

## REGULAR TRIANGULATIONS AND STEINER POINTS

M. YU. ZVAGEL'SKIĬ, A. V. PROSKURNIKOV, AND YU. R. ROMANOVSKIĬ

ABSTRACT. Gel'fand, Zelevinskiĭ, and Kapranov showed that the regular triangulations of a “primary” convex polytope can be viewed as the vertices of another convex polytope, which is said to be secondary. Billera, Filliman, and Sturmfels gave a geometric construction of the secondary polytope, based upon Gale transforms. We apply this construction to describing regular triangulations of nonconvex polytopes. We also discuss the problem of triangulating nonconvex polytopes with Steiner points.

### §1. INTRODUCTION

Gel'fand, Zelevinskiĭ, and Kapranov [1] showed that the regular (or coherent) triangulations of a point configuration  $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$  can be viewed as the vertices of a certain polytope  $\Sigma(\mathcal{A}) \subset \mathbb{R}^n$ , which is called the secondary polytope. Billera, Filliman, and Sturmfels [2] gave a geometric construction of  $\Sigma(\mathcal{A})$  based upon Gale transforms (see §2 for a brief survey of the basic notions and results).

In §3, we apply this construction to solving the following problem. A collection  $S$  of simplexes with vertices in  $\mathcal{A}$  is said to be *regular* if there exists a regular triangulation  $T$  of  $\mathcal{A}$  such that each element  $\sigma \in S$  is a  $k$ -face of a  $d$ -dimensional simplex  $\tau \in T$ . For a given collection  $S$ , it is required either to find its regular triangulation  $T$ , or to prove that  $S$  is nonregular. In Theorem 1, we give a polynomial time algorithm for solving this problem, which immediately implies an algorithm for finding regular triangulations of nonconvex polytopes. Indeed, by a regular triangulation of a nonconvex simplicial polytope  $P \subset \mathbb{R}^d$  we mean a regular triangulation  $T$  corresponding to the configuration  $\mathcal{A}$  of its vertices and the collection  $S$  of its faces. In this case our algorithm (announced in [3]) verifies the regular triangulability of  $P$  without new points (henceforth, Steiner points). We emphasize that, without the condition of regularity, such a verification becomes an NP-complete problem [4].

In §4, we study nonregular configurations of edges (1-dimensional simplexes) in  $\mathbb{R}^d$ . Such a configuration may arise, for example, as the 1-skeleton of a nonconvex polytope  $P \subset \mathbb{R}^d$ . A simple argument shows that the polytope  $P$  has no regular triangulations if its 1-skeleton is not regular. In this situation, it is necessary to subdivide some edges of  $P$  by Steiner points. Our strategy for finding a regular subdivision is based on the following almost obvious fact: a configuration of edges is nonregular if and only if it contains at least one minimal nonregular configuration  $S$  (the minimality means that the complement  $S \setminus \{\sigma\}$  is regular for all  $\sigma \in S$ ). This strategy reduces to the iterated detecting and subdividing a minimal nonregular configuration while such configurations exist. An extensive series of computer experiments shows that minimal nonregular configurations admit one-point regular subdivisions. In more detail, our empirical observation says

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that the optimal (in the number of extra points) regular subdivision of each minimal nonregular configuration  $S$  requires only one Steiner point, which can be put on any of the edges in  $S$ . In Theorem 2, we give a rigorous mathematical proof of this conjecture if  $S$  contains only three edges. In the proof, we explicitly describe the interval in which the Steiner point should be inserted. At the end of §4, we give an example of a minimal nonregular configuration, which contains arbitrarily many edges and admits a one-point regular subdivision.

Of course, the regularity of the 1-skeleton does not imply the regular triangulability of the entire polytope  $P$ . However, it turns out that the regularity of its  $(d-2)$ -skeleton guarantees the triangulability of  $P$ . This fact is a direct consequence of Shewchuk's results (see [5, 6]). Usually, the Shewchuk triangulation is not regular, but in some natural sense it is close to a regular triangulation of the  $(d-2)$ -skeleton. A useful link between Shewchuk's results and our approach is discussed at the end of §3.

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## §2. REGULAR TRIANGULATIONS AND SECONDARY POLYTOPES

A *triangulation*  $T$  of a point set  $\mathcal{A} = \{a_1, \dots, a_n\}$  is a set of simplexes such that their vertices are in  $\mathcal{A}$ , their union equals the convex hull  $\text{conv}(\mathcal{A})$ , and the intersection of any pair of simplexes is their common (possibly empty) face. By definition, some points of  $\mathcal{A}$  may fail to be a vertex of any simplex (Figure 1). For a fixed triangulation  $T$  of  $\mathcal{A}$ , every vector  $\psi \in \mathbb{R}^n$  induces a unique piecewise linear function  $g_{\psi, T}$  on the “primary” polytope  $\text{conv}(\mathcal{A})$ . This function is defined by the values  $g_{\psi, T}(a_i) = \psi_i$  at the vertices  $a_i$  of  $T$  and by the requirement that  $g_{\psi, T}$  should be an affine function on each simplex of  $T$ . A triangulation  $T$  of  $\mathcal{A}$  is said to be *regular* if there exists a vector  $\psi \in \mathbb{R}^n$  such that  $g_{\psi, T}$  is strictly convex over  $T$ . This means that  $g_{\psi, T}$  is convex and its restrictions to distinct maximal cells of  $T$  are distinct affine functions.

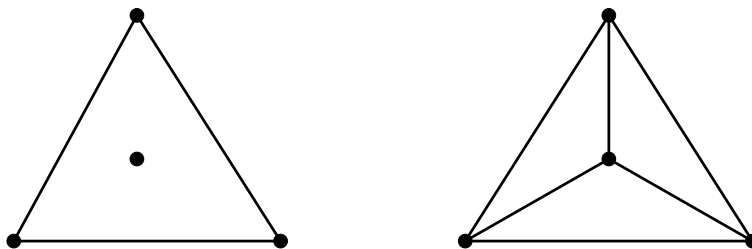


FIGURE 1. Two triangulations of a set of 4 points in the plane.

There is a simple relationship between regular triangulations in  $\mathbb{R}^d$  and convex hulls in  $\mathbb{R}^{d+1}$ . Indeed, pick a function  $\psi: \mathcal{A} \rightarrow \mathbb{R}$  (we identify  $\mathbb{R}^n$  with the vector space  $\mathbb{R}^{\mathcal{A}}$  of real-valued function on  $\mathcal{A}$ ), lift each point  $a_i \in \mathcal{A}$  to the graph of  $\psi$ , i.e., to the point  $(a_i, \psi_i) \in \mathbb{R}^{d+1}$ , and compute the convex hull of the lifted points. The lower part of this convex hull can be viewed as the graph of a convex piecewise linear function. For almost every  $\psi \in \mathbb{R}^n$ , the projection of this graph down to  $\mathbb{R}^d$  gives a triangulation  $T(\psi)$  of  $\mathcal{A}$ . Directly from this definition, it follows that  $T(\psi)$  is regular, and every regular triangulation can be constructed in this way. The most famous example of a regular

triangulation is the Delaunay triangulation ( $DT$ ), which can be obtained by lifting the input points to the paraboloid of revolution:  $\psi_i = a_i^2$ . This triangulation has the following characteristic property: the circumscribed sphere of each simplex in  $DT$  is empty [7]. It turns out that every regular triangulation  $T(\psi)$  has a similar property: if we interpret each vertex  $a_i$  as a sphere of radius  $r_i = \sqrt{a_i^2 - \psi_i}$  and replace the circumsphere of a simplex by the sphere orthogonal to its “vertices”, then this orthosphere must be empty for all simplexes in  $T(\psi)$  (see [8, 9, 10] for the details).

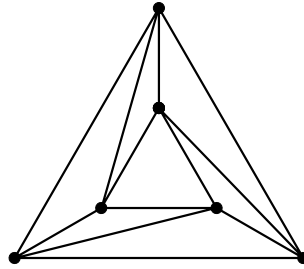


FIGURE 2. A nonregular triangulation in  $\mathbb{R}^2$ .

