

REMARKS ON THE QUANTUM ADIABATIC THEOREM

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§1. INTRODUCTION

The adiabatic theorem (it would be more accurate to say the “quantum adiabatic theorem”) was discovered in 1928 in the paper [BF] by Born and Fock. In a somewhat loose formulation requiring certain additional assumptions, it says the following.

For $t \in \Delta = [0, a] \subset \mathbf{R}$, let $H_0(t)$ be a t -dependent linear operator on a complex Banach space X . Suppose the spectrum $\sigma(t)$ of $H_0(t)$ consists of two components $\sigma_1(t)$, $\sigma_2(t)$ such that $\text{dist}(\sigma_1(t), \sigma_2(t)) \geq \alpha > 0$ for $t \in \Delta$. Let $P_1(t)$ and $P_2(t)$ denote the spectral projections of $H_0(t)$ that correspond to the components $\sigma_1(t)$ and $\sigma_2(t)$.

Along with $H_0(t)$, consider the operator

$$H(t, \varepsilon) = H_0(t) + \varepsilon H_1(t, \varepsilon), \quad \varepsilon > 0,$$

where $H_1(t, \varepsilon)$ is bounded uniformly in ε .

Next, let $U(t, s, \varepsilon) : X \rightarrow X$ be the resolving operator for the equation

$$(1.1) \quad i\varepsilon \frac{\partial \psi}{\partial t} = H(t, \varepsilon)\psi,$$

so that

$$(1.2) \quad i\varepsilon \frac{\partial U(t, s, \varepsilon)}{\partial t} = H(t, \varepsilon)U(t, s, \varepsilon), \quad U(s, s, \varepsilon) = I.$$

Then

$$(1.3) \quad (I - P_j(t))U(t, s, \varepsilon)P_j(s) = O(\varepsilon), \quad \varepsilon \rightarrow 0.$$

Kato’s paper [K1] was the first mathematical publication devoted to the adiabatic theorem. Among the large series of subsequent studies, we mention the papers [ASY] by Avron, Seiler, and Yaffe and [N] by Nenciu, and the books [K2] by Kreĭn and [DK] by Daletskiĭ and Kreĭn. These references make it possible to trace a great number of publications where the general ideas related to the adiabatic theorem were discussed.

Mainly, the investigations of the adiabatic theorem have been dealing with formula (1.3) (under various assumptions) and with its refinements; for instance, this refers to formulas of the type

$$(1.4) \quad (I - P_{j,k})(t, \varepsilon)U(t, s, \varepsilon)P_{j,k}(s, \varepsilon) = O(\varepsilon^k), \quad k = 1, 2, \dots,$$

with various constructions of $P_{j,k}$. This formula means that, in the dynamics in question, the solution of equation (1.1) with initial condition

$$\psi(s) \in P_{j,k}(s, \varepsilon)X$$

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stays with high accuracy in the subspace $P_{j,k}(t, \varepsilon)X$. However, quite often, the question about the explicit asymptotic behavior of the projections $P_{j,k}(t, \varepsilon)$ for small ε used to be left open; only the existence of such projections used to be proved (see, e.g., [N]).

The books [K2] and [DK] mentioned before stand apart: in them asymptotic representations for the solutions of (1.1) were constructed, and after that, as a consequence, it was shown that the solutions split into noninteracting branches in the sense of formula (1.4). In [K2] this idea was realized in a restricted form: the results do not apply to the case of selfadjoint operators $H(t, \varepsilon)$ in Hilbert space. This restriction was lifted in [DK].

Our results are close to those of [DK] and lead readily to formulas of type (1.4).

