Алгебра и анализ
Том. 16 (2004), вып. 4

St. Petersburg Math. J.
Vol. 16 (2005), No. 4, Pages 719–747
S 1061-0022(05)00876-9
Article electronically published on June 23, 2005

ON MAPS OF A SPHERE TO A SIMPLY CONNECTED SPACE WITH FINITELY GENERATED HOMOTOPY GROUPS

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Abstract. It is proved that the homotopy class of a map of a sphere to a simply connected CW-complex with finitely generated homotopy groups depends polynomially on the induced homomorphism of the groups of zero-dimensional singular chains.

Introduction. Main result

Terminology and notation. We denote $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathcal{P}$ is the set of all primes.

A *pointed set* is a set with a distinguished element; the latter is denoted by $*$. Any Abelian group is a pointed set with $* = 0$. Any pointed space is a pointed set. For a pointed set T, let $\langle T \rangle$ denote the (free) Abelian group with generators 't', $t \in T$, and a single relation '$*$' $= 0$.

A map $f\colon T \to T'$ of pointed sets is said to be *bound* if $f(*) = *$. In this case we introduce the homomorphism

$$\langle f \rangle\colon \langle T \rangle \to \langle T' \rangle, \quad \langle f \rangle(\text{'}t\text{'}) = \text{'}f(t)\text{'}, \ \ t \in T.$$

For $q \in \mathbb{N}_0$ and a pointed space X, let $\Pi_q X$ be the pointed set of all continuous bound maps $a\colon S^q \to X$ ($*(S^q) = \{*\}$).

For a map $a \in \Pi_q X$, $[a] \in \pi_q X$ is its homotopy class. We put

$$\Psi_q X := \operatorname{Hom}(\langle S^q \rangle, \langle X \rangle).$$

By an *equivalence* we mean a weak homotopy equivalence. A space X is *admissible* if there exists a CW-complex Y and an equivalence $h\colon X \to Y$.

1. Theorem. *Suppose $m \in \mathbb{N}$ and X is a simply connected admissible pointed space with finitely generated homotopy groups. For $r \in \mathbb{N}_0$, let $Q_r \subset (\Psi_m X)^{\otimes r}$ be the subgroup generated by the elements $\langle a \rangle^{\otimes r}$, $a \in \Pi_m X$. Then for any sufficiently large $r \in \mathbb{N}_0$ there exists a homomorphism $l\colon Q_r \to \pi_m X$ such that $l(\langle a \rangle^{\otimes r}) = [a]$ for each $a \in \Pi_m X$.*

Discussion. The condition that the homotopy groups are finitely generated seems to be superfluous. We show that the simple connectivity condition is essential (if $m > 1$). (Perhaps, homotopy simplicity would suffice.) We use the action of the fundamental group of a pointed space on its higher homotopy groups.

Let $p \in \Pi_1 X$, $q \in \Pi_m X$, and let $p_0 = * \in \Pi_1 X$, $p_1 = p$. For $r \in \mathbb{N}_0$, we put

$$B = \left(\bigvee_{s=0}^{r} S^1 \right) \vee S^m, \ \ A_{e_0 \dots e_r} = \left(\bigvee_{s=0}^{r} p_{e_s} \right) \vee q\colon B \to X, \ \ e_0, \dots, e_r \in \{0, 1\}.$$

2000 *Mathematics Subject Classification.* Primary 55P15.

Key words and phrases. Homotopy class, CW-complex, pointed set, fundamental group.

Partially supported by the Russian Science Support Foundation and the grant NSh–1914.203.1.

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Suppose $j_0, \dots, j_r \in \Pi_1 B$ and $k \in \Pi_m B$ are canonical embeddings. We choose a map $h \in \Pi_m B$ with $[h] = [j_0] \cdots [j_r][k]$ and put

$$a_{e_0 \dots e_r} = A_{e_0 \dots e_r} \circ h \in \Pi_m X, \quad e_0, \dots, e_r \in \{0, 1\}.$$

It is easily seen that there exist homomorphisms

$$V_0, \dots, V_r, W : \langle B \rangle \to \langle X \rangle$$

such that

$$\langle A_{e_0 \dots e_r} \rangle = e_0 V_0 + \dots + e_r V_r + W, \quad e_0, \dots, e_r \in \{0, 1\}.$$

Put

$$v_s = V_s \circ \langle h \rangle \in \Psi_m X, \quad s = 0, \dots, r, \;\; w = W \circ \langle h \rangle \in \Psi_m X.$$

We have

$$\langle a_{e_0 \dots e_r} \rangle = e_0 v_0 + \dots + e_r v_r + w, \quad e_0, \dots, e_r \in \{0, 1\}.$$

This implies the relation

$$\sum_{e_0, \dots, e_r \in \{0,1\}} (-1)^{e_0 + \dots + e_r} \langle a_{e_0 \dots e_r} \rangle^{\otimes r} = 0.$$

If the required homomorphism exists, then

$$\sum_{e_0, \dots, e_r \in \{0,1\}} (-1)^{e_0 + \dots + e_r} [a_{e_0 \dots e_r}] = 0.$$

It is easily seen that

$$[a_{e_0 \dots e_r}] = [p]^{e_0 + \dots + e_r} [q], \quad e_0, \dots, e_r \in \{0, 1\}.$$

Suppose that p, q are chosen in such a way that $[p][q] = -[q]$ and $[q]$ is of infinite order in $\pi_m X$ (for example, this is possible if m is even and $X = \mathbb{R}P^m$). Then

$$\sum_{e_0, \dots, e_r \in \{0,1\}} (-1)^{e_0 + \dots + e_r} [a_{e_0 \dots e_r}] = 2^{r+1} [q] \neq 0. \qquad \square$$

Outline of the proof of Theorem 1. The proof consists of two parts, "primary" and "rational", in conformity with the structure of $\pi_m X$ (see §14). In the primary part, the space X is replaced by a space with primary finite homotopy groups (§9) and the Serre method is used: the homotopy groups below the mth group are killed gradually, and then the Hurewicz theorem is applied to the mth group (§8). In the rational part, we pass from the space X to its loop space, in which the rational homotopy class of a spheroid is determined by its homology class in view of the Cartan–Serre theorem. (More precisely, instead of the loop space we use a certain model of its multiple suspension, a version of the cobar construction; see §§12, 13.) $\square$

§1. Preliminaries

Main maps. Suppose $q \in \mathbb{N}_0$ and X is a pointed space. The maps

$$J_{q,X} : \Pi_q X \to \Psi_q X, \;\; a \mapsto \langle a \rangle, \quad \text{and} \quad P_{q,X} : \Pi_q X \to \pi_q X, \;\; a \mapsto [a],$$

are called the *main* maps. Sometimes, we do not mention q and X and simply write J and P, or J' and P', etc.

Induced maps. Suppose $q \in \mathbb{N}_0$ and $f : X \to X'$ is a continuous bound map of pointed spaces. Let

$$H_q f : H_q X \to H_q X', \quad \pi_q f : \pi_q X \to \pi_q X';$$
$$\Pi_q f : \Pi_q X \to \Pi_q X', \quad \Psi_q f : \Psi_q X \to \Psi_q X',$$

be the induced maps. (We put $(\Pi_q f)(a) = f \circ a$, $a \in \Pi_q X$, and $(\Psi_q f)(w) = \langle f \rangle \circ w$, $w \in \Psi_q X$.)

More functors. Suppose $q \in \mathbb{N}_0$, and T is a pointed set. We denote by $\Phi_q T$ the pointed set of all bound maps $A : S^q \to T$ $(*(S^q) = \{*\})$. If U is an Abelian group, so is $\Phi_q U$.

For a bound map $f : T \to T'$, we introduce the bound map

$$\Phi_q f : \Phi_q T \to \Phi_q T', \quad A \mapsto f \circ A.$$

Free Abelian groups. For a set T, let $\langle \underline{T} \rangle$ be the Abelian group with free generators '$\underline{t}$', $t \in T$.

We introduce the homomorphism

$$\langle \underline{T} \rangle \to \mathbb{Z}, \quad x \mapsto \overline{x}, \quad \text{`}\overline{t}\text{'} := 1, \quad t \in T.$$

A map $f : T \to T'$ induces the homomorphism

$$\langle \underline{f} \rangle : \langle \underline{T} \rangle \to \langle \underline{T'} \rangle, \quad \langle \underline{f} \rangle(\text{`}\underline{t}\text{'}) := \text{`}\underline{f(t)}\text{'}, \quad t \in T.$$

Reduction modulo q. Suppose $q \in \mathbb{N}$, U is an Abelian group, and $u \in U$. We put $U/q := U \otimes \mathbb{Z}_q$, $u|_q := u \otimes 1 \in U/q$. For a homomorphism $h : U \to V$ of Abelian groups, we introduce the homomorphism

$$h/q : U/q \to V/q, \quad (h/q)(u|_q) = h(u)|_q, \quad u \in U.$$

p-Special Abelian groups. Suppose $p \in \mathcal{P}$. We say that an Abelian group is *p-special* if it is finite and its order is a power of p.

Increasing maps. By an *increasing* map we mean a nonstrictly increasing map.

Simplicial objects and morphisms. For $q \in \mathbb{N}_0$, we put

$$[q] := \{0, \dots, q\}.$$

For $p \in \mathcal{P}$, a *simplicial p-special Abelian group* is a simplicial Abelian group $\mathbf{U}$ such that the groups $\mathbf{U}_q$, $q \in \mathbb{N}_0$, are p-special.

A *simplicial pointed set* and a *simplicial bound map* are a simplicial object and a simplicial morphism (respectively) of the category of pointed sets and bound maps. (Simplicial pointed set = pointed simplicial set; the geometric realization of a simplicial pointed set is a pointed space; the same applies to simplicial bound maps.)

A *simplicial finite set* is a simplicial set $\mathbf{T}$ such that the sets $\mathbf{T}_q$, $q \in \mathbb{N}_0$, are finite. A *simplicial finite pointed set* is understood similarly.

For simplicial pointed sets $\mathbf{T}$ and $\mathbf{T}'$, a simplicial bound map $\mathbf{f} : \mathbf{T} \to \mathbf{T}'$ is called an *embedding* if the maps $\mathbf{f}_q$, $q \in \mathbb{N}_0$, are injective.

Convenient spaces etc. A *convenient space* is the realization of a simplicial countable set. A *convenient map*, a *convenient pair*, etc. are understood accordingly.

Cubes and simplexes. Suppose $r \in \mathbb{N}_0$. We put $I = [0, 1]$, $I^r = I^{\times r}$. We make obvious identifications: $I^p \times I^q = I^{p+q}$ $(p, q \in \mathbb{N}_0)$, $I^1 = I$. Also, we put

$$\Delta^r = \{(z_1, \dots, z_r) \in I^{\times r} : z_1 \le \dots \le z_r\}.$$

For $t = (t_1, \dots, t_r) \in I^r$ we put $t^{\triangle} = (z_1, \dots, z_r) \in \Delta^r$, where $z_s = t_s \dots t_r$, $s = 1, \dots, r$.

Spheres. Suppose $r \in \mathbb{N}_0$. Convention: $S^r = I^r/\partial I^r$. For the projection $I^r \to S^r$ we write $t \mapsto t^\circ$ ($t \in I^r$).

Suspension. Suppose $r \in \mathbb{N}_0$ and X is a pointed space. We put

$$S^r X = I^r \times X/(\partial I^r \times X \cup I^r \times \{*\}).$$

For the projection $I^r \times X \to S^r X$ we write $(t,x) \mapsto t^\circ x$ ($t \in I^r$, $x \in X$). Let $q \in \mathbb{N}$. The map

$$Z^r : \Pi_q X \to \Pi_{q+r} X, \quad Z^r(a)((t,s)^\circ) = t^\circ a(s^\circ), \quad t \in I^r, \ \ s \in I^q,$$

is called the *suspension transformation*. We recall that the suspension homomorphism is defined as follows:

$$z^r : \pi_q X \to \pi_{q+r} X, \quad z^r([a]) = [Z^r(a)], \quad a \in \Pi_q X.$$

Path and loop spaces. For a space X, by ΓX we denote the space of all paths $u : I \to X$. A continuous map $f : X \to X'$ induces the map

$$\Gamma f : \Gamma X \to \Gamma X', \quad (\Gamma f)(u) := f \circ u.$$

For a pointed space X we have $\Omega X \subset \Gamma X$.

Nameless arrows. Suppose X is a pointed space and $q \in \mathbb{N}$. We refer to the bijection

$$D : \Pi_{q+1} X \to \Pi_q \Omega X, \quad D(a)(s^\circ)(t) := a((t,s)^\circ), \quad s \in I^q, \ \ t \in I,$$

and the isomorphism

$$d : \pi_{q+1} X \to \pi_q \Omega X, \quad [a] \mapsto [D(a)], \quad a \in \Pi_{q+1} X,$$

as to the *nameless bijection* and the *nameless homomorphism*.

Weak continuity. A map $f : X \to Y$ of spaces is said to be *weakly continuous* if for any compact Hausdorff space T and any continuous map $k : T \to X$ the map $f \circ k : T \to Y$ is continuous.

§2. Comparison of maps of a set to Abelian groups

In this section we introduce polynomial dependence relations between maps of a set to Abelian groups and study the properties of such relations.

