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BOUNDARY VALUES OF CAUCHY TYPE INTEGRALS

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Abstract. Results by A. G. Poltoratskiĭ and A. B. Aleksandrov about nontangential boundary values of pseudocontinuable H^2-functions on sets of zero Lebesgue measure are used for the study of operators on L^2-spaces on the unit circle. For an arbitrary bounded operator X acting from one such L^2-space to another and having the property that the commutator of it with multiplication by the independent variable is a rank one operator, it is shown that X can be represented as a sum of multiplication by a function and a Cauchy transformation in the sense of angular boundary values.

This article is devoted to the nontangential, or angular, boundary values for certain classes of functions holomorphic in the unit disk of the complex plane. For functions in the Hardy classes H^p, the existence of angular boundary values almost everywhere with respect to Lebesgue measure is a classical fact. However, much less is known what happens if we replace the Lebesgue measure by a singular measure. In the paper [1] by Poltoratskiĭ, a series of results in this context were obtained for functions that are ratios of two Cauchy transforms of complex measures on the unit circle. These results are closely related to the existence problem for boundary values of functions in subspaces K_θ of the Hardy space H^2 (these are precisely the subspaces invariant under the backward shift operator). Actually, in [1] the existence of angular boundary values was proved for Cauchy transforms in the case of the spaces $L^2(\sigma_\alpha)$ for special measures σ_α linked with an inner function θ. These examples will be discussed after Theorem 1. Aleksandrov [2] proved (see Theorem 2 below) that if the operator $K_\theta \to L^0(\mu)$, taking continuous functions in K_θ to their boundary values, is continuous, then nontangential boundary values exist μ-almost everywhere for all functions in K_θ (here μ is a measure on the unit circle, and $L^0(\mu)$ denotes the space of all measurable functions with the topology of convergence in μ-measure). Our main result has the same meaning: the boundedness of a Cauchy transformation yields the existence of angular boundary values. With the help of Aleksandrov's result just mentioned, we shall describe the continuous operators that act from K_θ to the space of μ-measurable functions on the circle and "almost commute" with multiplication by the independent variable. Our main theorem can also be viewed in the framework of scattering theory: in a simple (scalar) case, we show that a wave operator exists also without any assumptions (which are present in the classical results) about the absolute continuity of the spectrum, i.e., we admit that the spectral measure of a unitary operator under consideration may have a singular part.

For Cauchy type integrals we use the notation

$$\mathcal{K}_\alpha(\lambda) = \int \frac{d\alpha(z)}{1-\bar{z}\lambda},$$

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where α is a complex measure on the unit circle $\mathbb{T}$. The angular (nontangential) boundary value at a point z of the unit circle of a function f defined in the unit disk is the limit of $f(\lambda)$ as λ tends to z inside the domain $\{\lambda : |\arg(1-\bar{z}\lambda)| < \varepsilon\}$, where ε is a certain (arbitrary) number in the interval $(0, \pi/2)$, and the symbol arg stands for the value in $(-\pi, \pi]$ of the argument of a complex number. In the paper we usually work with functions on the unit disk and their boundary values from the inside of the unit disk. At one point, we shall work with boundary values both from the inside and from the outside.

The main result of the present paper reads as follows.

Main theorem. *Let μ and ν be two positive Borel measures on the unit circle $\mathbb{T}$, and suppose $X : L^2(\mu) \to L^2(\nu)$ is a bounded linear operator the commutator of which with multiplication by the independent variable z is of rank one*:

$$XM_z - M_z X = (\,\cdot\,, \varphi)\psi, \quad \varphi \in L^2(\mu), \;\; \psi \in L^2(\nu), \tag{1}$$

i.e., $X(zu) - zXu = (u, \varphi)\psi$ for $u \in L^2(\mu)$. Then for every $u \in L^2(\mu)$ the Cauchy type integral

$$\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(\lambda) = \int \frac{\bar{z}u(z)\overline{\varphi(z)}\,d\mu(z)}{1-\bar{z}\lambda}$$

has nontangential boundary values $\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(z)$ (from the inside of the unit disk) at ν-almost all points z such that $\psi(z) \neq 0$. Define an operator $B : L^2(\mu) \to L^2(\nu)$ by

$$(Bu)(z) = \psi(z)\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(z), \quad u \in L^2(\mu) \tag{2}$$

(*we set $(Bu)(z) = 0$ if $\psi(z) = 0$*).

Then:

1) *B is a bounded operator*;

2) *$A = X - B$ commutes with multiplication by z*: *$AM_z = M_z A$*;

3) *$\|A\| \le \|X\|$*;

4) *the restrictions of the singular parts of μ and ν to the sets where $\varphi \neq 0$, $\psi \neq 0$* (*respectively*) *are mutually singular.*

In the proof of the main theorem we essentially use the results of [1, 2].

The well-known description of the operators $A : L^2(\mu) \to L^2(\nu)$ commuting with multiplication by z will be used repeatedly in the paper. Any such operator is multiplication by a function in the following sense. Represent each of the measures μ and ν as the sum of the absolutely continuous and the singular part relative to the other measure: $\mu = \mu_1 + \mu_2$, $\nu = \nu_1 + \nu_2$, where μ_1 and ν_1 are mutually absolutely contionuous, and μ_2, ν_2 are singular relative to ν, μ, respectively. The spaces $L^2(\mu)$ and $L^2(\nu)$ split into direct sums: $L^2(\mu) = L^2(\mu_1) \oplus L^2(\mu_2)$, $L^2(\nu) = L^2(\nu_1) \oplus L^2(\nu_2)$. In the matrix of A corresponding to these decompositions, only the entry representing an operator $L^2(\mu_1) \to L^2(\nu_1)$ may be nonzero; this operator is multiplication by a certain function in the usual sense. A similar statement is true in the general situation where A is an operator intertwining a pair of normal operators N_1 and N_2: $AN_1 = N_2 A$ (see, e.g., [3, §1.5.1]). In our case N_1 and N_2 are operators of multiplication by z on the spaces $L^2(\mu)$ and $L^2(\nu)$, respectively. In particular, if measures μ and ν are mutually singular, then there is no nonzero operator A commuting with multiplication by z.