Loosely speaking (without mentioning some important additional conditions), our result says that, after finitely many elementary steps, the initial equation (1.1) can be transformed to

$$(1.5) \quad i\varepsilon \frac{dz}{dt} = \begin{pmatrix} M_{11}^{(N)}(t, \varepsilon) & \varepsilon^N M_{12}^{(N)}(t, \varepsilon) \\ \varepsilon^N M_{21}^{(N)}(t, \varepsilon) & M_{22}^{(N)}(t, \varepsilon) \end{pmatrix} z.$$

Additionally, it must be assumed that $H(t, \varepsilon)$ admits a full asymptotic power expansion as $\varepsilon \rightarrow 0$. Then N in (1.5) can be made any natural number. The operators $M_{ij}^{(N)}$ also admit asymptotic power expansion as $\varepsilon \rightarrow 0$, and the spectra of the leading terms

$$M_{11}^{(N)}(t, \varepsilon) \sim M_1(t), \quad M_{22}^{(N)}(t, \varepsilon) \sim M_2(t)$$

coincide with $\sigma_1(t)$ and $\sigma_2(t)$, respectively. The components of the vector

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

belong to the fixed (i.e., independent of t and ε) subspaces

$$Z_1 = P_1(0)X, \quad Z_2 = P_2(0)X.$$

It is natural that, under appropriate conditions, equation (1.5) can be approximated as closely as we wish by the equation

$$(1.6) \quad \varepsilon \frac{dz}{dt} = \begin{pmatrix} M_{11}^{(N)}(t, \varepsilon) & 0 \\ 0 & M_{22}^{(N)}(t, \varepsilon) \end{pmatrix} z$$

and thus splits into two independent equations in the spaces Z_1 and Z_2 . Clearly, (1.4) will readily follow from this result, and explicit formulas for the P_{jk} can be obtained from the above-mentioned elementary construction that leads to (1.5). Such formulas will be given in §3.

We believe that equation (1.5) is more efficient than the equations treated in [DK]. First, (1.5) is simpler, because in [DK] the role of Z_1 and Z_2 was played by “moving” subspaces $P_1(t)X$ and $P_2(t)X$, so that, even after dropping the terms of small order, the analog of (1.5) cannot be split into equations in subspaces of smaller dimension.

Second, our argument is simpler, because we start with a formula of different structure. Moreover, in contrast to [DK], in the course of deduction we do not intend to replace the dependence of $M_{ij}^{(N)}(t, \varepsilon)$ on ε by a minimal polynomial dependence correlated with the choice of N , because this minimality leads to no advantage.

The independent equations resulting from (1.6) deserve a comment that is usually ignored. Consider one of these equations:

$$(1.7) \quad i\varepsilon \frac{dz_1}{dt} = M_{11}^{(N)}(t, \varepsilon)z_1.$$

Like the general equation (1.1), this equation may in turn admit efficient simplification as $\varepsilon \rightarrow 0$. This simplification may or may not depend on specific assumptions about

the spectrum structure of the operator $M_{11}^{(N)}$, and such assumptions may differ from the requirement that the spectrum be split into parts. For instance, the terms of the order of ε^2 and higher involved in $M_{11}^{(N)}$ can often be taken into account by using the elementary perturbation theory (see, e.g., [K2]).

Now we formulate the conditions under which we justify the further constructions that lead to equations (1.5) and (1.7) and to their consequences. These conditions will also ensure the validity of all the remarks made above. We choose a simpler (strong) version of the conditions in order that all the results depending on them be evident without special proofs and follow from calculations alone. A certain technique, which can be regarded as well developed, allows one to easily relax our assumptions if necessary, but we do not aim at this here.

So, we list our assumptions.

1. For $t \in \Delta$ and $\varepsilon \in \delta = [0, \beta)$, $\beta > 0$, the operator $H(t, \varepsilon)$ is uniformly bounded together with all derivatives in t (in the sense of the norm).

2. $H(t, \varepsilon)$ admits a full asymptotic expansion

$$H(t, \varepsilon) = H_0(t) + i\varepsilon H^{(1)}(t) + (i\varepsilon)^2 H^{(2)}(t) + \dots,$$

and the coefficients in this expansion also satisfy condition 1.