Notation. Suppose T is a set, U is an Abelian group, and $e : T \to U$ is a map. Consider the homomorphism

$$e^+ : \langle \underline{T} \rangle \to U, \quad e^+(‘\underline{t}’) := e(t), \quad t \in T.$$

For $t \in T$, we put

$$t^e = \{x \in \langle \underline{T} \rangle : \overline{x} = 1, e^+(x) = e(t)\}.$$

For $r \in \mathbb{N}_0$, let

$$[\![e]\!]_r \subset \langle \underline{T} \rangle^{\otimes r}$$

be the subgroup generated by the elements $x^{\otimes r}$, $x \in t^e$, $t \in T$.

Definition. Suppose $e : T \to U$ and $f : T \to V$ are maps to Abelian groups. For $r \in \mathbb{N}_0$, we write

$$e \xrightarrow{r}\!\!\triangleright f$$

if there exists a homomorphism $k : [\![e]\!]_r \to V$ such that $k(x^{\otimes r}) = f(t)$ for any $t \in T$ and $x \in t^e$. We write

$$e \longrightarrow\!\!\triangleright f$$

if $e \xrightarrow{r}\!\!\triangleright f$ for some $r \in \mathbb{N}_0$.

2.1. Lemma. *Suppose $e : T \to U$ and $f : T \to V$ are maps to Abelian groups, $r \in \mathbb{N}_0$, and $e \xrightarrow{r}\!\!\triangleright f$. Then $e \xrightarrow{r+1}\!\!\triangleright f$.*

Proof. Since $e \overset{r}{\longrightarrow\!\triangleright} f$, there exists a homomorphism $k\colon [\![e]\!]_r \to V$ such that $k(x^{\otimes r}) = f(t)$ for any $t \in T$ and $x \in t^e$. We introduce the homomorphism

$$L\colon \langle \underline{T} \rangle \otimes [\![e]\!]_r \to V, \quad L(x \otimes z) := \overline{x}k(z).$$

We have

$$\langle \underline{T} \rangle \otimes [\![e]\!]_r \subset \langle \underline{T} \rangle \otimes \langle \underline{T} \rangle^{\otimes r} = \langle \underline{T} \rangle^{\otimes (r+1)}.$$

If $t \in T$ and $x \in t^e$, then $L(x^{\otimes(r+1)}) = \overline{x}k(x^{\otimes r}) = f(t)$, because $\overline{x} = 1$, $k(x^{\otimes r}) = f(t)$. Therefore, $e \overset{r+1}{\longrightarrow\!\triangleright} f$. □

2.2. Lemma. *Suppose U and V are Abelian groups, $e\colon T \to U$ is a map, and $h\colon U \to V$ is a homomorphism. Then* a) $e \overset{1}{\longrightarrow\!\triangleright} h \circ e$; b) *if h is a monomorphism, then $h \circ e \overset{1}{\longrightarrow\!\triangleright} e$.*

Proof. We have $[\![e]\!]_1 \subset \langle \underline{T} \rangle^{\otimes 1} = \langle \underline{T} \rangle$. Let $k : [\![e]\!]_1 \to U$ be the restriction of the homomorphism e^+. For $t \in T$ and $x \in t^e$ we have $k(x^{\otimes 1}) = e^+(x) = e(t)$. (Thus, $e \overset{1}{\longrightarrow\!\triangleright} e$.) The homomorphism $h \circ k\colon [\![e]\!]_1 \to V$ yields $e \overset{1}{\longrightarrow\!\triangleright} h \circ e$. If h is a monomorphism, then $[\![h \circ e]\!]_1 = [\![e]\!]_1$ (this is easy to check) and the homomorphism $k\colon [\![h \circ e]\!]_1 \to U$ yields $h \circ e \overset{1}{\longrightarrow\!\triangleright} e$. □

2.3. Lemma. *Suppose $e\colon T \to U$, $f\colon T \to V$, and $g\colon T \to W$ are maps to Abelian groups, and $r, s \in \mathbb{N}_0$. Suppose $e \overset{r}{\longrightarrow\!\triangleright} f$ and $f \overset{s}{\longrightarrow\!\triangleright} g$. Then $e \overset{rs}{\longrightarrow\!\triangleright} g$.*

Proof. Consider the homomorphism

$$c\colon \langle \underline{T} \rangle \to \mathbb{Z}, \quad c(x) := \overline{x}.$$

We have a homomorphism $c^{\otimes r}\colon \langle \underline{T} \rangle^{\otimes r} \to \mathbb{Z}^{\otimes r} = \mathbb{Z}$. Since $e \overset{r}{\longrightarrow\!\triangleright} f$, there exists a homomorphism $k\colon [\![e]\!]_r \to V$ such that $k(x^{\otimes r}) = f(t)$ for every $t \in T$ and $x \in t^e$. We introduce the homomorphisms

$$h\colon [\![e]\!]_r \to \mathbb{Z} \oplus V, \quad h(z) := (c^{\otimes r}(z), k(z))$$

and

$$F\colon \langle \underline{T} \rangle \to \mathbb{Z} \oplus V, \quad F(x) = (\overline{x}, f^+(x)).$$

For $t \in T$ and $x \in t^e$, we have $h(x^{\otimes r}) = (1, f(t)) = F(‘\underline{t}’)$. Therefore, $\operatorname{im} h \subset \operatorname{im} F$. Since the Abelian group $[\![e]\!]_r$ is free (this is a subgroup of the free Abelian group $\langle \underline{T} \rangle^{\otimes r}$), there exists a homomorphism $b\colon [\![e]\!]_r \to \langle \underline{T} \rangle$ such that $F \circ b = h$. For $t \in T$ and $x \in t^e$ we have $b(x^{\otimes r}) \in t^f$, because $F(b(x^{\otimes r})) = h(x^{\otimes r}) = (1, f(t))$.

Note that

$$[\![e]\!]_r^{\otimes s} \subset (\langle \underline{T} \rangle^{\otimes r})^{\otimes s} = \langle \underline{T} \rangle^{\otimes (rs)}.$$

We have a homomorphism

$$b^{\otimes s}\colon [\![e]\!]_r^{\otimes s} \to \langle \underline{T} \rangle^{\otimes s}.$$

Clearly, $[\![e]\!]_{rs} \subset [\![e]\!]_r^{\otimes s}$. We have $b^{\otimes s}([\![e]\!]_{rs}) \subset [\![f]\!]_s$; indeed, if $t \in T$ and $x \in t^e$, then $b^{\otimes s}(x^{\otimes rs}) = b(x^{\otimes r})^{\otimes s} \in [\![f]\!]_s$ because $b(x^{\otimes r}) \in t^f$. Since $f \overset{s}{\longrightarrow\!\triangleright} g$, there exists a homomorphism $l\colon [\![f]\!]_s \to W$ such that $l(y^{\otimes s}) = g(t)$ for any $t \in T$ and $y \in t^f$. Consider the homomorphism

$$m\colon [\![e]\!]_{rs} \to W, \quad m(Z) = l(b^{\otimes s}(Z)).$$

For $t \in T$ and $x \in t^e$ we have

$$m(x^{\otimes rs}) = l(b^{\otimes s}(x^{\otimes rs})) = l(b(x^{\otimes r})^{\otimes s}) = g(t)$$

because $b(x^{\otimes r}) \in t^f$. Thus, $e \overset{rs}{\longrightarrow\!\triangleright} f$. □

2.4. Lemma. *Suppose $g : Z \to T$ is a map, $e : T \to U$ and $f : T \to V$ are maps to Abelian groups, and $r \in \mathbb{N}_0$. If $e \overset{r}{\longrightarrow\!\triangleright} f$, then $e \circ g \overset{r}{\longrightarrow\!\triangleright} f \circ g$.*

Proof. Note that $\langle\underline{g}\rangle(y) \in g(z)^e$ for $z \in Z$ and $y \in z^{e\circ g}$. We have a homomorphism

$$\langle\underline{g}\rangle^{\otimes r} : \langle\underline{Z}\rangle^{\otimes r} \to \langle\underline{T}\rangle^{\otimes r}.$$

Next, $\langle\underline{g}\rangle^{\otimes r}([\![e \circ g]\!]_r) \subset [\![e]\!]_r$; indeed, for $z \in Z$ and $y \in z^{e\circ g}$ we have $\langle\underline{g}\rangle^{\otimes r}(y^{\otimes r}) = \langle\underline{g}\rangle(y)^{\otimes r} \in [\![e]\!]_r$ because $\langle\underline{g}\rangle(y) \in g(z)^e$. Since $e \xrightarrow{r}\!\!\triangleright f$, there is a homomorphism $k\colon [\![e]\!]_r \to V$ such that $k(x^{\otimes r}) = f(t)$ for each $t \in T$ and $x \in t^e$. Consider the homomorphism

$$l\colon [\![e \circ g]\!]_r \to V, \quad w \mapsto k(\langle\underline{g}\rangle^{\otimes r}(w)).$$

For $z \in Z$ and $y \in z^{e\circ g}$ we can write

$$l(y^{\otimes r}) = k(\langle\underline{g}\rangle^{\otimes r}(y^{\otimes r})) = k(\langle\underline{g}\rangle(y)^{\otimes r}) = f(g(z))$$

because $\langle\underline{g}\rangle(y) \in g(z)^e$. Thus, $e \circ g \xrightarrow{r}\!\!\triangleright f \circ g$. □

2.5. Lemma. *Suppose $e : T \to U$ and $f : T \to V$ are maps to Abelian groups and $r \in \mathbb{N}_0$. If $e|_D \xrightarrow{r}\!\!\triangleright f|_D$ for each finite set $D \subset T$, then $e \xrightarrow{r}\!\!\triangleright f$.*

Proof. We must show that there exists a homomorphism $k\colon [\![e]\!]_r \to V$ such that $k(x^{\otimes r}) = f(t)$ for each $t \in T$ and $x \in t^e$. Take an arbitrary number $n \in \mathbb{N}_0$ and elements $t_i \in T$, $x_i \in t_i^e$, $i = 1, \dots, n$. Let $P \subset \langle\underline{T}\rangle^{\otimes r}$ be the subgroup generated by the elements $x_i^{\otimes r}$, $i = 1, \dots, n$. It suffices to show that there exists a homomorphism $k'\colon P \to V$ such that $k'(x_i^{\otimes r}) = f(t_i)$, $i = 1, \dots, n$. There is a finite set $D \subset T$ such that $t_i \in D$ and $x_i \in \langle\underline{D}\rangle$ ($\subset \langle\underline{T}\rangle$) for each $i = 1, \dots, n$. We have $P \subset \langle\underline{D}\rangle^{\otimes r} \subset \langle\underline{T}\rangle^{\otimes r}$. Since $e|_D \xrightarrow{r}\!\!\triangleright f|_D$, the desired homomorphism k' exists. □

2.6. Lemma. *Suppose $e : T \to U$ and $f_j : T \to V_j$, $j \in J$, are maps to Abelian groups and $r \in \mathbb{N}_0$. Put*

$$f = \prod_{j\in J} f_j : T \to \prod_{j\in J} V_j.$$

If $e \xrightarrow{r}\!\!\triangleright f_j$, $j \in J$, then $e \xrightarrow{r}\!\!\triangleright f$.

2.7. Lemma. *Suppose $e_j : T_j \to U_j$ and $f_j : T_j \to V_j$, $j \in J$, are maps to Abelian groups and $r \in \mathbb{N}_0$. Put*

$$E = (e_j)_{j\in J}\colon \prod_{j\in J} T_j \to \prod_{j\in J} U_j, \quad F = (f_j)_{j\in J}\colon \prod_{j\in J} T_j \to \prod_{j\in J} V_j.$$

If $e_j \xrightarrow{r}\!\!\triangleright f_j$, $j \in J$, then $E \xrightarrow{r}\!\!\triangleright F$.

Proof. We take an arbitrary $i \in J$ and consider the commutative diagram

$$\begin{array}{ccccc}
\prod_{j\in J} U_j & \xleftarrow{E} & \prod_{j\in J} T_j & \xrightarrow{F} & \prod_{j\in J} V_j \\
{\scriptstyle p'}\big\downarrow & & {\scriptstyle p}\big\downarrow & & {\scriptstyle p''}\big\downarrow \\
U_i & \xleftarrow{e_i} & T_i & \xrightarrow{f_i} & V_i,
\end{array}$$

where p, p', and p'' are projections. By Lemma 2.6, it suffices to show that $E \xrightarrow{r}\!\!\triangleright p'' \circ F$. By assertion a) of Lemma 2.2, $E \xrightarrow{1}\!\!\triangleright p' \circ E = e_i \circ p$. Since $e_i \xrightarrow{r}\!\!\triangleright f_i$, Lemma 2.4 implies the relation $e_i \circ p \xrightarrow{r}\!\!\triangleright f_i \circ p = p'' \circ F$. Therefore, $E \xrightarrow{r}\!\!\triangleright p'' \circ F$ by Lemma 2.3. □

2.8. Lemma. *Suppose U is an Abelian group and $r \in \mathbb{N}_0$. If*

$$R\colon U \to U^{\otimes r}, \quad u \mapsto u^{\otimes r},$$

then $\mathrm{id} = \mathrm{id}_U \xrightarrow{r}\!\!\triangleright R$.