The standard example of a nonregular triangulation in  $\mathbb{R}^2$  is depicted in Figure 2, where the two triangles are homothetic with respect to a common center. This nonregular triangulation is not stable with respect to small perturbations. It lies on the boundary between the sets of regular and nonregular triangulations. Rotating the inner triangle clockwise (counterclockwise), we obtain a regular (respectively, nonregular) triangulation (Figure 3). These examples help to develop an intuitive understanding of the regular triangulations, based on a physical interpretation (going back to Maxwell) of this notion in terms of rubber bands and their equilibrium states (see Ziegler’s book [11] for more information).

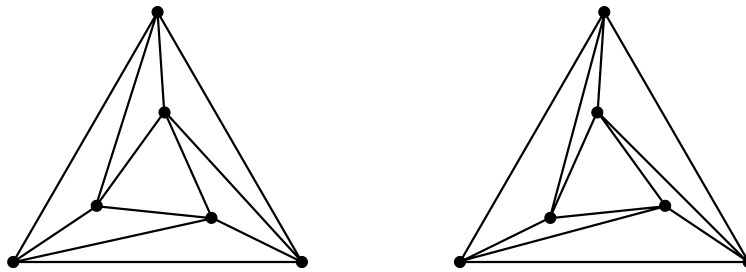


FIGURE 3. Deformations of the previous nonregular triangulation.

It is easy to check that a triangulation  $T$  of  $\mathcal{A}$  is regular if and only if the closed polyhedral cone

$$C(T) = \{\psi \in \mathbb{R}^n \mid g_{\psi,T} \text{ is convex; } g_{\psi,T}(a_i) \leq \psi_i \text{ if } a_i \text{ is not a vertex of } T\}$$

in  $\mathbb{R}^n$  has a nonempty interior. Distinct regular triangulations have distinct full-dimensional cones. The interior of such a cone consists of the vectors  $\psi$  that generate the same regular triangulation by lifting  $\mathcal{A}$  to  $\mathbb{R}^{d+1}$  and taking the convex hull. The Gel’fand–Zelevinskiĭ–Kapranov theorem (see [1]) asserts that the collection

$$\mathcal{F}(\mathcal{A}) = \{C(T) \mid T \text{ is a triangulation of } \mathcal{A}\}$$

coincides with the normal fan (Figure 4) of a certain polytope  $\Sigma(\mathcal{A})$  in  $\mathbb{R}^n$ , which is called the *secondary polytope*. In particular, this theorem means that the regular triangulations of  $\mathcal{A}$  are in one-to-one correspondence with the vertices of the secondary polytope  $\Sigma(\mathcal{A})$ , because the vertices of any polytope are in one-to-one correspondence with the maximal cells of its normal fan.

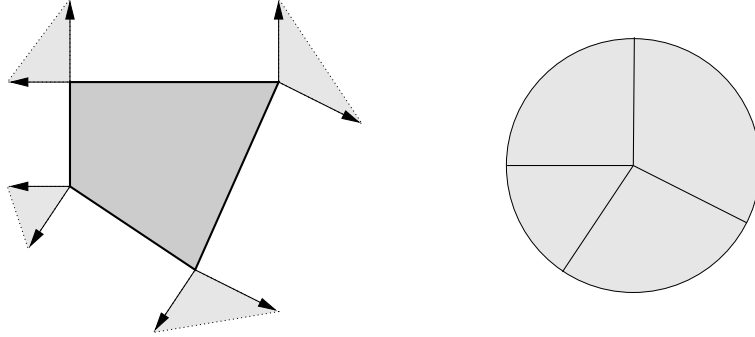


FIGURE 4. The normal fan of a polytope.

A constructive geometric description of the *secondary fan*  $\mathcal{F}(\mathcal{A})$  was given in [2]. Let  $V(\mathcal{A})$  denote a linear subspace of  $\mathbb{R}^n$  consisting of affine-valued vectors, i.e.,

$$V(\mathcal{A}) = \{\psi \in \mathbb{R}^n \mid \psi_i = g(a_i) \text{ for all } i = 1, \dots, n\},$$

where  $g$  ranges over all affine functions on  $\mathbb{R}^d$ . As a linear complement of  $V(\mathcal{A})$  in  $\mathbb{R}^d$ , we can take the space

$$D(\mathcal{A}) = \left\{ \psi \in \mathbb{R}^n \mid \sum_{i=1}^n \psi_i = 0, \sum_{i=1}^n \psi_i a_i = 0 \right\}$$

of affine dependences among the points of  $\mathcal{A}$ . In other words,  $\mathbb{R}^n$  is decomposed into the direct sum

$$\mathbb{R}^n = V(\mathcal{A}) \oplus D(\mathcal{A}).$$

Let  $\mathbf{B} : \mathbb{R}^n \rightarrow D(\mathcal{A})$  be the projection compatible with this decomposition, and let  $\{e_1, \dots, e_n\}$  be the standard basis in  $\mathbb{R}^n$ . The *Gale transform* of the point configuration  $\mathcal{A}$  is the vector configuration  $\mathcal{B} = \{b_1, \dots, b_n\} \subset D(\mathcal{A})$  with  $b_i = \mathbf{B}e_i$ . It is clear that the convex hull  $\text{conv}\{a_{\tau_1}, \dots, a_{\tau_{d+1}}\}$  is a  $d$ -dimensional simplex (henceforth, a  $d$ -simplex) if and only if the set of vectors  $\{b_{\tau_1^*}, \dots, b_{\tau_{n-d-1}^*}\}$  is a basis of  $D(\mathcal{A})$ . Here  $\tau$  and  $\tau^*$  are mutually complementary collections of indices, i.e.  $\tau \cap \tau^* = \emptyset$  and  $\tau \cup \tau^* = \{1, \dots, n\}$ .

The Billera–Filliman–Sturmfels theorem (see [2]) asserts that for any triangulation  $T$ , the cone  $C(T)$  is decomposed into the direct sum

$$C(T) = \text{Ker}(\mathbf{B}) \oplus C'(T)$$

of the linear subspace  $\text{Ker}(\mathbf{B}) = V(\mathcal{A})$  and the intersection  $C'(T) = \bigcap_{\tau \in T} C_{\tau^*}$  of the conical hulls  $C_{\tau^*} = \text{Cone}\{b_{\tau_1^*}, \dots, b_{\tau_{n-d-1}^*}\}$  in  $D(\mathcal{A})$ . Moreover, if  $T$  is regular and  $\psi$  is in the interior of  $C'(T)$ , then  $T = \{\tau \mid \psi \in C_{\tau^*}\}$ . Here the collection  $\tau = (\tau_1, \dots, \tau_{d+1})$  of indices is identified with the  $d$ -simplex  $\text{conv}\{a_{\tau_1}, \dots, a_{\tau_{d+1}}\}$ .

In particular, this theorem implies that the secondary polytope  $\Sigma(\mathcal{A})$  is of dimension

$$N = n - d - 1,$$

and its  $N$ -dimensional normal fan is the collection

$$\mathcal{F}'(\mathcal{A}) = \{C'(T) \mid T \text{ is a triangulation of } \mathcal{A}\}$$

of cones, which contain no nontrivial linear subspaces. Such cones are said to be *pointed*. Therefore, the fan  $\mathcal{F}'(\mathcal{A})$  is also said to be pointed. Thus, the algorithm for computing all regular triangulations of  $\mathcal{A}$  reduces to the following operations: find the Gale transform  $\mathcal{B}$  of  $\mathcal{A}$ , then find the pointed secondary fan  $\mathcal{F}'(\mathcal{A})$  as the multiintersection of the conical hulls  $\text{Cone}\{b_{\mu_1}, \dots, b_{\mu_N}\}$ , where  $\{b_{\mu_1}, \dots, b_{\mu_N}\}$  ranges over all subsets of  $\mathcal{B}$  that are bases of  $D(\mathcal{A})$ . The maximal cells of  $\mathcal{F}'(\mathcal{A})$  are in one-to-one correspondence with the regular triangulations of the point set  $\mathcal{A}$ .

### §3. REGULAR TRIANGULATIONS OF SIMPLICIAL COMPLEXES

Let  $S$  be a collection of simplexes with vertices in a point set  $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$ . In §1, we assumed tacitly that  $S$  is a *simplicial complex*. This means that any two simplexes in  $S$  intersect along a common (possibly empty) face. From now on, we shall assume this explicitly. For example, such a complex can be thought of as the boundary complex of a simplicial nonconvex polytope  $P \subset \mathbb{R}^d$ .