3. The spectrum of $H_0(t)$ splits into two components $\sigma_1(t), \sigma_2(t)$ such that

$$\text{dist}(\sigma_1(t), \sigma_2(t)) \geq \alpha > 0.$$

4. The resolving operator $U^{(0)}(t, s, \varepsilon)$ of the equation

$$i\varepsilon \frac{\partial \psi}{\partial t} = H_0(t)\psi,$$

i.e.,

$$(1.8) \quad i\varepsilon \frac{\partial U^{(0)}(t, s, \varepsilon)}{\partial t} = H_0(t)U^{(0)}(t, s, \varepsilon), \quad U^{(0)}(s, s, \varepsilon) = \mathbf{I},$$

also satisfies conditions of type 1 for $t, s \in \Delta$, $\varepsilon \in \delta$. We note that, in principle, the corresponding properties of $U^{(0)}(t, s, \varepsilon)$ can be deduced from appropriate assumptions concerning $H_0(t)$ (see, e.g., [K2]).

Also, as a comment to condition 4, we note that if this condition is fulfilled, then the resolving operators of the equations

$$(1.9) \quad \begin{aligned} i\varepsilon \frac{\partial U^{(1)}}{\partial t} &= (H_0(t) + i\varepsilon H^{(1)})U^{(1)}, & U^{(1)}(s, s, \varepsilon) &= \mathbf{I}, \\ i\varepsilon \frac{\partial U}{\partial t} &= H(t, \varepsilon)U, & U(s, s, \varepsilon) &= \mathbf{I}, \end{aligned}$$

possess the same properties as $U^{(0)}$ (for instance, see [K2] again). These operators can be characterized consecutively by the following Volterra equations, which are easy to study:

$$(1.10) \quad \begin{aligned} U^{(1)}(t, s, \varepsilon) &= U^{(0)}(t, s, \varepsilon) + \int_s^t U^{(0)}(t, \tau, \varepsilon) H^{(1)}(\tau) U^{(1)}(t, \tau, \varepsilon) d\tau, \\ U(t, s, \varepsilon) &= U^{(1)}(t, s, \varepsilon) \\ &+ \frac{1}{i\varepsilon} \int_s^t U^{(1)}(t, \tau, \varepsilon) \left[H(\tau, \varepsilon) - H_0(\tau) - i\varepsilon H^{(1)}(\tau) \right] U(t, \tau, \varepsilon) d\tau. \end{aligned}$$

§2. SPLITTING OF EQUATIONS

2.1. The system. Condition 3 implies that the subspaces $X_j(t)$, $j = 1, 2, \dots$, are isomorphic for all $t \in \Delta$, i.e., there exists an operator $T(t, s)$ such that

$$(2.1) \quad P_j(t)T(t, s) = T(t, s)P_j(s), \quad T(s, s) = I.$$

The operator $T(t, s)$ can be chosen to be invertible and C^∞ -smoothly depending on t and s , together with the inverse. It is well known that such T can be described by the "rotation" equation

$$(2.2) \quad T_t(t, s) = [P'_1(t)P_1(t) + P'_2(t)P_2(t)]T(t, s)$$

(for the details, see, e.g., [DK]).

This allows us to transform the equation

$$i\varepsilon \frac{d\psi}{dt} = H(t, \varepsilon)\psi$$

to a system for the vector $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, $y_j(t) \in Y_j = X_j(0)$:

$$(2.3) \quad \psi = T[p_1, p_2] \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad T = T(t, 0), \quad p_j = P_j(0).$$