Proof. Consider the homomorphism $(\mathrm{id}^+)^{\otimes r}\colon \langle \underline{U}\rangle^{\otimes r} \to U^{\otimes r}$. For $u \in U$ and $x \in u^{\mathrm{id}} \subset \langle \underline{U}\rangle$, we have
$$(\mathrm{id}^+)^{\otimes r}(x^{\otimes r}) = \mathrm{id}^+(x)^{\otimes r} = u^{\otimes r} = R(u).$$
Thus, $\mathrm{id} \xrightarrow{r}\!\!\triangleright R$. □

2.9. Claim. *Suppose $e : T \to U$ and $f : T \to V$ are maps to Abelian groups, and $r \in \mathbb{N}_0$. Suppose $\operatorname{Tors} U = 0$ and $e \xrightarrow{r}\!\!\triangleright f$. Let $Q \subset (\mathbb{Z} \oplus U)^{\otimes r}$ be the subgroup generated by the elements $(1, e(t))^{\otimes r}$, $t \in T$. Then there exists a homomorphism $l\colon Q \to V$ such that $l((1, e(t))^{\otimes r}) = f(t)$ for each $t \in T$.*

Proof. Consider the homomorphism
$$E\colon \langle \underline{T}\rangle \to \mathbb{Z} \oplus U, \quad `\underline{t}' \mapsto (1, e(t)), \ \ t \in T.$$
We have a homomorphism
$$E^{\otimes r}\colon \langle \underline{T}\rangle^{\otimes r} \to (\mathbb{Z} \oplus U)^{\otimes r}.$$
Not that $E^{\otimes r}([\![e]\!]_r) = Q$, because for $t \in T$ and $x \in t^e$ (in particular, for $x = `\underline{t}'$) we have
$$E^{\otimes r}(x^{\otimes r}) = E(x)^{\otimes r} = E(`\underline{t}')^{\otimes r} = (1, e(t))^{\otimes r}.$$
Since $e \xrightarrow{r}\!\!\triangleright f$, there exists a homomorphism $k\colon [\![e]\!]_r \to V$ such that $k(x^{\otimes r}) = f(t)$ for each $t \in T$ and $x \in t^e$. It suffices to show that there exists a homomorphism $l\colon Q \to V$ such that $l(E^{\otimes r}(z)) = k(z)$ for each $z \in [\![e]\!]_r$. Indeed, in this case for $t \in T$ we have
$$l((1, e(t))^{\otimes r}) = l(E(`\underline{t}')^{\otimes r}) = l(E^{\otimes r}(`\underline{t}'^{\otimes r})) = k(`\underline{t}'^{\otimes r}) = f(t),$$
as required.

It suffices to show that $[\![e]\!]_r \cap \ker E^{\otimes r} \subset \ker k$. We take an arbitrary $z \in [\![e]\!]_r \cap \ker E^{\otimes r}$ and show that $z \in \ker k$. Since $z \in [\![e]\!]_r$, there exist numbers $n \in \mathbb{N}_0$, $a_i \in \mathbb{Z}$ and elements $t_i \in T$, $x_i \in t_i^e$, $i = 1, \ldots, n$, such that
$$z = \sum_{i=1}^{n} a_i x_i^{\otimes r}.$$
Let $B \subset \mathbb{Z} \oplus U$ be the subgroup generated by the elements $E(`\underline{t_i}')$, $i = 1, \ldots, n$. It is free because $\operatorname{Tors} U = 0$. Consequently, there exists a homomorphism
$$d\colon B \to \langle \underline{T}\rangle$$
such that $E(d(b)) = b$ for any $b \in B$. Put $y_i = d(E(`\underline{t_i}'))$, $i = 1, \ldots, n$. Then $y_i \in t_i^e$, $i = 1, \ldots, n$. Note that $B^{\otimes r} \subset (\mathbb{Z} \oplus U)^{\otimes r}$. We have a homomorphism $d^{\otimes r}\colon B^{\otimes r} \to \langle \underline{T}\rangle^{\otimes r}$. Since
$$y_i = d(E(`\underline{t_i}')) = d(E(x_i)), \quad i = 1, \ldots, n,$$
it follows that
$$y_i^{\otimes r} = d(E(x_i))^{\otimes r} = d^{\otimes r}(E^{\otimes r}(x_i^{\otimes r})), \quad i = 1, \ldots, n.$$
Thus,
$$\sum_{i=1}^{n} a_i y_i^{\otimes r} = \sum_{i=1}^{n} a_i d^{\otimes r}(E^{\otimes r}(x_i^{\otimes r})) = d^{\otimes r}(E^{\otimes r}(z)) = 0$$
because $z \in \ker E^{\otimes r}$. Since
$$k(x_i^{\otimes r}) = f(t) = k(y_i^{\otimes r}), \quad i = 1, \ldots, n,$$
we see that
$$k(z) = \sum_{i=1}^{n} a_i k(x_i^{\otimes r}) = \sum_{i=1}^{n} a_i k(y_i^{\otimes r}) = 0.$$
□

2.10. Lemma. *Suppose* $e : T \to U$ *and* $f : T \to V$ *are maps to Abelian groups and* $r \in \mathbb{N}_0$. *Suppose* $\operatorname{Tors} U = 0$ *and* $e \xrightarrow{r}\!\!\triangleright f$. *Let*

$$P \subset \bigoplus_{s=0}^{r} U^{\otimes s}$$

be the subgroup generated by the elements $(e(t)^{\otimes s})_{s=0}^{r}$, $t \in T$. *Then there exists a homomorphism* $k\colon P \to V$ *such that* $k((e(t)^{\otimes s})_{s=0}^{r}) = f(t)$ *for any* $t \in T$.

Proof. Let $Q \subset (\mathbb{Z} \oplus U)^{\otimes r}$ be the subgroup generated by the elements $(1, e(t))^{\otimes r}$, $t \in T$. Since $\operatorname{Tors} U = 0$ and $e \xrightarrow{r}\!\!\triangleright f$, Claim 2.9 yield the existence of a homomorphism $l\colon Q \to V$ such that $l((1, e(t))^{\otimes r}) = f(t)$ for any $t \in T$.

Let $i = (1, 0) \in \mathbb{Z} \oplus U$, and let $j\colon U \to \mathbb{Z} \oplus U$ be the canonical embedding. For each $s = 0, \dots, r$ we introduce the homomorphism

$$g_s\colon U^{\otimes s} \to (\mathbb{Z} \oplus U)^{\otimes r},$$

$$g_s(u_1 \otimes \dots \otimes u_s) := \sum_{\substack{t_0, \dots, t_s \in \mathbb{N}_0:\\ t_0 + \dots + t_s = r - s}} i^{\otimes t_0} \otimes j(u_1) \otimes i^{\otimes t_1} \otimes \dots \otimes j(u_s) \otimes i^{\otimes t_s}.$$

Put

$$G = \bigoplus_{s=0}^{r} g_s\colon \bigoplus_{s=0}^{r} U^{\otimes s} \to (\mathbb{Z} \oplus U)^{\otimes r}.$$

It is easily seen that $G((u^{\otimes s})_{s=0}^{r}) = (1, u)^{\otimes r}$ for $u \in U$. In particular, $G((e(t)^{\otimes s})_{s=0}^{r}) = (1, e(t))^{\otimes r}$ for any $t \in T$. Thus, $G(P) = Q$. If $k(z) = l(G(z))$, $z \in P$, then for $t \in T$ we have

$$k((e(t)^{\otimes s})_{s=0}^{r}) = l(G((e(t)^{\otimes s})_{s=0}^{r})) = l((1, e(t))^{\otimes r}) = f(t). \qquad \square$$

2.11. Corollary. *Suppose* T *is a pointed set,* $e : T \to U$ *and* $f : T \to V$ *are bound maps to Abelian groups, and* $r \in \mathbb{N}_0$. *Suppose* $\operatorname{Tors} U = 0$ *and* $e \xrightarrow{r}\!\!\triangleright f$. *Let*

$$P \subset \bigoplus_{s=1}^{r} U^{\otimes s}$$

be the subgroup generated by the elements $(e(t)^{\otimes s})_{s=1}^{r}$, $t \in T$. *Then there exists a homomorphism* $k\colon P \to V$ *such that* $k((e(t)^{\otimes s})_{s=1}^{r}) = f(t)$ *for any* $t \in T$.

Proof. Let

$$P' \subset \bigoplus_{s=0}^{r} U^{\otimes s}$$

be the subgroup generated by the elements $(e(t)^{\otimes s})_{s=0}^{r}$, $t \in T$. Since $\operatorname{Tors} U = 0$ and $e \xrightarrow{r}\!\!\triangleright f$, by Lemma 2.10 there exists a homomorphism $k' : P' \to V$ such that $k'((e(t)^{\otimes s})_{s=0}^{r}) = f(t)$ for any $t \in T$. Let

$$q\colon \bigoplus_{s=0}^{r} U^{\otimes s} \to \bigoplus_{s=1}^{r} U^{\otimes s}$$

be the projection. Clearly, $q(P') = P$. Put

$$i = (1, 0, \dots, 0) \in \bigoplus_{s=0}^{r} U^{\otimes s}.$$

Then $i = (0^{\otimes s})_{s=0}^{r} = (e(*)^{\otimes s})_{s=0}^{r}$. Therefore, $k'(i) = f(*) = 0$. Since i generates $\ker q$, there exists a homomorphism $k\colon P \to V$ such that $k(q(z)) = k'(z)$ for any $z \in P'$. For $t \in T$ we have

$$k((e(t)^{\otimes s})_{s=1}^{r}) = k(q((e(t)^{\otimes s})_{s=0}^{r})) = k'((e(t)^{\otimes s})_{s=0}^{r}) = f(t). \qquad \square$$

§3. Comparison of maps to p-special Abelian groups

Our aim in this section is to prove Lemma 3.5.

Notation. Suppose $q \in \mathbb{Z}$, U is an Abelian group, and $u \in U$. Put

$$\mathbf{1}_{qU}(u) = \begin{cases} 1 & \text{if } u \in qU, \\ 0 & \text{otherwise.} \end{cases}$$

For $q, z \in \mathbb{Z}$ we denote $\mathbf{1}_{(q)}(z) = \mathbf{1}_{q\mathbb{Z}}(z)$.

3.1. Lemma. *Suppose $p \in \mathcal{P}$, $m \in \mathbb{N}$, and $z \in \mathbb{Z}$. Then*

$$\binom{z-1}{p^m - 1} \equiv \mathbf{1}_{(p^m)}(z) \pmod p.$$

Proof. If $z \not\equiv 0 \pmod{p^m}$, the claim follows from Kummer's theorem on binomial coefficients (see [2, Appendix 3]). Otherwise, we use the identity

$$\binom{z-1}{p^m - 1} = \sum_{k=0}^{p^m-1} (-1)^{p^m-1-k} \binom{z}{k}.$$

If $z \equiv 0 \pmod{p^m}$, then

$$\binom{z}{k} \equiv 0 \pmod p, \quad k = 1, \dots, p^m - 1$$

(by Kummer's theorem), which gives what we need. □

3.2. Lemma. *Suppose $p \in \mathcal{P}$, $k \in \mathbb{N}$, and $x, y \in \mathbb{Z}$. Then*

$$x \equiv y \pmod{p^k} \implies x^p \equiv y^p \pmod{p^{k+1}}.$$

Proof. Indeed,

$$x^p - y^p = (x-y)(x^{p-1} + x^{p-2}y + \dots + y^{p-1}) \equiv 0 \pmod{p^{k+1}}$$

because the first factor is divisible by p^k and the second is divisible by p. □

3.3. Corollary. *Suppose $p \in \mathcal{P}$, $n \in \mathbb{N}$, and $x, y \in \mathbb{Z}$. Then*

$$x \equiv y \pmod p \implies x^{p^{n-1}} \equiv y^{p^{n-1}} \pmod{p^n}.$$

3.4. Lemma. *Suppose $p \in \mathcal{P}$, $m, n \in \mathbb{N}$, and $z \in \mathbb{Z}$. Then*

$$\binom{z-1}{p^m - 1}^{p^{n-1}} \equiv \mathbf{1}_{(p^m)}(z) \pmod{p^n}.$$

Proof. This follows from Lemma 3.1 and Corollary 3.3. □

3.5. Lemma. *Suppose $p \in \mathcal{P}$, T is a finite set, and $e : T \to U$ and $f : T \to V$ are maps to p-special Abelian groups. If e is injective, then $e \multimap f$.*

Proof. For some m and n, we have $p^m U = 0$ and $p^n V = 0$. Put $q := p^m - 1$, $r := p^{n-1}$, and $s := \operatorname{card} T$. We assume that $T = \{1, \dots, s\}$. Put

$$E = \{x \in \langle \underline{T} \rangle : \overline{x} = 1\}.$$

For $k \in \mathbb{N}$ we introduce the homomorphism

$$b_k : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Q}, \quad b_k(v, z) := z/k - v.$$

Let

$$b = \bigotimes_{k=1}^{q} b_k : (\mathbb{Z} \oplus \mathbb{Z})^{\otimes q} \to \mathbb{Q}^{\otimes q} = \mathbb{Q}.$$

For $z \in \mathbb{Z}$ we have
$$b((1,z)^{\otimes q}) = \prod_{k=1}^{q} \Big(\frac{z}{k} - 1\Big) = \binom{z-1}{q}.$$
Put
$$B = b^{\otimes r} \colon (\mathbb{Z} \oplus \mathbb{Z})^{\otimes\, qr} = ((\mathbb{Z} \oplus \mathbb{Z})^{\otimes q})^{\otimes r} \to \mathbb{Q}^{\otimes r} = \mathbb{Q}.$$
For $z \in \mathbb{Z}$, Lemma 3.4 implies
$$B((1,z)^{\otimes\, qr}) = \binom{z-1}{q}^{r} \equiv \mathbf{1}_{(p^m)}(z) \pmod{p^n}.$$

