f(z)||\, dm(z)}{R^2} &\le \frac{\sum_{k=0}^{n} \int_{2^k \le |z| \le 2^{k+1}} |u(z) - \log|f(z)||\, dm(z) + O(1)}{4^n} \\ &\le \frac{C_1 \sum_{k=0}^{n} 4^k \log \psi(2^k)}{4^n} \le 4C_1 \log \psi(2^n).\end{aligned}$$

2.2. Approximation of $u_2(z)$. By construction, $\mu^{(2)}(\mathbb{C}) = +\infty$. We put $T_n = \sup\{R > 0 : \mu^{(2)}(\overline{D_0(R)}) \le 5n\}$, $A_n = \{\zeta : T_n \le |\zeta| \le T_{n+1}\}$. Let (A_n, μ_n) be a partition of the measure $\mu^{(2)}$ such that $\mu^{(2)} = \sum_{k=1}^{+\infty} \mu_k$, $\operatorname{supp} \mu_n \subset A_n$, and $\mu_n(A_n) = 5$. We introduce r_n by the identities $\log r_n = \frac{1}{5} \int_{A_n} \log|\zeta|\, d\mu_n(\zeta)$, $n \in \mathbb{N}$, and consider the formal product

$$f_2(z) = \prod_{n=1}^{+\infty} \Bigl(1 - \frac{z}{r_n}\Bigr)^5.$$

Property 4) of the measures $\mu_k^{(j)}$ implies that

$$\Psi_1(T_n) \le T_{n+1} \le \Psi_6(T_n). \tag{2.3}$$

Since $r_{n+1}/r_{n-1} \ge T_{n+1}/T_n \ge \psi(T_n) \to +\infty$, $n \to +\infty$, the function f_2 is entire. Let

$$\begin{aligned} d_k(z) &\equiv \int_{A_k} \Bigl(\log\Bigl|1 - \frac{z}{\zeta}\Bigr| - \log\Bigl|1 - \frac{z}{r_k}\Bigr|\Bigr)\, d\mu_k(\zeta) \\ &= \int_{A_k} \Bigl(l \log\Bigl|1 - \frac{\zeta}{z}\Bigr| - \log\Bigl|1 - \frac{r_k}{z}\Bigr|\Bigr)\, d\mu_k(\zeta). \end{aligned} \tag{2.4}$$

Here we have used the choice of r_k. Fix $n \in \mathbb{N}$, and let $2^n \in [T_N, T_{N+1})$. Then for $k \ge N+2$ and $\zeta \in A_k$ we have $|\zeta| \ge T_k \ge |z|T_k/T_{N+1}$, $r_k \ge |z|T_k/T_{N+1}$. Consequently, $\bigl|\log|1 - \frac{z}{\zeta}|\bigr| \le 2|z|/|\zeta|$, $\bigl|\log|1 - \frac{z}{r_k}|\bigr| \le 2|z|/r_k$. Therefore,

$$\begin{aligned} \int_{2^n \le |z| \le 2^{n+1}} \sum_{k=N+2}^{\infty} |d_k(z)|\, dm(z) &\le \int_{2^n \le |z| \le 2^{n+1}} \sum_{k=N+2}^{\infty} 20 \frac{T_{N+1}}{T_k}\, dm(z) \\ &\le C_2 \frac{T_{N+1}}{T_{N+2}} 4^n = o(4^n), \quad n \to \infty. \end{aligned} \tag{2.5}$$

Similarly, $|d_k(z)| \le 20 T_{k+1} 2^{-n}$ for $k \le N-2$. Using (2.4), we obtain

$$\int_{2^n \le |z| \le 2^{n+1}} \sum_{k=1}^{N-2} |d_k(z)|\, dm(z) \le o(4^n), \quad n \to \infty. \tag{2.6}$$

We estimate $\int_{2^n \le |z| \le 2^{n+1}} |d_k(z)|\, dm(z)$ for $k \in \{N-1, N, N+1\}$. By definition,

$$\begin{aligned} &\int_{2^n \le |z| \le 2^{n+1}} |d_{N+1}(z)|\, dm(z) \\ &\qquad = \int_{2^n \le |z| \le 2^{n+1}} \Bigl| \int_{A_{N+1}} \Bigl(\log\Bigl|1 - \frac{z}{\zeta}\Bigr| - \log\Bigl|1 - \frac{z}{r_{N+1}}\Bigr|\Bigr)\, d\mu_{N+1}(\zeta) \Bigr| dm(z). \end{aligned}$$

If $|\zeta| \geq 2|z|$ and $\zeta \in A_{N+1}$, then $\big|\log|1-\frac{z}{\zeta}|\big| \leq \log 2$. Otherwise, we have $T_{N+1} \leq |\zeta| < 2|z| < 2^{n+2} \leq 4T_{N+1}$. Therefore, applying Fubini's theorem and changing the variables by the rule $T_{N+1}\eta = \zeta$, $T_{N+1}\xi = z$, we obtain

(2.7)

$$\begin{aligned}
&\int_{2^n\leq|z|\leq 2^{n+1}}\int_{A_{N+1}}\Big|\log\Big|1-\frac{z}{\zeta}\Big|\Big|\,d\mu_{N+1}(\zeta)dm(z)\\
&\quad\leq 2\int_{2^n\leq|z|\leq 2^{n+1}}dm(z)+\int_{2^n\leq|z|\leq 2^{n+1}}\int_{T_{N+1}\leq|\zeta|\leq 4T_{N+1}}\Big|\log\Big|1-\frac{z}{\zeta}\Big|\Big|\,d\mu_{N+1}(\zeta)dm(z)\\
&\qquad=O(4^n)+T_{N+1}^2\int_{1\leq|\eta|\leq 4}\int_{2^n/T_{N+1}\leq|\xi|\leq 2^{n+1}/T_{N+1}}\Big|\log\Big|1-\frac{\xi}{\eta}\Big|\Big|\,dm(\xi)\,d\mu_{N+1}(T_{N+1}\eta)\\
&\qquad=O(4^n)+O(4^n)\int_{1\leq|\eta|\leq 4}\int_{|\xi|\leq 2}\Big|\log\Big|1-\frac{\xi}{\eta}\Big|\Big|\,dm(\xi)\,d\mu_{N+1}(T_{N+1}\eta).
\end{aligned}$$

However, elementary calculations show that

$$\int_{|\xi|\leq 2}\Big|\log\Big|1-\frac{\xi}{\eta}\Big|\Big|\,dm(\xi)\leq C_3$$

for $1 \leq \eta \leq 4$. Recalling that $\mu_{N+1}(\mathbb{C}) = 2$, from (2.7) we deduce the estimate

$$\int_{2^n\leq|z|\leq 2^{n+1}}\int_{A_{N+1}}\Big|\log\Big|1-\frac{z}{\zeta}\Big|\Big|\,d\mu_{N+1}(\zeta)\,dm(z)=O(4^n),\quad n\to+\infty. \tag{2.8}$$