A *regular triangulation*  $T$  of  $S$  is a regular triangulation of  $\mathcal{A}$  that contains  $S$  in the sense that each  $\sigma \in S$  is a  $k$ -face of a  $d$ -simplex  $\tau \in T$ . A simplicial complex  $S$  is said to be *regular* if it has at least one regular triangulation. In this section, we solve the following problem: for a given simplicial complex  $S$  with vertices in  $\mathcal{A}$ , either find a regular triangulation  $T$  of  $S$ , or verify that  $S$  is not regular.

With a  $k$ -simplex  $\sigma = \{\sigma_1, \dots, \sigma_{k+1}\}$  we associate the closed polyhedral cone

$$C_{\sigma^*} = \text{Cone}\{b_{\sigma_1^*}, \dots, b_{\sigma_{n-k-1}^*}\}, \quad \sigma \cap \sigma^* = \emptyset, \sigma \cup \sigma^* = \{1, \dots, n\}.$$

**Lemma 1.** *The cone  $C_{\sigma^*}$  is composed of all cells  $C(T)$  that correspond to distinct regular triangulations  $T$  containing  $\sigma$ .*

*Proof.* Suppose that the interior of the intersection  $C_{\sigma^*} \cap C'(T)$  is not empty and contains a vector  $\psi$ . Then, by the Carathéodory theorem [11], there exists an  $N$ -tuple  $\tau^* \subset \sigma^*$  such that  $\psi \in C_{\tau^*}$ . By the Billera–Filliman–Sturmfels [2] theorem (see §2), the corresponding  $d$ -simplex  $\tau$  is in  $T$  and  $C'(T) \subset C_{\tau^*}$ . Since  $C_{\tau^*} \subset C_{\sigma^*}$ , we have  $C'(T) \subset C_{\sigma^*}$ . Since  $\sigma \subset \tau$ , the triangulation  $T$  contains  $\sigma$ . □

**Theorem 1.** *The regularity of a given simplicial complex  $S$  with vertices in a point set  $\mathcal{A} = \{a_1, \dots, a_n\}$  can be verified in time polynomially dependent on  $n$ .*

*Proof.* The algorithm is designed in the following way.

1. Find the Gale transform  $\mathcal{B} = \{b_1, \dots, b_n\}$  of the point set  $\mathcal{A}$ .
2. Find the cone  $C_{\sigma^*}$  for each simplex  $\sigma$  of the given collection  $S$ .
3. Look for an interior point  $\psi'$  of the cone  $C'(S) = \bigcap_{\sigma \in S} C_{\sigma^*}$ .
4. If the interior of the cone  $C'(S)$  is empty, then  $S$  is not regular. Otherwise, construct a regular triangulation of  $S$  by computing the convex hull of the lifted points  $(a_i, \psi_i) \in \mathbb{R}^{d+1}$  and projecting its lower part onto  $\mathbb{R}^d$ . Here  $\psi$  is a preimage of  $\psi'$  under the map  $\mathbf{B}$ .

Below we estimate the complexity of each step 1–4. Note that our bounds are not sharp, but all of them are polynomial.

**1. Finding the Gale transform.** Without loss of generality, we assume that the set  $\{a_1, \dots, a_{d+1}\}$  is affinely independent. Then the set  $\{b_{d+2}, \dots, b_n\}$  is a basis of  $D(\mathcal{A})$ . We identify  $\mathbb{R}^N$  and its standard basis with  $D(\mathcal{A})$  and  $\{b_{d+2}, \dots, b_n\}$ , respectively (recall

that  $N = n - d - 1$ ). Finding the Gale transform  $\mathcal{B}$  is equivalent to computing the matrix

$$B = \begin{vmatrix} b_1^1 & b_2^1 & \cdots & b_{d+1}^1 \\ b_1^2 & b_2^2 & \cdots & b_{d+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ b_1^N & b_2^N & \cdots & b_{d+1}^N \end{vmatrix}$$

whose columns contain the coordinates of  $b_1, b_2, \dots, b_{d+1}$  with respect to the standard basis. This matrix is of size  $N \times (d + 1)$  and is computed in time  $O(n)$  by solving the following  $N$  linear systems:

$$\sum_{j=1}^{d+1} b_j^i = -1, \quad \sum_{j=1}^{d+1} b_j^i a_j = -a_{d+1+i} \quad (i = 1, \dots, N).$$

**2. Finding the cone  $C_{\sigma^*}$  for each  $\sigma \in S$ .** For each input simplex  $\sigma \in S$ , finding the cone  $C_{\sigma^*}$  reduces to representing this cone in the form of an intersection of half-spaces, i.e., in the form of a system of homogeneous linear inequalities in  $\mathbb{R}^N$ :

$$l_{\sigma i}(x) = \sum_{j=1}^N c_{\sigma i}^j x_j \geq 0 \quad (i = 1, \dots, m_{\sigma}),$$

where the  $i$ th inequality corresponds to the  $i$ th hyperface of  $C_{\sigma^*}$ , and  $m_{\sigma}$  is the number of hyperfaces. For each pair of integers  $i \leq m_{\sigma}$  and  $j \leq N$ , the coefficient  $c_{\sigma i}^j$  is a minor determinant of the matrix  $\|BE\|$ , where  $E$  is the unit matrix of size  $N \times N$ . Therefore, each  $c_{\sigma i}^j$  can be computed in time  $t_{\sigma i}^j = O(1)$ , and the complexity of the entire step is

$$\sum_{\sigma \in S} \sum_{i=1}^{m_{\sigma}} \sum_{j=1}^N t_{\sigma i}^j = O(NM), \quad M = \sum_{\sigma \in S} m_{\sigma},$$

where  $M$  depends on  $n$  polynomially because the number of  $k$ -simplexes  $\sigma \in S$  is not greater than  $\binom{n}{k+1}$ , and  $m_{\sigma} \leq \binom{n-k-1}{d-k+1} = O(n^{d-k+1})$ . Now it is clear that this step is of polynomial complexity. For the practical use, we can outline a method for computing the exact value of  $m_{\sigma}$ . If we are given a  $(d - 1)$ -simplex  $\sigma = (\sigma_1, \dots, \sigma_d)$ , then the number  $m_{\sigma}$  of hyperfaces of the cone  $C_{\sigma^*} \subset \mathbb{R}^N$  is equal to the product  $m_{\sigma} = pq$  of the number  $p$  of positive coefficients in the nontrivial linear combination

$$c_1 b_{\sigma_1^*} + \cdots + c_{N+1} b_{\sigma_{N+1}^*} = 0$$

and the number  $q = N + 1 - p$  of negative coefficients. Of course, this is valid for the generic case:  $c_i \neq 0$  for all  $i = 1, \dots, N + 1$ . The hyperfaces  $F$  of  $C_{\sigma^*}$  are in one-to-one correspondence with the pairs of indices  $ij$  such that  $c_i c_j < 0$ . This correspondence has the form

$$F_{ij} = \text{Cone}(\{b_{\sigma_1^*}, \dots, b_{\sigma_{N+1}^*}\} \setminus \{b_i, b_j\}).$$

For a  $(d - 2)$ -simplex  $\sigma = \{\sigma_1, \dots, \sigma_{d-1}\}$ , it is also easy to enumerate all hyperfaces of  $C_{\sigma^*}$ . For this, we choose two independent linear combinations

$$\begin{aligned} c_1^1 b_{\sigma_1^*} + \cdots + c_{N+2}^1 b_{\sigma_{N+2}^*} &= 0, \\ c_1^2 b_{\sigma_1^*} + \cdots + c_{N+2}^2 b_{\sigma_{N+2}^*} &= 0 \end{aligned}$$

and consider the vector configuration  $V = \{v_1, \dots, v_{N+2}\} \subset \mathbb{R}^2$  defined by the rule  $v_i = (c_i^1, c_i^2)$ . This configuration is dual (see [11]) to the configuration  $\{b_{\sigma_1^*}, \dots, b_{\sigma_{N+2}^*}\} \subset \mathbb{R}^N$ . The affine Gale diagram of  $V$  is a straight line with black and white points: the black points are the traces of the direct rays on the line chosen, and the white points are the traces of the inverse rays (see Figure 5). On this diagram, we find all triplets  $ijk$  of

the form depicted in Figure 6 (for example, there are five such triplets in Figure 5). These triplets are in one-to-one correspondence with the hyperfaces  $F$  of  $C_{\sigma^*}$ , and the correspondence has a similar form:

$$F_{ijk} = \text{Cone}(\{b_{\sigma_1^*}, \dots, b_{\sigma_{N+2}^*}\} \setminus \{b_i, b_j, b_k\}).$$

