

BI-LIPSCHITZ-EQUIVALENT ALEKSANDROV SURFACES, I

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ABSTRACT. In this first paper of two, it is proved that two compact Aleksandrov surfaces with bounded integral curvature and without peak points are bi-Lipschitz-equivalent if they are homeomorphic. Also, conditions under which two tubes with finite negative part of integral curvature are bi-Lipschitz-equivalent are considered. In the second paper an estimate depending only on several geometric characteristics is found for a bi-Lipschitz constant.

§1. BASIC DEFINITIONS AND STATEMENTS

Recently, Bonk and Lang [BL] proved that if a complete Riemannian manifold M is homeomorphic to the plane \mathbb{R}^2 and $\int_M K^+ dS < 2\pi$, $\int_M K^- dS < \infty$, then the Lipschitz distance $d_L(M, \mathbb{R}^2)$ between M and \mathbb{R}^2 satisfies the inequality

$$d_L(M, \mathbb{R}^2) \leq \ln \left(\frac{2\pi + \int_M K^- dS}{2\pi - \int_M K^+ dS} \right)^{\frac{1}{2}}.$$

This inequality is sharp if curvature does not change its sign. Here $K^+ = \max\{K, 0\}$, $K^- = \max\{-K, 0\}$, and dS is the area element. Later we shall recall the definition of the Lipschitz metric.

In fact, this result was obtained in [BL] for the class of Aleksandrov surfaces, which is wider than the class of Riemannian manifolds.

This paper is inspired by the paper [BL]; we investigate surfaces that are not necessarily simply connected. In contrast to the case of surfaces homeomorphic to \mathbb{R}^2 , in more general cases no standard model is available. For this reason, we try to estimate the Lipschitz distance between two homeomorphic surfaces. Our estimates will be far from optimal (and in this paper we even restrict ourselves to the proof of the fact that the distances are finite). Our consideration splits naturally into two parts: asymptotics at infinity and the study of compact surfaces.

We assume that the reader is familiar with the basic notions of the theory of two-dimensional manifolds of bounded total (integral) curvature, as exposed, for instance, in [AZ] or [Resh].

Hereafter, by an Aleksandrov surface we mean a *complete* two-dimensional manifold of bounded curvature and with boundary; the boundary (which may be empty) is assumed to consist of finitely many curves with finite variation of turn.

We introduce the following notation: M is an Aleksandrov surface with metric d ; ω is its curvature, which is a signed measure; ω^+ and ω^- are the positive and the negative

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parts of ω , and $\Omega = \omega^+ + \omega^-$ is the variation of the curvature. For any Riemannian manifold we have $\omega^+ = \int_M K^+ dS$, $\omega^- = \int_M K^- dS$.

A point p carrying the curvature 2π and a boundary point carrying the turn π are called peak points.

Recall that the *dilatation* $\text{dil } f$ of a Lipschitz map $f: X \rightarrow Y$, where (X, d_X) and (Y, d_Y) are metric spaces, is defined by the formula

$$\text{dil } f = \sup_{x, x' \in X, x \neq x'} \frac{d_Y(f(x), f(x'))}{d_X(x, x')}.$$

A homeomorphism f is said to be *bi-Lipschitz* if both f and f^{-1} are Lipschitz. The quantity

$$d_L(X, Y) = \inf_{f: X \rightarrow Y} \ln(\max\{\text{dil}(f), \text{dil}(f^{-1})\})$$

is called the *Lipschitz distance* between X and Y ; here the infimum is taken over all Lipschitz homeomorphisms $f: X \rightarrow Y$. Metric spaces X, Y are bi-Lipschitz-equivalent if and only if $d_L(X, Y) < \infty$.

Theorem 1. *Suppose two compact Aleksandrov surfaces are homeomorphic and have no peak points. Then these surfaces are bi-Lipschitz-equivalent.*

This theorem is trivial for Riemannian manifolds, because two homeomorphic smooth 2-manifolds are always diffeomorphic. Theorem 1 gives no upper estimate for the Lipschitz distance in terms of a finite set of geometric characteristics of the surfaces, like their diameters, total curvatures, systolic constants, etc. However, such an estimate does exist; we are going to publish this result in a separate paper.

Let T be a tube, i.e., an Aleksandrov surface homeomorphic to a closed disk with center removed and such that for every sequence of points $p_i \in T$ whose images in the disk converge to the center, the condition $d(a, p_i) \rightarrow \infty$ is fulfilled as $i \rightarrow \infty$; here a is any fixed point. Such a sequence is said to be *divergent* (or *going to infinity*).

Let σ denote the turn of the tube boundary ∂T . The number $v(T) = -\omega(T) - \sigma$ is called the *growth speed* of the tube T . The Cohn-Vossen inequality says that this quantity is nonnegative. The growth speed of a tube is positive if and only if $\lim_{i \rightarrow \infty} \frac{l(\gamma_i)}{d(a, p_i)} > 0$, where $l(\gamma_i)$ is the length of the shortest noncontractible loop with vertex p_i . Under the condition $\Omega(T) < \infty$, this limit is well defined and does not exceed 2.

As was proved in [Hub, Ver], any complete Aleksandrov surface satisfying $\omega^-(M) < \infty$ is homeomorphic to a closed surface with finitely many points removed; appropriate closed neighborhoods of these points are tubes. Thus, any Aleksandrov surface can be cut into a compact part and several tubes.