The symbol $[p_1, p_2]$ must be understood as a matrix. The latter formula can be written as follows:

$$(2.4) \quad \psi = T(t)(p_1 y_1(t) + p_2 y_2(t)) = T(t)(y_1(t) + y_2(t)).$$

The resulting system takes the form

$$(2.5) \quad i\varepsilon \frac{dy}{dt} = M(t, \varepsilon)y, \quad M(t, \varepsilon) = \begin{pmatrix} M_{11}(t, \varepsilon) & M_{12}(t, \varepsilon) \\ M_{21}(t, \varepsilon) & M_{22}(t, \varepsilon) \end{pmatrix}.$$

Here the M_{kl} are operators acting from Y_l to Y_k ; their properties are similar to those imposed in condition 1 on the operator H . This becomes clear when we deduce the system:

$$\begin{aligned} \psi' &= T'(p_1 y_1 + p_2 y_2) + T(p_1 y'_1 + p_2 y'_2), \\ i\varepsilon [T'(p_1 y_1 + p_2 y_2) + T(p_1 y'_1 + p_2 y'_2)] &= H(t, \varepsilon)T(p_1 y_1 + p_2 y_2), \\ i\varepsilon (p_1 y'_1 + p_2 y'_2) &= T^{-1}H(t, \varepsilon)T(p_1 y_1 + p_2 y_2) - i\varepsilon T^{-1}T'(p_1 y_1 + p_2 y_2), \\ i\varepsilon y'_1 &= p_1 T^{-1}H(t, \varepsilon)T(p_1 y_1 + p_2 y_2) - i\varepsilon p_1 T^{-1}T'(p_1 y_1 + p_2 y_2), \\ i\varepsilon y'_2 &= p_2 T^{-1}H(t, \varepsilon)T(p_1 y_1 + p_2 y_2) - i\varepsilon p_2 T^{-1}T'(p_1 y_1 + p_2 y_2). \end{aligned}$$

Thus,

$$(2.6) \quad M_{kl}(t, \varepsilon) = p_k T^{-1} (H(t, \varepsilon) - i\varepsilon T' T^{-1}) T p_l.$$

Now, we observe that

$$T^{-1}(t)H_0(t)T(t) = p_1 M_1(t)p_1 + p_2 M_2(t)p_2,$$

where $M_1(t)$ and $M_2(t)$ are operators in Y_1 and Y_2 , respectively. Therefore, the system can be rewritten as follows:

$$(2.7) \quad i\varepsilon \frac{dy}{dt} = M(t, \varepsilon)y, \quad M(t, \varepsilon) = \begin{pmatrix} M_1(t) + i\varepsilon \tilde{M}_{11}(t, \varepsilon) & i\varepsilon \tilde{M}_{12}(t, \varepsilon) \\ i\varepsilon \tilde{M}_{21}(t, \varepsilon) & M_2(t) + i\varepsilon \tilde{M}_{22}(t, \varepsilon) \end{pmatrix}.$$

2.2. A lemma. Consider the equation

$$i\varepsilon \frac{d\xi_k}{dt} = B^{(k)} \xi_k,$$

$$B^{(k)} = \begin{pmatrix} B_{11}^{(k)}(t, \varepsilon) & (i\varepsilon)^k B_{12}^{(k)}(t, \varepsilon) \\ (i\varepsilon)^k B_{21}^{(k)}(t, \varepsilon) & B_{22}^{(k)}(t, \varepsilon) \end{pmatrix},$$

where the B_{kl} are operators from Y_l to Y_k depending smoothly on t and admitting asymptotic power expansion as $\varepsilon \rightarrow 0$. We assume that the spectra of $B_1(t) = B_{11}^{(k)}(t, 0)$ and $B_2(t) = B_{22}^{(k)}(t, 0)$ are uniformly separated away from each other.

Under these conditions, the following is true.