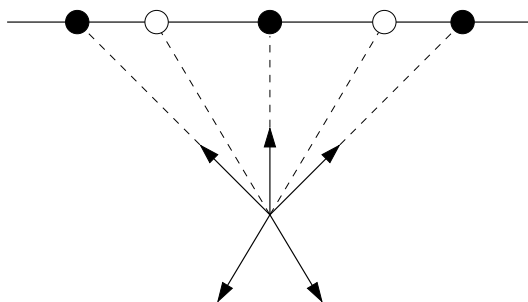


FIGURE 5. Affine diagram of a vector configuration in  $\mathbb{R}^2$ .



FIGURE 6. Triplets of points corresponding to the hyperfaces  $F_{ijk}$  of  $C_{\sigma^*}$ .

**3. Looking for an interior point of  $C'(S)$ .** The interior of  $C'(S)$  is not empty if and only if there exists a solution of the following system of  $M$  nonhomogeneous inequalities:  $l_{\sigma_i}(x) \geq 1$  ( $i \leq m_\sigma$ ,  $\sigma \in S$ ). Khachiyan [12] proved that the consistency of a system of  $M$  inequalities in  $\mathbb{R}^N$  can be verified in time  $O(N^4M(N^2 + M))$ . In practice, an interior point of  $C'(S)$  can be found as an extremal point of the polyhedral set

$$X = \{x \in \mathbb{R}^N \mid l_{\sigma_i}(x) \geq 1 \text{ for all } i \leq m_\sigma, \sigma \in S\}.$$

Every extremal point of  $X$  is a solution of the linear programming problem  $l(x) \rightarrow \min$  provided  $x \in X$ , where the objective function  $l(x)$  is a nonnegative combination of the left-hand sides  $l_{\sigma_i}(x)$ , i.e.,  $l(x) = \sum w_{\sigma_i} l_{\sigma_i}(x)$ ,  $w_{\sigma_i} \geq 0$ . This problem can be solved in expected time  $O(NM)$ .

**4. Finding a regular triangulation.** Let  $(x_1, \dots, x_N)$  be a solution of the above nonhomogeneous system of inequalities. Then the vector  $\psi' = x_1 b_{d+2} + \dots + x_N b_n$  is in the interior of  $C'(S)$ . By Lemma 1, it generates a regular triangulation of  $S$ . The same triangulation is generated by the preimage  $\psi = x_1 e_{d+2} + \dots + x_N e_n$  of  $\psi'$  under the map **B**. The coordinates of  $\psi$  with respect to the standard basis  $\{e_1, \dots, e_n\}$  of  $\mathbb{R}^n \simeq \mathbb{R}^A$  are given by

$$\psi_i = 0 \quad (i \leq d + 1), \quad \psi_i = x_{i-d-1} \quad (i \geq d + 2).$$

The convex hull of  $n$  points  $(a_i, \psi_i) \in \mathbb{R}^{d+1}$  is computed in time  $O(n^{\lfloor (d+1)/2 \rfloor + 1})$  (see [13]). In practice, the computation of convex hulls can be avoided: the regular triangulation can be constructed by using the fast flip algorithm by Edelsbrunner and Shah [8].  $\square$

*Remark 1.* There is a simple sufficient condition guaranteeing the nonregularity of  $S$ . Let  $S'$  be a collection of simplexes such that every  $\sigma' \in S'$  is a  $k'$ -face of a  $k$ -simplex  $\sigma \in S$ , where  $k' \leq k$ . If  $S'$  is nonregular, then so is  $S$ . Indeed, if the function  $\psi \in \mathbb{R}^A$  generates a regular triangulation of  $S$ , then the restriction  $\psi' \in \mathbb{R}^{A'}$  of  $\psi$  to the subset  $A' \subset A$  of vertices of  $S'$  generates a regular triangulation of  $S'$ . This fact leads to the following useful corollary: if the 1-skeleton (i.e., the collection of edges) of a nonconvex polytope  $P$  is nonregular, then  $P$  has no regular triangulations. Such a polytope cannot be triangulated regularly without adding Steiner points to its edges.

*Remark 2.* In his recent paper [6], Shewchuk introduced the notion of a *constrained regular triangulation (CRT)*. This is a triangulation of  $S$  (nonregular in general), which is close (in a certain natural sense) to an *a priori* fixed regular triangulation  $RT$  of the vertex configuration  $\mathcal{A}$ . For example, if  $RT$  is the Delaunay triangulation  $DT$  of  $\mathcal{A}$ , then the  $CRT$  is none other than the  $CDT$ , i.e. the constrained Delaunay triangulation of  $S$  (see [5]). Shewchuk's results have the following elegant consequence: if the  $(d-2)$ -skeleton  $S$  of a nonconvex  $d$ -polytope  $P$  is regular, then  $P$  is triangulable without Steiner points. Actually, the regularity of  $S$  means that  $S$  has a regular triangulation  $RT$ . Of course, in general,  $RT$  is not a triangulation of  $P$ . However, by the Shewchuk theorem, every polytope  $P$  whose  $(d-2)$ -skeleton is in  $RT$  admits a special triangulation  $CRT$ , which can be viewed as a modification of  $RT$ . A useful link between this theorem and our approach is as follows. Our algorithm may verify the regular triangulability of the entire polytope  $P$ . If we have no luck, the same algorithm may verify the regular triangulability of its  $(d-2)$ -skeleton. If this skeleton turns out to be regular, we can apply the Shewchuk method and get a constrained regular triangulation of  $P$ . Finally, if this skeleton is nonregular, we can put Steiner points on the edges of  $P$  to get a regular subdivision. A simple strategy for finding regular subdivisions is discussed in §4.

#### §4. NONREGULAR COMPLEXES AND STEINER POINTS

Throughout this section,  $S$  is a configuration of edges (line segments) in  $\mathbb{R}^d$ , and  $\mathcal{A}$  is the set of their vertices. Such a configuration can be thought of as a graph embedded in  $\mathbb{R}^d$ , for example, the 1-skeleton of a nonconvex polytope. An edge  $\sigma$  in a nonregular configuration  $S$  is said to be *essential* if the complement  $S_\sigma = S \setminus \{\sigma\}$  is regular. A nonregular configuration  $S$  is said to be *minimal* if all edges in  $S$  are essential.

**Lemma 2.** *A configuration  $S$  is nonregular if and only if it contains a minimal nonregular configuration  $S' \subset S$ .*

*Proof.* If  $S$  contains a minimal nonregular configuration  $S'$ , then  $S$  cannot be regularly triangulated without adding Steiner points to  $S'$  (see Remark 1 in §3). Therefore,  $S$  is nonregular. Conversely, suppose that  $S$  is nonregular. If  $S$  is not minimal, then there exists at least one inessential edge  $\sigma \in S$ . If the complement  $S_\sigma = S \setminus \{\sigma\}$  is not minimal, then  $S_\sigma$  (in turn) has an inessential edge, which can also be eliminated, and so on. Obviously, after a finite number of such eliminations we get a minimal nonregular configuration  $S' \subset S$ .  $\square$

By a *subdivision* of  $S$  we mean a configuration  $\dot{S}$  that can be obtained by adding Steiner points to the edges of  $S$ . Lemma 2 provides a simple strategy for finding a regular subdivision  $\dot{S}$  of the nonregular configuration  $S$ . This strategy reduces to the iterated detecting and subdividing a minimal nonregular configuration while such configurations exist. An extensive series of computer experiments shows that minimal nonregular configurations admit *one-point regular subdivisions*.