Theorem 2. *Suppose that two complete Aleksandrov surfaces M_1, M_2 are homeomorphic, satisfy the condition $\omega^-(M_i) < \infty$, and contain neither points with curvature 2π , nor boundary points with turn π . If all tubes of these surfaces have nonzero growth speed, then these surfaces are bi-Lipschitz-equivalent.*

Remark 1. We say that two tubes are equivalent if their intersection contains a tube. It is not difficult to check that equivalent tubes have equal growth speeds. This means that the above theorem does not depend on how the surfaces are cut into compact parts and tubes.

Simple examples show that tubes having zero speed (even having no peak points) may fail to be bi-Lipschitz-equivalent. For instance, no two of the surfaces obtained by rotation of the following graphs are bi-Lipschitz-equivalent:

$$\{y = \sqrt{x}, x \geq 1\}, \quad \{y = 1, x \geq 0\}, \quad \{y = e^{-x}, x \geq 0\}.$$

However, the surfaces of rotation obtained from the third graph and the graph of the function $\{y = e^{-2x}, x \geq 0\}$ are bi-Lipschitz-equivalent.

Also, we note that a tube having zero growth speed cannot be bi-Lipschitz-equivalent to a tube with nonzero growth speed.

A tube is *rotationally symmetric* if its isometry group contains a subgroup whose restriction to the boundary is transitive. The problem of bi-Lipschitz equivalence of tubes reduces to the same problem for rotationally symmetric tubes. More precisely, the following fact is true.

Theorem 3. *Every tube T satisfying the condition $\omega^-(T) < \infty$ and having no peak points is bi-Lipschitz-equivalent to a rotationally symmetric tube with the same growth speed.*

Remark 2. If $v(T) > 0$, then T contains a smaller tube T_1 such that the Lipschitz distance between T_1 and a plane from which a disk of length $l = l(\partial T_1)$ is removed does not exceed a constant depending on v and l only.

This can be proved by a minor modification of the arguments in [BL]; so we omit the proof.

We see that the problem of bi-Lipschitz classification of tubes with zero growth speed reduces to a question about functions of one variable. Intuitively, a zero speed tube can be imagined as “having a peak point (with curvature 2π) at infinity”. From this point of view, it is natural to expect some analogy between classification of zero speed tubes and bi-Lipschitz classification of neighborhoods of finite peak points. Note that, for smooth surfaces with isolated singularities, the classification problem for neighborhoods of peak points was considered by Grieser [Gr].

Sketch of the proof. The idea of our proof of Theorem 1 is simple and can be outlined as follows. Preliminarily, a simply connected region Δ on an Aleksandrov surface will be called a *generalized triangle* if Δ is bounded by a simple closed curve with three selected points. These points are *vertices* of the triangle, and the intervals of the curve between the vertices are its *sides*. We always assume that the lengths of the sides satisfy the triangle inequality and that the sides are geodesic polygonal lines. Actually, we shall only consider generalized triangles having small variations of curvature and having sides of small turn. The word “generalized” will often be omitted.

By a *partition* of a 2-manifold M we mean a set of generalized triangles in M such that their interiors do not overlap and their union is M . A *triangulation* of M is a partition such that for any two triangles of it their intersection is either a side or a vertex.

We shall partition both Aleksandrov surfaces M_1 and M_2 into generalized triangles in such a way that these triangles have small variations of curvature and the sides have small turn, but the angles are bounded away from zero. In particular, all points carrying essential portion of curvature are included in the set of vertices of the triangles. We prove that each triangle of the partition is bi-Lipschitz-equivalent to its comparison triangle, i.e., to a planar triangle with the same side lengths; such a bi-Lipschitz mapping can be chosen to preserve the length of sides. For this, we use the method introduced by Bakel'man [Bak] for constructing Chebyshev coordinates on Aleksandrov surfaces. Replacing each generalized triangle of our partition by its comparison triangle and attaching the resulting planar triangles together in accordance with the same combinatorial pattern, we get two surfaces P_1, P_2 equipped with polyhedral metrics. By construction, each polyhedron P_i is bi-Lipschitz-equivalent to the surface M_i , $i = 1, 2$. Finally, observe that the polyhedra P_1 and P_2 are bi-Lipschitz-equivalent to each other. This completes the proof.

Theorem 2 follows from Theorem 1 and the next lemma.

Lemma 1. *Every tube T with nonzero growth speed, without peak points, and with $\omega^-(T) < \infty$ is bi-Lipschitz-equivalent to \mathbb{R}^2 with a disk removed (as always, we assume that the boundary of the tube in question is a curve with finite variation of turn and that there are no peak points on the tube and on its boundary). Moreover, there is a bi-Lipschitz equivalence that induces an affine map on the boundary of the tube.*

In the case where the curvature and turn of the boundary are not large, the lemma can be proved by a minor modification of the arguments used in [BL]; for this it suffices to suppose that $\omega^+(T) + \tau^+(\partial T) < \pi$, where $\tau^+(\partial T)$ is the positive turn of the tube boundary (from the side of the tube).

To prove the lemma in the general case, it suffices to cut the tube into an annulus and a tube satisfying the above condition and to apply Theorem 1 to the annulus.

Also, Lemma 1 follows from Theorems 3 and 1. Indeed, Theorem 3 implies easily that every tube with nonzero growth speed is bi-Lipschitz-equivalent to a cone over a circle of appropriate length with a round neighborhood of the vertex removed. Obviously, the latter surface is bi-Lipschitz-equivalent to the plane with a disk removed.

To prove Theorem 3, again we use a modification of Bakel'man's construction [Bak].

§2. TRIANGLES WITH SMALL CURVATURE

We begin with simple statements about planar triangles. Let $\triangle ABC$ be a flat triangle. Denote by a, b, c its sides opposite the angles $\angle A, \angle B, \angle C$, respectively. The side lengths will be denoted by the same letters as sides.