As a direct consequence of the theorem for the case $\varphi = \bar{z}$, $\psi = 1$, we obtain the following interpretation of the question about the boundedness of the Cauchy transformation relative to a pair of measures on $\mathbb{T}$. It follows that, for any natural definition of the Cauchy transformation (i.e., if the property described in item 1 below is fulfilled), boundedness implies the existence of boundary values. Conditions on pairs of measures for which the Cauchy transformation is bounded in the corresponding pair of L^2-spaces can be found in [4]–[6].

Corollary. *Let μ, ν be two measures on $\mathbb{T}$. The following assertions are equivalent*:

1) *there exists a bounded operator $X : L^2(\mu) \to L^2(\nu)$ such that $XM_z - M_zX = (\,\cdot\,, \bar{z})1$;*

2) *for any function $u \in L^2(\mu)$, the Cauchy type integral $\mathcal{K}_{u\mu}$ has angular boundary values ν-almost everywhere, and the operator $B : L^2(\mu) \to L^2(\nu)$ taking a function u to the boundary values of $\mathcal{K}_{u\mu}$ is bounded.*

Remark 1. An analog of statement 4) of the theorem is obviously not true for absolutely continuous parts of the measures. A typical example is the Riesz projection P_+ on the Hardy space H^2, where $\mu = \nu$ is the Lebesgue measure on the unit circle. We have $P_+M_z - M_zP_+ = (\,\cdot\,, \bar{z})1$, i.e., $\varphi = \bar{z}$, $\psi = 1$; the functions φ and ψ are supported on the same set.

Remark 2. It may be of interest to consider our main theorem in its special case where the spaces $L^2(\mu)$ and $L^2(\nu)$ are finite-dimensional. Let μ and ν be measures on the unit circle whose supports are finite sets $\{p_j\}$ and $\{q_i\}$, respectively. The matrices of operators from $L^2(\mu)$ to $L^2(\nu)$ will be taken relative to the natural orthonormal bases in these spaces. Consider an arbitrary oprator $X : L^2(\mu) \to L^2(\nu)$; let (x_{ij}) be the matrix of it. Clearly, the matrix of $XM_z - M_zX$ has the form $((p_j - q_i)x_{ij})$. In particular, this implies that $XM_z = M_zX$ if and only if $x_{ij} = 0$ whenever $p_j \neq q_i$, i.e., X is an operator of multiplication by a function.

Define operators A and B by their matrices (a_{ij}) and (b_{ij}): if $p_j = q_i$ we set $a_{ij} = x_{ij}$ and $b_{ij} = 0$; if $p_j \neq q_i$ then, by contrast, $a_{ij} = 0$ and $b_{ij} = x_{ij}$. Clearly, $X = A + B$, $AM_z = M_zA$, and the operator B is determined by the operator $XM_z - M_zX$. By construction, every row and every column of the matrix (a_{ij}) contain at most one nonzero entry; hence $\|A\| = \max|a_{ij}| \le \|X\|$.

Now suppose that $XM_z - M_zX = (\,\cdot\,, \varphi)\psi$ is a rank one operator; then the martix of this operator has the form $(\bar{\varphi}_j\psi_i)$, where the φ_j and ψ_i are the coefficients of φ and ψ relative to the natural othonormal bases. If $p_j = q_i$, then the entry with the indices i, j of the matrix of $XM_z - M_zX$ is zero, and therefore at least one of the functions φ, ψ vanishes at $p_j = q_i$. (An analog of this property for singular measures on the circle is contained in assertion 4) of the main theorem.) Thus, the expression on the right-hand side of (2) is well defined (it is assumed to be equal to 0 if $\psi(z) = 0$). It is not difficult to check that the definitions of the operator B via matrix entries and by formula (2) coincide.

Remark 3. The assertions of the theorem are also true under the assumption that X is a continuous operator from $L^2(\mu)$ to the space $L^0(\nu)$ of all measurable functions with the topology of convergence in ν-measure. This immediately follows from the existence of a weight $w(z) > 0$ for which X will act continuously to the weighted space $L^2(w\nu)$. The existence of such a weight w follows, e.g., from the results of chapter III.H in [7] about factorization of operators on L^p-spaces. Obviously, the properties of factorization defined there (see Definitions 4 and 9) are equivalent to the continuous action of the operator under consideration into the "weak" or the usual L^p-space with some weight. Let X be a continuous operator from some Hilbert space H to the space $L^0(\nu)$. Hilbert spaces are spaces of "type 2", and, by Theorem 6, there exists a measurable function γ such that all functions γXh, $h \in H$, belong to the weak space $L_{2,\infty}$. Therefore, X acts into some weighted space L^p with $p < 2$. By Corollary 11, this implies that X acts into a weighted L^2-space.

The operator A in the main theorem will be constructed as a certain limit of the sequence $M_z^nXM_z^{-n}$ as $n \to +\infty$. Hence A can be viewed as a "wave operator" occurring in scattering theory. It is conjectured that an analog of the main theorem exists for arbitrary unitary operators U_1, U_2 on Hilbert spaces H_1, H_2 and an operator $X : H_1 \to H_2$ such that $U_2X - XU_1$ is of trace class. In the classical scattering theory, the existence of wave operators is based upon the existence of boundary values of Cauchy type integrals almost everywhere relative to Lebesgue measure. The proof of our main theorem will follow the same ideas, but the measure will not be assumed absolutely continuous. For completeness, we give a proof of Lemma 1 below (in this lemma, the main theorem is reduced to the question about the existence of boundary values). Thus, in this paper we do not make references to standard facts of scattering theory, and all necessary information is contained in the paper. A presentation of the basics of classical scattering theory for the case of an absolutely continuous spectral measure can be found, e.g., in [8].