Lemma. *There exists an operator S_k ,*

$$\xi_k = S_k \xi_{k+1},$$

such that

$$i\varepsilon \frac{d\xi_{k+1}}{dt} = B^{(k+1)} \xi_{k+1},$$

where

$$(2.8) \quad B^{(k+1)} = \begin{pmatrix} B_{11}^{(k+1)}(t, \varepsilon) & (i\varepsilon)^{k+1} B_{12}^{(k+1)}(t, \varepsilon) \\ (i\varepsilon)^{k+1} B_{21}^{(k+1)}(t, \varepsilon) & B_{22}^{(k+1)}(t, \varepsilon) \end{pmatrix}.$$

The operator S_k can be chosen in the form

$$S_k = \begin{pmatrix} \mathbf{I} & (i\varepsilon)^k S_{12}^{(k)}(t, \varepsilon) \\ (i\varepsilon)^k S_{21}^{(k)}(t, \varepsilon) & \mathbf{I} \end{pmatrix},$$

where, in turn, $S_{12}^{(k)}$ and $S_{21}^{(k)}$ admit asymptotic power expansions as $\varepsilon \rightarrow 0$.

Proof. We start with the observation that for small ε the operator S_k^{-1} exists and is given by the formula

$$S_k^{-1} = \begin{pmatrix} \left(\mathbf{I} - (i\varepsilon)^{2k} S_{12}^{(k)} S_{21}^{(k)} \right)^{-1} & 0 \\ 0 & \left(\mathbf{I} - (i\varepsilon)^{2k} S_{21}^{(k)} S_{12}^{(k)} \right)^{-1} \end{pmatrix} \\ \times \begin{pmatrix} \mathbf{I} & -(i\varepsilon)^k S_{12}^{(k)} \\ -(i\varepsilon)^k S_{21}^{(k)} & \mathbf{I} \end{pmatrix}.$$

An explicit calculation shows that ξ_{k+1} satisfies the required equation with the operator

$$B^{(k+1)} = S_k^{-1} [B^{(k)} S_k - i\varepsilon S_k'].$$

We put

$$(2.9) \quad S_{12}^{(k)} B_2 - B_1 S_{12}^{(k)} = B_{12}^{(k)}, \quad S_{21}^{(k)} B_1 - B_2 S_{21}^{(k)} = B_{21}^{(k)}.$$

Relations (2.9) can be viewed as equations for $S_{12}^{(k)}$ and $S_{21}^{(k)}$. Since the spectra of B_1 and B_2 are uniformly separated away from each other, these equations are uniquely solvable, and the solutions admit the representations

$$S_{12} = \frac{1}{2\pi i} \oint_{\Gamma_1} R_{B_1}(\lambda) B_{12} R_{B_2}(\lambda) d\lambda,$$

$$S_{21} = \frac{1}{2\pi i} \oint_{\Gamma_2} R_{B_2}(\lambda) B_{21} R_{B_1}(\lambda) d\lambda$$

(see [K2]). Here

$$R_{B_1}(\lambda) = (B_1 - \lambda I)^{-1}, \quad R_{B_2}(\lambda) = (B_2 - \lambda I)^{-1},$$

and the contours Γ_1 and Γ_2 encircle the spectra of B_1 and B_2 , respectively.

Using (2.9), it is easy to check that the contributions to the nondiagonal entries of the matrix $B^{(k+1)}$ that have the leading order ε^k cancel, so that finally the operator-valued matrix $B^{(k+1)}$ takes the form (2.8). The lemma is proved. \square

The following statement is deduced immediately from the above lemma.

Theorem. *The sequence of elementary transformations S_k ,*

$$y = S_1 S_2 \cdots S_k \xi_{k+1},$$

transforms the initial equation (1.1) to

$$(2.10) \quad i\varepsilon \frac{d\xi_{k+1}}{dt} = \begin{pmatrix} M_{11}^{(k+1)}(t, \varepsilon) & (i\varepsilon)^{(k+1)} M_{12}^{(k+1)}(t, \varepsilon) \\ (i\varepsilon)^{(k+1)} M_{21}^{(k+1)}(t, \varepsilon) & M_{22}^{(k+1)}(t, \varepsilon) \end{pmatrix} \xi_{k+1}.$$

The leading terms of the components M_{11} and M_{22} do not change in the course of transformations.