Lemma 2. *Suppose two planar triangles $\triangle ABC$ and $\triangle \overline{ABC}$ are such that $b = \overline{b}$, $c = \overline{c}$, $\angle A \leq \overline{\angle A} \leq L \cdot \angle A$, and $L \cdot \overline{\angle A} - \angle A \leq \pi(L - 1)$, where $L > 1$. Then the optimal bi-Lipschitz constant of the affine transformation mapping of $\triangle ABC$ onto $\triangle \overline{ABC}$ does not exceed L .*

For the proof, it suffices to find the eigenvalues of the corresponding affine transformation.

Corollary 4. *If $\triangle ABC$ and $\triangle \overline{ABC}$ satisfy the conditions*

$$\begin{aligned} L^{-1} \leq c/\overline{c} \leq L, \quad L^{-1} \leq b/\overline{b} \leq L, \\ \epsilon \leq \angle A \leq \pi - \epsilon, \quad \epsilon \leq \overline{\angle A} \leq \pi - \epsilon, \end{aligned}$$

then the optimal bi-Lipschitz constant of the affine transformation mapping of $\triangle ABC$ onto $\triangle \overline{ABC}$ does not exceed $(L \frac{\pi}{2\epsilon})^2$.

Lemma 3. *Suppose two planar triangles $\triangle ABC$ and $\triangle \overline{ABC}$ are such that $b = \overline{b}$ and $c = \overline{c}$. Let $D \in BC$, let \overline{D} denote the image of the point D under the affine transformation that maps $\triangle ABC$ onto $\triangle \overline{ABC}$, and let $\angle BAD = \alpha$, $\angle CAD = \beta$, $\overline{\angle BAD} = \alpha_1$, $\overline{\angle CAD} = \beta_1$. Suppose that $\alpha < \frac{\pi}{2}$, $\beta < \frac{\pi}{2}$ and*

$$0 < \overline{\angle BAC} - \alpha < \frac{\pi}{2}, \quad 0 < \overline{\angle BAC} - \beta < \frac{\pi}{2}.$$

Then

$$|\alpha - \alpha_1| \leq |\angle BAC - \overline{\angle BAC}|, \quad |\beta - \beta_1| \leq |\angle BAC - \overline{\angle BAC}|.$$

Proof. Suppose $\overline{\angle BAC} > \angle BAC$. We shall show that in this case $\alpha_1 \geq \alpha$ and $\beta_1 \geq \beta$. After that, the relations $\alpha + \beta = \angle BAC$ and $\alpha_1 + \beta_1 = \overline{\angle BAC}$ will imply the lemma.

Since \overline{D} is the image of D , we have

$$\frac{\sin \alpha}{\sin \beta} = \frac{\sin \alpha_1}{\sin \beta_1}.$$

Suppose that $\alpha_1 < \alpha$. Consider the point Z in \overline{BC} such that $\angle \overline{BAZ} = \alpha$. Then

$$\frac{\sin \alpha}{\sin \beta} < \frac{\sin(\overline{BAZ})}{\sin(\overline{CAZ})},$$

whence $\sin \beta > \sin(\overline{BAC} - \alpha)$. But $0 < \beta < (\overline{BAC} - \alpha) < \frac{\pi}{2}$. This means that $\sin \beta < \sin(\overline{BAC} - \alpha)$, a contradiction.

The case where $\overline{BAC} < \angle BAC$ can be treated similarly. □

Now we return to Aleksandrov surfaces. For a generalized triangle $\Delta \subset M$, we denote

$$\Omega'(\Delta) = \Omega(\text{int}(\Delta)) + \sigma(\Delta),$$

where $\sigma(\Delta)$ is the sum of variations of turn for the sides of Δ from inside.

Lemma 4. *For every $\theta > 0$, there exist numbers $\delta(\theta) > 0$ and $L(\theta) > 1$ with the following property: if every angle of a generalized triangle ΔABC is less than or equal to θ , and if $\Omega'(\Delta ABC) < \delta(\theta)$, then there exists a bi-Lipschitz map of the triangle ABC onto its comparison triangle with constant $L(\theta)$ and such that its restriction to the boundary of ΔABC preserves the length.*

This lemma seems obvious, and it is somewhat surprising that we could find neither an appropriate reference, nor an immediate proof.

Proof. 1. It suffices to prove the lemma for polyhedral metrics. So, we assume that our metric is polyhedral. We use a construction that is a minor modification of that used by Bakel'man in [Bak]. Suppose that our ΔABC (equipped with a polyhedral metric) is extended up to a complete surface M homeomorphic to \mathbb{R}^2 and flat outside the triangle. (For that, we can cut out a comparison triangle $\overline{\Delta ABC}$ from \mathbb{R}^2 and then attach ΔABC in place of $\overline{\Delta ABC}$.) Then we extend the sides \overline{AB} and \overline{AC} to rays $\overline{AB^*}$ and $\overline{AC^*}$. Let S denote the sector $B^* \overline{AC^*}$.

We shall partition the sector S (more precisely, some region of it containing ΔABC) in flat parallelograms that touch one another along entire sides in such a way that four parallelograms adjoin at each vertex (except those belonging to the boundary of S). This allows us to immediately introduce Chebyshev coordinates in the sector S . After some additional deformation straightening the side BC , these coordinates give the required bi-Lipschitz map.

