

**ASYMPTOTIC DEPENDENCE OF THE EXTREME EIGENVALUES  
OF TRUNCATED TOEPLITZ MATRICES ON THE RATE AT WHICH  
THE SYMBOL ATTAINS ITS EXTREMUM**

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In this paper we investigate the asymptotic behavior of extreme eigenvalues for truncated Toeplitz  $(N \times N)$ -matrices  $T_N$  with real-valued symbol  $a \in L_\infty(\mathbb{S})$  such that the function  $a(t) - \inf_{t \in \mathbb{S}} a(t)$  has finitely many zeros on the unit circle  $\mathbb{S}$  of the complex plane, and the maximal order of a zero equals  $\nu > 0$ .

Our main results are Theorems 2.1 and 2.2 saying that, as  $N \rightarrow +\infty$ , the lowest eigenvalues of the matrices  $T_N$  tend to  $\inf_{t \in \mathbb{S}} a(t)$  with the rate  $1/N^\nu$ .

Our research can be viewed as a response to the paper [1], where a similar result was obtained only for even  $\nu$ , and in some cases estimates were obtained in terms of the even numbers neighboring with  $\nu$ . Those results were based on the fundamental results of [2, 3], which can also be found in the monograph [4]. Our method does not employ those results, and it allows us to obtain upper and lower estimates of sharp order for any  $\nu > 0$ .

§1. NOTATION

We shall use the following notation:  $\mathbb{N}$  and  $\mathbb{Z}$  are the sets of all natural numbers and all integers, respectively;  $\mathbb{R}$  and  $\mathbb{C}$  are the fields of real and complex numbers, respectively;  $\mathbf{i}$  is the imaginary unit,  $\mathbf{e}$  is the base of the natural logarithm;  $[a, b]_{\mathbb{Z}} = \{c \in \mathbb{Z} : a \leq c \leq b\}$  is a segment of integers;  $\mathbb{S} = \{t \in \mathbb{C} : |t| = 1\}$  is the unit circle on the complex plane;  $s$  is Lebesgue measure on  $\mathbb{S}$ .

If  $x \in \mathbb{R}$ , then  $[x] = \sup_{y \in \mathbb{Z}, y \leq x} y$  is the integral part of  $x$ , and  $\lceil x \rceil = \inf_{y \in \mathbb{Z}, y \geq x} y$ .

For brevity, we write  $L_2$ ,  $L_\infty$ , and  $l_2$  instead of  $L_2(\mathbb{S})$ ,  $L_\infty(\mathbb{S})$ , and  $l_2(\mathbb{Z})$ , respectively.

By  $\Lambda$  we denote the Laurent transformation acting from  $L_2$  to  $l_2$ , which sends a function  $f$  to the sequence of its Laurent–Fourier coefficients; the latter will be denoted by  $f_n$ ,  $n \in \mathbb{Z}$ :

$$f_n = \frac{1}{2\pi} \int_{t \in \mathbb{S}} f(t)t^{-n} ds.$$

It is well known that the operator  $\Lambda$  is invertible and  $\|f\|_{L_2} = \sqrt{2\pi} \|\Lambda f\|_{l_2}$ .

For any measurable set  $X \subset \mathbb{S}$ , we introduce the following seminorm for functions  $f \in L_2$ :

$$\|f\|_X = \sqrt{\int_X |f|^2 ds}.$$

The space of polynomials on  $\mathbb{S}$  with complex coefficients and of degree at most  $m \in \mathbb{N}$  will be denoted by  $\mathcal{P}_m$ .

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2000 *Mathematics Subject Classification.* Primary 47B35.

*Key words and phrases.* Toeplitz matrix, eigenvalue asymptotics.

For  $k, l \in \mathbb{Z}$ ,  $k \leq l$ , we introduce the restriction operator  $P_k^l : l_2 \rightarrow \mathbb{C}^{l-k+1}$ ,  $P_k^l a = a|_{[k,l]_{\mathbb{Z}}}$ .

**Definition.** Suppose  $X \subset \mathbb{S}$ ,  $f : X \rightarrow \mathbb{R}$ . We say that  $f$  has a zero of order  $\nu > 0$  at a point  $t_0 \in \mathbb{S}$  if there exists a neighborhood  $U$  of  $t_0$  and positive numbers  $C_1$  and  $C_2$  such that

$$C_1 \leq \frac{|f(t)|}{|t - t_0|^\nu} \leq C_2$$

for every  $t \in U$ .