The proof is obvious.

Remark. Since the operator

$$\begin{pmatrix} M_1(t) & 0 \\ 0 & M_2(t) \end{pmatrix}$$

is similar to the operator $H_0(t)$, assumption 4 ensures the existence and uniform boundedness of the resolving operator for equation (2.10). In particular, the error brought by the nondiagonal entries in (2.10) to the resolving operator is of the order of ε^k . Up to this error, the resolving operator for (2.10) is characterized by the diagonal equation

$$(2.11) \quad i\varepsilon \frac{d\xi}{dt} = \begin{pmatrix} M_{11}^{(k+1)}(t, \varepsilon) & 0 \\ 0 & M_{22}^{(k+1)}(t, \varepsilon) \end{pmatrix} \xi.$$

The estimate will survive if we eliminate the terms of the order of ε^{k+1} and higher from $M_{ii}^{(k+1)}(t, \varepsilon)$.

§3. REMARKS

3.1. Adiabatic theorem. We compare equations (1.1) and (2.10). Let $V^{(k)}(t, s, \varepsilon)$ denote the resolving operator of equation (2.10), and let

$$\Gamma_k(t, \varepsilon) = T[p_1, p_2] S_1 S_2 \cdots S_k.$$

Obviously,

$$U(t, s, \varepsilon) = \Gamma_k(t, \varepsilon) V^{(k)}(t, s, \varepsilon) \Gamma_k^{-1}(s, \varepsilon).$$

The operator Γ_k is uniformly bounded and smooth together with its inverse Γ_k^{-1} . The operator $V^{(k)}$ has two almost invariant subspaces

$$Y_{1,k} = \text{span} \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix}, \quad Y_{2,k} = \text{span} \begin{pmatrix} 0 \\ \xi_2 \end{pmatrix};$$

both of them are invariant relative to the diagonal equation (2.11). Let $Q_{j,k}$ be the corresponding projections. Denoting the resolving operator for (2.11) by $V_d^{(k)}$, we obtain

$$(I - Q_{j,k}) V_d^{(k)}(t, s, \varepsilon) Q_{j,k} = 0.$$

Obviously, we have

$$\|V^{(k)} - V_d^{(k)}\| = O(\varepsilon^k)$$

uniformly in t, s, ε . Therefore,

$$(I - Q_{j,k}) V^{(k)}(t, s, \varepsilon) Q_{j,k} = O(\varepsilon^k).$$

We introduce two additional disjoint projections in X :

$$P_{j,k} = \Gamma_k Q_{j,k} \Gamma_k^{-1}.$$

Then

$$(I - P_{j,k}(t, \varepsilon)) U(t, s, \varepsilon) P_{j,k}(s, \varepsilon) = O(\varepsilon^k).$$

This is the adiabatic theorem; observe that in fact the projections $P_{j,k}(s, \varepsilon)$ are constructed efficiently.

We take a good look at the leading order ($k = 1$):

$$(3.1) \quad (I - P_{j,1}(t, \varepsilon)) U(t, s, \varepsilon) P_{j,1}(s, \varepsilon) = O(\varepsilon).$$

The projections $P_{j,1}$ are given by the explicit formulas

$$P_{1,1} = T [p_1, p_2] S_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S_1^{-1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} T^{-1},$$

$$P_{2,1} = T [p_1, p_2] S_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} S_1^{-1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} T^{-1}.$$

If in S_1 and S_1^{-1} we drop the terms with ε (i.e., we replace S_1 and S_1^{-1} by I), then the error in (3.1) will not change, and the $p_{j,1}$ will change by the rule

$$p_{j,1} \mapsto T p_j T^{-1} = P_j(t), \quad j = 1, 2.$$

Thus, we arrive at the classical version (1.3) of the adiabatic theorem.