2. To simplify exposition, we assume that ΔABC is a usual (not generalized) triangle with zero turn of its sides. The general case differs from this model by unessential details only. We note that the difference of the corresponding angles of the triangles ΔABC and $\overline{\Delta ABC}$ is not greater than some function of $\Omega'(\Delta ABC)$ (this function can be written explicitly) which goes to zero together with Ω' . The proof of this fact is standard and is based on the ‘‘arc and chord’’ theorem (see ([AZ, Chapter 9, Lemma 5]), the Gauss–Bonnet theorem, and the comparison theorem for (nongeneralized) triangles.

As a result, we can choose δ so small (smallness depends on θ only) that no angle of the triangle $\overline{\Delta ABC}$ exceeds $\frac{2}{3}\theta$.

3. Let $\angle C$ be the smallest angle of $\overline{\Delta ABC}$, and let $\angle A$ be the greatest angle. Then the angles $\angle B$ and $\angle C$ are acute and none of them exceeds $\frac{1}{2}\pi - \frac{1}{3}\theta$.

4. Consider the line l containing the ray $\overline{AB^*}$ and begin to shift it (continuously) to the sector S , keeping it parallel to itself. At first, this line will cut a flat ‘‘oblique’’ semistrip from S . We continue this process until vertices of the metric appear on l for the first time. Let A_{11} be the intersection of l and $\overline{AC^*}$, and let A_{12}, \dots, A_{1n_1} be the vertices of the metric lying on $l \cap S$, enumerated ‘‘from left to right’’.

From every point A_{1i} we emanate a geodesic $A_{1i}B_{1i}$ that goes outside the semistrip mentioned above and is such that $|A_{1i}B_{1i}| = |A_{1j}B_{1j}|$ for all i, j and $\angle B_{1i}A_{1i}A_{1,i+1} +$

$\angle A_{1i}A_{1,i+1}B_{1,i+1} = \pi$. If the length of $|A_{1i}B_{1i}|$ is sufficiently small, then we get a strip consisting of flat parallelograms $A_{1i}B_{1i}B_{1,i+1}A_{1,i+1}$. We choose $|A_{1i}B_{1i}|$ so that the vertices of the metric appear for the first time on the polygonal line $B_{11}B_{12}B_{1,3} \dots$. Continuing this process, we get a partition of the sector S into parallelograms (an infinite oblique semistrip is taken to be the last parallelogram in each row). To make the parallelograms adjoin along entire sides, it suffices to split some of them into narrower parallelograms.

5. If $\delta < \theta$, the process described above can be continued until we exhaust the entire sector S , which produces a bijective mapping of the sector S onto the first quadrant S_0 of the plane with oblique coordinates Ouv , where the coordinate angle $\angle uOv$ is equal to the angle $\angle A = \alpha$ of the sector S .

Indeed, there could be only one obstruction: at some step the polygonal line $\Lambda_k = A_{k1}, A_{k2}, \dots$ might touch itself or the ray AB^* . In both cases we would obtain a closed polygonal line bounding a region G homeomorphic to some disk. Applying the Gauss–Bonnet formula to G , we would come to a contradiction with the smallness of the curvature of our triangle (in comparison with its angles).

6. Now we can introduce Chebyshev coordinates in S . For the coordinates (u, v) of a point p , we take the lengths of the polygonal lines connecting this point with the rays AC^* , AB^* and such that in every parallelogram crossed by these lines they go along intervals parallel to the corresponding sides of the parallelogram. Let S_0 be the first coordinate quadrant of the plane \mathbb{R}^2 , and let $\varphi: S \rightarrow S_0$ be the coordinate mapping constructed as described above.

7. On S_0 we define a metric by the linear element

$$ds^2 = du^2 + 2 \cos \tau(u, v) du dv + dv^2,$$

where

$$\tau(u, v) = \alpha - \omega(\varphi^{-1}(D_{uv})),$$

and D_{uv} is the parallelogram $[0 \leq u' < u, 0 \leq v' < v]$.

This linear element turns S_0 into a metric space (S_0, d_1) .

8. The map φ is an isometry of S onto (S_0, d_1) . This can be checked without difficulty by considering all parallelograms of our partition of S consecutively in their natural order.

Next, we consider the standard flat metric d_2 defined by the linear element $ds^2 = du^2 + 2 \cos \alpha du dv + dv^2$ on S_0 . The map $\text{id}: (S_0, d_1) \rightarrow (S_0, d_2)$ is linear on every parallelogram. The inequality $|\tau(x, y) - \alpha| \leq \Omega'(S) < \delta$ shows that this map is bi-Lipschitz with constant $\delta\mu$. Here and in what follows we denote by μ positive constants (possibly, different) that depend only on θ .

The image of the side BC of the triangle $\triangle ABC$ under the map $\varphi_1 = \text{id} \circ \varphi$ is a polygonal line $\Lambda = Y_0 Y_1 \dots Y_l$ with $Y_0 = B' = \varphi_1 B$, $Y_l = C' = \varphi_1 C$. We want to prove that there is a μ -bi-Lipschitz transformation ζ that takes the region Q bounded by this polygonal line and the shortest $A'B'$, $A'C'$ (where $A' = \varphi_1 A$ is the vertex of the sector S_0) onto the triangle $\triangle A'B'C'$. After that, to finish the proof of the lemma, it will suffice to apply Lemma 2 to the flat triangles $\triangle A'B'C'$ and $\triangle \overline{ABC}$.

The transformation ζ is defined as follows. Denote by X_i the intersection of the ray $A'Y_i$ and the side $B'C'$ of the triangle $\triangle A'B'C'$. Now we map each triangle $\triangle A'Y_i Y_{i+1}$ affinely onto the corresponding triangle $\triangle A'X_i X_{i+1}$. We show that it gives the desired μ -bi-Lipschitz map.