For $t \in T$, consider the homomorphism
$$c^t \colon \langle \underline{T} \rangle \to \mathbb{Z}, \quad `\underline{t}' \mapsto 1, \quad `\underline{z}' \mapsto 0, \quad z \in T \setminus \{t\}.$$
For $a \in E$ and $t \in T$ we introduce the homomorphism
$$l^t_a \colon \langle \underline{T} \rangle \to \mathbb{Z} \oplus \mathbb{Z}, \quad x \mapsto (\overline{x}, c^t(x) - c^t(a)\overline{x})$$
and put
$$D^t_a = B \circ (l^t_a)^{\otimes\, qr} \colon \langle \underline{T} \rangle^{\otimes\, qr} \to \mathbb{Q}.$$
If $a \in E$, $t \in T$, and $x \in E$, then
$$\begin{aligned} D^t_a(x^{\otimes\, qr}) &= B((l^t_a)^{\otimes\, qr}(x^{\otimes\, qr})) = B(l^t_a(x)^{\otimes\, qr}) \\ &= B((1, c^t(x) - c^t(a))^{\otimes\, qr}) \in \mathbf{1}_{(p^m)}(c^t(x) - c^t(a)) + p^n\mathbb{Z}. \end{aligned}$$
For $a \in E$, let
$$D_a = \bigotimes_{t=1}^{s} D^t_a \colon \langle \underline{T} \rangle^{\otimes\, qrs} = (\langle \underline{T} \rangle^{\otimes\, qr})^{\otimes s} \to \mathbb{Q}^{\otimes s} = \mathbb{Q}.$$
For $a, x \in E$ we have
$$D_a(x^{\otimes\, qrs}) = \prod_{t=1}^{s} D^t_a(x^{\otimes\, qr}) \in \mathbf{1}_{p^m \langle \underline{T} \rangle}(x - a) + p^n\mathbb{Z}.$$
Let $P \subset \langle \underline{T} \rangle^{\otimes\, qrs}$ be the subgroup generated by the elements $x^{\otimes\, qrs}$, $x \in E$. For $a \in E$ we denote
$$d_a := D_a| \colon P \to \mathbb{Z}$$
$(D_a(P) \subset \mathbb{Z})$. If $a, x \in E$, then
$$d_a(x^{\otimes\, qrs}) \equiv \mathbf{1}_{p^m \langle \underline{T} \rangle}(x - a) \pmod{p^n}.$$

There is a set $A \subset E$ such that for any $x \in E$ there exists a unique $a \in A$ such that $x - a \in p^m \langle \underline{T} \rangle$. The set A is finite.

We introduce the homomorphism
$$K \colon P \to V, \quad K(Z) := \sum_{t \in T,\, a \in A \cap t^e} d_a(Z) f(t),$$
and take an arbitrary $t_0 \in T$. Let $x_0 \in t^e_0$. Since $p^n V = 0$, it follows that
$$K(x_0^{\otimes\, qrs}) = \sum_{t \in T,\, a \in A \cap t^e} d_a(x_0^{\otimes\, qrs}) f(t) = \sum_{t \in T,\, a \in A \cap t^e} \mathbf{1}_{p^m \langle \underline{T} \rangle}(x_0 - a) f(t).$$
There is $a_0 \in A$ such that $x_0 - a_0 \in p^m \langle \underline{T} \rangle$.

Since $a_0 \in A \subset E$, we see that $\overline{a_0} = 1$. Next, we have $p^m U = 0$; consequently, $e^+(a_0) = e^+(x_0) = e(t_0)$. Therefore, $a_0 \in A \cap t^e_0$. In the last-written sum the term with $t = t_0$ and $a = a_0$ is equal to $f(t_0)$, because $x_0 - a_0 \in p^m \langle \underline{T} \rangle$. There are no other terms with $a = a_0$, because if $t \neq t_0$, then $a_0 \notin t^e$ (recall that $e^+(a_0) = e(t_0) \neq e(t)$ since e is injective). The terms with $a \neq a_0$ are zero because $x_0 - a_0 \in p^m \langle \underline{T} \rangle$; consequently,

$x_0 - a \notin p^m\langle \underline{T}\rangle$ (since $a_0, a \in A$ and $a \neq a_0$). Therefore, $K(x_0^{\otimes qrs}) = f(t_0)$. It follows that $e \xrightarrow{qrs}\!\!\triangleright f$. □

§4. The map $z_{\mathbf{T}}$

Notation. Suppose $\mathbf{T}$ is a simplicial set,

$$c\colon \coprod_{q=0}^{\infty} \Delta^q \times \mathbf{T}_q \to |\mathbf{T}|$$

is the natural projection, $q \in \mathbb{N}_0$, and $z \in \Delta^q$. Consider the map

$$z_{\mathbf{T}}: \mathbf{T}_q \to |\mathbf{T}|, \quad t \mapsto z_t := c(z,t).$$

Notation. For $q \in \mathbb{N}_0$ and $i \in [q]$, we put $\tau_i^q = (0,\dots,0,1,\dots,1) \in \Delta^q$ (i zeroes). For $q, r \in \mathbb{N}_0$ and an increasing map $d\colon [r] \to [q]$, we introduce the following affine map:

$$d_*: \Delta^r \to \Delta^q, \quad \tau_j^r \mapsto \tau_{d(j)}^q, \quad j \in [r].$$

4.1. Lemma. *Suppose $z \in \operatorname{Int}\Delta^q$. Then $z_{\mathbf{T}}$ is injective.*

Proof. We take arbitrary $t, t' \in \mathbf{T}_q$ with $z_t = z_{t'}$ and show that $t = t'$. There exist $r \in \mathbb{N}_0$, $w \in \Delta^r$, and increasing maps $d, d'\colon [r] \to [q]$ such that $\mathbf{T}(d)(t) = \mathbf{T}(d')(t')$ and $d_*(w) = d'_*(w) = z$ (see [1, I.2.13]). Put $E = \{j \in [r] : d(j) = d'(j)\}$. It is not difficult to realize that the relation $d_*(w) = d'_*(w)$ implies that w belongs to the convex hull of the vertices τ_j^r, $j \in E$. Since $d_*(w) = z \in \operatorname{Int}\Delta^q$, $d|_E$ is surjective. Thus, there exists an increasing map $c\colon [q] \to [r]$ such that $c([q]) \subset E$, $d \circ c = \mathrm{id}_{[q]}$; consequently, $d' \circ c = \mathrm{id}_{[q]}$. We have

$$t = \mathbf{T}(c)(\mathbf{T}(d)(t)) = \mathbf{T}(c)(\mathbf{T}(d')(t')) = t'. \qquad \square$$

4.2. Lemma. *Suppose $\mathbf{T}$ is a simplicial set and $D \subset |\mathbf{T}|$ is a finite set. Then there exist $q \in \mathbb{N}_0$ and $z \in \operatorname{Int}\Delta^q$ such that $D \subset z_{\mathbf{T}}(\mathbf{T}_q)$.*

Proof. For $m \in \mathbb{N}_0$ and $u = (u_1,\dots,u_m) \in \Delta^m$, we put $\|u\| = \{0, u_1, \dots, u_m, 1\} \subset I$. If $\|u\| \subset \|v\|$ for some $u \in \Delta^m$, $v \in \Delta^n$ ($m, n \in \mathbb{N}_0$), then there exists an increasing map $d\colon [n] \to [m]$ such that $u = d_*(v)$, and $u_t = v_{\mathbf{T}(d)(t)} \in v_{\mathbf{T}}(\mathbf{T}_n)$ for any $t \in \mathbf{T}_m$. Thus, it suffices to choose z with $\|z\|$ sufficiently large. Namely, each point of D is y_t for some $y \in \Delta^p$ and $t \in \mathbf{T}_p$ ($p \in \mathbb{N}_0$), and we require that $\|y\| \subset \|z\|$. □

§5. Simplicial Abelian groups

Commonplaces. Suppose $\mathbf{U}$ is a simplicial Abelian group. Then $|\mathbf{U}|$ is an Abelian group with weakly continuous addition and subtraction. For $q \in \mathbb{N}_0$ and $z \in \Delta^q$, the map $z_{\mathbf{U}}: \mathbf{U}_q \to |\mathbf{U}|$ is a homomorphism. For $q \in \mathbb{N}_0$, the set $\Pi_q|\mathbf{U}|$ is an Abelian group.

5.1. Lemma. *Suppose $m \in \mathbb{N}_0$, and $\mathbf{U}$ is a simplicial Abelian group. Then*

$$J = J_{m,|\mathbf{U}|} \xrightarrow{1}\!\!\triangleright \mathrm{id}_{\Pi_m|\mathbf{U}|}\,.$$

Proof. Let $j : \Pi_m|\mathbf{U}| \to \Phi_m|\mathbf{U}|$ be the inclusion homomorphism. Consider the homomorphisms

$$r: \langle|\mathbf{U}|\rangle \to |\mathbf{U}|, \quad \text{`}v\text{'} \mapsto v, \quad v \in |\mathbf{U}|,$$

and

$$h\colon \Psi_m|\mathbf{U}| \to \Phi_m|\mathbf{U}|, \quad h(w)(z) := r(w(\text{`}z\text{'})), \quad z \in S^m.$$

We have $h \circ J = j$ because

$$h(J(a))(z) = h(\langle a\rangle)(z) = r(\langle a\rangle(\text{`}z\text{'})) = r(\text{`}a(z)\text{'}) = a(z) = j(a)(z)$$

for any $a \in \Pi_m|\mathbf{U}|$ and $z \in S^m$. By assertion a) of Lemma 2.2, $J \xrightarrow{1}\!\!\triangleright h \circ J = j$. By assertion b) of Lemma 2.2, $j \xrightarrow{1}\!\!\triangleright \mathrm{id}$. Therefore, $J \xrightarrow{1}\!\!\triangleright \mathrm{id}$ by Lemma 2.3. □

5.2. Lemma. *Suppose* $m \in \mathbb{N}$ *and* $\mathbf{U}$ *is a simplicial Abelian group. Then the main map* $P\colon \Pi_m|\mathbf{U}| \to \pi_m|\mathbf{U}|$ *is a homomorphism.*

See [3, Lecture 4, Supplement, Proposition 5].

§6. Comparison of simplicial maps

Definition. Suppose $\mathbf{T}$ is a simplicial set, $\mathbf{U}$ and $\mathbf{V}$ are simplicial Abelian groups, and $\mathbf{e} : \mathbf{T} \to \mathbf{U}$ and $\mathbf{f} : \mathbf{T} \to \mathbf{V}$ are simplicial maps. For $r \in \mathbb{N}_0$ we write $\mathbf{e} \overset{r}{\to\!\triangleright} \mathbf{f}$ if $\mathbf{e}_q \overset{r}{\to\!\triangleright} \mathbf{f}_q$ for every $q \in \mathbb{N}_0$. We write $\mathbf{e} \to\!\triangleright \mathbf{f}$ if there exists $r \in \mathbb{N}_0$ such that $\mathbf{e} \overset{r}{\to\!\triangleright} \mathbf{f}$.

6.1. Lemma. *If* $\mathbf{e} \overset{r}{\to\!\triangleright} \mathbf{f}$, *then* $|\mathbf{e}| \overset{r}{\to\!\triangleright} |\mathbf{f}|$.

Proof. Let $D \subset |\mathbf{T}|$ be an arbitrary finite set, and let $j : D \to |\mathbf{T}|$ be the inclusion. By Lemma 4.2, there exist $q \in \mathbb{N}_0$ and $z \in \operatorname{Int} \Delta^q$ such that $D \subset z_{\mathbf{T}}(\mathbf{T}_q)$. There exists a map $s : D \to \mathbf{T}_q$ such that $z_{\mathbf{T}} \circ s = j$. Consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{U}_q & \xleftarrow{\mathbf{e}_q} & \mathbf{T}_q & \xrightarrow{\mathbf{f}_q} & \mathbf{V}_q \\ {\scriptstyle z_{\mathbf{U}}}\downarrow & & {\scriptstyle z_{\mathbf{T}}}\downarrow & & {\scriptstyle z_{\mathbf{V}}}\downarrow \\ |\mathbf{U}| & \xleftarrow{|\mathbf{e}|} & |\mathbf{T}| & \xrightarrow{|\mathbf{f}|} & |\mathbf{V}|. \end{array}$$

The maps $z_{\mathbf{U}}$ and $z_{\mathbf{V}}$ are homomorphisms. We have

$$|\mathbf{e}| \circ z_{\mathbf{T}} = z_{\mathbf{U}} \circ \mathbf{e}_q \overset{1}{\to\!\triangleright} \mathbf{e}_q$$

by assertion b) of Lemma 2.2, because $z_{\mathbf{U}}$ is a monomorphism by Lemma 4.1. By assumption, $\mathbf{e}_q \overset{r}{\to\!\triangleright} \mathbf{f}_q$. By assertion a) of Lemma 2.2, $\mathbf{f}_q \overset{1}{\to\!\triangleright} z_{\mathbf{V}} \circ \mathbf{f}_q = |\mathbf{f}| \circ z_{\mathbf{T}}$. Therefore, $|\mathbf{e}| \circ z_{\mathbf{T}} \overset{r}{\to\!\triangleright} |\mathbf{f}| \circ z_{\mathbf{T}}$ by Lemma 2.3. Lemma 2.4 shows that

$$|\mathbf{e}| \circ j = |\mathbf{e}| \circ z_{\mathbf{T}} \circ s \overset{r}{\to\!\triangleright} |\mathbf{f}| \circ z_{\mathbf{T}} \circ s = |\mathbf{f}| \circ j.$$