**Conjecture.** *Each minimal nonregular configuration  $S$  of edges in  $\mathbb{R}^d$  admits a regular subdivision  $\dot{S}$  with exactly one Steiner point. Furthermore, this point can be put on any of the edges in  $S$ .*

In other words, our conjecture says that the optimal (in the number of additional points) regular subdivision of each minimal nonregular configuration requires only one Steiner point. In Theorem 2 we give a rigorous mathematical proof of this statement provided  $S$  contains only three edges.

**Lemma 3.** *Let  $\mathcal{A}', \mathcal{A}''$  be two finite point configurations in  $\mathbb{R}^d$ , and let  $T', T''$  be their regular triangulations. If  $\text{conv}(\mathcal{A}') \cap \text{conv}(\mathcal{A}'') = \emptyset$ , then the union  $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}''$  has a regular triangulation  $T$  such that  $T' \cup T'' \subset T$ .*

*Remark 3.* Here we do not assume that the affine hulls of the sets  $\mathcal{A}'$  and  $\mathcal{A}''$  have full dimension  $d$ .

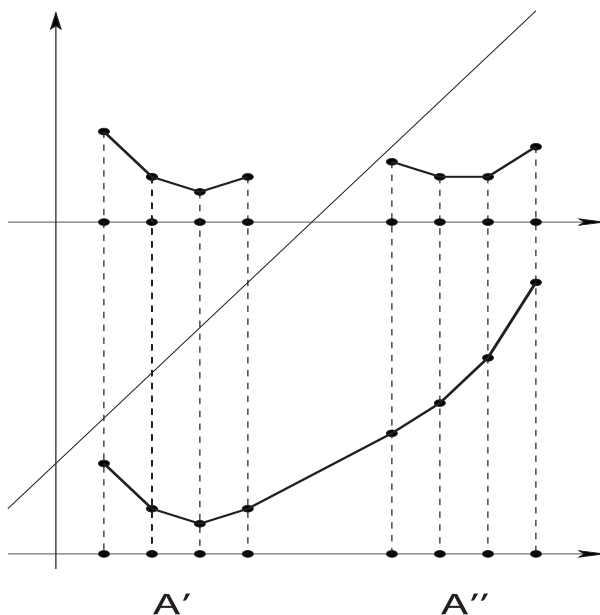


FIGURE 7. Composition of two regular triangulations.

*Proof.* Let  $\psi': \mathcal{A}' \rightarrow \mathbb{R}$  and  $\psi'': \mathcal{A}'' \rightarrow \mathbb{R}$  be the functions generating the regular triangulations  $T'$  and  $T''$ , and let  $h: \mathbb{R}^d \rightarrow \mathbb{R}$  be an affine function negative on  $\mathcal{A}'$  and positive on  $\mathcal{A}''$ . For a real number  $\lambda$ , we define a function  $\psi: \mathcal{A} \rightarrow \mathbb{R}$  by the rule

$$\psi = \begin{cases} \psi'(x) & \text{if } x \in \mathcal{A}', \\ \psi''(x) + \lambda h(x) & \text{if } x \in \mathcal{A}'', \end{cases}$$

and generate the corresponding regular triangulation  $T = T(\psi)$ . The graphs of the functions  $g_{\psi', T'}$ ,  $g_{\psi'', T''}$ ,  $h$  and  $g_{\psi, T}$  are shown in Figure 7. If  $\lambda$  is chosen to be sufficiently large, then  $T' \cup T'' \subset T$ . Actually, it suffices to verify that for each  $\sigma \in T' \cup T''$  there is a number  $\lambda(\sigma)$  such that  $\sigma \in T$  for all  $\lambda > \lambda(\sigma)$ .

1. Suppose  $\sigma \in T'$  and  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  is a global affine function such that  $g(x) = g_{\psi', T'}(x)$  for all  $x \in \sigma$ . By construction,  $\sigma \in T$  if  $\psi(x) > g(x)$  for all  $x \in \mathcal{A}''$ . But this condition

is fulfilled for all  $\lambda > \lambda(\sigma)$ , where

$$\lambda(\sigma) = \max_{x \in \mathcal{A}''} \frac{g(x) - \psi''(x)}{h(x)}.$$

2. Suppose  $\sigma \in T''$  and  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  is a global affine function such that  $g(x) = g_{\psi'', T''}(x)$  for all  $x \in \sigma$ . By construction,  $\sigma \in T$  if  $\psi(x) > g(x) + \lambda h(x)$  for all  $x \in \mathcal{A}'$ . But this condition is fulfilled for all  $\lambda > \lambda(\sigma)$ , where

$$\lambda(\sigma) = \max_{x \in \mathcal{A}'} \frac{\psi'(x) - g(x)}{h(x)}. \quad \square$$

Lemma 3 has the following obvious consequence.

**Corollary 1.** *If  $S$  is a minimal nonregular configuration of edges in  $\mathbb{R}^d$ , then the convex hull of every subset  $S' \subset S$  intersects the convex hull of its complement  $S \setminus S'$ .*

The proof of the following theorem is constructive and can be viewed as an explicit method for regular subdividing of any nonregular triplet of edges in  $\mathbb{R}^d$ .

**Theorem 2.** *Each nonregular triplet  $S$  of edges in  $\mathbb{R}^d$  admits a regular subdivision  $\hat{S}$  with exactly one Steiner point, which can be put on any of the edges in  $S$ .*

*Proof.* Let  $S = \{a, b, c\}$  be a nonregular triplet of edges, let  $\mathcal{A}$  be the vertex configuration of  $S$ , and let  $\mathbb{R}^d$  be the affine hull of  $\mathcal{A}$ . By Corollary 1, every edge in  $S$  must intersect the convex hull of two other edges. This property admits a “nonsymmetric” reformulation: the edge  $c$  must intersect the convex hull  $D_{ab} = \text{conv}(a \cup b)$  and two more domains

$$D_a = \{x \in \mathbb{R}^d \mid a \cap \text{conv}(\{x\} \cup b) \neq \emptyset\},$$

$$D_b = \{x \in \mathbb{R}^d \mid b \cap \text{conv}(\{x\} \cup a) \neq \emptyset\}.$$

It follows that the dimension  $d$  can only be either 3 or 2. The corresponding cases are illustrated in Figures 8 and 9.

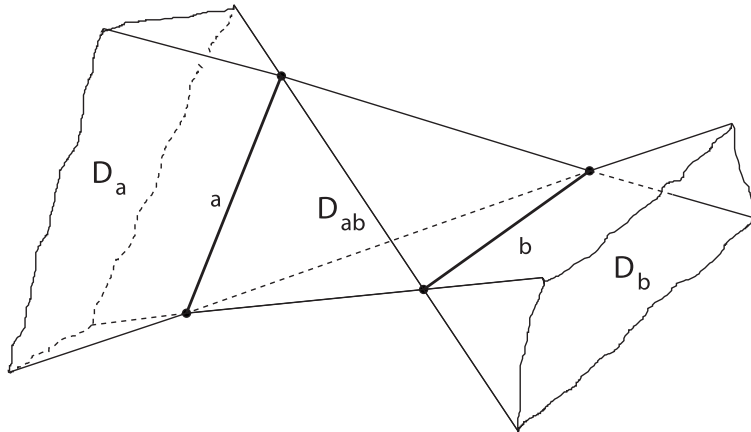


FIGURE 8. Three domains  $D_{ab}$ ,  $D_a$ ,  $D_b$  in  $\mathbb{R}^3$ .

Let  $d = 3$ . In this case, every triplet of edges  $S$  satisfying the above property is nonregular. Assume the contrary: there is a regular triangulation  $T$  of  $S$  generated by a function  $\psi: \mathcal{A} \rightarrow \mathbb{R}$ . Without loss of generality we assume that  $\psi$  takes the value 0 at the ends of the edges  $a$  and  $b$ . Notice that the union  $D_{ab} \cup D_a \cup D_b$  is composed of all straight lines  $L$  that intersect the edges  $a$  and  $b$  (see Figure 10).