Indeed, every interval $Y_i Y_{i+1}$ of the polygonal line Λ is the affine image of an interval located in one of the parallelograms. By our construction, the parallelograms in (S_0, d_2) form “horizontal” rows. Adding “horizontal” and “vertical” rays, we can assume that each point Y_i is a vertex of a parallelogram. Since δ is small in comparison with θ , every

interval $Y_i Y_{i+1}$ is a diagonal of the corresponding parallelogram. Let a ray N_i be the “upper boundary” of the k th row, so that $Y_i \in N_i$, and let $\tilde{N}_i = \varphi_1^{-1}(N_i)$. It is not difficult to see that the variation of turn of the polygonal line \tilde{N}_i is not greater than $\Omega'(M) < \delta$ (this is a rough estimate). We apply the Gauss–Bonnet formula to the region bounded by the shortest AB , by intervals of the shortest AC , BC , and by the curve \tilde{N}_i . This shows immediately that the angle between \tilde{N}_i and the shortest BC differs from the angle $\angle B$ by a quantity not exceeding 2δ . Therefore, the angle between \tilde{N}_i and the part of Λ that starts on \tilde{N}_i , and, thus, also the angle between the same part of Λ and the ray $A'B'$, differ from the angle $\angle B'$ of the triangle $\triangle A'B'C'$ by at most 3δ (that follows easily from Lemma 3). Now it is easily seen that

$$\left| \frac{A'Y_i}{A'X_i} - 1 \right| < \frac{10\delta}{\alpha}$$

(under the condition that δ is small in comparison with α). Applying Lemma 4 to the couples $\triangle A'X_i X_{i+1}$ and $\triangle A'Y_i Y_{i+1}$, we see that the triangles $\triangle A'B'C'$ and $\triangle ABC$ are bi-Lipschitz-equivalent with some constant μ . Unfortunately, the restriction of the map we constructed to the side BC is not an isometry. Nevertheless, this restriction distorts distances with coefficient not exceeding μ ; therefore, we can correct our map in every triangle $\triangle A'X_i X_{i+1}$ in such a way that it remains to be a bi-Lipschitz equivalence (with some constant μ') but becomes an isometry on the boundary of the triangle. Thus, we have constructed the required map. \square

Lemma 5. *There exists $\xi > 0$ such that if a generalized triangle $\triangle ABC$ in an Aleksandrov space satisfies the conditions*

$$\angle ABC \geq \frac{\pi}{5}, \quad \angle ACB \geq \frac{\pi}{5}, \quad \angle BAC > 0, \quad \Omega'(\triangle ABC) < \xi,$$

then there exists a bi-Lipschitz map of $\triangle ABC$ onto its comparison triangle with the following property: the restriction of it to the boundary takes the vertices of $\triangle ABC$ to the corresponding vertices and preserves the length.

In contrast to the preceding lemma, now we allow one of the angles to be arbitrarily small; on the other hand, now we cannot estimate the bi-Lipschitz constant.

Proof. To prove the lemma, we cut our triangle into triangles in such a way that each of them is bi-Lipschitz-equivalent to its comparison triangle. We may assume that $\alpha = \angle BAC < \frac{1}{100}$. Otherwise, by Lemma 4, $\triangle ABC$ is $L(\frac{1}{100})$ -bi-Lipschitz-equivalent to its comparison triangle $\triangle \bar{A}\bar{B}\bar{C}$. Next, we choose $\chi = \min\{\delta(\frac{1}{100}\pi), \frac{1}{1000}\}$ (see Lemma 4).

It is easily seen that on the sides BC , AC , and AB there exist points A_1, C_1, B_1 such that

$$|AB_1| = |AC_1|, \quad |BA_1| = |BB_1|, \quad |CA_1| = |CC_1|.$$

Consider some shortest paths (with respect to the induced metric of the triangle) connecting these points. They partition $\triangle ABC$ into four triangles (see explanation below). We repeat the same construction for the triangle $\triangle AB_1C_1$, i.e., choose points A_2, C_2, B_2 on the sides B_1C_1 , AC_1 and AB_1 in such a way that

$$|AB_2| = |AC_2|, \quad |B_1A_2| = |B_1B_2|, \quad |C_1A_2| = |C_1C_2|.$$

We continue this process. It is not difficult to show that all the angles of all resulting triangles are bounded away from zero and π ; for instance, they lie between 0.05π and 0.95π . Therefore, by Lemma 4, all such triangles except for $\triangle AB_i C_i$ are $L(0.05\pi)$ -bi-Lipschitz-equivalent to their comparison triangles.

Note that the partition process can be continued up to infinity, and $\sum |B_i C_i| = \sum (|B_i B_{i+1}| + |C_i C_{i+1}|) \leq |AB| + |AC|$, so that $|B_i C_i| \rightarrow 0$ as $i \rightarrow \infty$, and $B_i, C_i \rightarrow A$

because $\alpha = \angle BAC > 0$. Consequently, there exists $i = i_0$ such that $\Omega'(\triangle AB_{i_0}C_{i_0}) < \delta(\alpha)$, which implies that $\triangle AB_{i_0}C_{i_0}$ is bi-Lipschitz-equivalent to its comparison triangle (again by Lemma 4).