Lemma 1. *Suppose $X : L^2(\mu) \to L^2(\nu)$ is a bounded linear operator satisfying* (1) *and such that for any $u \in L^2(\mu)$ the function $\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}$ has angular boundary limits ν-almost everywhere on the set where $\psi \neq 0$. Define an operator $B : L^2(\mu) \to L^2(\nu)$ by formula* (2). *Then B is a bounded operator*; *for $A = X - B$ we have $AM_z = M_zA$ and $\|A\| \le \|X\|$.*

Proof. It is not difficult to check the following relation, where $0 < r < 1$ and the series converge in norm:

$$X - \sum_{n=1}^{\infty} r^n M_z^{n-1}(XM_z - M_zX)M_z^{-n} = (1-r)\sum_{n=0}^{\infty} r^n M_z^n X M_z^{-n}. \tag{3}$$

We compute the sum on the left-hand side of identity (3). For $u \in L^2(\mu)$ and $\xi \in \mathbb{T}$, we get

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty} r^n M_z^{n-1}(XM_z - M_zX)M_z^{-n}u\right)(\xi)\\
&\quad= \sum_{n=1}^{\infty} r^n (z^{-n}u, \varphi)\psi(\xi)\xi^{n-1} = \psi(\xi)\sum_{n=1}^{\infty} r^n \xi^{n-1}\left(\int \bar{z}^n u(z)\overline{\varphi(z)}\,d\mu(z)\right)\\
&\quad= \psi(\xi)\int\left(\sum_{n=1}^{\infty} r^n\xi^{n-1}\bar{z}^n\right)u(z)\overline{\varphi(z)}\,d\mu(z) = r\psi(\xi)\int\frac{\bar{z}u(z)\overline{\varphi(z)}\,d\mu(z)}{1-\bar{z}r\xi}\\
&\quad= r\psi(\xi)\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(r\xi).
\end{aligned}$$

These expressions converge pointwise to $\psi(\xi)\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(\xi)$ as $r \to 1$. The norm of the operator on the right-hand side of (3) does not exceed $(1-r)\sum_{n=0}^{\infty} r^n\|X\| = \|X\|$. Hence a bounded operator A can be defined via the limit expressions of (3); then $\|A\| \le \|X\|$.

By construction, we have $X = A + B$, where the operator B is given by formula (2). Moreover, if $u \in L^2(\mu)$, then

$$\begin{aligned}
&((BM_z - M_zB)u)(\xi) = \psi(\xi)(\mathcal{K}_{u\bar{\varphi}\mu}(\xi) - \xi\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(\xi))\\
&\quad= \psi(\xi)\int\frac{(u(z)\overline{\varphi(z)} - \xi\bar{z}u(z)\overline{\varphi(z)})\,d\mu(z)}{1-\bar{z}\xi} = \psi(\xi)\int u(z)\overline{\varphi(z)}\,d\mu(z).
\end{aligned}$$

This means that $BM_z - M_zB = (\,\cdot\,,\varphi)\psi$, and from relation (1) it follows that $AM_z = M_zA$. □

It can be seen from the proof that it suffices to assume the existence of only radial (instead of angular) boundary values.

Since the functions $\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}$ belong to the Hardy classes H^p with any $p \in (0,1)$, they have angular boundary values almost everyhere relative to the Lebesgue measure m on the unit circle. Thus, we have proved the main theorem in the case where ν is an absolutely continuous measure, which is a classical result of scattering theory (in the special case of a rank one commutator).

Corollary. *The main theorem is true in the case where ν is an absolutely continuous measure.*

By Lemma 1, in order to prove the main theorem, we should establish the existence of angular boundary values for the function $\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}$, $u \in L^2(\mu)$, ν_s-almost everywhere on the set where $\psi \neq 0$ (ν_s denotes the singular part of ν). For this, it suffices to prove that nontangential boundary values of $\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}$ exist ν-almost everywhere in the special case where ν is singular and $\psi(z) \neq 0$ at ν-almost all z. Indeed, if X is the operator from the main theorem, then an analog of relation (1) is also true for the measure $\chi\nu$ and the operator $u \mapsto \chi \cdot Xu$ regarded as a map from $L^2(\mu)$ to $L^2(\chi\nu)$, where χ is the indicator of a set of zero Lebesgue measure on which $\psi \neq 0$ ν_s-almost everywhere. In a similar way, the absolutely continuous and singular parts of μ can also be treated separately. Without loss of generality we may assume that $\varphi \neq 0$ μ-almost everywhere. Thus, *for a complete proof of the main theorem we need to prove the following Lemmas* 2 *and* 3.

Lemma 2. *Let μ, ν be two singular measures on the unit circle, and $X : L^2(\mu) \to L^2(\nu)$ a bounded linear operator satisfying* (1). *Assume that $\varphi(z) \neq 0$ for μ-almost all z, and $\psi(z) \neq 0$ for ν-almost all z. Then the functions $\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}$, $u \in L^2(\mu)$, have nontangential boundary values ν-almost everywhere, and the measures μ and ν are mutually singular.*

Lemma 3. *Suppose that μ is an absolutely continuous measure, $d\mu = w dm$, where $w \in L^1$, $w \geq 0$, ν is a singular measure, $\psi(z) \neq 0$ for ν-almost all z, and $X : L^2(\mu) \to L^2(\nu)$ is a bounded linear operator for which condition* (1) *is fulfilled. Then the functions $\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}$, $u \in L^2(\mu)$, have angular boundary values ν-almost everywhere.*

In both lemmas, μ and ν are mutually singular measures, which implies that there is no nonzero operator intertwining the operators of multiplication by z on the spaces $L^2(\mu)$ and $L^2(\nu)$. Therefore, for the decomposition $X = A + B$ from the main theorem in the special cases corresponding to Lemmas 2 and 3, we have $A = 0$ and $X = B$, and hence the operator X is determined only by the boundary valules of the Cauchy type integrals: $(Xu)(z) = \psi(z)\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(z)$ for ν-almost all z, $u \in L^2(\mu)$.

Before proving these lemmas, we make some remarks in connection with corollaries to our main theorem about different ways of defining boundary values of Cauchy type integrals.