3.2. Equation in the leading error order. A direct comparison in full of the formulas obtained above and the Daletskiĭ–Kreĭn formulas would be difficult because these two types of formulas have somewhat different structure. However, this problem disappears if we restrict comparison to the leading and the first order in ε .

It is not hard to check that, in these orders, the transformation of (1.1) is described by the relations

$$\psi = T [p_1, p_2] S y, \quad S = \begin{pmatrix} I & i\varepsilon S_{12} \\ i\varepsilon S_{21} & I \end{pmatrix}.$$

Here S_{12} and S_{21} are determined by

$$(3.2) \quad S_{12} M_2 - M_1 S_{12} = M_{12}^{(1)}, \quad S_{21} M_1 - M_2 S_{21} = M_{21}^{(1)},$$

where

$$M_{kl}^{(1)} = p_k T^{-1} (H^{(1)} - T' T^{-1}) T p_l.$$

In our approximation, the differential equation for y takes the form

$$(3.3) \quad i\varepsilon \frac{dy}{dt} = \begin{pmatrix} M_1 + i\varepsilon M_{11}^{(1)} & 0 \\ 0 & M_2 + i\varepsilon M_{22}^{(1)} \end{pmatrix} y.$$

More precisely,

$$i\varepsilon \frac{dy}{dt} = \left(\begin{pmatrix} M_1 + i\varepsilon M_{11}^{(1)} & 0 \\ 0 & M_2 + i\varepsilon M_{22}^{(1)} \end{pmatrix} + O(\varepsilon^2) \right) y.$$

In (3.3), we make the change of variables $Y = Ty$:

$$i\varepsilon Y' = H_0 Y + i\varepsilon T' T^{-1} Y + i\varepsilon [P_1 (H^{(1)} - T' T^{-1}) P_1 + P_2 (H^{(1)} - T' T^{-1}) P_2] Y.$$

Observe that

$$\begin{aligned} P_1(T'T^{-1})P_1 + P_2(T'T^{-1})P_2 \\ = P_1(P_1'P_1 + P_2'P_2)P_1 + P_2(P_1'P_1 + P_2'P_2)P_2 = P_1P_1'P_1 + P_2P_2'P_2 = 0. \end{aligned}$$

Thus,

$$(3.4) \quad i\varepsilon Y' = H_0Y + i\varepsilon T'T^{-1}Y + i\varepsilon(P_1H^{(1)}P_1 + P_2H^{(1)}P_2)Y.$$

In the same approximation, the relationship between Y and the solution ψ is given by the formula

$$(3.5) \quad \psi = [P_1, P_2]TST^{-1}Y,$$

and it is easy to check that

$$TST^{-1} = I + i\varepsilon V, \quad V = P_1V_{12}P_2 + P_2V_{21}P_1.$$

The operators V_{12} and V_{21} satisfy the equations

$$\begin{aligned} V_{12}H_0P_2 - H_0P_1V_{12} &= P_1(H^{(1)} - T'T^{-1})P_2, \\ V_{21}H_0P_1 - H_0P_2V_{21} &= P_2(H^{(1)} - T'T^{-1})P_1, \end{aligned}$$

which follow from (3.2).

Taken together with the formulas for S and equation (3.4) for Y , the representation (3.5) coincides (up to notation) with the Daletskiĭ–Kreĭn formulas (see [DK, Chapter 8]).