We complete the proof of the lemma by using backward induction on i . The triangle $\triangle AB_{i_0}C_{i_0}$ is bi-Lipschitz-equivalent to its comparison triangle. We suppose that the same is true for $\triangle AB_{i+1}C_{i+1}$, where $i \leq i_0$, and prove this for $\triangle AB_iC_i$. The latter is cut into four triangles, and each of them is bi-Lipschitz-equivalent to its comparison triangle. Consider the similar partition for the comparison triangle $\triangle \overline{AB}_i\overline{C}_i$. Since the curvature of $\triangle ABC$ is small as compared to the angles of the triangles under consideration (except for $\angle BAC$), the triangles of the partition of $\triangle \overline{AB}_i\overline{C}_i$ are almost the same as the comparison triangles for the corresponding triangles of our partition of $\triangle AB_iC_i$. Therefore, it is easy to construct bi-Lipschitz mappings for these couples of triangles, which completes the proof. \square

§3. COMPACT ALEKSANDROV SURFACES

For the proof of Theorem 1 we need the following statement.

Lemma 6. *Let M be a compact Aleksandrov surface without peak points. For any $\xi > 0$ there exists a partition of M into generalized triangles such that any triangle \triangle of this partition has positive angles and $\Omega'(\triangle) < \xi$.*

Proof. First, we triangulate a small neighborhood of each point p such that $\Omega(p) \leq \frac{1}{2}\xi$, where ξ is chosen in accordance with Lemma 5. We assume that these neighborhoods are bounded by polygonal lines and do not overlap. It is easy to triangulate these neighborhoods so as to satisfy the conditions of the lemma.

To partition the remaining part of M , we use the following statement in [AZ, Chapter 3, Theorem 2]. Any compact subset of M with polyhedral boundary can be covered by a finite system of arbitrarily small pairwise nonoverlapping simple triangles such that in every triangle no side is equal to the sum of two other sides.

It is clear that these triangles can be chosen to be so small that for each of them we have $\Omega' < \frac{1}{2}\xi$. It remains to deform these triangles to generalized triangles so as to remove the zero angles without breaking the other assumptions of the lemma. It suffices to remove one zero angle and to use induction on the number of zero angles. Suppose there is at least one zero angle. Since we have no peak points, we can find two adjacent angles (say, $\angle BOA$ and $\angle AOC$) such that the first of them is zero and the second is nonzero. Observe that the curve AOC may either consist of two sides, or coincide with an entire side of the generalized triangle in question. Now it suffices to replace a very short initial interval of the curve OA by a two-component polygonal line ODE with $E \in OA$, lying in the sector $\angle AOC$, forming a very small but nonzero angle with OA , and such that the variation of the turn of the polygonal line $ODEA$ exceeds that of the shortest line OA only slightly. \square

Proof of Theorem 1. For a compact Aleksandrov surface M , consider a partition of it satisfying Lemma 6. We want to prove that every triangle of this partition is bi-Lipschitz-equivalent to a flat triangle with the same side lengths. Moreover, the corresponding bi-Lipschitz maps can be chosen so that their restrictions to the boundaries of the triangles be isometries. For a triangle with two angles of at least $\frac{1}{5}\pi$ each, this is true by Lemma 5. Therefore, we can assume that the triangle \triangle under consideration has two angles less than $\frac{1}{5}\pi$ each. Then the third angle is at least $\frac{3}{5}\pi - \Omega'(\triangle) \geq \frac{3}{5}\pi - \frac{1}{500}$. Let $\triangle = \triangle ABC$, and let $\angle B$ be the greatest angle.

Since the curvature of the triangle \triangle is small, the angles of any lune formed by two shortest curves and contained in \triangle do not exceed $\frac{1}{500}$. Therefore, there is a point

$X \in AC$ such that the angles made by any shortest BX with AB and CB are at least $\frac{3}{10}\pi - \frac{1}{500}$. A simple calculation shows that both triangles $\triangle ABX$ and $\triangle CBX$ satisfy the conditions of Lemma 5. Consequently, each of these triangles is bi-Lipschitz-equivalent to its comparison triangle. It is easily seen that the same is true for $\triangle ABC$. Thus, M is bi-Lipschitz-equivalent to some surface with polyhedral metric.

Now, it remains to use the fact that if two polyhedral surfaces X and Y are homeomorphic, then there exists a piecewise linear homeomorphism $g: X \rightarrow Y$ between them. In accordance with [RS], then we can find isomorphic subdivisions X' and Y' of these triangulations (with flat triangles) such that g is linear on every triangle of X' and transforms such a triangle to the corresponding triangle of Y' . Thus, any two homeomorphic polyhedral Aleksandrov spaces are bi-Lipschitz-equivalent; hence, the same is true for two arbitrary homeomorphic Aleksandrov spaces satisfying the conditions of the theorem. \square

§4. ROTATIONALLY SYMMETRIC TUBES

A tube T is *rotationally symmetric* if there is an isometry group of T acting transitively on ∂T .

Proof of Theorem 3. If a tube has nonzero speed, then, by Remark 2, it is bi-Lipschitz-equivalent to \mathbb{R}^2 with a disk removed. We have already mentioned that this fact can be proved by a minor modification of the method used in [BL]. Therefore, we may assume that the tube T has zero speed. Moreover, it suffices to prove the theorem for polyhedral tubes with finitely many vertices. Also, we can restrict ourselves to the tubes T satisfying the following conditions:

1) the boundary Γ of T is either a geodesic loop or a polygon no angle of which exceeds π ;

2) $\Omega(T) + \sigma(\Gamma) < \epsilon = \frac{1}{1000}$, where $\sigma(\Gamma)$ is the variation of turn for the boundary of T .

Indeed, from the initial T we can cut off a tubular neighborhood of the boundary in such a way that the remaining tube satisfy conditions 1) and 2) and the annulus we cut off be bi-Lipschitz-equivalent to a flat annulus (Theorem 1).