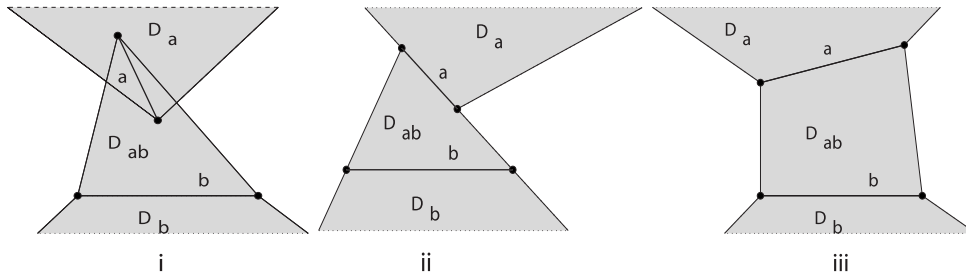


FIGURE 9. Three domains  $D_{ab}, D_a, D_b$  in  $\mathbb{R}^2$ .

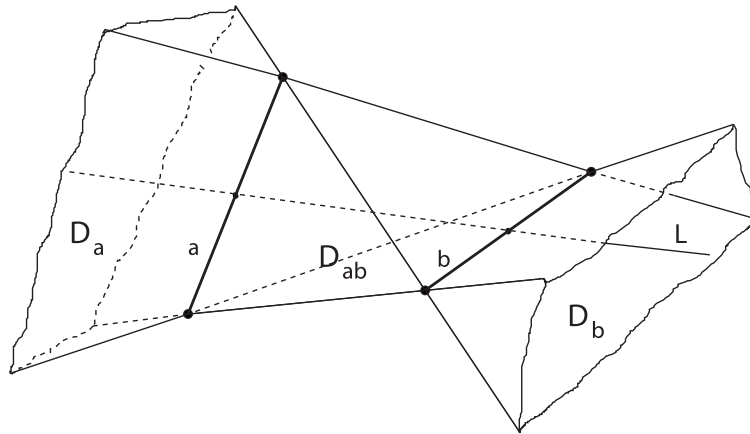


FIGURE 10. The union  $D_{ab} \cup D_a \cup D_b$  is composed of straight lines  $L$ .

Since the piecewise linear function  $g_{\psi,T}$  is convex and  $g_{\psi,T}(a \cap L) = g_{\psi,T}(b \cap L) = 0$ , we see that  $g_{\psi,T}$  is nonpositive on the segment  $L \cap D_{ab}$  and nonnegative on the rays  $L \cap D_a$  and  $L \cap D_b$ . This means that  $g_{\psi,T}$  is nonpositive on  $D_{ab}$  and nonnegative on  $D_a \cup D_b$ . By assumption, the edge  $c$  must intersect all three domains  $D_{ab}, D_a, D_b$ , and the function  $g_{\psi,T}$  must be linear on  $c$ . Therefore,  $g_{\psi,T}|_c = 0$ . By construction,  $g_{\psi,T}|_a = 0$  and  $g_{\psi,T}|_b = 0$ . Hence,  $g_{\psi,T}$  is identically zero. But this contradicts the strict convexity of  $g_{\psi,T}$ .

Now we have the following nonregularity criterion: the triplet  $S$  of edges in  $\mathbb{R}^3$  is nonregular if and only if the edge  $c$  intersects all three domains  $D_{ab}, D_a, D_b$ . This criterion says that the Steiner point  $s$  must be put on the edge  $c = [v_1, v_2]$  in the interval  $(r_1, r_2)$  depicted in Figure 11. More precisely, let  $c_1 = [v_1, s]$  and  $c_2 = [s, v_2]$ . Then the quadruple  $\dot{S} = \{a, b, c_1, c_2\}$  is regular if and only if  $s \in (r_1, r_2)$ . Actually, if  $s \notin (r_1, r_2)$ , then one of the triplets  $\{a, b, c_1\}$  or  $\{a, b, c_2\}$  satisfies the nonregularity criterion. Therefore, the quadruple  $\dot{S}$  is all the more nonregular. Conversely, assume that  $s \in (r_1, r_2)$  and define a function  $\psi: \mathcal{A} \cup \{s\} \rightarrow \mathbb{R}$  in the following way:  $\psi = 0$  at the ends of the edges  $a$  and  $b$ ;  $\psi > 0$  at the ends of the edge  $c$ ;  $\psi < 0$  at the Steiner point  $s$ . Consider an auxiliary piecewise linear function  $f: c \rightarrow \mathbb{R}$  such that  $f(x) = \psi(x)$  for  $x \in \{v_1, s, v_2\}$ . It is easy to verify that the regular triangulation  $T = T(\psi)$  generated by  $\psi$  includes the tetrahedra  $\tau_1 = \text{conv}(a \cup c_1)$  and  $\tau_2 = \text{conv}(b \cup c_2)$  if the graph of  $f$  has the form shown in Figure 12. Since the tetrahedra  $\tau_1$  and  $\tau_2$  are in  $T$ , we see that  $T$  is a regular triangulation of  $\dot{S}$ , and  $\dot{S}$  is the desired regular subdivision of  $S$ .

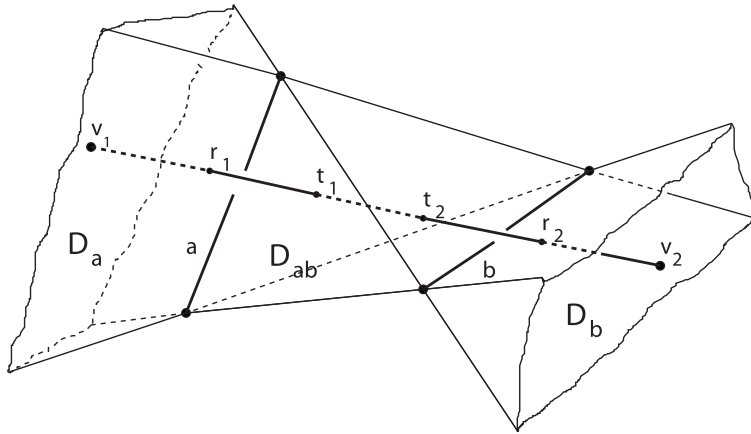


FIGURE 11. The Steiner point must be put in the interval  $(r_1, r_2)$ .

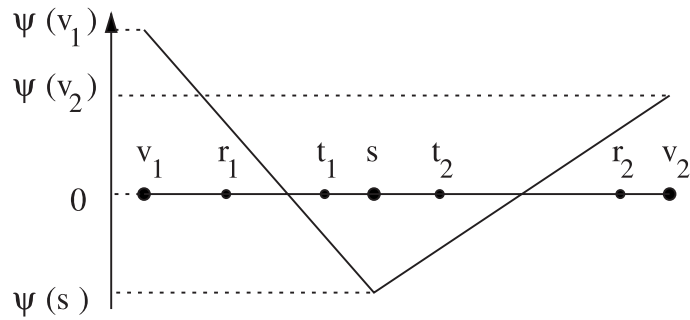


FIGURE 12. The graph must intersect the intervals  $(r_1, t_1)$  and  $(t_2, r_2)$ .

Now, let  $d = 2$ . By the above property, the edge  $c$  must intersect three domains  $D_{ab}, D_a, D_b$  depicted in Figure 9. Clearly, this is possible if and only if the edges  $a$  and  $b$  are arranged as in Figure 9i. Without loss of generality, we assume that the edge  $c$  intersects both of the shaded regions depicted in Figure 13.

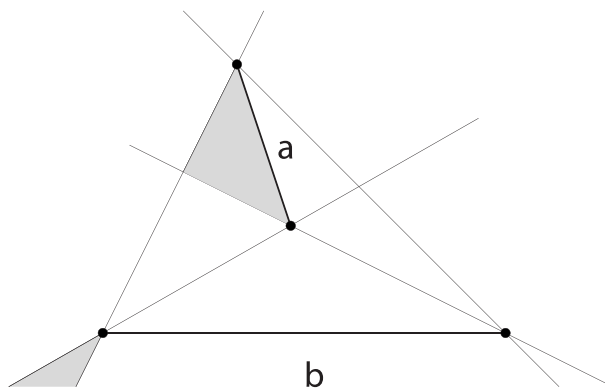


FIGURE 13. The edge  $c$  intersects each of the shaded regions

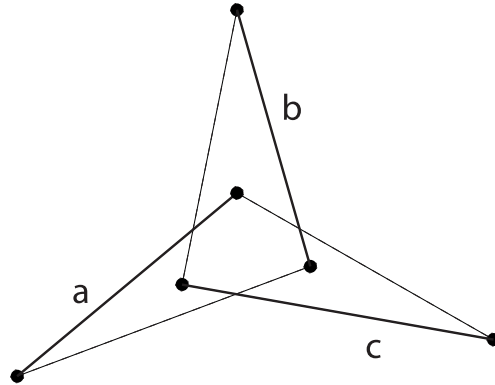


FIGURE 14. Arrangement of edges in a nonregular configuration.