## §2. MAIN RESULTS

Suppose  $a : \mathbb{S} \rightarrow \mathbb{R}$ ,  $a \in L_\infty$ ,  $\inf_{t \in \mathbb{S}} a(t) = 0$ . For an arbitrary  $N \in \mathbb{N}$ , let  $T_N$  be the truncated Toeplitz matrix with elements  $(T_N)_{j,k} = a_{j-k}$ ,  $j, k \in [0, N-1]_{\mathbb{Z}}$ . Let  $\lambda_j^{(N)}$ ,  $j \in [1, N]_{\mathbb{Z}}$ , be the full collection of eigenvalues (with regard to multiplicities) of the matrix  $T_N$ , written in ascending order.

In this section we formulate our main results. Their proofs will be given in §§ 3–5.

**Theorem 2.1.** *Suppose that  $a(t)$  has a zero of order  $\nu > 0$  at the point  $t_0 = 1$ , and that for each neighborhood  $U$  of  $t_0$  the inequality  $\inf_{t \in \mathbb{S} \setminus U} a(t) > 0$  is fulfilled. Then for each  $j \in \mathbb{N}$  there exist positive constants  $C_-$  and  $C_+$  such that if  $N \geq j$ , then for the eigenvalue  $\lambda_j^{(N)}$  of the matrix  $T_N$  we have*

$$\frac{C_-}{N^\nu} \leq \lambda_j^{(N)} \leq \frac{C_+}{N^\nu}.$$

**Theorem 2.2.** *Suppose that  $a(t)$  has zeros of order  $\nu_l > 0$  at points  $t_l \in \mathbb{S}$ ,  $l \in [1, n]_{\mathbb{Z}}$ , and that for each collection of neighborhoods  $U_l$  of the points  $t_l$  we have  $\inf_{t \in X} a(t) > 0$ , where  $X = \mathbb{S} \setminus \bigcup_{l=1}^n U_l$ . Denote  $\nu_{\max} = \max_{l \in [1, n]_{\mathbb{Z}}} \nu_l$ . Then for each  $j \in \mathbb{N}$  there exist positive constants  $C_-$  and  $C_+$  such that if  $N \geq j$ , then for the eigenvalue  $\lambda_j^{(N)}$  of the matrix  $T_N$  we have*

$$\frac{C_-}{N^{\nu_{\max}}} \leq \lambda_j^{(N)} \leq \frac{C_+}{N^{\nu_{\max}}}.$$

In §§ 3 and 4 it is assumed that  $a(t)$  satisfies the hypothesis of Theorem 2.1.

## §3. THE UPPER BOUND

For any pair of numbers  $m, \alpha \in \mathbb{N}$ , we define a polynomial  $p_{m,\alpha} \in \mathcal{P}_{m(\alpha-1)}$  as follows:

$$(3.1) \quad p_{m,\alpha}(t) = \frac{C_m(\alpha)}{\alpha^{m-\frac{1}{2}}} (t^{\alpha-1} + t^{\alpha-2} + \dots + 1)^m,$$

where  $C_m(\alpha) > 0$  is a number such that  $\|p_{m,\alpha}\|_{L_2} = \sqrt{2\pi}$ .

**Lemma 3.1.** *For any  $m \in \mathbb{N}$  there exist numbers  $C_1 > 0$  and  $C_2 > 0$  such that  $C_1 \leq C_m(\alpha) \leq C_2$  for all  $\alpha \in \mathbb{N}$ .*

*Proof.* Observing that  $p_{m,\alpha} = \frac{C_m(\alpha)}{\alpha^{m-\frac{1}{2}}}\left(\frac{t^\alpha-1}{t-1}\right)^m$  and that  $2|\theta|/\pi \leq |\sin \theta| \leq |\theta|$  on the segment  $[-\pi/2, \pi/2]$ , we obtain