In what follows we assume that conditions 1) and 2) are fulfilled. The proof we give here is similar to the proof of Lemma 4. The only difference is that now we shall construct a partition of the tube into flat trapezoids rather than parallelograms; these trapezoids will be arranged in annular layers, and all trapezoids in one layer will have equal heights. Each layer will be bi-Lipschitz-equivalent to a surface of revolution, the bi-Lipschitz constants will be uniformly bounded, and the restrictions of the corresponding bi-Lipschitz maps to the boundaries will preserve length.

For this, from every "angular" point X_i , $i = 1, \dots, m$, of the geodesic polygonal Γ (i.e., from the points at which Γ has nonzero turn) we emanate a geodesic that forms equal angles with the branches of Γ starting at X_i (i.e., this geodesic goes along the bisector of the angle between the branches). We choose a small number $h_1 > 0$ and, on every such geodesic, select a point X_{1i} such that $|X_i X_{1i}| = (\cos \alpha_i)^{-1} h_1$, where 2α is the turn of Γ at X_i .

Now we connect cyclically the points X_{1i} by shortest paths $X_{1i} X_{1i+1}$. For sufficiently small t_1 , the quadrangles $X_i X_{1i} X_{1i+1} X_{i+1}$ are flat nonoverlapping trapezoids having height h_1 and angles α_i and α_{i+1} at the "bottom" base. (If Γ is a loop with zero turn everywhere except the vertex, we get one trapezoid glued with itself along $X_1 X_{11}$.)

Now we shall increase h_1 until one of the following events happens:

a) On the polygonal line $\Gamma_1 = X_{11} X_{12} \dots X_{1m}$, a vertex of the metric appears for the first time;

b) the shortest paths $X_i X_{1i}$ and $X_{i+1} X_{1i+1}$ meet at a point $X_{1i} = X_{1i+1}$ for some i .

Of course, several such new vertices and meetings can occur simultaneously. It is not difficult to see that a) and b) are the only possible singular events because curvature is small.

We repeat the construction just described starting with the polygonal line Γ_1 . (The number of its vertices may happen to be larger or smaller than that of the polygonal line Γ .) We get the second row of trapezoids, and so on. Like in the proof of Lemma 4, assumption 2) guarantees that this process will not stop until all the vertices of the metric are exhausted. Moreover, the turn of the polygonal line Γ_k is the sum of the turn of the polygonal line Γ_{k-1} and the curvatures of the additional vertices, so that $\sigma(\Gamma_k) \leq \Omega(T) + \sigma(\Gamma)$. In particular, each of the angles of the resulting trapezoids differs from $\pi/2$ by at most $\frac{1}{1000}$.

Let $L_k = L(\Gamma_k)$ denote the length of the polygonal line Γ_k , $k = 0, 1, \dots$. Here it is assumed that $\Gamma = \Gamma_0$. Now we consider an annulus C_i that lies on the cone with total angle $\theta = t_k^{-1}(L_k - L_{k-1})$ around its vertex O and is bounded by the circles of radii $\theta^{-1}L_k$ and $\theta^{-1}L_{k+1}$ centered at O . The lengths of these circles are L_k and L_{k+1} , and the distance between them is equal to h_k . (If $L_k = L_{k+1}$, then the cone degenerates to a cylinder, which simplifies our considerations.)

Our next (and final) aim is to show that every annular layer between Γ_k and Γ_{k+1} can be mapped onto C_k in a bi-Lipschitz way, with a constant depending only on ϵ . The number k of the layer is assumed to be fixed; in what follows we omit k in the notation. Putting $a_i = |X_k i X_{k+1} i|$, $b_i = |X_{k+1} i X_{k+1} i+1|$, $a = \sum a_i$, $b = \sum b_i$, we consider a trapezoid $ABCD$ with $|AD| = a$, $|BC| = b$, $\angle DAB = \pi$, and mark points $A = A_0, A_1, \dots, A_m = D$ and $B = B_0, B_1, \dots, B_m = C$ on its bases in such a way that $|A_i A_{i+1}| = a_i$, $|B_i B_{i+1}| = b_i$. Simple calculations show that the angles β_i between the intervals $A_i B_i$ and the trapezoid bases are uniformly bounded away from zero and π by a constant depending only on ϵ . Indeed, since $|a_j - b_j| = h |\tan \alpha_j + \tan \alpha_{j+1}| \leq 2h(|\alpha_j| + |\alpha_{j+1}|)$ (the latter inequality is valid because ϵ is small), we have

$$|\cot \beta_i| = h^{-1} \left| \sum_{j=1}^{i+1} (a_j - b_j) \right| \leq 4 \sum_{j=1}^m |\alpha_j| \leq \epsilon.$$

Now it is easy to construct a map of each trapezoid of our partition onto the corresponding trapezoid $A_i B_i A_{i+1} B_{i+1}$, and hence a map of each annular layer onto the corresponding trapezoid $ABCD$. Finally, we can map the trapezoid $ABCD$ onto the corresponding annulus C_i on the cone. (All these maps can be constructed in such a way that their restrictions to the boundaries preserve lengths.) \square

Final Remarks: Alexandrov polyhedra. Together with Aleksandrov surfaces, it is possible to consider polyhedra glued of such surfaces.

An Aleksandrov polyhedron is a connected 2-polyhedron P glued from a finite number of Aleksandrov surfaces and satisfying the following conditions: (i) the gluings are made along the boundary curves, and (ii) the parts attached to one another have equal lengths (more precisely, gluing is made along isometries of boundaries).