In Lemma 1 the operator A was constructed as a certain limit of the sequence $M_z^n X M_z^{-n}$, where $n \to +\infty$; this gives a representation $X = A + B$, where the operator B is defined by formula (2) via boundary values of Cauchy type integrals (the unit circle is approached from the inside of the unit disk). Similarly, we can consider the limit as $n \to -\infty$, which is the same as the limit of the sequence $M_z^{-n} X M_z^n$ as $n \to +\infty$. Then an analog B' of the operator B may be defined by the same formula (2) involving the boundary functions from the outside of the unit disk. Indeed, this follows from the formulas

$$X + \sum_{n=1}^{\infty} r^n M_z^{-n}(XM_z - M_z X)M_z^{n-1} = (1-r)\sum_{n=0}^{\infty} r^n M_z^{-n} X M_z^n$$

and

$$\left(\sum_{n=1}^{\infty} r^n (M_z^{n-1}u, \varphi) M_z^{-n}\psi\right)(\xi) = -\psi(\xi)\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(\xi/r), \quad u \in L^2(\mu),$$

in the same way as in the proof of Lemma 1. The existence of angular boundary values of the function $\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}$ when the unit circle is approached from the outside of the unit disk is a consequence of the same property for approaching from the inside: we have

$$\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(1/\bar{\lambda}) = -\overline{\lambda\mathcal{K}_{\bar{u}\varphi\mu}(\lambda)},$$

and by considering the function $z\bar{u}\varphi/\bar{\varphi} \in L^2(\mu)$ in place of u, we can see that the function on the right has boundary values from the inside. The norm of the operator $X - B'$ (as well as the norm $\|A\| = \|X - B\|$) does not exceed the norm of X.

It is easily seen that the operator $B - B'$ commutes with multiplication by z; hence it can be viewed as an operator of multiplication by some function. This means that for any $u \in L^2(\mu)$ the function $(B - B')u$ vanishes almost everywhere relative to the part of ν that is singular with respect to μ. For absolutely continuous measures this is equivalent to the well-known fact that boundary values of the Cauchy transform $\mathcal{K}_\alpha$ of a complex measure α on the unit circle from the outside and from the inside coincide m-almost everywhere on the set where the density of α relative to Lebesgue measure equals zero. For a singular measure ν this observation leads us to the following assertion.

Proposition. *Suppose μ, ν are measures on the unit circle, $\varphi \in L^2(\mu)$, and ν is a singular measure. If there exists a function $\psi \in L^2(\nu)$ such that $\psi \neq 0$ ν-almost everywhere, and there exists a bounded operator X for which relation* (1) *is fulfilled, then for any function $u \in L^2(\mu)$ the angular boundary values of the Cauchy type integral $\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(\lambda)$* (*when the circle is approached from the inside and from the outside*) *exist and coincide ν-almost everywhere.*

To prove this, we can assume without loss of generality that $\varphi \neq 0$ μ-almost everywhere; then by assertion 4) of the main theorem μ and ν are mutually singular. Thus, whichever way we define the Cauchy type integral, we obtain the same operator. In particular, $B = B'$, and the boundary values of $\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(\lambda)$, $u \in L^2(\mu)$, from the inside and from the outside must coincide ν-almost everywhere.

Remark. This result remains true if X acts continuously from $L^2(\mu)$ to the space $L^0(\nu)$ and $\psi \in L^0(\nu)$; see Remark 3 to the main theorem.

The proof of Lemma 2 will be based on the results of [1, 2] about boundary values of functions in the subspaces K_θ of the Hardy space H^2.

Let θ be an inner function on the unit circle $\mathbb{T}$, i.e., $\theta \in H^2$ and $|\theta(z)| = 1$ for m-almost all $z \in \mathbb{T}$, where m is Lebesgue measure on $\mathbb{T}$. Then K_θ denotes the orthogonal complement of the subspace θH^2 in H^2:

$$K_\theta = H^2 \ominus \theta H^2.$$

Suppose that $\theta(0) = 0$, take a complex number α, $|\alpha| = 1$, and consider the function $\frac{1+\bar{\alpha}\theta}{1-\bar{\alpha}\theta}$. Its real part is positive, and hence it can be represented in the form

$$\frac{1 + \bar{\alpha}\theta(\lambda)}{1 - \bar{\alpha}\theta(\lambda)} = \int \frac{1 + \bar{z}\lambda}{1 - \bar{z}\lambda}\, d\sigma_\alpha(z) \tag{4}$$

for some positive probability measure σ_α on $\mathbb{T}$, $\sigma_\alpha\mathbb{T} = 1$. Since θ is an inner function, it is easily seen that the measure σ_α is singular relative to the Lebesgue measure. From relation (4) it follows that

$$\mathcal{K}_{\sigma_\alpha}(\lambda) = \int \frac{d\sigma_\alpha(z)}{1 - \bar{z}\lambda} = (1 - \bar{\alpha}\theta(\lambda))^{-1}. \tag{5}$$

Clark [9] constructed a unitary operator $V_\alpha : K_\theta \to L^2(\sigma_\alpha)$ taking the functions in a dense set of K_θ to their boundary values. Poltoratskiĭ [1] proved the following theorem (which holds true without the assumption $\theta(0) = 0$).

Theorem 1. *An arbitrary function $h \in K_\theta$ has nontangential boundary values $h(z)$ for σ_α-almost all z, and the operator V_α takes the function h to the boundary function.*

If $\theta(0) = 0$, then $h \in K_\theta$ can be recovered by the function $f = V_\alpha h$ with the help of the formula

$$h(\lambda) = (1 - \bar{\alpha}\theta(\lambda)) \int \frac{f(z)\, d\sigma_\alpha(z)}{1 - \bar{z}\lambda} = \frac{\mathcal{K}_{f\sigma_\alpha}(\lambda)}{\mathcal{K}_{\sigma_\alpha}(\lambda)} \tag{6}$$

for complex numbers λ in the unit disk. Note that the measure σ_α is supported on the set where the angular boundary values of θ exist and are equal to α. Therefore, the measures σ_α corresponding to different numbers α are mutually singular.