Similarly, if $r_{N+1} \geq 2^{n+2} \geq 2|z|$, then $\big|\log|1-z/r_{N+1}|\big| \leq \log 2$; otherwise, $T_{N+1} \leq r_{N+1} < 2^{n+2}$, whence ($T_{N+1}\xi = z$)

$$\begin{aligned}
&\int_{2^n\leq|z|\leq 2^{n+1}}\Big|\log\Big|1-\frac{z}{r_{N+1}}\Big|\Big|\,dm(z)\\
&\qquad=O(4^n)+O\bigg(4^n\int_{1/2\leq|\xi|\leq 1}\Big|\log\Big|1-\frac{\xi}{\eta}\Big|\Big|\,dm(\xi)\bigg)=O(4^n),\quad n\to+\infty.
\end{aligned}\tag{2.9}$$

Now, we estimate the integral of $|d_N(z)|$ over the annulus $\{2^n \leq |z| \leq 2^{n+1}\}$. Putting $\varphi(x) \equiv x/\psi(x)$, we have

$$\begin{aligned}
&\int_{A_N}\int_{2^n\leq|z|\leq 2^{n+1}}\Big|\log\Big|\frac{z-\zeta}{\zeta}\Big|\Big|\,dm(z)\,d\mu_N(\zeta)\\
&\quad\leq\int_{2^{n-1}\leq|\zeta|\leq 2^{n+2}}\bigg(\int_{D_\zeta(\varphi(|\zeta|))}+\int_{\substack{2^n\leq|z|\leq 2^{n+1}\\ z\notin D_\zeta(\varphi(|\zeta|))}}\bigg)\Big|\log\Big|\frac{z-\zeta}{\zeta}\Big|\Big|\,dm(z)\,d\mu_N(\zeta)\\
&\qquad+\bigg(\int_{T_N\leq|\zeta|\leq 2^{n-1}}+\int_{2^{n+2}\leq|\zeta|\leq T_{N+1}}\bigg)\int_{2^n\leq|z|\leq 2^{n+1}}\Big|\log\Big|\frac{z-\zeta}{\zeta}\Big|\Big|\,dm(z)\,d\mu_N(\zeta)\\
&\quad\equiv I_{N,1}+I_{N,2}+I_{N,3}+I_{N,4}.
\end{aligned}$$

Integrating by parts and using the relation $\varphi(x)=o(x)$, $x\to+\infty$, we obtain

$$\begin{aligned}
I_{N,1} &= \int_{2^{n-1}\le|\zeta|\le 2^{n+2}} \left(2\pi\int_0^{\varphi(|\zeta|)} \log\frac{|\zeta|}{s}\, s\,ds\right) d\mu_N(\zeta)\\
&= 2\pi\int_{2^{n-1}\le|\zeta|\le 2^{n+2}} \left(\frac{\varphi^2(|\zeta|)}{2}\log\psi(|\zeta|) + \int_0^{\varphi(|\zeta|)} s\,ds\right) d\mu_N(\zeta)\\
&= (\pi+o(1))\int_{2^{n-1}\le|\zeta|\le 2^{n+2}} \varphi^2(|\zeta|)\log\psi(|\zeta|)\,d\mu_N(\zeta) = o(4^n), \quad n\to+\infty;\\
I_{N,2} &\le \int_{2^{n-1}\le|\zeta|\le 2^{n+2}} \int_{\substack{2^n\le|z|\le 2^{n+1}\\ z\notin D_\zeta(\varphi(|\zeta|))}} \log\frac{|\zeta|}{\varphi(|\zeta|)}\, dm(z)\,d\mu_n(\zeta)\\
&\le \log\psi(2^{n+1})4^{n+1}\pi\mu_N(\{2^{n-1}\le|\zeta|\le 2^{n+2}\}) = O(4^n\log\psi(2^n)),\\
&\qquad\qquad n\to+\infty.
\end{aligned}$$

In the expressions for $I_{N,3}$ and $I_{N,4}$, the relationships between ζ and z yield, respectively,

$$\begin{aligned}
\left|\log\left|\frac{z-\zeta}{\zeta}\right|\right| &\le \log\frac{2^{n+1}+T_N}{T_N} \le \log\Bigl(1+2\frac{T_{N+1}}{T_N}\Bigr)\\
&\le \log\psi(T_N)+O(1),\\
\left|\log\left|\frac{z-\zeta}{\zeta}\right|\right| &\le 2\left|\frac{z}{\zeta}\right| \le 1.
\end{aligned}$$

Therefore, $I_{N,3}+I_{N,4}=O(4^n\log\psi(2^n))$, $n\to+\infty$. In a similar way we can obtain the estimate

$$\int_{2^n\le|z|\le 2^{n+1}} \left|\log\left|1-\frac{z}{r_N}\right|\right| dm(z) = O(4^n\log\psi(2^n)), \quad n\to+\infty.$$

Combining the estimates for $I_{N,j}$, $j\in\{1,\dots,4\}$, and the latter inequality, we obtain

$$\int_{2^n\le|z|\le 2^{n+1}} |d_N(z)|\,dm(z) = O\bigl(4^n\log\psi(2^n)\bigr), \quad n\to+\infty. \tag{2.10}$$

From the definition (2.4) it follows that

$$\begin{aligned}
&\int_{2^n\le|z|\le 2^{n+1}} |d_{N-1}(z)|\,dm(z)\\
&\qquad\le \int_{2^n\le|z|\le 2^{n+1}} \int_{A_{N-1}} \left|\log\left|1-\frac{\zeta}{z}\right| - \log\left|1-\frac{r_{N-1}}{z}\right|\right| d\mu_{N-1}(\zeta)\,dm(z).
\end{aligned}$$

Much as in the treatment of the integral of $d_{N+1}(z)$, for $|\zeta|\le|z|/2$, $\zeta\in A_{N-1}$, we observe that $\bigl|\log|1-z/\zeta|\bigr|\le\log 2$, whence

$$\begin{aligned}
&\int_{2^n\le|z|\le 2^{n+1}} \int_{A_{N-1}} \left|\log\left|1-\frac{\zeta}{z}\right|\right| d\mu_{N-1}(\zeta)\,dm(z)\\
&\le \int_{2^n\le|z|\le 2^{n+1}} \int_{2^{n-1}\le|\zeta|\le T_N} \left|\log\left|1-\frac{\zeta}{z}\right|\right| d\mu_{N-1}(\zeta)\,dm(z) + O(4^n)\\
&\le \int_{T_N/2\le|\zeta|\le T_N} \int_{2^n\le|z|\le 2^{n+1}} \left|\log\left|1-\frac{\zeta}{z}\right|\right| dm(z)\,d\mu_{N-1}(\zeta) + O(4^n)\\
&\le T_N^2 \int_{\frac{1}{2}\le|\eta|\le 1} \int_{2^n/T_N\le|\xi|\le 2^{n+1}/T_N} \left|\log\left|1-\frac{\eta}{\xi}\right|\right| dm(T_N\xi)\,d\mu_{N-1}(T_N\eta) + O(4^n)\\
&= O(T_N^2)+O(4^n) = O(4^n), \quad n\to+\infty.
\end{aligned}$$