$$\begin{aligned} \int_{t \in \mathbb{S}} \left| \frac{t^\alpha - 1}{t - 1} \right|^{2m} ds &= \int_{-\pi}^{\pi} \frac{\sin^{2m} \frac{\alpha\theta}{2}}{\sin^{2m} \frac{\theta}{2}} d\theta \\ &= 2 \int_{-\pi/2}^{\pi/2} \frac{\sin^{2m}(\alpha\theta)}{\sin^{2m} \theta} d\theta \leq 2 \int_{-\pi/2}^{\pi/2} \frac{\sin^{2m}(\alpha\theta)}{\left(\frac{2}{\pi}\theta\right)^{2m}} d\theta \\ &= \frac{\pi^{2m} \alpha^{2m-1}}{2^{2m-1}} \int_{-\alpha\pi/2}^{\alpha\pi/2} \frac{\sin^{2m} \theta}{\theta^{2m}} d\theta \leq \frac{\pi^{2m} \alpha^{2m-1}}{2^{2m-1}} \int_{-\infty}^{+\infty} \frac{\sin^{2m} \theta}{\theta^{2m}} d\theta; \\ \int_{t \in \mathbb{S}} \left| \frac{t^\alpha - 1}{t - 1} \right|^{2m} ds &\geq 2\alpha^{2m-1} \int_{-\alpha\pi/2}^{\alpha\pi/2} \frac{\sin^{2m} \theta}{\theta^{2m}} d\theta \\ &\geq 2\alpha^{2m-1} \int_{-\pi/2}^{\pi/2} \frac{\sin^{2m} \theta}{\theta^{2m}} d\theta. \end{aligned}$$

This yields the estimates

$$\begin{aligned} 2C_m^2(\alpha) \int_{-\pi/2}^{\pi/2} \frac{\sin^{2m} \theta}{\theta^{2m}} d\theta &\leq \frac{C_m^2(\alpha)}{\alpha^{2m-1}} \int_{t \in \mathbb{S}} \left| \frac{t^\alpha - 1}{t - 1} \right|^{2m} ds = \|p_{m,\alpha}\|_{L_2}^2 = 2\pi \\ &\leq \frac{\pi^{2m} C_m^2(\alpha)}{2^{2m-1}} \int_{-\infty}^{+\infty} \frac{\sin^{2m} \theta}{\theta^{2m}} d\theta. \end{aligned}$$

Thus,

$$\sqrt{\frac{2^{2m}}{\pi^{2m-1}} \left( \int_{-\infty}^{+\infty} \frac{\sin^{2m} \theta}{\theta^{2m}} d\theta \right)^{-1}} \leq C_m(\alpha) \leq \sqrt{\pi \left( \int_{-\pi/2}^{\pi/2} \frac{\sin^{2m} \theta}{\theta^{2m}} d\theta \right)^{-1}}.$$

The lemma is proved. □

**Lemma 3.2.** *There exists a constant  $C_+ > 0$  such that for all  $N \in \mathbb{N}$  the lowest eigenvalue  $\lambda_1^{(N)}$  of the matrix  $T_N$  satisfies the estimate*

$$\lambda_1^{(N)} \leq \frac{C_+}{N^\nu}.$$

*Proof.* Let  $C > 0$  be a number such that  $a(t) \leq C|t-1|^\nu$  for any  $t \in \mathbb{S}$ . Let  $m = \lceil \frac{\nu+1}{2} \rceil$ ,  $N \in \mathbb{N}$ ,  $\alpha = \lceil \frac{N}{m} \rceil$ . We set  $p = p_{m,\alpha}$ , where the polynomial  $p_{m,\alpha}$  is defined by (3.1). Then  $m(\alpha-1) \leq N-1$  and  $p \in \mathcal{P}_{N-1}$ . Therefore,

$$\begin{aligned} &(T_N P_0^{N-1} \Lambda p, P_0^{N-1} \Lambda p) \\ &= \sum_{n=0}^{N-1} \left( \sum_{k=0}^{N-1} a_{n-k}(\Lambda p)_k \right) \overline{(\Lambda p)_n} = \sum_{n=-\infty}^{+\infty} \left( \sum_{k=-\infty}^{+\infty} a_{n-k}(\Lambda p)_k \right) \overline{(\Lambda p)_n} \\ &= \frac{1}{2\pi} \int_{t \in \mathbb{S}} a(t) |p(t)|^2 ds \leq \frac{CC_m^2(\alpha)}{2\pi\alpha^{2m-1}} \int_{t \in \mathbb{S}} |t-1|^\nu \left| \frac{t^\alpha-1}{t-1} \right|^{2m} ds. \end{aligned}$$

In the same way as in the proof of Lemma 3.1, we obtain

$$\frac{CC_m^2(\alpha)}{2\pi\alpha^{2m-1}} \int_{t \in \mathbb{S}} |t-1|^\nu \left| \frac{t^\alpha-1}{t-1} \right|^{2m} ds \leq \frac{CC_m^2(\alpha)\pi^{2m-1-\nu}}{2^{2(m-\nu)}\alpha^\nu} \int_{-\infty}^{+\infty} \frac{\sin^{2m} \theta}{\theta^{2m-\nu}} d\theta.$$