Let α_1, α_2 be two different complex numbers with modulus 1. Theorem 1 and formula (6) yield the existence of a (unique) operator X and imply the assertion of our main theorem in the special case $\mu = \sigma_{\alpha_1}$, $\nu = \sigma_{\alpha_2}$, $\varphi = \bar{z}$, $\psi = 1$; moreover, X is a scalar multiple of a unitary operator. Indeed, from formulas (5) and (6) and Clark's result [9] mentioned above it follows that the operator $u \mapsto (1 - \bar{\alpha}_1\theta)\mathcal{K}_{u\sigma_{\alpha_1}}$, $u \in L^2(\sigma_{\alpha_1})$, maps the space $L^2(\sigma_{\alpha_1})$ unitarily onto K_θ. By Theorem 1, the functions in K_θ have angular boundary values σ_{α_2}-almost everywhere, and the operator taking the functions in K_θ to their boundary functions regarded as elements of $L^2(\sigma_{\alpha_2})$ is unitary. Since the angular boundary values of θ exist and are equal to α_2 σ_{α_2}-almost everywhere, we see that the operator from $L^2(\sigma_{\alpha_1})$ to $L^2(\sigma_{\alpha_2})$ defined by the formula $u \mapsto (1 - \bar{\alpha}_1\alpha_2)\mathcal{K}_{u\sigma_{\alpha_1}}$ (in the sense of angular boundary values) is also unitary. The unitary operator constructed above differs from the operator B defined by formula (2) only by the scalar factor $1 - \bar{\alpha}_1\alpha_2$. Since the measures $\mu = \sigma_{\alpha_1}$ and $\nu = \sigma_{\alpha_2}$ are mutually singular, the only operator $A : L^2(\mu) \to L^2(\nu)$ commuting with multiplication by z is zero. Hence the only operator possessing the required properties is the operator $X = B$.

In what follows we deal with the case of $\alpha = 1$, and instead of σ_1 and V_1 we use the symbols σ and V. The unitary operator $V : K_\theta \to L^2(\sigma)$ will be referred to as the *standard identification* of these spaces.

With the help of Theorem 1, Aleksandrov [2] proved the following statement.

Theorem 2. *Let τ be a finite positive Borel measure on the unit circle $\mathbb{T}$. The set $K_\theta \cap C_A$ of all continuous functions belonging to K_θ is dense in K_θ. Consider the natural map from $K_\theta \cap C_A$ to $L^0(\tau)$ taking the functions in $K_\theta \cap C_A$ to their restriction to the support of τ, where $L^0(\tau)$ is the space of all measurable functions on $\mathbb{T}$ with the topology of convergence in τ-measure. Suppose that this map admits a continuous extension to a map from K_θ to $L^0(\tau)$. Then every function $h \in K_\theta$ has finite nontangential boundary values $h(z)$ τ-almost everywhere on $\mathbb{T}$.*

Now we intend to adapt Theorem 2 to our purposes. Below we prove Theorem 3; it gives a description of the action of an arbitrary continuous linear operator $Y : K_\theta \to L^0(\tau)$ "almost commuting" with multiplication by z, i.e., for which $h, zh \in K_\theta$ implies $Y(zh) = zYh$.

The next lemma describes a property equivalent to the assumptions of Theorem 2 under the extra condition $\theta(0) = 0$. The latter is equivalent to the fact that the constant functions belong to K_θ.

Lemma 4. *Let θ be an inner function such that $\theta(0) = 0$. Assume that $Y : K_\theta \to L^0(\tau)$ is a continuous linear map for which $h, zh \in K_\theta$ implies $Y(zh) = zYh$.*

1) *If $Y1 = 0$, then Y is the zero operator.*

2) *If $Y1 = 1$, then for any function $h \in K_\theta \cap C_A$ we have $(Yh)(z) = h(z)$ for τ-almost all z.*

Proof. If $h \in K_\theta$, then the functions $h - h(0)$ and $(h - h(0))/z$ lie in K_θ. By assumption we obtain

$$Yh = h(0) \cdot (Y1) + zY((h - h(0))/z). \tag{7}$$

Let p_n be the nth Taylor polynomial of the function h at the origin (its degree does not exceed n), and let $g_n = (h - p_n)/z^{n+1}$. Since the space K_θ is invariant under the backward shift $h \mapsto (h - h(0))/z$, it is easily seen that $g_n \in K_\theta$. By induction on n, from relation (7) we obtain the following formula:

$$Yh = p_n \cdot (Y1) + z^{n+1} Y g_n.$$

Indeed, for $n = 0$ this coincides with (7): $p_0 = h(0)$, $g_0 = (h - h(0))/z$. Now we put $Yh = p_n \cdot (Y1) + z^{n+1} Y g_n$ and check this formula for $n + 1$. By relation (7) applied to g_n we obtain $Yg_n = g_n(0) \cdot (Y1) + zY((g_n - g_n(0))/z)$. Therefore,

$$Yh = p_n \cdot (Y1) + z^{n+1} Y g_n = p_n \cdot (Y1) + z^{n+1}(g_n(0) \cdot (Y1) + zY((g_n - g_n(0))/z)).$$

Clearly, $p_n + z^{n+1} g_n(0) = p_{n+1}$ and $(g_n - g_n(0))/z = g_{n+1}$. Thus, $Yh = p_{n+1} \cdot (Y1) + z^{n+2} Y g_{n+1}$, as required.

By construction, $g_n \to 0$ in K_θ. By the assumption about continuity of the map Y we have $Yg_n \to 0$ in $L^0(\tau)$. This yields $z^{n+1} Y g_n \to 0$ in $L^0(\tau)$, which is equivalent to the convergence $p_n \cdot (Y1) \to Yh$. If $Y1 = 0$, this immediately yields $Yh = 0$. Now consider the case where $Y1 = 1$, i.e., $p_n \to Yh$ in $L^0(\tau)$. It is well known that the sequence of Abel means of the Taylor polynomials p_n of a continuous function h converges uniformly to h in the unit disk. Clearly, the limit functions must coincide, i.e., $(Yh)(z) = h(z)$ for τ-almost all z. The lemma is proved. □

Theorem 3. *Take an arbitrary inner function θ. Assume that $Y : K_\theta \to L^0(\tau)$ is a continuous linear map such that if $h, zh \in K_\theta$, then $Y(zh) = zYh$.*

Then there exists a function $\gamma \in L^0(\tau)$ such that every function $h \in K_\theta$ has finite nontangential boundary values $h(z)$ at τ-almost all $z \in \mathbb{T}$ for which $\gamma(z) \neq 0$, and for such z we have $(Yh)(z) = \gamma(z)h(z)$ τ-almost everywhere. For the points z where $\gamma(z) = 0$, the relation $(Yh)(z) = 0$ is true τ-almost everywhere.