3.3. Something like gauge invariance. Expression (2.1) determines the operator $T(t) = T(t, 0)$ as a “rotation” operator linking spectral projections for different values of t . Next, we introduce the operator

$$(3.6) \quad \Gamma_0 = T[p_1, p_2],$$

which establishes the similarity of H_0 and the operator

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

Generalizing the situation somewhat, suppose that there exists an operator $\Gamma_0(t)$ such that it is uniformly bounded and smoothly depends on t together with its inverse, and satisfies

$$H(t)\Gamma_0(t) = \Gamma_0(t) \begin{pmatrix} M_1(t) & 0 \\ 0 & M_2(t) \end{pmatrix},$$

where $M_1(t)$ and $M_2(t)$ are bounded operators with uniformly separated spectra. This leads to a system for y :

$$\psi = \Gamma_0 y, \quad i\varepsilon y' = \Gamma_0^{-1}H\Gamma_0 y - i\varepsilon \Gamma_0^{-1}\Gamma_0' y,$$

and, almost without modifications, to the asymptotic procedure of splitting the equations. This viewpoint provides additional freedom: if such an operator $\Gamma_0(t)$ exists, then the same properties will be shared by the operator

$$\hat{\Gamma}_0(t) = \Gamma_0 E, \quad E = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix},$$

where e_1 and e_2 are bounded, smooth, and smoothly invertible operators commuting with M_1 and M_2 , respectively. In particular, such modification can be applied also to the operator (3.6). The relationship between ψ and y will change:

$$\psi = T[p_1, p_2] E y,$$

and the equation for y will also change. Of course, the presence of E influences the calculations in the course of the splitting procedure. A natural question arises: how to

neutralize E in the asymptotic formulas that appear if only the diagonal entries are kept in the asymptotic equations?

The equation satisfied by the new y has the form

$$i\varepsilon \frac{dy}{dt} = -i\varepsilon E^{-1} E' y + E^{-1} M E y.$$

Of course, this modifies the earlier operator M ; we denote the new M by M^E :

$$M^E = \begin{pmatrix} M_1 + i\varepsilon M_{11}^E & i\varepsilon M_{12}^E \\ i\varepsilon M_{21}^E & M_2 + i\varepsilon M_{22}^E \end{pmatrix},$$

$$M_{kl}^E = -e_k^{-1} e_l' \delta_{kl} + e_k^{-1} M_{kl}^{(1)} e_l, \quad M_{kl}^{(1)} = p_k T^{-1} \left(H^{(1)} - i\varepsilon T' T^{-1} \right) T p_l.$$

Now we perform the first step of the splitting of equations, adding the operator S_1^E to the transformation Γ :

$$\psi = T [p_1, p_2] E S_1^E \tilde{z},$$

and choosing S_1^E so that the nondiagonal entries of M^E vanish in the orders under consideration:

$$S_{12}^E M_2 - M_1 S_{12}^E = e_1^{-1} M_{12}^E e_2, \quad S_{21}^E M_1 - M_2 S_{21}^E = e_2^{-1} M_{21}^E e_1.$$

This implies

$$S_{12}^E = e_1^{-1} S_{12} e_2, \quad S_{21}^E = e_2^{-1} S_{21} e_1,$$

whence

$$S_1^E = E^{-1} S_1 E.$$

It remains to look at the resulting diagonal entries of the equation for \tilde{z} :

$$i\varepsilon \frac{d\tilde{z}}{dt} = \begin{pmatrix} M_1 + i\varepsilon M_{11}^{(1)} - i\varepsilon e_1^{-1} e_1' & 0 \\ 0 & M_2 + i\varepsilon M_{22}^{(1)} - i\varepsilon e_2^{-1} e_2' \end{pmatrix} \tilde{z}.$$

The nondiagonal entries are of the order of ε^2 , and therefore are dropped. Putting

$$\tilde{z} = E^{-1} z,$$

we obtain

$$i\varepsilon \frac{dz}{dt} = \begin{pmatrix} M_1 + i\varepsilon M_{11} & 0 \\ 0 & M_2 + i\varepsilon M_{22} \end{pmatrix} z.$$

This transformation returns us to the initial diagonal equation, and the additional factor E introduced under passage from ψ to y cancels luckily. Thus, the arbitrariness in the choice of Γ_0 does not influence the asymptotic formulas for solutions. In fact, these calculations are similar to the calculations that lead to the classical transport equations in the WKB method.

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