Therefore, $|\mathbf{e}| \overset{r}{\to\!\triangleright} |\mathbf{f}|$ by Lemma 2.5. □

6.2. Corollary. *Under the assumptions of Lemma* 6.1, $\Pi_m|\mathbf{e}| \overset{r}{\to\!\triangleright} \Pi_m|\mathbf{f}|$ *for every* $m \in \mathbb{N}_0$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} \Pi_m|\mathbf{U}| & \xleftarrow{\Pi_m|\mathbf{e}|} & \Pi_m|\mathbf{T}| & \xrightarrow{\Pi_m|\mathbf{f}|} & \Pi_m|\mathbf{V}| \\ {\scriptstyle j'}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle j''}\downarrow \\ \Phi_m|\mathbf{U}| & \xleftarrow{\Phi_m|\mathbf{e}|} & \Phi_m|\mathbf{T}| & \xrightarrow{\Phi_m|\mathbf{f}|} & \Phi_m|\mathbf{V}|, \end{array}$$

where j, j', and j'' are inclusions. Clearly, j' and j'' are homomorphisms. By assertion a) of Lemma 2.2,

$$\Pi_m|\mathbf{e}| \overset{1}{\to\!\triangleright} j' \circ \Pi_m|\mathbf{e}| = \Phi_m|\mathbf{e}| \circ j.$$

By Lemma 6.1, we have $|\mathbf{e}| \overset{r}{\to\!\triangleright} |\mathbf{f}|$. Thus, $\Phi_m|\mathbf{e}| \overset{r}{\to\!\triangleright} \Phi_m|\mathbf{f}|$ by Lemma 2.7. Consequently,

$$\Phi_m|\mathbf{e}| \circ j \overset{r}{\to\!\triangleright} \Phi_m|\mathbf{f}| \circ j = j'' \circ \Pi_m|\mathbf{f}|$$

by Lemma 2.4. Assertion b) of Lemma 2.2 yields $j'' \circ \Pi_m|\mathbf{f}| \overset{1}{\to\!\triangleright} \Pi_m|\mathbf{f}|$. Therefore, $\Pi_m|\mathbf{e}| \overset{r}{\to\!\triangleright} \Pi_m|\mathbf{f}|$ by Lemma 2.3. □

§7. The Eilenberg–Mac Lane construction

Generalities. Suppose $n \in \mathbb{N}$, and V is an Abelian group. The following objects are well defined (see [10, §23]): the simplicial Abelian groups $\mathbf{K}(V,n) = \mathbf{K}$ and $\mathbf{L}(V,n) = \mathbf{L}$, and a simplicial homomorphism $\mathbf{c}(V,n) = \mathbf{c} \colon \mathbf{L} \to \mathbf{K}$. For $q \in \mathbb{N}_0$, $\mathbf{K}_q$ is the group of classical n-cocycles of Δ^q with coefficients in V, $\mathbf{L}_q$ is the group of classical $(n-1)$-cochains of Δ^q with coefficients in V, and $\mathbf{c}_q$ is the restriction of the coboundary homomorphism.

For $q, r \in \mathbb{N}_0$ and an increasing map $d : [r] \to [q]$,

$$\mathbf{K}(d) : \mathbf{K}_q \to \mathbf{K}_r$$

is the homomorphism induced by $d_* : \Delta^r \to \Delta^q$. The same applies to $\mathbf{L}$. We have $\pi_q|\mathbf{K}| = 0$, $q \in \mathbb{N} \setminus \{n\}$. There is a canonical isomorphism $V \to \pi_n|\mathbf{K}|$. Composing it with the Hurewicz homomorphism, we get an isomorphism $i \colon V \to H_n|\mathbf{K}|$, called the *standard isomorphism.* The space $|\mathbf{L}|$ is contractible. Since the $\mathbf{c}_q$ are epimorphisms for $q \in \mathbb{N}_0$, $\mathbf{c}$ is a Kan fibration by [8, Lemma III.2.8]; consequently, $|\mathbf{c}|$ is a Serre fibration by the Quillen theorem (see [8, Theorem I.10.10]).

7.1. Lemma. *Suppose $n \in \mathbb{N}$, $\mathbf{T}$ is a simplicial pointed set, V is an Abelian group, and $g : H_n|\mathbf{T}| \to V$ is a homomorphism. Put $\mathbf{K} = \mathbf{K}(V,n)$. Let $i : V \to H_n|\mathbf{K}|$ be the standard isomorphism. Then there exists a simplicial bound map $\mathbf{f} : \mathbf{T} \to \mathbf{K}$ such that $H_n|\mathbf{f}| = i \circ g$.*

This follows from the universal coefficient theorem and "the universal cohomology class theorem" (see [10, Theorem 24.4]).

7.2. Lemma. *Suppose $p \in \mathcal{P}$, $\mathbf{T}$ is a simplicial finite pointed set, $\mathbf{U}$ is a simplicial p-special Abelian group, V is a p-special Abelian group, $n \in \mathbb{N}$, $\mathbf{e} : \mathbf{T} \to \mathbf{U}$ is an embedding, and $\mathbf{f} : \mathbf{T} \to \mathbf{K}(V,n)$ is a simplicial bound map. Then $\mathbf{e} \multimap \mathbf{f}$.*

Proof. Put $\mathbf{K} = \mathbf{K}(V,n)$. By Lemma 3.5, there exists $r \in \mathbb{N}_0$ such that $\mathbf{e}_n \overset{r}{\multimap} \mathbf{f}_n$. For an arbitrary $q \in \mathbb{N}_0$, we consider the commutative diagram

$$\begin{array}{ccccc}
\mathbf{U}_q & \xleftarrow{\mathbf{e}_q} & \mathbf{T}_q & \xrightarrow{\mathbf{f}_q} & \mathbf{K}_q \\
{\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\
\prod_{d\in D} \mathbf{U}_n & \xleftarrow{E=(\mathbf{e}_n)_{d\in D}} & \prod_{d\in D} \mathbf{T}_n & \xrightarrow{F=(\mathbf{f}_n)_{d\in D}} & \prod_{d\in D} \mathbf{K}_n,
\end{array}$$

where D is the set of all increasing maps $d : [n] \to [q]$ and

$$g = \prod_{d\in D} \mathbf{T}(d), \quad g' = \prod_{d\in D} \mathbf{U}(d), \quad h = \prod_{d\in D} \mathbf{K}(d).$$

By assertion a) of Lemma 2.2, $\mathbf{e}_q \overset{1}{\multimap} g' \circ \mathbf{e}_q = E \circ g$. By Lemma 2.7, $E \overset{r}{\multimap} F$. Thus, $E \circ g \overset{r}{\multimap} F \circ g = h \circ \mathbf{f}_q$ by Lemma 2.4. It is not difficult to check that h is a monomorphism. Thus, $h \circ \mathbf{f}_q \overset{1}{\multimap} \mathbf{f}_q$ by assertion b) of Lemma 2.2. Therefore, $\mathbf{e}_q \overset{r}{\multimap} \mathbf{f}_q$ by Lemma 2.3. □

§8. Applying Serre's method

Definition. Let $m \in \mathbb{N}_0$. Suppose we have a commutative diagram

(1)
$$\begin{array}{ccc}
\tilde{\mathbf{U}} & \xleftarrow{\tilde{\mathbf{e}}} & \tilde{\mathbf{T}} \\
{\scriptstyle \mathbf{s}}\downarrow & & {\scriptstyle \mathbf{r}}\downarrow \\
\mathbf{U} & \xleftarrow{\mathbf{e}} & \mathbf{T},
\end{array}$$

where $\mathbf{T}$ and $\tilde{\mathbf{T}}$ are simplicial pointed sets, $\mathbf{r}$ is a simplicial bound map, $\mathbf{U}$ and $\tilde{\mathbf{U}}$ are simplicial Abelian groups, $\mathbf{s}$ is a simplicial homomorphism, and $\mathbf{e}$ and $\tilde{\mathbf{e}}$ are embeddings. Consider the commutative diagram

$$\begin{array}{ccc}
\Pi_m|\tilde{\mathbf{U}}| & \xleftarrow{\Pi_m|\tilde{\mathbf{e}}|} & \Pi_m|\tilde{\mathbf{T}}| \\
{\scriptstyle \Pi_m|\mathbf{s}|}\big\downarrow & & {\scriptstyle \Pi_m|\mathbf{r}|}\big\downarrow \\
\Pi_m|\mathbf{U}| & \xleftarrow{\Pi_m|\mathbf{e}|} & \Pi_m|\mathbf{T}|.
\end{array}$$

By a *gear* for diagram (1) we mean a bound map $G\colon \Pi_m|\mathbf{T}| \to \Pi_m|\tilde{\mathbf{T}}|$ such that $\Pi_m|\mathbf{r}| \circ G = \mathrm{id}_{\Pi_m|\mathbf{T}|}$ and $\Pi_m|\mathbf{e}| \mathrel{-\!\!\rhd} \Pi_m|\tilde{\mathbf{e}}| \circ G$.

8.1. Claim. *Suppose $p \in \mathcal{P}$, $m, n \in \mathbb{N}$, and we are given a commutative diagram*

$$(2)\qquad \begin{array}{ccccc}
\tilde{\mathbf{U}} & \xleftarrow{\tilde{\mathbf{e}}} & \tilde{\mathbf{T}} & \xrightarrow{\mathbf{g}} & \mathbf{L} \\
{\scriptstyle \mathbf{s}}\big\downarrow & & {\scriptstyle \mathbf{r}}\big\downarrow & & {\scriptstyle \mathbf{c}}\big\downarrow \\
\mathbf{U} & \xleftarrow{\mathbf{e}} & \mathbf{T} & \xrightarrow{\mathbf{f}} & \mathbf{K},
\end{array}$$

where $\mathbf{T}$ is a simplicial finite pointed set, $\mathbf{U}$ is a simplicial p-special Abelian group, $\mathbf{e}$ is an embedding, $\mathbf{f}$ is a simplicial bound map, $\mathbf{K} = \mathbf{K}(\mathbb{Z}_p, n)$, $\mathbf{L} = \mathbf{L}(\mathbb{Z}_p, n)$, $\mathbf{c} = \mathbf{c}(\mathbb{Z}_p, n)$, $\tilde{\mathbf{T}}$ is a simplicial pointed set, $\mathbf{r}$ and $\mathbf{g}$ are simplicial bound maps, the right square is Cartesian, $\tilde{\mathbf{U}} = \mathbf{U} \times \mathbf{L}$, $\mathbf{s}$ is the projection, and $\tilde{\mathbf{e}} = (\mathbf{e} \circ \mathbf{r}) \times \mathbf{g}$. Suppose $m \neq n$. Then there exists a gear $G\colon \Pi_m|\mathbf{T}| \to \Pi_m|\tilde{\mathbf{T}}|$ for the left square of diagram (2).

Proof. Consider the commutative diagram

$$(3)\qquad \begin{array}{ccccccc}
\Pi_m|\mathbf{U}| \times \Pi_m|\mathbf{L}| & \xleftarrow{\Pi_m|\mathbf{s}| \times \Pi_m|\mathbf{t}|} & \Pi_m|\tilde{\mathbf{U}}| & \xleftarrow{\Pi_m|\tilde{\mathbf{e}}|} & \Pi_m|\tilde{\mathbf{T}}| & \xrightarrow{\Pi_m|\mathbf{g}|} & \Pi_m|\mathbf{L}| \\
 & & {\scriptstyle \Pi_m|\mathbf{s}|}\big\downarrow & & {\scriptstyle \Pi_m|\mathbf{r}|}\big\downarrow & & {\scriptstyle \Pi_m|\mathbf{c}|}\big\downarrow \\
 & & \Pi_m|\mathbf{U}| & \xleftarrow{\Pi_m|\mathbf{e}|} & \Pi_m|\mathbf{T}| & \xrightarrow{\Pi_m|\mathbf{f}|} & \Pi_m|\mathbf{K}|,
\end{array}$$

where $\mathbf{t}\colon \tilde{\mathbf{U}} \to \mathbf{L}$ is the projection.

We have $p\mathbf{K}_q = 0$ for every $q \in \mathbb{N}_0$. Therefore, $p|\mathbf{K}| = 0$, whence $p\Pi_m|\mathbf{K}| = 0$. Similarly, $p\Pi_m|\mathbf{L}| = 0$. Thus, we can view $\Pi_m|\mathbf{K}|$ and $\Pi_m|\mathbf{L}|$ as vector spaces over the field $\mathbb{Z}_p$, and $\Pi_m|\mathbf{c}|$ as a linear map. Since $\pi_m|\mathbf{K}| = 0$ and $|\mathbf{c}|$ is a Serre fibration, $\Pi_m|\mathbf{c}|$ is surjective. Hence, there exists a linear map $F\colon \Pi_m|\mathbf{K}| \to \Pi_m|\mathbf{L}|$ such that $\Pi_m|\mathbf{c}| \circ F = \mathrm{id}_{\Pi_m|\mathbf{K}|}$.

Since the right square of diagram (2) is Cartesian, so is the right square of diagram (3). We define the required map

$$G\colon \Pi_m|\mathbf{T}| \to \Pi_m|\tilde{\mathbf{T}}|$$

by the conditions $\Pi_m|\mathbf{r}| \circ G = \mathrm{id}_{\Pi_m|\mathbf{T}|}$ and $\Pi_m|\mathbf{g}| \circ G = F \circ \Pi_m|\mathbf{f}|$ (compatibility: $\Pi_m|\mathbf{f}| = \Pi_m|\mathbf{c}| \circ F \circ \Pi_m|\mathbf{f}|$).