If $\theta(0) = 0$, then $\gamma = Y1$.

Proof. First, assume that $\theta(0) = 0$; then $1 \in K_\theta$. The sets where $(Y1)(z) = 0$ and $(Y1)(z) \neq 0$ may be considered separately. This corresponds to the special cases where the function $Y1$ identically equals zero, and $(Y1)(z) \neq 0$ for τ-almost all z, respectively. In the former case, the required fact that Y is the zero operator follows immediately from assertion 1) of Lemma 4. Now suppose that $Y1 \neq 0$ τ-almost everywhere, and consider the map $Y_0 : h \mapsto Yh/Y1$, $h \in K_\theta$. The assumptions of Lemma 4 are fulfilled for the operator Y_0, and $Y_0 1 = 1$. Then by item 2) of Lemma 4, Y_0 sends any continuous function belonging to K_θ to its values τ-almost everywhere. By Theorem 2, this implies that every function $h \in K_\theta$ has nontangential boundary values $h(z)$ for τ-almost all $z \in \mathbb{T}$, and Y_0 takes h to the boundary function. Therefore, $(Yh)(z) = (Y1)(z)h(z)$ for τ-almost all z, as required.

Now let θ be an arbitrary inner function. The general case will be reduced to that already analyzed, where the inner function in question vanishes at the point 0. Define an inner function θ_1 by the formula $\theta_1(\lambda) = \frac{\theta(\lambda) - \theta(0)}{1 - \overline{\theta(0)}\theta(\lambda)}$. It is easily seen that $h \in K_{\theta_1}$ if and only if $(1 - \overline{\theta(0)}\theta)h \in K_\theta$. Consider the map $Y_1 : K_{\theta_1} \to L^0(\tau)$, $Y_1 h = Y((1 - \overline{\theta(0)}\theta)h)$, $h \in K_{\theta_1}$. Since $\theta_1(0) = 0$ and the relation $h, zh \in K_{\theta_1}$ obviously implies $Y_1(zh) = zY_1 h$, from the first part of this proof it follows that there exists a function γ_1 for which $(Y_1 h)(z) = \gamma_1(z)h(z)$ in the sense of angular boundary values. Since $\bar{z}\theta_1 \in K_{\theta_1}$, the function $\bar{z}\theta_1$ has angular boundary values τ-almost everywhere on the set where $\gamma_1 \neq 0$.

Clearly, this is also true for θ and $1-\overline{\theta(0)}\theta$. Since $|\theta(0)|<1$, the function $(1-\overline{\theta(0)}\theta)^{-1}$ is also well defined in the sense of angular boundary values. Thus, the statement of the theorem follows from the definition of the operator $Y1$ with $\gamma=(1-\overline{\theta(0)}\theta)^{-1}\gamma_1$. □

Remark 1. If $Y: K_\theta \to L^0(\tau)$ is a continuous map, then there exists a measurable function w such that $w>0$ τ-almost everywhere and Y maps K_θ continuously into the weighted space $L^2(w\tau)$; see Remark 3 to the main theorem.

Remark 2. Lemma 4 allowed us to deduce the assumptions of Theorem 2 from the condition $Y1=1$ and the hypothesis of Lemma 4 about "commutation" of Y and multiplication by z in K_θ. In the original proof [2] of Theorem 2, two stages can be distinguished. First, the assumption of the theorem that Y sends any continuous function in K_θ to its boundary values, is used to prove that θ is a "divisor" of the function $\mathcal{K}_{\nu-gm}$ on the unit disk, where g is the function that determines the functional $h\mapsto\int(Yh)\,d\nu$, $h\in K_\theta$. The remaining part of the proof in [2] leans upon this statement without further use of the assumptions.

The intermediate property in [2] can be deduced from the assumptions of Lemma 4 in a more direct way, i.e., without reduction to the assumptions of Theorem 2. Indeed, Y can be regarded as an operator from K_θ to $L^2(\nu)$, where $d\nu=wd\tau$; see Remark 1. Since $h(0)=(h,1)$, relation (7) with $Y1=1$ implies the following identity for the adjoint operator: $Y^*(zv)-zY^*v=(v,\bar z)1-(v,Y(\bar z\theta))\theta$, $v\in L^2(\nu)$. Putting $v=(z-\lambda)^{-1}$, where λ is a point of the unit disk, and then taking the values at the point λ, we obtain the desired relation $\mathcal{K}_\nu(\lambda)-g(\lambda)=\theta(\lambda)\mathcal{K}_{\bar z\bar s\nu}(\lambda)$, where $g=Y^*1\in K_\theta$, $s=Y(\bar z\theta)\in L^2(\nu)$.

Finally, arguing along the lines of the rest of the proof in [2], we obtain the existence of angular boundary values as in Theorem 2 without use of Lemma 4, which provides a somewhat different way to deduce the conclusion of Theorem 2 from the assumptions of Lemma 4 for the case where $Y1=1$.

Proof of Lemma 2. If necessary, we reshape the operator $(\,\cdot\,,\varphi)\psi$ so as to have $\|\varphi\|=1$. The map $u\mapsto\bar z u/\varphi$, $u\in L^2(\mu)$, identifies the space $L^2(\mu)$ isometrically with the space $L^2(\sigma)$, where $d\sigma=|\varphi|^2d\mu$. Then the function $\bar z\in L^2(\sigma)$ corresponds to $\varphi\in L^2(\mu)$ and has norm 1. From the latter statement it follows that $\sigma\mathbb{T}=1$. We construct an inner function θ by formula (4) with $\alpha=1$ and consider the standard identification $V: K_\theta\to L^2(\sigma)$, where $\sigma=\sigma_1$. Let $Y: K_\theta\to L^2(\nu)$ be the corresponding transplantation of the operator X: $Yh=X(z\varphi Vh)$, $h\in K_\theta$. Applying relation (1) to $z\varphi Vh$, $h\in K_\theta$, we obtain