Finally, from (2.8)–(2.10) and the latter relation we get

$$\int_{2^n\le|z|\le 2^{n+1}} \bigl|u_2(z)-\log|f_2(z)|\bigr|\,dm(z)=O(4^n\log\psi(2^n)),\quad n\to+\infty. \tag{2.11}$$

2.3. Approximation of $u_1(z)$. We recall that $Q_k=\{\zeta : R_k\le|\zeta|\le R_k\psi(R_k)\}$, $\operatorname{supp}\mu_k^{(1)}\subset Q_k$, $\mu_k^{(1)}(Q_k)\in 2\mathbb{Z}_+$. Let

$$P_k=\log Q_k=\{s=\sigma+it : \log R_k\le\sigma\le\log R_k+\log\psi(R_k), 0\le t<2\pi\},$$

and let l_k denote the ratio of the larger side of the rectangle P_k to the smaller. For $k\ge k_0$ we have $l_k=\log\psi(R_k)/(2\pi)>1$. Consider the measure $\mu_k^{(1)}$. If $\mu_k^{(1)}(\{p\})\ge 2$ at some point p, from this measure we subtract the measure $\tilde\mu_k^{(1)}$ equal to $2[\mu_k^{(1)}(\{p\})/2]$ at every such p. The measure $\tilde\mu^{(1)}=\sum_k\tilde\mu_k^{(1)}$ is discrete, integer-valued, and finite on the compact sets in $\mathbb{C}$. By the Weierstrass theorem, there exists an entire function f_3 with zeros of the corresponding multiplicity on the support of the measure $\tilde\mu^{(1)}$ (this support is an at most countable set of isolated points). We have $\mu_{\log|f_3|}=\tilde\mu^{(1)}$. For every $k\in\mathbb{N}$, the measure $\mu^k=\mu_k^{(1)}-\tilde\mu^{(1)}\big|_{Q_k}$ satisfies the condition $\mu^k(\{p\})<2$ at every point $p\in Q_k$. By the choice of the origin, on the rays emanating from it and on the circles centered at it there is at most one point p such that $0<\mu^k(\{p\})<2$, and at the same time the μ^k-measure of the remaining part of either a ray or a circle is equal to zero. Under these conditions, the measures ν^k defined by $d\nu^k(s)=d\mu^k(e^s)$, $s\in P_k$ (i.e., $\nu^k(S)=\mu^k(\exp S)$ for every Borel set S), satisfy the hypothesis of Theorem 3 with $\Pi=P_k$, $k\in\mathbb{N}$. Applying that theorem, we obtain a system of rectangles P_{km} and of measures ν_{km}, $1\le m\le N_k$. We have $\nu_{km}(P_{km})=2$, and every point s with $|\operatorname{Im}s|\le 2\pi$ belongs to the interiors of at most four rectangles P_{km}. We enumerate (P_{km},ν_{km}) by natural numbers, in an arbitrary way. As a result, we obtain $(P^{(k)},\nu^{(k)})$ with $\nu^{(k)}(P^{(k)})=2$ and $\operatorname{supp}\nu^{(k)}\subset P^{(k)}$; then also the remaining properties listed in Theorem 3 are fulfilled.

Let $\omega_l^{(1)}$, $\omega_l^{(2)}$ be the solutions of the system of equations

$$\begin{cases} \omega_l^{(1)}+\omega_l^{(2)}=\displaystyle\int_{P^{(l)}}\omega\,d\nu^{(l)}(\omega),\\ (\omega_l^{(1)})^2+(\omega_l^{(2)})^2=\displaystyle\int_{P^{(l)}}\omega^2\,d\nu^{(l)}(\omega),\end{cases} \tag{2.12}$$

and let

$$\omega_l=\frac12\int_{P^{(l)}}\omega\,d\nu^{(l)}(\omega) \tag{2.13}$$

be the center of mass of $P^{(l)}$, $l\in\mathbb{N}$. From (2.12) we see that $\omega_l^{(1)}+\omega_l^{(2)}=2\omega_l$,

$$\int_{P^{(l)}}\omega^2\,d\nu^{(l)}(\omega)=(\omega_l^{(1)})^2+(2\omega_l-\omega_l^{(1)})^2=2(\omega_l^{(1)}-\omega_l)^2+2\omega_l^2.$$

Therefore, by (2.13), for $j\in\{1,2\}$ we obtain

$$\begin{aligned}|\omega_l^{(j)}-\omega_l|&=\Bigl|\frac12\int_{P^{(l)}}\omega^2\,d\nu^{(l)}(\omega)-\omega_l^2\Bigr|^{\frac12}\\ &=\Bigl|\frac12\int_{P^{(l)}}(\omega-\omega_l)^2\,d\nu^{(l)}(\omega)\Bigr|^{\frac12}\le\operatorname{diam}P^{(l)}\equiv d_l.\end{aligned}$$

Since $\omega_l\in P^{(l)}$, we have

$$\sup_{\omega\in P^{(l)}}|\omega-\omega_l^{(j)}|\le 2d_l,\ \ j\in\{1,2\},\qquad \sup_{\omega\in P^{(l)}}|\omega-\omega_l|\le d_l. \tag{2.14}$$

Put $\zeta_l^{(j)} = \exp\{\omega_l^{(j)}\}$ and

(2.15)
$$
\begin{aligned}
V(z) &= \sum_l \int_{Q^{(l)}} \Big(\log\Big|1-\frac{z}{\zeta}\Big| - \frac{1}{2}\log\Big|1-\frac{z}{\zeta_l^{(1)}}\Big| - \frac{1}{2}\log\Big|1-\frac{z}{\zeta_l^{(2)}}\Big|\Big)\, d\mu^{(l)}(\zeta)\\
&= \sum_l \int_{P^{(l)}} \Big(\log|1-ze^{-\omega}| - \frac{1}{2}\log\Big|1-ze^{-\omega_l^{(1)}}\Big| - \frac{1}{2}\log\Big|1-ze^{-\omega_l^{(2)}}\Big|\Big)\, d\nu^{(l)}(\omega)\\
&\equiv \sum_l \Delta_l(z).
\end{aligned}
$$