This means that (up to the mirror image) the edges of each planar nonregular configuration  $S = \{a, b, c\}$  must be arranged as in Figure 14. But this is only a necessary condition. Note that the regularity of  $S$  is equivalent to the regularity of the hexagon depicted in Figure 15. We prolong the sides  $a', b', c'$  and construct three auxiliary points  $1'', 2'', 3''$  as shown in Figure 16.

Then the above hexagon (and hence the triplet  $S$ ) is nonregular if and only if the triangles  $\Delta_i = ii'i''$  ( $i = 1, 2, 3$ ) have at least one common point (Figure 17). Actually, if  $\Delta_1 \cap \Delta_2 \cap \Delta_3 = \emptyset$ , in  $\mathbb{R}^2 \times \mathbb{R}$  we can draw three rays that start from the vertices  $1, 2, 3$  and that are projected to  $\mathbb{R}^2$  as shown in Figure 18 (left). Dropping each interior vertex  $1', 2', 3'$  to the corresponding ray, we get the graph of a function  $\psi: \mathcal{A} \rightarrow \mathbb{R}$  such that  $T = T(\psi)$  is a regular triangulation of  $S$ . Conversely, assume that  $\Delta_1 \cap \Delta_2 \cap \Delta_3 \neq \emptyset$ . Arguing by contradiction, suppose there is a regular triangulation  $T = T(\psi)$  of  $S$  generated by a function  $\psi: \mathcal{A} \rightarrow \mathbb{R}$ . This function can be chosen so that  $\psi_i = 0, \psi_{i'} < 0$  for all  $i = 1, 2, 3$ . Then we can recover three similar rays. This time, their projections to  $\mathbb{R}^2$  must have the form shown in Figure 18 (right). But this contradicts the condition  $\psi_{i'} < 0$ .

Finally, we divide the edge  $c$  by a Steiner point  $s$  into a pair of edges  $c_1$  and  $c_2$ . The above criterion of nonregularity in  $\mathbb{R}^2$  says that the triplets  $S_1 = \{a, b, c_1\}$  and  $S_2 = \{a, b, c_2\}$  are both regular if and only if  $s$  is in the interval  $(r_1, r_2)$  shown in

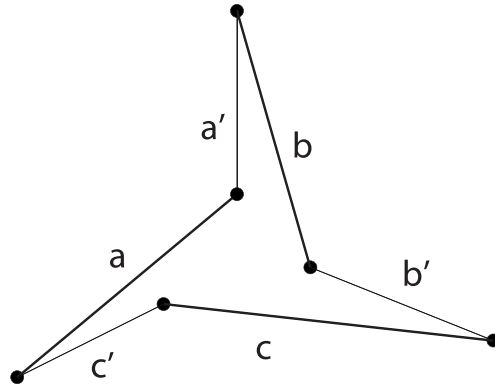


FIGURE 15. An auxiliary hexagon.

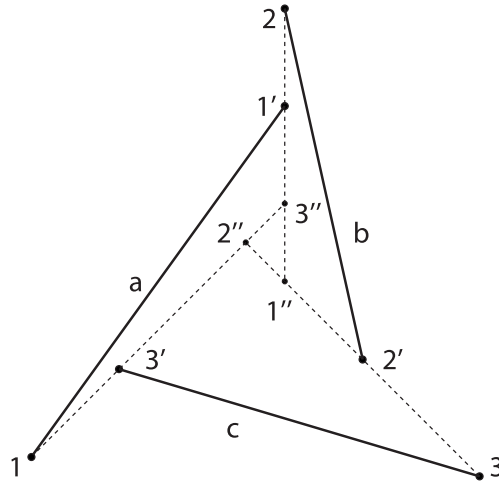


FIGURE 16. Three auxiliary points  $1''$ ,  $2''$ ,  $3''$ .

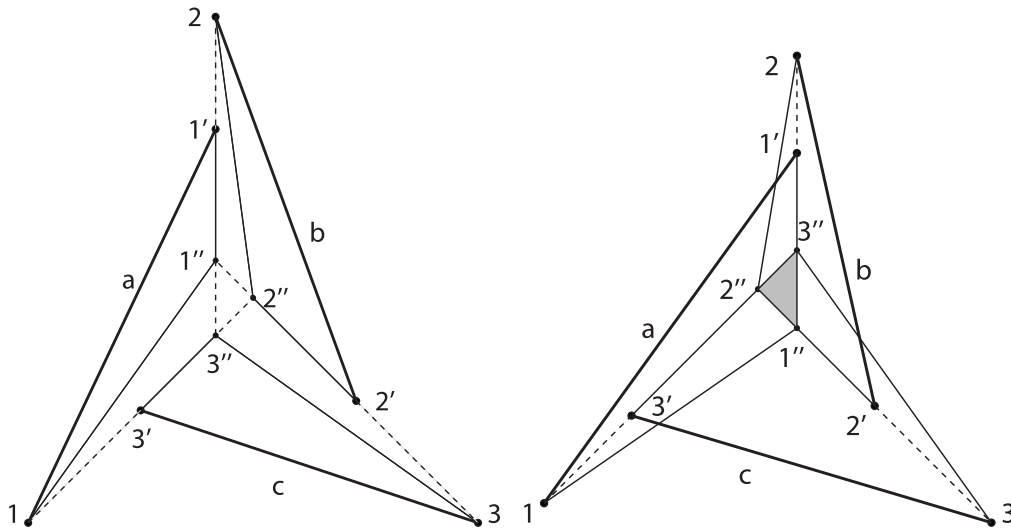


FIGURE 17. Regular and nonregular configurations.

Figure 19. It is easy to check that the quadruple  $\dot{S} = \{a, b, c_1, c_2\}$  is also regular if  $s \in (r_1, r_2)$ . Indeed, there are two possible dispositions of the Steiner point  $s$  with respect to  $\text{conv}(a, b)$ . They are depicted in Figure 20. If  $s \notin \text{conv}(a, b)$ , then a regular triangulation of the quadruple  $\dot{S}$  is obtained from the regular triangulation of the triplet  $S_1$  by adding several (three or two) triangles with the common vertex 3. If  $s \in \text{conv}(a, b)$ , then we can start with a regular triangulation of the triplet  $S_2$ . Obviously, it contains the triangle  $\Delta = s11'$ . Subdividing  $\Delta$  into three triangles with the common vertex  $3'$ , again we get a regular triangulation of the quadruple  $\dot{S}$ .  $\square$

**Example.** The simplest polytope that has no triangulation without Steiner points is Schönhardt's prism [14], obtained from a triangular prism by rotating the upper base with

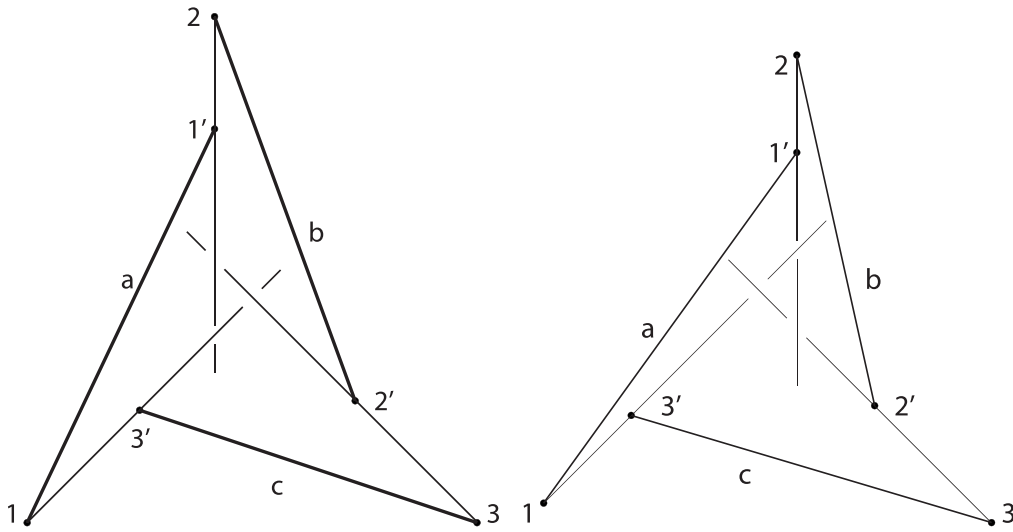


FIGURE 18. Projections of rays to  $\mathbb{R}^2$  (view from above).