Using Lemma 3.1, this estimate, and the choice of  $\alpha$ , we see that there exists a constant  $C_+ > 0$  independent of  $N$  and such that

$$(T_N P_0^{N-1} \Lambda p, P_0^{N-1} \Lambda p) \leq \frac{C_+}{N^\nu}.$$

Since  $\|P_0^{N-1} \Lambda p\| = \|\Lambda p\|_{l_2} = 1$ , the lowest eigenvalue of the matrix  $T_N$  does not exceed  $C_+/N^\nu$ .  $\square$

**Lemma 3.3.** *For any  $j \in \mathbb{N}$  there exists a constant  $C_+ > 0$  such that if  $N \geq j$ , then the eigenvalue  $\lambda_j^{(N)}$  of the matrix  $T_N$  satisfies the estimate*

$$\lambda_j^{(N)} \leq \frac{C_+}{N^\nu}.$$

*Proof.* Let  $C, m$  be the same as in the proof of Lemma 3.2. We take  $N \in \mathbb{N}$ ,  $N \geq mj$ ,  $\alpha = \lfloor \frac{N}{mj} \rfloor$ . For  $l \in [1, j]_{\mathbb{Z}}$  we set  $p^{(l)} = t^{[m(\alpha-1)+1](l-1)} p_{m,\alpha}$ , where the polynomial  $p_{m,\alpha}$  is defined by (3.1). It is easily seen that all  $p^{(l)}$  belong to  $\mathcal{P}_{N-1}$ . Observe that for different  $l$  the polynomials  $p^{(l)}$  have no terms of equal degree; consequently, they form an orthogonal basis of a  $j$ -dimensional subspace of  $L_2$ . Let  $\gamma_1, \gamma_2, \dots, \gamma_j \in \mathbb{C}$ ,  $p = \sum_{l=1}^j \gamma_l p^{(l)}$ , and  $\|p\| = \sqrt{2\pi}$ . Omitting the steps made in the proof of Lemma 3.2, we can write

$$\begin{aligned} & (T_N P_0^{N-1} \Lambda p, P_0^{N-1} \Lambda p) \\ &= \frac{1}{2\pi} \int_{t \in \mathbb{S}} a(t) |p(t)|^2 ds = \frac{1}{2\pi} \sum_{l=1}^j |\gamma_l|^2 \int_{t \in \mathbb{S}} a(t) |p^{(l)}(t)|^2 ds \\ &= \frac{1}{2\pi} \int_{t \in \mathbb{S}} a(t) |p_{m,\alpha}(t)|^2 ds \leq \frac{CC_m^2(\alpha) \pi^{2m-1-\nu}}{2^{2(m-\nu)} \alpha^\nu} \int_{-\infty}^{+\infty} \frac{\sin^{2m} \theta}{\theta^{2m-\nu}} d\theta. \end{aligned}$$

Using Lemma 3.1, this estimate, and the choice of  $\alpha$ , we see that there exists a constant  $\tilde{C}_+ > 0$  independent of  $N$  and of the coefficients  $\gamma_1, \gamma_2, \dots, \gamma_j$  and such that

$$(T_N P_0^{N-1} \Lambda p, P_0^{N-1} \Lambda p) \leq \frac{\tilde{C}_+}{N^\nu}.$$

We choose  $C_+ > \tilde{C}_+$  so that this inequality is fulfilled also for all  $N \in [j, mj-1]_{\mathbb{Z}}$ . Since  $\|P_0^{N-1} \Lambda p\| = \|\Lambda p\|_{l_2} = 1$ , we have proved the following statement: for any normalized vector  $v$  in some  $j$ -dimensional subspace of  $\mathbb{C}^N$  we have  $(T_N v, v) \leq C_+/N^\nu$ . This implies that the matrix  $T_N$  has at least  $j$  eigenvalues on the segment  $[0, C_+/N^\nu]$ . In other words, the  $j$ th eigenvalue satisfies the desired estimate.  $\square$

#### §4. THE LOWER BOUND

We start with some auxiliary statements needed in the sequel.