Under the assumption that the series (2.15) converges absolutely, we need to prove that

$$\int_{2^n\le|z|\le 2^{n+1}} |V(z)|\, dm(z) = O(4^n \log\psi(2^n)).$$

Let $\mathcal{L}^+$ be the set of ls such that $Q^{(l)} \subset D_0(2^{n-2})$, let $\mathcal{L}^-$ consist of all l with $Q^{(l)} \subset \{z : |z| > 2^{n+1}\}$, and let $\mathcal{L}^0 = \mathbb{N} \setminus (\mathcal{L}^- \cup \mathcal{L}^+)$. We denote $L(\omega) = \log(1-ze^{-\omega})$. For $l \in \mathcal{L}^- \cup \mathcal{L}^+$, $\omega \in P^{(l)}$, and $2^n \le |z| \le 2^{n+1}$, the function $L(\omega)$ is analytic. We shall use the following identities:

$$
\begin{aligned}
L(\omega) - L(\omega_l^{(1)}) &= \int_{\omega_l^{(1)}}^{\omega} L'(s)\, ds\\
&= L'(\omega_l)(\omega-\omega_l^{(1)}) + \int_{\omega_l^{(1)}}^{\omega} L''(s)(\omega-s)\, ds\\
&= L'(\omega_l)(\omega-\omega_l^{(1)})\\
&\quad + \frac{1}{2}L''(\omega_l)(\omega-\omega_l^{(1)})^2 + \frac{1}{2}\int_{\omega_l^{(1)}}^{\omega} L'''(s)(\omega-s)^2\, ds.
\end{aligned}
\tag{2.16}
$$

From the first identity in (2.16) we obtain

$$
\begin{aligned}
|\Delta_l(z)| &= \Big|\mathrm{Re}\int_{P^{(l)}} \Big(L(\omega) - L(\omega_l^{(1)}) - \frac{1}{2}(L(\omega_l^{(2)}) - L(\omega_l^{(2)})\Big)\, d\nu^{(l)}(\omega)\Big|\\
&\le \int_{P^{(l)}} \Big|\int_{\omega_l^{(1)}}^{\omega} \frac{z}{e^s - z}\, ds\Big|\, d\nu^{(l)}(\omega) + \frac{1}{2}\int_{P^{(l)}} \Big|\int_{\omega_l^{(1)}}^{\omega_l^{(2)}} \frac{z}{e^s-z}\, ds\Big|\, d\nu^{(l)}(\omega)\\
&\le |z| \sup_{s\in P^{(l)}} \frac{1}{|e^s - z|}\Big(\int_{P^{(l)}} (|\omega-\omega_l^{(1)}| + \frac{1}{2}|\omega_l^{(1)} - \omega_l^{(2)}|)\, d\nu^{(l)}(\omega)\Big)\\
&\le \frac{3d_l|z|}{\inf_{s\in P^{(l)}} |e^s - z|}.
\end{aligned}
\tag{2.17}
$$

Let $l \in \mathcal{L}^-$. The rectangle $P^{(l)}$ is of the form $\{s = \sigma + it : \sigma_l^- \le \sigma \le \sigma_l^+, 0 \le t_l^- \le t \le t_l^+ \le 2\pi\}$, i.e., the lengths of its sides are equal to $\sigma_l^+ - \sigma_l^-$ and $t_l^+ - t_l^-$. Let λ_l be the ratio of the maximal of these numbers to the minimal ($\lambda_l \ge 1$). By condition 4) in Theorem 3, for $\lambda_l > 3$ we have $t_l^+ - t_l^- = 2\pi$, $\sigma_l^+ - \sigma_l^- = 2\pi\lambda_l$. First, we estimate the quantities $|\Delta_l(z)|$ with $\lambda_l > 3$. Then $d_l = 2\pi\sqrt{1+\lambda_l^2} \le 2\log\psi(e^{\sigma_l^-})$, because $\log R_k \le \sigma_l^- < \sigma_l^+ \le \log R_k + \log\psi(R_k)$ for some $k \in \mathbb{N}$. From (2.17) it follows that

$$\inf_{s\in P^{(l)}} |e^s - z| \ge \inf_{s\in P^{(l)}} \frac{e^{\mathrm{Re}\, s}}{2} \ge \frac{e^{\sigma_l^-}}{2}, \quad l \in \mathcal{L}^-.$$

But

$$\int_{\sigma_l^-}^{\sigma_l^+} e^{-s}\, ds = e^{-\sigma_l^-}(1 - e^{-\sigma_l^+ + \sigma_l^-}) \ge e^{-\sigma_l^-}/2.$$

Therefore,

$$|\Delta_l(z)| \le \frac{6d_l|z|}{e^{\sigma_l^-}} \le 24|z| \int_{\sigma_l^-}^{\sigma_l^+} e^{-\sigma} \log\psi(e^{\sigma})\, d\sigma. \tag{2.18}$$

Since $\psi(x)$ varies slowly, so does $\log\psi(x)$; consequently,

$$\log\psi(x2^{k+1}) \le (1+\varepsilon)\log\psi(x2^k) \le (1+\varepsilon)^{k+1}\log\psi(x), \quad \varepsilon > 0,\ x \ge x_\varepsilon.$$

Applying the above estimates and the fact that every point s belongs to the interiors of at most 4 of the rectangles $P^{(l)}$, from (2.18) we deduce the inequality

$$\begin{aligned}\sum_{\substack{l\in\mathcal{L}^-\\ \lambda_l>3}} |\Delta_l(z)| &\le 24|z| \int_{\log 2^{n+1}}^{+\infty} e^{-\sigma}\log\psi(e^{\sigma})\, d\sigma \\ &\le 24|z| \sum_{k=0}^{+\infty} \int_{\log 2^{n+1+k}}^{\log 2^{n+2+k}} e^{-\sigma}\log\psi(e^{\sigma})\, d\sigma \\ &\le 24|z| \sum_{k=0}^{+\infty} \frac{\log\psi(2^{n+k+2})}{2^{n+k+2}} \\ &\le 24|z|\log\psi(2^n) \sum_{k=0}^{+\infty} \frac{(1+\varepsilon)^{k+2}}{2^{n+2+k}} \le 96\frac{\log\psi(2^n)}{1-\varepsilon}.\end{aligned} \tag{2.19}$$

Using the first equation in (2.12), it is not difficult to obtain the following representation for Δ_l:

$$\Delta_l(z) = \int_{P^{(l)}} \Bigl(\log\Bigl|1-\frac{e^{\omega}}{z}\Bigr| - \frac{1}{2}\log\Bigl|1-\frac{e^{\omega_l^{(1)}}}{z}\Bigr| - \frac{1}{2}\log\Bigl|1-\frac{e^{-\omega_l^{(2)}}}{z}\Bigr|\Bigr)\, d\nu^{(l)}(\omega). \tag{2.20}$$