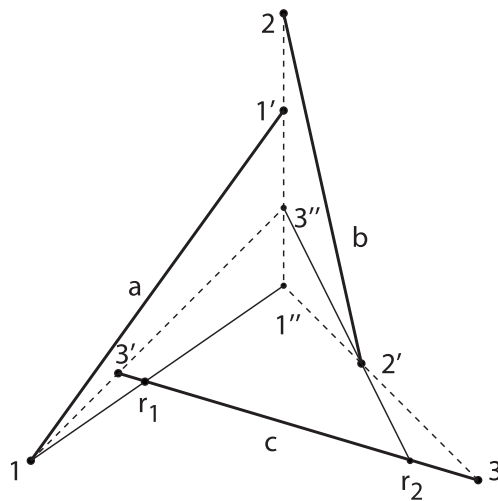


FIGURE 19. The Steiner point must be put in the interval  $(r_1, r_2)$ .

respect to the lower one in such a way that every lateral face folds into two triangles with a cuspidal edge between them (Figure 21). It is easily seen that these three cuspidal edges constitute a unique minimal nonregular configuration. Subdividing any cuspidal edge by one Steiner point, we arrive at a configuration that does allow a regular triangulation of the entire polytope. In Figure 21 an example of such a subdivision is depicted: the edge 26 is subdivided by the point 0, which yields a regular triangulation consisting of six tetrahedra 0123, 0125, 0145, 0134, 0346, 0456.

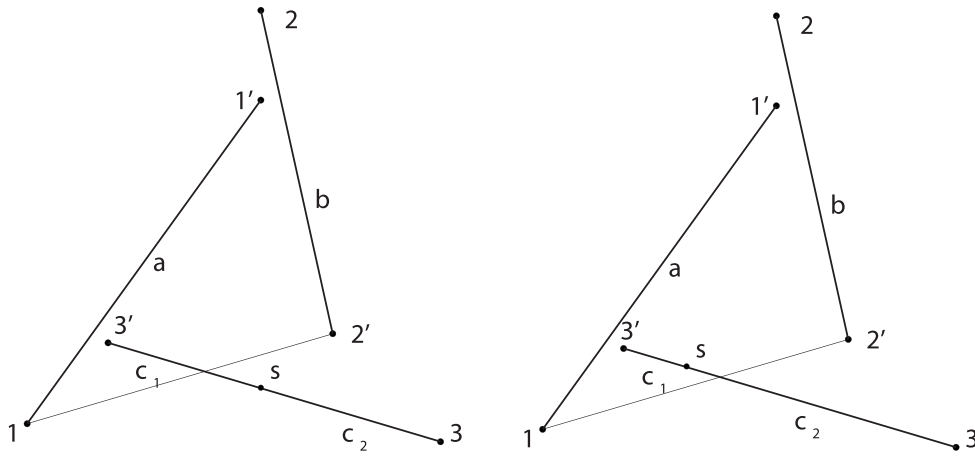


FIGURE 20. Various dispositions of the Steiner point  $s$ .

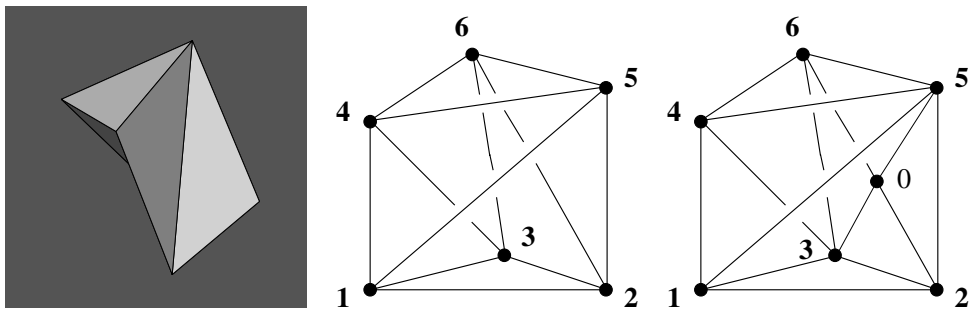


FIGURE 21. Schönhardt's prism: general view, 1-skeleton, and subdivision.

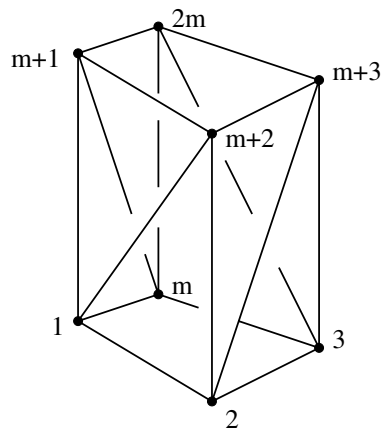


FIGURE 22. Polygonal prism ( $m \geq 3$  is an arbitrary natural number).

Similar facts are also true for the polygonal prism and a cyclically symmetric triangulation of its lateral parallelograms (see Figure 22). Their diagonals

$$S = \{a_m a_{m+1}, a_1 a_{m+2}, \dots, a_{m-1} a_{2m}\}$$

constitute a minimal nonregular configuration, which admits a one-point regular subdivision. Assume the contrary: there is a regular triangulation  $T = T(\psi)$  of  $S$  generated by a function  $\psi: \mathcal{A} \rightarrow \mathbb{R}$ . Then the strict convexity of the function  $g_{\psi, T}$  gives the chain of inequalities

$$\frac{\psi_i + \psi_{m+(i+1)}}{2} = g_{\psi, T}(s_i) < \frac{\psi_{(i+1)} + \psi_{m+i}}{2}, \quad s_i = \frac{a_i + a_{m+(i+1)}}{2}.$$

Setting  $(m+1) = 1$  and summing over all  $i = 1, \dots, m$ , we get a contradiction:

$$\sum_{i=1}^n \psi_i < \sum_{i=1}^n \psi_i.$$

The minimality of  $S$  can be verified with the help of Lemma 3. Dividing any diagonal  $\sigma = a_i a_{m+(i+1)}$  into two parts  $\sigma' = a_i s_i$  and  $\sigma'' = s_i a_{m+(i+1)}$ , we get a one-point regular subdivision  $\hat{S}$ .

Here we would like to mention the recent paper [15] by Rambau, in which he proved that no prism over an  $m$ -gon admits a triangulation that involves a cyclic set  $S$  of diagonals. He also proved that the nonconvex twisted prism over an  $m$ -gon cannot be triangulated without Steiner points. We emphasize that the proof of nontriangulability (in contrast to the proof of nonregularity) is quite nontrivial and is based on the elegant technique exposed in [16].

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DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, UNIVERSITETSKIĬ PROSPEKT 28, STARYĬ PETERHOF, ST. PETERSBURG 198904, RUSSIA  
*E-mail address:* `zvag@pdmi.ras.ru`

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, UNIVERSITETSKIĬ PROSPEKT 28, STARYĬ PETERHOF, ST. PETERSBURG 198904, RUSSIA  
*E-mail address:* `anton@AP9560.spb.edu`

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, UNIVERSITETSKIĬ PROSPEKT 28, STARYĬ PETERHOF, ST. PETERSBURG 198904, RUSSIA  
*E-mail address:* `romanov@YR2008.spb.edu`

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Translated by YU. R. ROMANOVSKIĬ