By assertion a) of Lemma 2.2, we see that

$$\Pi_m|\mathbf{e}| \stackrel{1}{\mathrel{-\!\!\rhd}} \Pi_m|\mathbf{e}| = \Pi_m|\mathbf{e}| \circ \Pi_m|\mathbf{r}| \circ G = \Pi_m|\mathbf{s}| \circ \Pi_m|\tilde{\mathbf{e}}| \circ G.$$

Lemma 7.2 and Corollary 6.2 imply $\Pi_m|\mathbf{e}| \mathrel{-\!\!\rhd} \Pi_m|\mathbf{f}|$. Assertion a) of Lemma 2.2 shows that $\Pi_m|\mathbf{f}| \stackrel{1}{\mathrel{-\!\!\rhd}} F \circ \Pi_m|\mathbf{f}|$. Thus, by Lemma 2.3,

$$\Pi_m|\mathbf{e}| \mathrel{-\!\!\rhd} F \circ \Pi_m|\mathbf{f}| = \Pi_m|\mathbf{g}| \circ G = \Pi_m|\mathbf{t}| \circ \Pi_m|\tilde{\mathbf{e}}| \circ G.$$

Therefore,

$$\Pi_m|\mathbf{e}| \mathrel{-\!\!\rhd} (\Pi_m|\mathbf{s}| \times \Pi_m|\mathbf{t}|) \circ \Pi_m|\tilde{\mathbf{e}}| \circ G$$

by Lemmas 2.1 and 2.6. Since $\Pi_m|\mathbf{s}| \times \Pi_m|\mathbf{t}|$ is an isomorphism, $\Pi_m|\mathbf{e}| \mathrel{-\!\!\rhd} \Pi_m|\tilde{\mathbf{e}}| \circ G$. $\square$

8.2. Claim. *Suppose $p \in \mathcal{P}$, $m \in \mathbb{N}$, $\mathbf{T}$ is a simply connected simplicial finite pointed set, $\mathbf{U}$ is a simplicial p-special Abelian group, and $\mathbf{e} : \mathbf{T} \to \mathbf{U}$ is an embedding. Suppose the groups $\pi_q|\mathbf{T}|$, $q \in \mathbb{N}$, are p-special. Then there exists a commutative diagram of the form (1), where $\tilde{\mathbf{T}}$ is an $(m-1)$-connected simplicial finite pointed set and $\tilde{\mathbf{U}}$ is a simplicial p-special Abelian group, and a gear $G\colon \Pi_m|\mathbf{T}| \to \Pi_m|\tilde{\mathbf{T}}|$ for that diagram.*

Proof. We proceed in a finite number of *steps*. At the ith ($i \in \mathbb{N}_0$) step we shall construct a commutative diagram

$$\begin{array}{ccc} \tilde{\mathbf{U}}^i & \xleftarrow{\tilde{\mathbf{e}}^i} & \tilde{\mathbf{T}}^i \\ {\scriptstyle \mathbf{s}^i}\downarrow & & {\scriptstyle \mathbf{r}^i}\downarrow \\ \mathbf{U} & \xleftarrow{\mathbf{e}} & \mathbf{T}, \end{array}$$

where $\tilde{\mathbf{T}}^i$ is a simply connected simplicial finite pointed set such that the groups $\pi_q|\tilde{\mathbf{T}}^i|$, $q \in \mathbb{N}$, are p-special, $\mathbf{r}^i$ is a simplicial bound map, $\tilde{\mathbf{U}}^i$ is a simplicial p-special Abelian group, $\mathbf{s}^i$ is a simplicial homomorphism, and $\tilde{\mathbf{e}}^i$ is an embedding, together with a gear $G^i : \Pi_m|\mathbf{T}| \to \Pi_m|\tilde{\mathbf{T}}^i|$ for that diagram.

The 0th step: we put

$$\tilde{\mathbf{T}}^0 = \mathbf{T}, \quad \mathbf{r}^0 = \mathbf{id}, \quad \tilde{\mathbf{U}}^0 = \mathbf{U}, \quad \mathbf{s}^0 = \mathbf{id}, \quad \tilde{\mathbf{e}}^0 = \mathbf{e}, \quad G^0 = \mathrm{id}\,.$$

Suppose that the ith ($i \in \mathbb{N}_0$) step is finished. If the simplicial set $\tilde{\mathbf{T}}^i$ is $(m-1)$-connected, we put

$$\tilde{\mathbf{T}} = \tilde{\mathbf{T}}^i, \quad \mathbf{r} = \mathbf{r}^i, \quad \tilde{\mathbf{U}} = \tilde{\mathbf{U}}^i, \quad \mathbf{s} = \mathbf{s}^i, \quad \tilde{\mathbf{e}} = \tilde{\mathbf{e}}^i, \quad G = G^i,$$

and we are done. Otherwise, we pass to the $(i+1)$st step. Put

$$n = \inf\{q \in \mathbb{N} : \pi_q|\tilde{\mathbf{T}}^i| \neq 0\}.$$

Then $n < m$. Let

$$\mathbf{K} = \mathbf{K}(\mathbb{Z}_p, n), \quad \mathbf{L} = \mathbf{L}(\mathbb{Z}_p, n), \quad \mathbf{c} = \mathbf{c}(V, n)\colon \mathbf{L} \to \mathbf{K}.$$

Since $\pi_n|\tilde{\mathbf{T}}^i|$ is a nonzero p-special Abelian group, there exists an epimorphism h : $\pi_n|\tilde{\mathbf{T}}^i| \to \mathbb{Z}_p$. By the Hurewicz theorem and Lemma 7.1, there exists a simplicial bound map $\mathbf{f} : \tilde{\mathbf{T}}^i \to \mathbf{K}$ such that $\pi_n|\mathbf{f}|$ is an epimorphism. There is a commutative diagram

$$\text{(4)} \qquad \begin{array}{ccccc} \tilde{\mathbf{U}}^{i+1} & \xleftarrow{\tilde{\mathbf{e}}^{i+1}} & \tilde{\mathbf{T}}^{i+1} & \xrightarrow{\mathbf{g}} & \mathbf{L} \\ {\scriptstyle \mathbf{s}'}\downarrow & & {\scriptstyle \mathbf{r}'}\downarrow & & {\scriptstyle \mathbf{c}}\downarrow \\ \tilde{\mathbf{U}}^i & \xleftarrow{\tilde{\mathbf{e}}^i} & \tilde{\mathbf{T}}^i & \xrightarrow{\mathbf{f}} & \mathbf{K}, \end{array}$$

where $\tilde{\mathbf{T}}^{i+1}$ is a simplicial pointed set, $\mathbf{r}'$ and $\mathbf{g}$ are simplicial bound maps, the right square is Cartesian, $\tilde{\mathbf{U}}^{i+1} = \tilde{\mathbf{U}}^i \times \mathbf{L}$, $\mathbf{s}'$ is the projection, and $\tilde{\mathbf{e}}^{i+1} = (\tilde{\mathbf{e}}^i \circ \mathbf{r}') \times \mathbf{g}$. Putting $\mathbf{r}^{i+1} = \mathbf{r}^i \circ \mathbf{r}'$ and $\mathbf{s}^{i+1} = \mathbf{s}^i \circ \mathbf{s}'$, we get the required commutative diagram

$$\text{(5)} \qquad \begin{array}{ccc} \tilde{\mathbf{U}}^{i+1} & \xleftarrow{\tilde{\mathbf{e}}^{i+1}} & \tilde{\mathbf{T}}^{i+1} \\ {\scriptstyle \mathbf{s}^{i+1}}\downarrow & & {\scriptstyle \mathbf{r}^{i+1}}\downarrow \\ \mathbf{U} & \xleftarrow{\mathbf{e}} & \mathbf{T}. \end{array}$$

By Claim 8.1, there exists a gear $G' : \Pi_m|\tilde{\mathbf{T}}^i| \to \Pi_m|\tilde{\mathbf{T}}^{i+1}|$ for the left square of diagram (4). Put $G^{i+1} = G' \circ G^i$. Clearly, $\Pi_m|\mathbf{r}^{i+1}| \circ G^i = \mathrm{id}$. Consider the diagram

$$\begin{array}{ccc} \Pi_m|\tilde{\mathbf{U}}^{i+1}| & \xleftarrow{\Pi_m|\tilde{\mathbf{e}}^{i+1}|} & \Pi_m|\tilde{\mathbf{T}}^{i+1}| \\ & & \uparrow{\scriptstyle G'} \\ \Pi_m|\tilde{\mathbf{U}}^{i}| & \xleftarrow{\Pi_m|\tilde{\mathbf{e}}^{i}|} & \Pi_m|\tilde{\mathbf{T}}^{i}| \\ & & \uparrow{\scriptstyle G^i} \\ \Pi_m|\mathbf{U}| & \xleftarrow{\Pi_m|\mathbf{e}|} & \Pi_m|\mathbf{T}|. \end{array}$$

We have

$$\Pi_m|\mathbf{e}| \multimap \Pi_m|\tilde{\mathbf{e}}^i| \circ G^i, \quad \Pi_m|\tilde{\mathbf{e}}^i| \multimap \Pi_m|\tilde{\mathbf{e}}^{i+1}| \circ G'.$$

Consequently, by Lemma 2.4,

$$\Pi_m|\tilde{\mathbf{e}}^i| \circ G^i \multimap \Pi_m|\tilde{\mathbf{e}}^{i+1}| \circ G' \circ G^i = \Pi_m|\tilde{\mathbf{e}}^{i+1}| \circ G^{i+1}.$$

Thus, $\Pi_m|\mathbf{e}| \multimap \Pi_m|\tilde{\mathbf{e}}^{i+1}| \circ G^{i+1}$ by Lemma 2.3. Therefore, G^{i+1} is a gear for diagram (5). Since the right square of diagram (4) is Cartesian and $|\mathbf{c}|$ is a Serre fibration, $|\mathbf{r}'|$ is also a Serre fibration, and $|\mathbf{g}|$ maps the fibers of $|\mathbf{r}'|$ homeomorphically to those of $|\mathbf{c}|$. Comparing the homotopy sequences of the fibrations $|\mathbf{r}'|$ and $|\mathbf{c}|$, we see that $\pi_q|\mathbf{r}'|$ is an isomorphism for $q \neq n$ and a monomorphism with cokernel of order p for $q = n$. Since the groups $\pi_q|\tilde{\mathbf{T}}^i|$, $q \in \mathbb{N}$, are p-special, so are the groups $\pi_q|\tilde{\mathbf{T}}^{i+1}|$, $q \in \mathbb{N}$. The step is completed.

We see that the order of the direct sum of the groups $\pi_q|\tilde{\mathbf{T}}^i|$, $q < m$, strictly decreases at each step. Thus, we shall stop at some step. □

8.3. Claim. *Suppose $p \in \mathcal{P}$, $m \in \mathbb{N}$, $\mathbf{T}$ is a simplicial finite pointed set, $\mathbf{U}$ is a simplicial p-special Abelian group, $\mathbf{e} : \mathbf{T} \to \mathbf{U}$ is an embedding, V is a p-special Abelian group, and $g : H_m|\mathbf{T}| \to V$ is a homomorphism. Then $\Pi_m|\mathbf{e}| \multimap g \circ \mathrm{Hur}_{m,|\mathbf{T}|} \circ P_{m,|\mathbf{T}|}$.*

Proof. Put $\mathbf{K} = \mathbf{K}(V, m)$. Let $i : V \to H_m|\mathbf{K}|$ be the standard isomorphism. By Lemma 7.1, there exists a simplicial bound map $\mathbf{f} : \mathbf{T} \to \mathbf{K}$ such that $H_m|\mathbf{f}| = i \circ g$. Consider the commutative diagram

$$\begin{array}{ccccccccc} \Pi_m|\mathbf{U}| & \xleftarrow{\Pi_m|\mathbf{e}|} & \Pi_m|\mathbf{T}| & \xrightarrow{P} & \pi_m|\mathbf{T}| & \xrightarrow{h} & H_m|\mathbf{T}| & & \\ & & \downarrow{\scriptstyle \Pi_m|\mathbf{f}|} & & \downarrow{\scriptstyle \pi_m|\mathbf{f}|} & & \downarrow{\scriptstyle H_m|\mathbf{f}|} & & \\ & & \Pi_m|\mathbf{K}| & \xrightarrow{P'} & \pi_m|\mathbf{K}| & \xrightarrow{h'} & H_m|\mathbf{K}| & \xleftarrow{i} & V, \end{array}$$

where $h = \mathrm{Hur}_{m,|\mathbf{T}|}$ and $h' = \mathrm{Hur}_{m,|\mathbf{K}|}$. From Lemma 7.2 and Corollary 6.2 it follows that $\Pi_m|\mathbf{e}| \multimap \Pi_m|\mathbf{f}|$. By Lemma 5.2, P' is a homomorphism. Assertion a) of Lemma 2.2 shows that

$$\Pi_m|\mathbf{f}| \overset{1}{\multimap} h' \circ P' \circ \Pi_m|\mathbf{f}| = H_m|\mathbf{f}| \circ h \circ P = i \circ g \circ h \circ P.$$