We use (2.20) in order to estimate Δ_l for $l \in \mathcal{L}^+$ with $\lambda_l > 3$. For the analytic function $L_2(\omega) = \log\bigl(1-\frac{e^{\omega}}{z}\bigr)$ and $z \in P^{(l)}$, $l \in \mathcal{L}^+$, identities (2.16) are true with $L = L_2$. Therefore, as in (2.17), we obtain the estimate

$$|\Delta_l(z)| \le \frac{3d_l}{\inf_{s\in P^{(l)}} |1 - ze^{-s}|} \le \frac{12e^{\sigma_l^+} d_l}{|z|}, \quad l \in \mathcal{L}^+.$$

Since $d_l \le \log\psi(2^{n-1})$, $l \in \mathcal{L}^+$, it follows that

$$\begin{aligned}\sum_{\substack{l\in\mathcal{L}^+\\ \lambda_l>3}} |\Delta_l(z)| &\le C_3 \frac{\log\psi(2^{n-1})}{|z|} \sum_{\substack{l\in\mathcal{L}^+\\ \lambda_l>3}} \int_{\sigma_l^-}^{\sigma_l^+} e^{\sigma}\, d\sigma \\ &\le 4C_3\frac{\log\psi(2^{n-1})}{|z|} \int_{\log R_1}^{\log 2^{n-1}} e^{\sigma}\, d\sigma \le C_4 \log\psi(2^{n-1}).\end{aligned} \tag{2.21}$$

For $l \in \mathcal{L}^+ \cup \mathcal{L}^-$, it remains to estimate the Δ_l with $1 \le \lambda_l \le 3$. Then $d_l^2 \asymp m(P^{(l)})$, in particular, $d_l \le 6\pi$. In this case, arguing as in [10] and using the second identity in

(2.16), we get

$$
\begin{aligned}
\Delta_l(z) = \operatorname{Re}\int_{P^{(l)}}\biggl(&(\omega-\omega_l^{(1)})L'(\omega_l^{(1)})-\frac12 L'(\omega_l^{(1)})(\omega_l^{(2)}-\omega_l^{(1)})\\
&+\int_{\omega_l^{(1)}}^{\omega} L''(s)(\omega-s)\,ds-\frac12\int_{\omega_l^{(1)}}^{\omega_l^{(2)}} L''(s)(\omega_l^{(2)}-s)\,ds\biggr)\,d\nu^{(l)}(\omega)\\
= \operatorname{Re}\biggl\{-z\int_{P^{(l)}}&\int_{\omega_l^{(1)}}^{\omega}\frac{e^s(\omega-s)}{(e^s-z)^2}\,ds\,d\nu^{(l)}(\omega)+\frac{z}{2}\int_{P^{(l)}}\int_{\omega_l^{(1)}}^{\omega_l^{(2)}}\frac{e^s(\omega_l^{(2)}-s)}{(e^s-z)^2}\,ds\,d\nu^{(l)}(\omega)\biggr\},
\end{aligned}
$$

whence

$$
|\Delta_l(z)|\le 6d_l^2|z|\sup_{s\in P^{(l)}}\Bigl|\frac{e^s}{(e^s-z)^2}\Bigr|. \tag{2.22}
$$

If $l\in\mathcal{L}^+$, then $|e^s-z|\ge |z|/2$, and from (2.22) we deduce the inequality

$$
|\Delta_l(z)|\le 24d_l^2\frac{e^{\sigma_l^+}}{|z|}\le\frac{C_4}{|z|}\iint_{P^{(l)}}e^{\sigma}\,d\sigma\,dt. \tag{2.23}
$$

Applying (2.23), we obtain

$$
\begin{aligned}
\sum_{\substack{l\in\mathcal{L}^+\\ \lambda_l\le 3}}|\Delta_l(z)|&\le\frac{C_4}{|z|}\sum_{\substack{l\in\mathcal{L}^-\\ \lambda_l\le 3}}\iint_{P^{(l)}}e^{\sigma}\,d\sigma\,dt\le\frac{4C_4}{|z|}\int_0^{2\pi}\int_{\log R_1}^{\log 2^{n-2}}e^{\sigma}\,d\sigma\,dt\\
&\le\frac{8\pi C_4 2^{n-2}}{|z|}=O(1).
\end{aligned} \tag{2.24}
$$

Now, let $l\in\mathcal{L}^-$; then $|e^s-z|\ge e^{\sigma}/2$, and therefore, (2.22) implies that

$$
|\Delta_l(z)|\le 24d_l^2|z|e^{-\sigma_l^-}\le C_5|z|\iint_{P^{(l)}}e^{-\sigma}\,d\sigma\,dt,
$$

$$
\sum_{l\in\mathcal{L}^-,\lambda_l\le 3}|\Delta_l(z)|\le 8\pi K_5|z|\int_{\log 2^{n+1}}^{\infty}e^{-\sigma}\,d\sigma=8\pi|z|2^{-n-1}\le C_6.
$$

Combining (2.19), (2.21), (2.24), and the latter estimate, we see that

$$
\sum_{l\in\mathcal{L}^+\cup\mathcal{L}^-}|\Delta_l(z)|\le C_7\log\psi(2^{n+1}). \tag{2.25}
$$

Now, let $l\in\mathcal{L}^0$, i.e., $Q^{(l)}\cap G_n\ne\varnothing$, where $G_n=\{z: 2^{n-2}\le|z|\le 2^{n+1}\}$. Among $l\in\mathcal{L}^0$, at most eight are such that $\lambda_l>3$. Indeed, for such l we have $Q^{(l)}=\{z: e^{\sigma_l^-}\le|z|\le e^{\sigma_l^+}\}$ and $\sigma_l^+-\sigma_l^->6\pi$, so that $e^{\sigma_l^+}-e^{\sigma_l^-}\ge 2^{n-1}(1-e^{-6\pi})$. Since the interiors of these $Q^{(l)}$ form an at most 4-fold cover, for all l except for at most four of them we have $e^{\sigma_l^-}\ge 2^{n-1}$, and, except for at most eight of them, $e^{\sigma_l^-}\ge e^{6\pi}2^{n-1}>2^{n+1}$, i.e., the intersection with G_n is empty.