A particular case of Aleksandrov polyhedra are 2-polyhedra of curvature bounded above, investigated in [BB]. The notions of essential edges and maximal faces introduced there are of purely topological nature and hence can be employed in our case. Each maximal face is an Aleksandrov surface (by Aleksandrov's gluing theorem). We do not exclude "boundary curves" consisting of one point (by definition, the turn of such a curve is $2\pi - \theta$, where θ is the total angle around this point). Also, we could allow the existence of edges without any faces adjacent to them. Nevertheless, we shall not do this here. For simplicity, we restrict ourselves to polyhedra without boundary edges.

Remark. This definition of the Aleksandrov polyhedra is of constructive nature. It is desirable to find an axiomatic definition, but we have not succeeded in doing this.

For Aleksandrov polyhedra, curvature can be defined naturally. Indeed, it suffices to define it for essential vertices and on subsets of essential edges, because off these vertices and edges the curvature of a set is assumed to be equal to the curvature of this set viewed as part of the maximal face. If γ is an essential edge, $e \subset \gamma$, and τ_i is the turn of γ viewed as a part of the boundary of the i th face adjacent to γ , then, by definition, $\omega(e) = \sum_i \tau_i(e)$, where the sum is over all faces adjacent to the edge γ . For a vertex p , curvature is defined by the formula

$$\omega(p) = (2 - \chi(\Sigma_p))\pi - s(\Sigma_p),$$

where χ and s are the Euler characteristic and the length of the graph Σ_p . Under this definition, the Gauss–Bonnet theorem remains valid in its usual form for the compact Aleksandrov polyhedra.

The issue concerning the positive and the negative parts of curvature is more complicated. Also, we need to explain what we mean by a tube. Having in mind the straightforward generalizations of the above theorems, we must exclude analogs of peak points and tubes with zero speed growth. The former means that we must assume that the maximal faces contain neither inner points with curvature 2π , nor boundary points at which the boundary has turn π . This restriction can be described more strictly in the language of curvatures of the essential edges and vertices. Namely, we need to suppose that there are no points p at which $\max_i \tau_i^+(p) = \pi$ on the essential edges. Here the maximum is taken over all faces adjacent to γ . For the vertex p , we consider its link $\Lambda(p)$. The vertices of a link correspond to the essential edges of the polyhedron (starting at p), and its edges correspond to the maximal faces to which these edges are adjacent; in particular, the edges of a link can be circles (containing no essential vertices or containing only one vertex). The space of directions at the point p induces a semimetric on $\Lambda(p)$ in a natural way. Now we require that this semimetric be a metric, i.e., that $\Lambda(p)$ be homeomorphic to the space of directions. In other words, we require that every edge of a link be of nonzero length in the angular metric.

Now we pass to tubes. Let Q be a part of a polyhedron, with complete infinite metric and homeomorphic to the direct product of a finite graph and \mathbb{R}^+ . It is reasonable to consider a cone, i.e., the single-point compactification $\bar{Q} = Q \cup \{p\}$ of the space Q , so that all paths going to the point p have infinite length.

By tubes we shall mean the subcones of \bar{Q} that are cones over the arcs of the graph connecting its two neighboring essential vertices. Thus, a tube can look like $S^1 \times \mathbb{R}^+$ or like $[a, b] \times R^+$.

With these definitions, the theorems proved above can easily be extended from surfaces to the Aleksandrov polyhedra.

REFERENCES

- [AZ] A. D. Aleksandrov and V. A. Zalgaller, *Two-dimensional manifolds of bounded curvature*. (*Foundations of the intrinsic geometry of surfaces*), Trudy Mat. Inst. Steklov. **63** (1962), 162 pp.; English transl., *Intrinsic geometry of surfaces*, Transl. Math. Monogr., vol. 15, Amer. Math. Soc., Providence, RI, 1967. MR0151930 (27:1911); MR0216434 (35:7267)
- [Bak] I. Ya. Bakel'man, *Chebyshev networks in manifolds of bounded curvature*, Trudy Mat. Inst. Steklov. **76** (1965), 124–129; English transl. in Proc. Steklov Inst. Math. No. 76 (1965) (1967). MR0209988 (35:883)
- [BB] Yu. D. Burago and S. V. Buyalo, *Metrics of upper bounded curvature on 2-polyhedra*. II, Algebra i Analiz **10** (1998), no. 4, 62–112; English transl., St. Petersburg Math. J. **10** (1999), no. 4, 619–650. MR1654071 (99j:53086)

- [BL] M. Bonk and U. Lang, *Bi-Lipschitz parametrization of surfaces*, Math. Ann. **327** (2003), 135–169. (Abstract DOI 10.1007/s00208-003-0443-8). MR2006006 (2004i:53100)
- [Gr] D. Grieser, *Quasiisometry of singular metrics*, Houston J. Math. **28** (2002), 741–752 (electronic). MR1953683 (2004b:53043)
- [Hub] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), 13–72. MR0094452 (20:970)
- [Resh] Yu. G. Reshetnyak, *Two-dimensional manifolds of bounded curvature*, Geometry, 4, Itogi Nauki i Tekhniki Ser. Sovrem. Probl. Mat. Fund. Naprav., t. 70, VINITI, Moscow, 1989, pp. 7–189; English transl., Encyclopaedia Math. Sci., vol. 70, Springer-Verlag, Berlin, 1993, pp. 3–163. MR1099202 (92b:53104); MR1263964
- [RS] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Ergeb. Math. Grenzgeb., vol. 69, Springer, Berlin etc., 1972. MR0350744 (50:3236)
- [Ver] A. Verner, *A condition for the finite connectedness of complete nonclosed surfaces*, Leningrad. Gos. Ped. Inst. Uchen. Zap. **395** (1970), 100–131. (Russian) MR0287489 (44:4693)

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