Since i is an isomorphism, $\Pi_m|\mathbf{f}| \overset{1}{\multimap} g \circ h \circ P$. Therefore, $\Pi_m|\mathbf{e}| \multimap g \circ h \circ P$ by Lemma 2.3. □

8.4. Claim. *Under the assumptions of Claim 8.2, $\Pi_m|\mathbf{e}| \multimap P_{m,|\mathbf{T}|}$.*

Proof. We apply Claim 8.2 and consider the commutative diagram

$$\begin{array}{ccccccc}
\Pi_m|\tilde{\mathbf{U}}| & \xleftarrow{\Pi_m|\tilde{\mathbf{e}}|} & \Pi_m|\tilde{\mathbf{T}}| & \xrightarrow{\tilde{P}} & \pi_m|\tilde{\mathbf{T}}| & \xrightarrow{\tilde{h}} & H_m|\tilde{\mathbf{T}}| \\
{\scriptstyle \Pi_m|\mathbf{s}|}\big\downarrow & & {\scriptstyle \Pi_m|\mathbf{r}|}\big\downarrow & & {\scriptstyle \pi_m|\mathbf{r}|}\big\downarrow & & \\
\Pi_m|\mathbf{U}| & \xleftarrow{\Pi_m|\mathbf{e}|} & \Pi_m|\mathbf{T}| & \xrightarrow{P} & \pi_m|\mathbf{T}|, & &
\end{array}$$

where $\tilde{h} = \mathrm{Hur}_{m,|\tilde{\mathbf{T}}|}$. By the Hurewicz theorem, $\tilde{h}$ is an isomorphism. We have $\Pi_m|\mathbf{e}| \mathrel{-\!\!\triangleright} \Pi_m|\tilde{\mathbf{e}}| \circ G$. By Claim 8.3,

$$\Pi_m|\tilde{\mathbf{e}}| \mathrel{-\!\!\triangleright} \pi_m|\mathbf{r}| \circ \tilde{h}^{-1} \circ \tilde{h} \circ \tilde{P} = \pi_m|\mathbf{r}| \circ \tilde{P} = P \circ \Pi_m|\mathbf{r}|.$$

Thus, by Lemma 2.4,

$$\Pi_m|\tilde{\mathbf{e}}| \circ G \mathrel{-\!\!\triangleright} P \circ \Pi_m|\mathbf{r}| \circ G = P.$$

Therefore, $\Pi_m|\tilde{\mathbf{e}}| \mathrel{-\!\!\triangleright} P$ by Lemma 2.3. □

8.5. Claim. *Suppose $p \in \mathcal{P}$, $m \in \mathbb{N}$, and $\mathbf{T}$ is a simply connected simplicial finite pointed set. If the groups $\pi_q|\mathbf{T}|$, $q \in \mathbb{N}$, are p-special, then $J_{m,|\mathbf{T}|} \mathrel{-\!\!\triangleright} P_{m,|\mathbf{T}|}$.*

Proof. Let $\mathbf{U}$ be the simplicial Abelian group with $\mathbf{U}_q = \langle \mathbf{T}_q \rangle / p$, $q \in \mathbb{N}_0$, and $\mathbf{U}(d) = \langle \mathbf{T}(d) \rangle / p$ for an increasing map $d : [r] \to [q]$ $(q, r \in \mathbb{N}_0)$. We introduce the embedding

$$\mathbf{e} \colon \mathbf{T} \to \mathbf{U}, \quad \mathbf{e}_q(t) := \text{`}t\text{'}|_p, \quad t \in \mathbf{T}_q, \ \ q \in \mathbb{N}_0,$$

and consider the commutative diagram

$$\begin{array}{ccccc}
\Psi_m|\mathbf{T}| & \xleftarrow{J} & \Pi_m|\mathbf{T}| & \xrightarrow{P} & \pi_m|\mathbf{T}| \\
{\scriptstyle \Psi_m|\mathbf{e}|}\big\downarrow & & {\scriptstyle \Pi_m|\mathbf{e}|}\big\downarrow & & \\
\Psi_m|\mathbf{U}| & \xleftarrow{J'} & \Pi_m|\mathbf{U}|. & &
\end{array}$$

By assertion a) of Lemma 2.2, $J \overset{1}{\mathrel{-\!\!\triangleright}} \Psi_m|\mathbf{e}| \circ J = J' \circ \Pi_m|\mathbf{e}|$. By Lemma 5.1, $J' \overset{1}{\mathrel{-\!\!\triangleright}} \mathrm{id}$. Thus, $J' \circ \Pi_m|\mathbf{e}| \overset{1}{\mathrel{-\!\!\triangleright}} \Pi_m|\mathbf{e}|$ by Lemma 2.4. Claim 8.4 implies $\Pi_m|\mathbf{e}| \mathrel{-\!\!\triangleright} P$. Therefore, $J \mathrel{-\!\!\triangleright} P$ by Lemma 2.3. □

§9. The most nonconstructive site

9.1. Lemma. *Suppose $p \in \mathcal{P}$, $m \in \mathbb{N}$, $\mathbf{T}$ is a simply connected simplicial pointed set, W is a p-special Abelian group, and $r \colon \pi_m|\mathbf{T}| \to W$ is a homomorphism. If the groups $\pi_q|\mathbf{T}|$, $q \in \mathbb{N}$, are finitely generated, then there exists a simply connected simplicial finite pointed set $\mathbf{T}'$, a homomorphism $r' \colon \pi_m|\mathbf{T}'| \to W$, and a simplicial bound map $\mathbf{h} \colon \mathbf{T} \to \mathbf{T}'$ such that $r' \circ \pi_m|\mathbf{h}| = r$ and the groups $\pi_q|\mathbf{T}'|$, $q \in \mathbb{N}$, are p-special.*

This follows from the results of [7] (see §15 below).

9.2. Claim. *Under the assumptions of Lemma 9.1, $J_{m,|\mathbf{T}|} \mathrel{-\!\!\triangleright} r \circ P_{m,|\mathbf{T}|}$.*

Proof. By Lemma 9.1, there exists a simply connected finite pointed set $\mathbf{T}'$, a homomorphism $r' : \pi_m|\mathbf{T}'| \to W$, and a simplicial bound map $\mathbf{h} : \mathbf{T} \to \mathbf{T}'$ such that $r' \circ \pi_m|\mathbf{h}| = r$ and the groups $\pi_q|\mathbf{T}'|$, $q \in \mathbb{N}$, are p-special. Consider the commutative diagram

$$\begin{array}{ccccccc}
\Psi_m|\mathbf{T}| & \xleftarrow{J} & \Pi_m|\mathbf{T}| & \xrightarrow{P} & \pi_m|\mathbf{T}| & & \\
{\scriptstyle \Psi_m|\mathbf{h}|}\big\downarrow & & {\scriptstyle \Pi_m|\mathbf{h}|}\big\downarrow & & {\scriptstyle \pi_m|\mathbf{h}|}\big\downarrow & & \\
\Psi_m|\mathbf{T}'| & \xleftarrow{J'} & \Pi_m|\mathbf{T}'| & \xrightarrow{P'} & \pi_m|\mathbf{T}'| & \xrightarrow{r'} & W.
\end{array}$$

By assertion a) of Lemma 2.2, $J \overset{1}{-\!\!\rhd} \Psi_m|\mathbf{h}| \circ J = J' \circ \Pi_m|\mathbf{h}|$. By Claim 8.5, $J' -\!\!\rhd P'$. Thus, $J' -\!\!\rhd r' \circ P'$ by assertion a) of Lemma 2.2 and Lemma 2.3. Thus, by Lemma 2.4,

$$J' \circ \Pi_m|\mathbf{h}| -\!\!\rhd r' \circ P' \circ \Pi_m|\mathbf{h}| = r' \circ \pi_m|\mathbf{h}| \circ P = r \circ P.$$

Therefore, $J -\!\!\rhd r \circ P$ by Lemma 2.3. □

§10. The Hurewicz invariant

10.1. Lemma. *If $m \in \mathbb{N}$ and $\mathbf{T}$ is a simplicial pointed set, then*

$$J_{m,|\mathbf{T}|} \overset{1}{-\!\!\rhd} \mathrm{Hur}_{m,|\mathbf{T}|} \circ P_{m,|\mathbf{T}|}.$$

Proof. Put $\mathbf{K} = \mathbf{K}(H_m|\mathbf{T}|, m)$. By Lemma 7.1, there exists a simplicial bound map $\mathbf{f} : \mathbf{T} \to \mathbf{K}$ such that $H_m|\mathbf{f}|$ is the standard isomorphism. Consider the commutative diagram

$$\begin{array}{ccccccc}
\Psi_m|\mathbf{T}| & \xleftarrow{J} & \Pi_m|\mathbf{T}| & \xrightarrow{P} & \pi_m|\mathbf{T}| & \xrightarrow{h} & H_m|\mathbf{T}| \\
{\scriptstyle \Psi_m|\mathbf{f}|}\downarrow & & {\scriptstyle \Pi_m|\mathbf{f}|}\downarrow & & {\scriptstyle \pi_m|\mathbf{f}|}\downarrow & & {\scriptstyle H_m|\mathbf{f}|}\downarrow \\
\Psi_m|\mathbf{K}| & \xleftarrow{J'} & \Pi_m|\mathbf{K}| & \xrightarrow{P'} & \pi_m|\mathbf{K}| & \xrightarrow{h'} & H_m|\mathbf{K}|,
\end{array}$$

where $h = \mathrm{Hur}_{m,|\mathbf{T}|}$ and $h' = \mathrm{Hur}_{m,|\mathbf{K}|}$. By assertion a) of Lemma 2.2,

$$J \overset{1}{-\!\!\rhd} \Psi_m|\mathbf{f}| \circ J = J' \circ \Pi_m|\mathbf{f}|.$$

By Lemma 5.1, $J' \overset{1}{-\!\!\rhd} \mathrm{id}_{\Pi_m|\mathbf{K}|}$. Thus, $J' \circ \Pi_m|\mathbf{f}| \overset{1}{-\!\!\rhd} \Pi_m|\mathbf{f}|$ by Lemma 2.4. Now, Lemma 5.2 shows that P' is a homomorphism. Assertion a) of Lemma 2.2 implies the relation

$$\Pi_m|\mathbf{f}| \overset{1}{-\!\!\rhd} h' \circ P' \circ \Pi_m|\mathbf{f}| = H_m|\mathbf{f}| \circ h \circ P.$$

Since $H_m|\mathbf{f}|$ is an isomorphism, $\Pi_m|\mathbf{f}| \overset{1}{-\!\!\rhd} h \circ P$. Therefore, $J \overset{1}{-\!\!\rhd} h \circ P$ by Lemma 2.3. □

§11. r-Point transformations

Definition. Suppose $r, m, n \in \mathbb{N}_0$, and X and Y are pointed spaces. A map $F : \Pi_m X \to \Pi_n Y$ is said to be *r-point* if for every point $z \in S^n$ there exists a set $T \subset S^m$ of at most r points such that for any $a, a' \in \Pi_m X$ the relation $a|_T = a'|_T$ implies $F(a)(z) = F(a')(z)$ (in other words, if the value of $F(a)$ at each point is determined by the values of a at some r points).

Our aim in this section is to prove Lemma 11.3.

Notation and a convention. Suppose $m \in \mathbb{N}_0$ and X is a pointed space. Put $\underline{\Psi}_m X = \mathrm{Hom}(\langle \underline{S^m} \rangle, \langle \underline{X} \rangle)$. The map

$$\underline{J}_{m,X} : \Pi_m X \to \underline{\Psi}_m X, \quad a \mapsto \langle \underline{a} \rangle,$$

is also called the *main map* (and is denoted by $\underline{J}$, $\underline{J}'$, etc. if this does not cause ambiguity).

11.1. Claim. *Suppose $F : \Pi_m X \to \Pi_n Y$ is an r-point map. Then $\underline{J}_{m,X} \overset{r}{-\!\!\rhd} \underline{J}_{m,Y} \circ F$.*

Proof. There are (possibly, discontinuous) maps

$$k_1, \dots, k_r : S^n \to S^m, \quad d_z : X^{\times r} \to Y, \quad z \in S^n,$$

such that

$$F(a)(z) = d_z(a(k_1(z)), \dots, a(k_r(z))), \quad z \in S^n, \ a \in \Pi_m X.$$

Consider the usual isomorphism

$$i : \langle \underline{X} \rangle^{\otimes r} \to \langle \underline{X^{\times r}} \rangle, \quad `\underline{x_1}\text{'} \otimes \dots \otimes `\underline{x_r}\text{'} \mapsto `\underline{(x_1, \dots, x_r)}\text{'}.$$

We have homomorphisms $\langle \underline{d_z}\rangle : \langle \underline{X^{\times r}}\rangle \to \langle \underline{Y}\rangle$, $z \in S^n$. We introduce the homomorphism

$$\begin{aligned} &h : (\underline{\Psi}_m X)^{\otimes r} \to \underline{\Psi}_n Y,\\ &\quad h(w_1 \otimes \cdots \otimes w_r)(‘\underline{z}’) := \langle \underline{d_z}\rangle (i(w_1(‘\underline{k_1(z)}’) \otimes \cdots \otimes w_r(‘\underline{k_r(z)}’))), \quad z \in S^n, \end{aligned}$$

and consider the commutative diagram

$$\begin{array}{ccccc} \Pi_m X & & \xrightarrow{F} & & \Pi_n Y \\ {\scriptstyle \underline{J}}\big\downarrow & & & & {\scriptstyle \underline{J}'}\big\downarrow \\ \underline{\Psi}_m X & \xrightarrow{R} & (\underline{\Psi}_m X)^{\otimes r} & \xrightarrow{h} & \underline{\Psi}_n Y, \end{array}$$

where $R(w) = w^{\otimes r}$, $w \in \underline{\Psi}_m X$. By Lemmas 2.8 and 2.4, $\underline{J} \xrightarrow{r}\triangleright R \circ \underline{J}$. Assertion a) of Lemma 2.2 shows that

$$R \circ \underline{J} \xrightarrow{1}\triangleright h \circ R \circ \underline{J} = \underline{J}' \circ F.$$

Therefore, $\underline{J} \xrightarrow{r}\triangleright \underline{J}' \circ F$ by Lemma 2.3. □

11.2. Claim. *If* $m \in \mathbb{N}_0$ *and* X *is a pointed space, then* $\underline{J} = \underline{J}_{m,X} \xrightarrow{1}\triangleright J_{m,X}$ *and* $J \xrightarrow{1}\triangleright \underline{J}$.