We denote this exceptional set of indices l by $\mathcal{L}^0_*$. We must estimate $\int_{G_n}|\Delta_l(z)|\,dm(z)$ for $l\in\mathcal{L}^0_*$. If $|\zeta|\ge 3\cdot 2^n$ ($\zeta\in Q^{(l)}$, $z\in G_n$), then $\bigl|\log|1-\frac{z}{\zeta}|\bigr|\le 2|z|/|\zeta|\le 3$. Otherwise, we have $|\zeta|\le 3\cdot 2^n$, and

$$
\begin{aligned}
\int_{D_\zeta(2^n)}\Bigl|\log\Bigl|1-\frac{z}{\zeta}\Bigr|\Bigr|\,dm(z)&\le\int_{D_\zeta(2^n)}\Bigl|\log\Bigl|\frac{z-\zeta}{2^n}\Bigr|\Bigr|+\Bigl|\log\Bigl|\frac{\zeta}{2^n}\Bigr|\Bigr|\,dm(z)\\
&\le 2\pi\int_0^{2^n}\log\frac{2^n}{\tau}\tau\,d\tau+\sup_{\zeta\in Q^{(l)}}\bigl|\log|\zeta 2^{-n}|\bigr|\pi 2^{2n}\le\pi^2 4^{n-1}+C_8\log\psi(2^n)\cdot 4^n.
\end{aligned} \tag{2.26}
$$

Thus,

$$
\begin{aligned}
&\int_{G_n} \sum_{l\in\mathcal{L}^*} \Big| \int_{Q^{(l)}} \log\Big|1-\frac{z}{\zeta}\Big|\, d\mu^{(l)}(\zeta)\Big|\, dm(z)\\
&\quad\le \sum_{l\in\mathcal{L}^*} \Big(\int_{Q^{(l)}\cap\{|\zeta|\ge 3\cdot 2^n\}} + \int_{Q^{(l)}\cap\{|\zeta|\le 3\cdot 2^n\}} \Big) \int_{G_n} \Big|\log\Big|1-\frac{z}{\zeta}\Big|\Big|\, dm(z) d\mu^{(l)}(\zeta)\\
&\quad\le \sum_{l\in\mathcal{L}^*} \Big(3\cdot 2m(G_n)\\
&\qquad\qquad + \int_{Q^{(l)}\cap\{|\zeta|\le 3\cdot 2^n\}} \Big(\int_{D_\zeta(2^n)} \Big|\log\Big|1-\frac{z}{\zeta}\Big|\Big|\, dm(z)\\
&\qquad\qquad\qquad + \int_{G_n\setminus D_\zeta(2^n)} \Big|\log\Big|1-\frac{z}{\zeta}\Big|\Big|\, dm(z)\Big) d\mu^{(l)}(\zeta)\Big)\\
&\quad\le K_9 4^n + K_9 \log\psi(2^n)\cdot 4^n\\
&\qquad + \int_{Q^{(l)}\cap\{|\zeta|\le 3\cdot 2^n\}} \int_{G_n\setminus D_\zeta(2^n)} \log\frac{|z|+|\zeta|}{|\zeta|}\, dm(z)\, d\mu^{(l)}(\zeta)\\
&\quad\le K_9 4^n \log\psi(2^n).
\end{aligned}
\tag{2.27}
$$

For $l\in\mathcal{L}^0\setminus\mathcal{L}^0_*$ we have $\lambda_l\le 3$, i.e., all the corresponding rectangles $P^{(l)}$ are "almost squares"; this allows us to apply the arguments used in [10, e–g]. Let $D_l = \operatorname{diam} Q^{(l)}$. Under the condition $\operatorname{dist}\{z, Q^{(l)}\} > 4D_l$, we can use the last identity in (2.16) and argue as in (2.22) to obtain

$$
\begin{aligned}
|\Delta_l(z)| &= \Big|\frac{1}{2}\operatorname{Re}\int_{P^{(l)}} z\int_{\omega_l^{(1)}}^{\omega} e^s \frac{z+e^s}{(e^s-z)^3}(\omega-s)^2\, ds\, d\nu^{(l)}(\omega)\\
&\qquad\qquad + \frac{1}{2}\operatorname{Re} z\int_{\omega_l^{(1)}}^{\omega_l^{(2)}} e^s\frac{z+e^s}{(e^s-z)^3}(\omega_l^{(2)}-s)^2\, ds\Big|\\
&\le \frac{d_l^3|z|^3}{|\zeta_l^{(1)}-z|^3} \le \frac{D_l^3}{|\zeta_l^{(1)}-z|^3}.
\end{aligned}
\tag{2.28}
$$

It follows that

$$
\begin{aligned}
&\int_{G_n}\sum_{l\in\mathcal{L}^0\setminus\mathcal{L}^0_*} |\Delta_l(z)|\, dm(z) \le \sum_{l\in\mathcal{L}^0,\lambda_l\le 3}\int_{G_n} |\Delta_l(z)|\, dm(z)\\
&\quad\le \sum_{l\in\mathcal{L}^0,\lambda_l\le 3}\Big(\int_{G_n\cap\{|z-\zeta_l^{(1)}|>3D_l\}} + \int_{|z-\zeta_l^{(1)}|\le 3D_l}\Big)|\Delta_l(z)|\, dm(z)\\
&\quad\le \sum_{l\in\mathcal{L}^0,\lambda_l\le 3}\Big(\int_{G_n\cap\{|z-\zeta_l^{(1)}|>3D_l\}} \frac{D_l^3}{|z-\zeta_l^{(1)}|^3}\, dm(z) + \int_{|z-\zeta_l^{(1)}|\le 3D_l} |\Delta_l(z)|\, dm(z)\Big).
\end{aligned}
\tag{2.29}
$$

For the first sum in (2.29) we have

$$\begin{aligned}
&\sum_{l\in\mathcal{L}^0,\lambda_l\le 3} D_l^3 \int_{\{|z-\zeta_l^{(1)}|>3D_l\}\cap G_n} \frac{1}{|z-\zeta_l^{(1)}|^3}\,dm(z)\\
&\quad\le \sum_{l\in\mathcal{L}^0,\lambda_l\le 3} D_l^3 2\pi \int_{3D_l}^{2^{n+1}} \frac{1}{t^3}t\,dt\\
&\quad\le \sum_{l\in\mathcal{L}^0,\lambda_l\le 3} 2D_l^2 \le \sum_{l\in\mathcal{L}^0,\lambda_l\le 3} m(Q^{(l)}).
\end{aligned}\tag{2.30}$$

It remains to estimate $\int_{|z-\zeta_l^{(1)}|\le 3D_l}|\Delta_l(z)|\,dm(z)$. The definition of $\Delta(z)$ and the relation $\log|\zeta_l^{(1)}|+\log|\zeta_l^{(2)}|=\int_{Q^{(l)}}\log|\zeta|\,d\mu^{(l)}(\zeta)$ imply

$$|\Delta_l(z)|=\int_{Q^{(l)}}\Bigl(\log\Bigl|\frac{\zeta-z}{3D_l}\Bigr|-\frac{1}{2}\log\Bigl|\frac{\zeta_l^{(1)}-z}{3D_l}\Bigr|-\frac{1}{2}\log\Bigl|\frac{\zeta_l^{(2)}-z}{3D_l}\Bigr|\Bigr)\,d\mu^{(l)}(\zeta).$$