$$X(z^2\varphi Vh)-zX(z\varphi Vh)=(z\varphi Vh,\varphi)\psi. \tag{8}$$

Now take a function $h\in K_\theta$ for which also $zh\in K_\theta$. This is equivalent to $(h,\bar z\theta)=0$ in K_θ, or $(Vh,\bar z)=0$ in $L^2(\sigma)$, or $(z\varphi Vh,\varphi)=0$ in $L^2(\mu)$. Since V is the operator of taking the boundary values, we have $zVh=V(zh)$, and formula (8) for such h can be written as $Y(zh)-zYh=0$. Therefore, we can apply Theorem 3 to Y with $\tau=\nu$. The theorem says that for any function $h\in K_\theta$ the angular boundary function $h(z)$ exists ν-almost everywhere on the set where $Y1\neq0$, and Y takes h to $(Y1)(z)h(z)$. We have $(Y(\bar z\theta))(z)=(Y1)(z)\bar z\theta(z)$, whence $zY(\bar z\theta)=\theta\cdot Y1$ (in particular, as was mentioned in the proof of Theorem 3, θ has nontangential boundary values ν-almost everywhere on the set where $Y1\neq0$). From the definition of the operator Y it follows that $Y1=X(z\varphi)$ and $Y(\bar z\theta)=X(z\varphi\bar z)=X\varphi$. Relation (1) applied to the function φ gives $X(z\varphi)-zX\varphi=(\varphi,\varphi)\psi=\psi$. Hence

$$\psi=Y1-zY(\bar z\theta)=Y1-\theta\cdot Y1=(1-\theta)\cdot Y1.$$

Since, by assumption, $\psi\neq0$ ν-almost everywhere, we conclude that $\theta(z)\neq1$ and $(Y1)(z)=\psi(z)/(1-\theta(z))\neq0$ for ν-almost all z. The measure μ is supported on the same set as σ, on which, by construction, $\theta = 1$ in the sense of nontangential boundary values. Thus, μ and ν are mutually singular measures.

The operator V^{-1} can be recovered by formula (6). This formula applied to $h = V^{-1}(\bar{z}u/\varphi)$ yields

$$(V^{-1}(\bar{z}u/\varphi))(\lambda) = (1-\theta(\lambda))\mathcal{K}_{\frac{\bar{z}u}{\varphi}\sigma}(\lambda) = (1-\theta(\lambda))\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(\lambda).$$

This function belongs to K_θ, and, by Theorem 3, it has nontangential boundary values ν-almost everywhere; above, we have shown that the function $1-\theta$ has nonzero angular boundary values ν-almost everywhere. Therefore, $\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}$ also has nontangential boundary values, and the proof of the lemma is complete. □

Proof of Lemma 3. Here we assume that the functions φ and ψ are chosen so that $\|\psi\| = 1$; consider the singular measure σ for which $d\sigma = |\psi|^2 d\nu$. Let θ be the inner function defined by formula (4) with $\alpha = 1$, and let V be the standard identification of the spaces K_θ and $L^2(\sigma)$, $\sigma = \sigma_1$. We define a unitary operator $W : L^2(\nu) \to K_\theta$ by $Wv = V^{-1}(v/\psi)$, $v \in L^2(\nu)$.

Take an operator $X : L^2(\mu) \to L^2(\nu)$ such that $XM_z - M_zX = (\,\cdot\,,\varphi)\psi$. The measures μ and ν are mutually singular, hence there is no nonzero operator $A : L^2(\mu) \to L^2(\nu)$ such that $AM_z = M_zA$. Therefore, in the sum $X = A+B$ in the main theorem for this case we necessarily have $A = 0$. For the adjoint operator X^* we have $M_{\bar{z}}X^* - X^*M_{\bar{z}} = (\,\cdot\,,\psi)\varphi$, whence $X^*M_z - M_zX^* = (\,\cdot\,,\bar{z}\psi)z\varphi$. Therefore, we can apply the already proved part of the theorem to the operator X^*. Since $A = 0$, X^* is a Cauchy transformation of the form (2) with the functions $\bar{z}\psi, z\varphi$ in place of φ, ψ, respectively. Hence for $v \in L^2(\nu)$ at m-almost all points z we have the formula

$$(X^*v)(z) = z\varphi(z)\mathcal{K}_{v\bar{\psi}\nu}(z) = z\varphi(z)\mathcal{K}_{\frac{v}{\psi}\sigma}(z) = z\varphi(z)(1-\theta(z))^{-1}(Wv)(z),$$

where the last identity follows from (6) with $f = v/\psi$. This implies that for any function $h \in K_\theta$, if we set $v = \psi Vh = W^{-1}h$, then $\frac{\varphi h}{1-\theta} = \bar{z}X^*v \in L^2(\mu)$. For $h = 1$ we obtain $\frac{\varphi}{1-\theta} \in L^2(\mu)$.

Take $u \in L^2(\mu)$. We have

$$(Xu, v) = (u, X^*v) = \int u(z)\frac{\overline{z\varphi(z)}}{1-\overline{\theta(z)}}\overline{(Wv)(z)}w(z)\,dm(z).$$

Since the reproducing kernel k_λ in K_θ has the form $k_\lambda(z) = \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\bar{\lambda}z}$, $|\lambda| < 1$, and W is a unitary operator, for $v = W^*k_\lambda$ this implies

$$\begin{aligned}(WXu)(\lambda) &= (WXu, k_\lambda) = (Xu, W^*k_\lambda)\\ &= \int u(z)\frac{\overline{z\varphi(z)}}{1-\overline{\theta(z)}}\overline{k_\lambda(z)}w(z)\,dm(z)\\ &= \int u(z)\frac{\overline{z\varphi(z)}}{1-\overline{\theta(z)}}\frac{1-\theta(\lambda)\overline{\theta(z)}}{1-\bar{z}\lambda}w(z)\,dm(z)\\ &= \int \frac{1-\theta(\lambda)\overline{\theta(z)}}{1-\overline{\theta(z)}}\frac{\bar{z}u(z)\overline{\varphi(z)}w(z)\,dm(z)}{1-\bar{z}\lambda}.\end{aligned} \tag{9}$$

Since $WXu \in K_\theta$, angular boundary limits of this expression exist σ-almost everywhere by Theorem 1.