**Proposition 4.1.** *If  $m \in \mathbb{N}$ ,  $p \in \mathcal{P}_{m-1}$ , then*

$$\max_{t \in \mathbb{S}} |p(t)| \leq \sqrt{\frac{m}{2\pi}} \|p\|_{L_2}.$$

*Proof.* Let  $p(t) = p_{m-1} t^{m-1} + \dots + p_0$ ,  $t \in \mathbb{S}$ . We have

$$|p(t)| = \left| \sum_{n=0}^{m-1} p_n t^n \right| \leq \sum_{n=0}^{m-1} |p_n|.$$

The Cauchy–Bunyakovskiĭ inequality yields

$$|p(t)| \leq \sqrt{m} \sqrt{\sum_{n=0}^{m-1} |p_n|^2} = \sqrt{m} \frac{1}{\sqrt{2\pi}} \|p\|_{L_2}.$$

The proposition is proved. □

**Proposition 4.2.** *Suppose  $m \in \mathbb{N}$ ,  $X \subset \mathbb{S}$ ,  $X$  is measurable, and  $s(X) \leq \frac{\pi}{m}$ . Then for each polynomial  $p \in \mathcal{P}_{m-1}$  we have*

$$\frac{1}{\sqrt{2}} \|p\|_{L_2} \leq \|p\|_{\mathbb{S} \setminus X}.$$

*Proof.* By Proposition 4.1,

$$\begin{aligned} \|p\|_{L_2}^2 &= \|p\|_{\mathbb{S} \setminus X}^2 + \int_X |p|^2 ds \leq \|p\|_{\mathbb{S} \setminus X}^2 + s(X) \frac{m}{2\pi} \|p\|_{L_2}^2, \\ \left(1 - s(X) \frac{m}{2\pi}\right) \|p\|_{L_2}^2 &\leq \|p\|_{\mathbb{S} \setminus X}^2, \quad \frac{1}{2} \|p\|_{L_2}^2 \leq \|p\|_{\mathbb{S} \setminus X}^2. \end{aligned} \quad \square$$

**Lemma 4.1.** *There exists a constant  $C_- > 0$  such that, for all  $N \in \mathbb{N}$ , each eigenvalue  $\lambda$  of the matrix  $T_N$  satisfies the estimate*

$$\frac{C_-}{N^\nu} \leq \lambda.$$

*Proof.* Let  $C > 0$  be such that  $a(t) \geq C|t - 1|^\nu$  for any  $t \in \mathbb{S}$ . Suppose  $N \in \mathbb{N}$ ,  $p \in \mathcal{P}_{N-1}$ ,  $\|p\| = \sqrt{2\pi}$ ,  $\delta = \frac{\pi}{2N}$ ,  $X = \{e^{i\theta} : \theta \in [-\delta, \delta]\}$ . Using Proposition 4.2, we obtain the estimate

$$\begin{aligned} (T_N P_0^{N-1} \Lambda p, P_0^{N-1} \Lambda p) &= \frac{1}{2\pi} \int_{t \in \mathbb{S}} a(t) |p(t)|^2 ds \\ &\geq \frac{C}{2\pi} \int_{t \in \mathbb{S} \setminus X} |t - 1|^\nu |p(t)|^2 ds \geq \frac{C}{2\pi} \left(2 \sin \frac{\delta}{2}\right)^\nu \int_{t \in \mathbb{S} \setminus X} |p(t)|^2 ds \\ &\geq \frac{C}{2\pi} \left(\frac{2\delta}{\pi}\right)^\nu \|p\|_{\mathbb{S} \setminus X}^2 \geq \frac{C}{4\pi} \frac{1}{N^\nu} \|p\|_{L_2}^2 = \frac{C_-}{N^\nu}, \end{aligned}$$

where  $C_- = C/2$ . Since  $\|P_0^{N-1} \Lambda p\| = 1$ , the lowest eigenvalue of  $T_N$  is at least  $C_-/N^\nu$ , but then the same is true for all other eigenvalues. □

### §5. PROOF OF THEOREMS 2.1 AND 2.2

Theorem 2.1 follows immediately from Lemmas 3.3 and 4.1.

The proof of Theorem 2.2 differs from that of Theorem 2.1 by technical details only. For obtaining the upper estimate (an analog of Lemma 3.3), it suffices to choose any point where the function  $a(t)$  has zero of order  $\nu_{\max}$ , and shift the polynomials  $p^{(l)}(t)$  so that they attain their maximum at this point. For the lower estimate (an analog of Lemma 4.1), as  $X$  we can take the union of sufficiently small neighborhoods of all zeros of the function  $a(t)$ .

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Received 31/JUL/2003

Translated by V. V. KAPUSTIN