Thus,

$$\begin{aligned}
&\int_{|z-\zeta_l^{(1)}|\le 3D_l}|\Delta_l(z)|\,dm(z)\\
&\quad\le \int_{Q^{(l)}}\Bigl(\int_{|z-\zeta_l^{(1)}|\le 3D_l}\Bigl|\log\Bigl|\frac{\zeta-z}{3D_l}\Bigr|\Bigr|+\frac{1}{2}\Bigl|\log\Bigl|\frac{\zeta_l^{(1)}-z}{3D_l}\Bigr|\Bigr|\\
&\qquad\qquad+\frac{1}{2}\Bigl|\log\Bigl|\frac{\zeta_l^{(2)}-z}{3D_l}\Bigr|\Bigr|\,dm(z)\Bigr)\,d\mu^{(l)}(\zeta)\\
&\quad=\int_{Q^{(l)}}\Bigl(\int_{|z-\zeta|\le D_l}+\int_{\substack{|z-\zeta|>D_l\\|z-\zeta_l^{(1)}|\le 3D_l}}\Bigr)\Bigl|\log\Bigl|\frac{\zeta-z}{3D_l}\Bigr|\Bigr|\,dm(z)\,d\mu^{(l)}(\zeta)\\
&\qquad+\int_{|z-\zeta_l^{(1)}|<D_l}\log\frac{3D_l}{|\zeta_l^{(1)}-z|}\,dm(z)\\
&\qquad+\Bigl(\int_{|z-\zeta_l^{(2)}|\le 3D_l}+\int_{\substack{|z-\zeta_l^{(2)}|>3D_l\\|z-\zeta_l^{(1)}|\le 3D_l}}\Bigr)\Bigl|\log\Bigl|\frac{\zeta_l^{(2)}-z}{3D_l}\Bigr|\Bigr|\,dm(z)\\
&\quad\le\int_{Q^{(l)}}\Bigl(\int_0^{D_l}\log\frac{3D_l}{\tau}\tau\,d\tau+(3D_l)^2\pi\log 3\Bigr)\,d\mu^{(l)}(\zeta)\\
&\qquad+2\int_0^{D_l}\log\frac{3D_l}{\tau}\tau\,d\tau+\pi(3D_l)^2\log 3\\
&\quad\le 9(3\pi\log 3+1)D_l^2\le C_{11}m(Q^{(l)}).
\end{aligned}$$

The latter inequality and (2.29) yield

$$\int_{G_n}\sum_{l\in\mathcal{L}^0\setminus\mathcal{L}^0_*}|\Delta_l(z)|\,dm(z)\le\sum_{l\in\mathcal{L}^0\setminus\mathcal{L}^0_*}K_{11}m(Q^{(l)})\le K_{12}2^{2n}.$$

Applying this inequality and (2.25) completes the proof of Theorem 1. Using Chebyshev's inequality, from (2.2) we can easily deduce Corollary 1.

§3. PROOF OF THEOREM 2′

Suppose that σ satisfies the hypothesis of the theorem. Without loss of generality, we may assume that $\psi(r)=\exp\bigl\{\int_1^r\frac{\sigma(t)}{t}\,dt\bigr\}$ is unbounded. Obviously, $\psi\in\Phi$. Let u_ψ

be defined by formula (1.10) with $\varphi = \psi$, $\psi \in \Phi$. Suppose that there exists an entire function f and a constant $\alpha \in [0, 1)$ satisfying

$$\begin{aligned} \int_{|z|<R} \bigl|u_\psi(z) - \log|f(z)| - \alpha\log|z|\bigr|\,dm(z) & \\ < \varepsilon R^2 \log\psi(R), \quad R \ge R_\varepsilon, & \end{aligned} \tag{3.1}$$

for arbitrary $\varepsilon > 0$. There is no loss of generality in assuming that $f(0) \neq 0$. By separating the term $\frac{1}{2}\log|1 - z/r_1|$ from u_ψ, we reduce the case of $\alpha \in [1/2, 1)$ to the case where $\alpha \in [0, 1/2)$. Therefore, we consider the latter case in detail. We introduce the counting Nevanlinna characteristics of the Riesz masses u_ψ and f:

$$N(r, u_\psi) = \int_0^r \frac{n(t, u_\psi)}{t}\,dt,$$
$$N(r, f) = \int_0^r \frac{n(t, f)}{t}\,dt,$$

where $n(r, f)$ is the number of zeros of f in $\overline{D}_0(r)$. By Jensen's formula (see [1, Chapter 3.9], we have

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} (u_\psi(re^{i\theta}) - \log|f(re^{i\theta})| - \alpha\log r)\,d\theta & \\ = N(r, u_\psi) - N(r, f) - \alpha\log r - \log|f(0)|. & \end{aligned}$$

If $R \ge \widetilde{R}_\varepsilon$ then, since the function $\log\psi(R)$ varies slowly, we obtain

$$\begin{aligned} &\int_R^{2R} \bigl|N(r, u_\psi) - N(r, f) - \alpha\log r - \log|f(0)|\bigr| r\,dr \\ &\quad\le \int_R^{2R} \frac{1}{2\pi}\int_0^{2\pi} \bigl|u_\psi(re^{i\theta}) - \log|f(re^{i\theta})| - \alpha\log r\bigr|\,d\theta r\,dr \\ &\quad\le \frac{1}{2\pi}\int_{|z|\le 2R} \bigl|u_\psi(z) - \log|f(z)| - \alpha\log|z|\bigr|\,dm(z) \\ &\quad< \frac{\varepsilon}{2\pi}(2R)^2\log\psi(2R) \\ &\quad< \frac{2\varepsilon(1+\varepsilon)}{\pi}R^2\log\psi(R). \end{aligned}$$

This implies that on $[R, 2R]$ there exists r^* such that

$$|N(r^*, u_\psi) - N(r^*, f) - \alpha\log r^*| \le \frac{2\varepsilon(1+\varepsilon)}{3}\log\psi(r^*) < \varepsilon\log\psi(r^*) \tag{3.2}$$

for $\varepsilon \in (0, 1/2)$. From (3.2) we shall derive the relation

$$|n(r, u_\psi) - n(r, f) - \alpha| \le \frac{1}{2}, \quad r \to +\infty. \tag{3.3}$$

Assume the contrary. If (3.3) fails, then there exists a sequence (τ_k), $\tau_k \to +\infty$ as $k \to +\infty$, such that either i) $n(\tau_k, u_\psi) - n(\tau_k, f) - \alpha > 1/2$, or ii) $n(\tau_k, f) - n(\tau_k, u_\psi) + \alpha > 1/2$, $k \to +\infty$. Consider case i). For arbitrary $t \in [\tilde{\tau}_k, \tau_k]$, where $\tilde{\tau}_k\psi(\tilde{\tau}_k) = \tau_k$, we have $n(t, u_\psi) - n(\tau_k, u_\psi) \ge -1/2$; therefore,

$$n(t, u_\psi) - n(t, f) - \alpha \ge n(\tau_k, u_\psi) - n(\tau_k, f) - \alpha + n(t, u_\psi) - n(\tau_k, u_\psi) > 0.$$

Since the values of $n(t, u_\psi) - n(t, f)$ are integral multiples of $1/2$, we have

$$n(t, u_\psi) - n(t, f) - \alpha \ge 1/2 - \alpha > 0, \quad t \in [\tilde{\tau}_k, \tau_k]. \tag{3.4}$$