Define $f = \frac{\bar{z}u\bar{\varphi}w}{1-\theta}$. Since $u, \frac{\varphi}{1-\theta} \in L^2(\mu)$ and $d\mu = wdm$, we obtain $f \in L^1$. Since σ is a singular measure and the complex measure fm is singular with respect to σ, we can apply Corollary 1 to Theorem 2.7 of [1]. It says that the function $(1-\theta)\mathcal{K}_{fm} = \mathcal{K}_{fm}/\mathcal{K}_\sigma$ has zero nontangential boundary values σ-almost everywhere. We have

$$(1-\theta(\lambda))\mathcal{K}_{fm}(\lambda) = (1-\theta(\lambda)) \int \frac{f(z)\,dm(z)}{1-\bar{z}\lambda} = \int \frac{1-\theta(\lambda)}{1-\theta(z)} \frac{\bar{z}u(z)\overline{\varphi(z)}w(z)\,dm(z)}{1-\bar{z}\lambda}.$$

Therefore, combining this with (9), we conclude that the sum

$$\begin{aligned}\int \frac{1-\theta(\lambda)\overline{\theta(z)}}{1-\overline{\theta(z)}} \frac{\bar{z}u(z)\overline{\varphi(z)}w(z)\,dm(z)}{1-\bar{z}\lambda} &+ \int \frac{1-\theta(\lambda)}{1-\theta(z)} \frac{\bar{z}u(z)\overline{\varphi(z)}w(z)\,dm(z)}{1-\bar{z}\lambda} \\ &= \int \left(\frac{1-\theta(\lambda)\overline{\theta(z)}}{1-\overline{\theta(z)}} + \frac{1-\theta(\lambda)}{1-\theta(z)} \right) \frac{\bar{z}u(z)\overline{\varphi(z)}w(z)\,dm(z)}{1-\bar{z}\lambda} \\ &= \int \frac{\bar{z}u(z)\overline{\varphi(z)}w(z)\,dm(z)}{1-\bar{z}\lambda} = \mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(\lambda)\end{aligned}$$

has angular boundary values σ-almost everywhere, or, equivalently, ν-almost everywhere. The lemma is proved, and thus the proof of the main theorem is complete. □

In conclusion, consider a similar question for measures on the complex plane $\mathbb{C}$. Let μ, ν be two measures on $\mathbb{C}$ with compact supports, and let $X : L^2(\mu) \to L^2(\nu)$ be a bounded linear operator for which relation (1) is fulfilled. It is natural to try to construct a decomposition $X = A + B$, where A commutes with multiplication by z, and B is a Cauchy transformation in some sense. Operators commuting with multiplication by z are operators of multiplication by functions in the same sense as was said before about the case of measures on the circle. For measures that are finite linear combinations of point masses, all that was said in Remark 2 to the main theorem remains true. The operator $B : L^2(\mu) \to L^2(\nu)$ is then given by the formula

$$(Bu)(\xi) = \psi(\xi) \int \frac{u(z)\overline{\varphi(z)}\,d\mu(z)}{z-\xi}, \quad u \in L^2(\mu). \tag{10}$$

As in formula (2), if $\psi(\xi) = 0$, then $(Bu)(\xi) = 0$. Formula (2) is a special case of (10) for measures concentrated on the unit circle ($|\xi| = |z| = 1$).

To extend the main theorem to the case of measures on the complex plane, we must give a meaning to the expression (10) in the case of arbitrary measures, which would allow us to define the operator B. As approximants of the kernel $\frac{1}{z-\xi}$ of the integral operator, we may try to take, for instance, the kernels $\frac{1}{z-\xi} \cdot \frac{|z-\xi|^2}{|z-\xi|^2+\varepsilon}$, $\varepsilon > 0$, and consider passage to the limit as $\varepsilon \to 0$.

Conjecture. *Assume that for a bounded operator $X : L^2(\mu) \to L^2(\nu)$ relation* (1) *is fulfilled. Then for any function $u \in L^2(\mu)$ the expressions*

$$(B_\varepsilon u)(\xi) = \int \frac{u(z)\overline{\varphi(z)}}{z-\xi} \cdot \frac{|z-\xi|^2}{|z-\xi|^2+\varepsilon}\,d\mu(z)$$

have a limit as $\varepsilon \to 0$ for ν-almost all ξ with $\psi(\xi) \neq 0$.

With the help of Bessel functions, we can construct an analog of formula (3), which implies an estimate for the norms of the operators $A_\varepsilon = X - B_\varepsilon$: $\|A_\varepsilon\| \le \|X\|$. If the conjecture is true, then we can take the limit of the operators B_ε as $\varepsilon \to 0$ for the role of B, and then set $A = X - B$.

This conjecture is true in the case where the measures μ, ν are supported on the unit circle. Indeed, is is not difficult to check that if $|\xi| = |z| = 1$, then

$$\frac{1}{z-\xi} \cdot \frac{|z-\xi|^2}{|z-\xi|^2+\varepsilon} = \frac{r}{1+r} \cdot \frac{\bar{z}}{1-\bar{z}r\xi} + \frac{1}{1+r} \cdot \frac{\bar{z}}{1-\bar{z}\xi/r},$$

where r is any of the two roots of the equation $(1-r)(1/r-1) = \varepsilon$, whence

$$\int \frac{u(z)\overline{\varphi(z)}}{z-\xi} \cdot \frac{|z-\xi|^2}{|z-\xi|^2+\varepsilon}\, d\mu(z) = \frac{r}{1+r}\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(r\xi) + \frac{1}{1+r}\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}(\xi/r).$$

If $\varepsilon > 0$, $\varepsilon \to 0$, then r is a positive number, $r \to 1$. Hence the limit of the latter expression is half the sum of the radial boundary values of the function $\mathcal{K}_{\bar{z}u\bar{\varphi}\mu}$ from the outside and from the inside of the unit disk, the existence of which was proved in this paper.

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