Choose $t_k \in [\tilde{\tau}_k, 2\tilde{\tau}_k]$ and $T_k \in [\tau_k/2, \tau_k]$ so that (3.2) be fulfilled for $r^* \in \{t_k, T_k\}$. Then, using the definition of the function $N(r, \cdot)$ and applying (3.2) and (3.4), for $\varepsilon \in (0, 1/4 - \alpha/2)$ we obtain

$$\begin{aligned}\varepsilon \log \psi(T_k) &> |N(T_k, u_\psi) - N(T_k, f) - \alpha \log T_k| \\ &\geq \int_{t_k}^{T_k} \frac{n(t, u_\psi) - n(t, f) - \alpha}{t}\, dt - |N(t_k, u_\psi) - N(t_k, f) - \alpha \log t_k| \\ &\geq \Big(\frac{1}{2} - \alpha\Big) \log \frac{T_k}{t_k} - \varepsilon \log \psi(t_k) \\ &= \Big(\frac{1}{2} - \alpha - \varepsilon\Big) \log \psi(t_k) > \varepsilon \log \psi(T_k), \quad k \to +\infty.\end{aligned}$$

Thus, case i) is impossible. Similarly, in case ii) we have $n(t, f) - n(t, u_\psi) + \alpha > 0$ for $t \in [\tau_k, \tau_k \psi(\tau_k)]$, whence $n(t, f) - n(t, u_\psi) + \alpha \geq \beta$, where β is a positive constant. Choosing $t_k \in [\tau_k, 2\tau_k]$ and $T_k \in [\tau_k \psi(\tau_k)/2, \tau_k \psi(\tau_k)]$ satisfying (3.2) instead of r^*, and assuming that $\varepsilon \in (0, \beta/2)$, in the same way as before we obtain

$$\begin{aligned}&|N(T_k, f) - N(T_k, u_\psi) + \alpha \log T_k| \\ &\geq \int_{t_k}^{T_k} \frac{n(t, f) - n(t, u_\psi) + \alpha}{t}\, dt - |N(t_k, f) - N(t_k, u_\psi) + \alpha \log t_k| \\ &\geq (\beta - \varepsilon) \log \psi(t_k).\end{aligned}$$

Therefore, case ii) is also impossible. Thus, relation (3.3) is proved.

Let ρ_k be the modulus of the kth zero of f (the zeros are enumerated in nondecreasing order of their moduli). Since the jumps of $n(t, f)$ take natural values, and the jumps of $n(t, u_\psi)$ take the value $\frac{1}{2}$, relation (3.3) is possible only in the case when, starting with some $k_0 \in \mathbb{N}$, between every two neighboring jump points $\rho_k \leq \rho_{k+1}$ of the function $n(t, f)$ there are points r_m, r_{m+1}, in particular, $\rho_k < \rho_{k+1}$. First, we consider the case where $\alpha = 0$. If $\rho_k < r_{2k-1}$, then for $r \in (\rho_k, r_{2k-1})$ we have $n(r, f) - n(r, u_\psi) \geq k - (2k-2)/2 = 1$. But if $\rho_k > r_{2k}$, then $n(r, u_\psi) - n(r, f) \geq 1$ for $r \in (\max\{r_{2k}, \rho_{k-1}\}, \rho_k)$. Therefore, $r_{2k-1} \leq \rho_k \leq r_{2k}$ starting with some $k \geq k_1$. Thus,

$$n(t, u_\psi) - n(t, f) = \begin{cases} \frac{1}{2} & \text{if } t \in [r_{2k-1}, \rho_k), \\ -\frac{1}{2} & \text{if } t \in [\rho_k, r_{2k}), \end{cases} \qquad k \geq k_1.$$

If $\rho_k \in [r_{2k-1}, \sqrt{r_{2k-1} r_{2k}}]$, then, choosing $r_k^* \in [\rho_k, 2\rho_k]$ and $t_k^* \in [r_{2k}/2, r_{2k}]$ such that (3.2) is satisfied with $r^* \in \{r_k^*, t_k^*\}$, we obtain

$$\begin{aligned}&N(t_k^*, f) - N(t_k^*, u_\psi) \\ &= \int_{r_k^*}^{t_k^*} \frac{n(s, f) - n(s, u_\psi)}{s}\, ds + N(r_k^*, f) - N(r_k^*, u_\psi) \\ &\geq \frac{1}{2} \log \frac{t_k^*}{r_k^*} - \varepsilon \log \psi(r_k^*) \geq \Big(\frac{1}{2} + o(1)\Big) \log \frac{t_k^*}{\sqrt{r_{2k-1} r_{2k}}} - \varepsilon \log \psi(r_k^*) \\ &\geq \Big(\frac{1}{4} - \varepsilon + o(1)\Big) \log \psi(t_k^*), \quad k \to +\infty,\end{aligned}$$

which contradicts (3.2). If $\rho_k \in [\sqrt{r_{2k-1} r_{2k}}, r_{2k}]$, we choose $r_k^* \in [r_{2k-1}, 2r_{2k-1}]$, $t_k^* \in [\rho_k/2, \rho_k]$ satisfying (3.2). Since $n(t, u_\psi) - n(t, f) = 1/2$ for $t \in [r_k^*, t_k^*]$, again we arrive at a contradiction with (3.2). Consequently, for $\alpha = 0$ the theorem is proved.

Now, let $\alpha \in (0, 1/2)$. Relation (3.3) is possible only if the expression under the modulus sign takes the values $-\alpha$ and $\frac{1}{2} - \alpha$. Since the numbers r_k strictly increase, and the jump of $n(r, u_\psi)$ equals $\frac{1}{2}$ for $r = r_k$ and the jump $n(r, f)$ equals 1 for $r = \rho_k$, we see that $\rho_k = r_{2k}$, $k \geq k_2$. Also, we have

$$n(t, u_\psi) - n(t, f) = \begin{cases} 0 & \text{if } t \in [r_{2k}, r_{2k+1}), \\ \frac{1}{2} & \text{if } t \in [r_{2k+1}, r_{2k+2}), \end{cases} \quad k \geq k_2. \tag{3.5}$$

Choosing $t_k^* \in [r_{2k-1}, 2r_{2k-1}]$ and $r_k^* \in [r_{2k}/2, r_{2k}]$ that satisfy (3.2), using (3.5), and arguing as above, we arrive at a contradiction with (3.2). Therefore, for any $\alpha \in [0, 1/2]$ there exists no entire function f with property (3.1); consequently, the same is true for $\alpha \in [0, 1)$. Theorem $2'$ is proved.

The proof of Theorem 2 is a word-for-word repetition of that of Theorem $2'$ for $\alpha = 0$, with the difference that $\psi \in \Phi$ is given by the hypothesis of the theorem.

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