Proof. Consider the homomorphisms

$$p : \langle \underline{X}\rangle \to \langle X\rangle, \quad ‘\underline{x}’ \mapsto ‘x’, \quad x \in X,$$

and

$$h : \underline{\Psi}_m X \to \Psi_m X, \quad h(W)(‘z’) := p(W(‘\underline{z}’)), \quad z \in S^m \setminus \{*\}.$$

It is easy to check that $h \circ \underline{J} = J$. By assertion a) of Lemma 2.2, $\underline{J} \xrightarrow{1}\triangleright J$.

If

$$K : \Pi_m X \to \mathbb{Z} \oplus \Psi_m X, \quad K(a) := (1, J(a)),$$

then $J \xrightarrow{1}\triangleright K$ by Lemmas 2.8, 2.4, 2.1, and 2.6. We introduce the homomorphisms

$$s : \langle X\rangle \to \langle \underline{X}\rangle, \quad ‘x’ \mapsto ‘\underline{x}’ - ‘\underline{*}’, \quad x \in X,$$

and

$$f : \mathbb{Z} \oplus \Psi_m X \to \underline{\Psi}_m X, \quad f(t, w)(‘\underline{z}’) := t‘\underline{*}’ + s(w(‘z’)), \quad z \in S^m.$$

It is easy to check that $f \circ K = \underline{J}$. By assertion a) of Lemma 2.2, $K \xrightarrow{1}\triangleright \underline{J}$. Therefore, $J \xrightarrow{1}\triangleright \underline{J}$ by Lemma 2.3. □

11.3. Lemma. *Under the assumptions of Claim* 11.1, $J_{m,X} \xrightarrow{r}\triangleright J_{m,Y} \circ F$.

This follows from Claims 11.1, 11.2 and Lemmas 2.4, 2.3.

§12. Cobar construction

The construction M^r and convolution. Suppose $r \in \mathbb{N}_0$, and X is a pointed space. Let

$$W := \bigcup_{s=0}^{r} \{(x_1, \dots, x_r) \in X^{\times r} : x_s = x_{s+1}\} \subset X^{\times r},$$

where $x_0 = x_{r+1} = *$. We put $M^r X = X^{\times r}/W$ (cf. [12]). This construction preserves the convenience of spaces.

We introduce the map

$$K : \Delta^r \times \Omega X \to X^{\times r}, \quad (z, u) \mapsto (u(z_1), \dots, u(z_r))$$

(as in iterated integrals; see [5]). It is easily seen that

$$K(\partial\Delta^r \times \Omega X \cup \Delta^r \times \{*\}) \subset W.$$

Consider the following continuous bound map:

$$k\colon S^r\Omega X \to M^r X, \quad t^\circ u \mapsto c(K(t^\triangle, u))\,(= c(u(z_1), \dots, u(z_r))),$$
$$t \in I^r, \quad u \in \Omega X, \quad (z_1, \dots, z_r) = t^\triangle \in \Delta^r,$$

where $c : X^{\times r} \to M^r X$ is the projection. The map k is called *convolution*.

12.1. Lemma. *Suppose $r \in \mathbb{N}_0$ and X is a simply connected convenient pointed space. Then the convolution $k\colon S^r\Omega X \to M^r X$ is $(2r+1)$-connected.*

The proof follows the lines of [6] (see §§16 and 17 below).

§13. Applying the Cartan–Serre theorem

Definition. Suppose $r \in \mathbb{N}_0$, $q \in \mathbb{N}$, and X is a pointed space. Let $k : S^r\Omega X \to M^r X$ be the convolution. Consider the commutative diagram

$$\begin{array}{ccccccc}
\Pi_{q+1}X & \xrightarrow{D} & \Pi_q\Omega X & \xrightarrow{Z^r} & \Pi_{q+r}S^r\Omega X & \xrightarrow{\Pi_{q+r}k} & \Pi_{q+r}M^r X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_{q+1}X & \xrightarrow{d} & \pi_q\Omega X & \xrightarrow{z^r} & \pi_{q+r}S^r\Omega X & \xrightarrow{\pi_{q+r}k} & \pi_{q+r}M^r X,
\end{array}$$

where the vertical arrows are the main maps. The *development transformation*

$$F\colon \Pi_{q+1}X \to \Pi_{q+r}M^r X$$

and the *development homomorphism*

$$f\colon \pi_{q+1}X \to \pi_{q+r}M^r X$$

are the compositions in the upper and the lower (respectively) lines of the diagram.

13.1. Claim. *The development transformation $F\colon \Pi_{q+1}X \to \Pi_{q+r}M^r X$ is r-point.*

Proof. Let $c : X^{\times r} \to M^r X$ be the projection. For $a \in \Pi_{q+1}X$, $s \in I^q$, and $t \in I^r$ we have

$$F(a)((t,s)^\circ) = c(a((z_1,s)^\circ), \dots, a((z_r,s)^\circ)),$$

where $(z_1, \dots, z_r) = t^\triangle \in \Delta^r$. □

13.2. Lemma. *Suppose $q \in \mathbb{N}$ and X is a pointed space. Then* $\ker \mathrm{Hur}_{q,\Omega X} = \mathrm{Tors}$.

This follows from the Cartan–Serre theorem; see [11, Appendix].

13.3. Claim. *Suppose $r \in \mathbb{N}_0$, $q \in \mathbb{N}$, and X is a simply connected convenient pointed space. Let $f\colon \pi_{q+1}X \to \pi_{q+r}M^r X$ be the development homomorphism. Suppose $r \geq q$. Then* $\ker(\mathrm{Hur}_{q+r,M^r X} \circ f) = \mathrm{Tors}$.

Proof. Let $k\colon S^r\Omega X \to M^r X$ be the convolution. Consider the commutative diagram

$$\begin{array}{ccccccc}
\pi_{q+1}X & \xrightarrow{d} & \pi_q\Omega X & \xrightarrow{z^r} & \pi_{q+r}S^r\Omega X & \xrightarrow{\pi_{q+r}k} & \pi_{q+r}M^r X \\
 & & {\scriptstyle h'}\downarrow & & {\scriptstyle h''}\downarrow & & {\scriptstyle h}\downarrow \\
 & & H_q\Omega X & \xrightarrow{s^r} & H_{q+r}S^r\Omega X & \xrightarrow{H_{q+r}k} & H_{q+r}M^r X,
\end{array}$$

where s^r is the suspension isomorphism and h, h', and h'' are the Hurewicz homomorphisms. We have

$$h \circ f = h \circ \pi_{q+r}k \circ z^r \circ d = H_{q+r}k \circ s^r \circ h' \circ d.$$

By Lemma 12.1, $H_{q+r}k$ is an isomorphism. By Lemma 13.2, $\ker h' = \mathrm{Tors}$. Therefore, $\ker h \circ f = \mathrm{Tors}$. □

13.4. Claim. *Suppose $m \in \mathbb{N}$, and X is a simply connected convenient pointed space. Let $q\colon \pi_m X \to \pi_m X/\operatorname{Tors}$ be the projection. Then $J_{m,X} \xrightarrow{m-1}\!\!\triangleright\, q \circ P$.*

Proof. Put $r = m-1$, $n = 2m-2$ (we assume that $m > 1$). Consider the diagram

$$\begin{array}{ccccccc}
\Psi_m X & \xleftarrow{J} & \Pi_m X & \xrightarrow{P} & \pi_m X & \xrightarrow{q} & \pi_m X/\operatorname{Tors} \\
 & & {\scriptstyle F}\downarrow & & {\scriptstyle f}\downarrow & & \\
\Psi_n M^r X & \xleftarrow{J'} & \Pi_n M^r X & \xrightarrow{P'} & \pi_n M^r X & & {\scriptstyle t}\downarrow \\
 & & & & {\scriptstyle h}\downarrow & & \\
 & & & & H_n M^r X & \xrightarrow{p} & H_n M^r X/\operatorname{Tors},
\end{array}$$

where t is a homomorphism such that $t \circ q = p \circ h \circ f$. This diagram is commutative. By Claim 13.1, F is an r-point transformation. Thus, $J \xrightarrow{r}\!\!\triangleright\, J' \circ F$ by Lemma 11.3. By Lemma 10.1, $J' \xrightarrow{1}\!\!\triangleright\, h \circ P'$. Therefore, $J' \circ F \xrightarrow{1}\!\!\triangleright\, h \circ P' \circ F$ by Lemma 2.4. Assertion a) of Lemma 2.2 implies that

$$h \circ P' \circ F \xrightarrow{1}\!\!\triangleright\, p \circ h \circ P' \circ F = t \circ q \circ P.$$

Since t is a monomorphism (by Claim 13.3), we have $t \circ q \circ P \xrightarrow{1}\!\!\triangleright\, q \circ P$ by assertion b) of Lemma 2.2. Therefore, $J \xrightarrow{r}\!\!\triangleright\, q \circ P$ by Lemma 2.3. ☐

§14. Completion of the proof

14.1. Claim. *Suppose $m \in \mathbb{N}$, and X is a simply connected convenient pointed space. If the groups $\pi_q X$, $q \in \mathbb{N}$, are finitely generated, then $J_{m,X} \multimap\!\!\triangleright\, P_{m,X}$.*

Proof. There is an isomorphism

$$s = q \times \prod_{p \in T} r_p \colon \pi_m X \to \pi_m X/\operatorname{Tors} \times \prod_{p \in T} W_p,$$

where $q : \pi_m X \to \pi_m X/\operatorname{Tors}$ is the projection, T is the set of prime divisors of the order of the group $\operatorname{Tors} \pi_m X$, and, for each $p \in T$, W_p is a p-special Abelian group and $r_p\colon \pi_m X \to W_p$ is a homomorphism. By Claim 13.4, $J \multimap\!\!\triangleright\, q \circ P$. By Claim 9.2, $J \multimap\!\!\triangleright\, r_p \circ P$ for each $p \in T$. By Lemmas 2.1 and 2.6, $J \multimap\!\!\triangleright\, s \circ P$. Since s is an isomorphism, $J \multimap\!\!\triangleright\, P$. ☐

14.2. Claim. *Suppose $m \in \mathbb{N}$, and X is a simply connected admissible pointed space. If the groups $\pi_q X$, $q \in \mathbb{N}$, are finitely generated, then $J_{m,X} \multimap\!\!\triangleright\, P_{m,X}$.*

Proof. We have a minimal fibrant simplicial pointed set $\mathbf{T}$ and a bound equivalence $h : X \to |\mathbf{T}|$ (see [10, §8]). Since $\mathbf{T}$ is a connected minimal fibrant simplicial set and the groups $\pi_q|\mathbf{T}|$, $q \in \mathbb{N}$, are countable, $\mathbf{T}$ is a simplicial countable set. Thus, $|\mathbf{T}|$ is a convenient pointed space. Consider the commutative diagram

$$\begin{array}{ccccc}
\Psi_m X & \xleftarrow{J} & \Pi_m X & \xrightarrow{P} & \pi_m X \\
{\scriptstyle \Psi_m h}\downarrow & & {\scriptstyle \Pi_m h}\downarrow & & {\scriptstyle \pi_m h}\downarrow \\
\Psi_m |\mathbf{T}| & \xleftarrow{J'} & \Pi_m |\mathbf{T}| & \xrightarrow{P'} & \pi_m |\mathbf{T}|.
\end{array}$$

By assertion a) of Lemma 2.2,

$$J \xrightarrow{1}\!\!\triangleright\, \Psi_m h \circ J = J' \circ \Pi_m h.$$

By Claim 14.1, $J' \mathrel{-\!\!\rhd} P'$. Thus, by Lemma 2.4,

$$J' \circ \Pi_m h \mathrel{-\!\!\rhd} P' \circ \Pi_m h = \pi_m h \circ P.$$

Therefore, $J \mathrel{-\!\!\rhd} \pi_m h \circ P$ by Lemma 2.3. Since $\pi_m h$ is an isomorphism, $J \mathrel{-\!\!\rhd} P$. $\square$

14.3. Claim. *Suppose $r, m \in \mathbb{N}_0$, X is a pointed space, V is an Abelian group, and $F : \Pi_m X \to V$ is a bound map. Let $Q \subset (\Psi_m X)^{\otimes r}$ be the subgroup generated by the elements $\langle a\rangle^{\otimes r}$, $a \in \Pi_m X$. Suppose $J = J_{m,X} \overset{r}{\mathrel{-\!\!\rhd}} F$. Then there exists a homomorphism $l : Q \to V$ such that $l(\langle a\rangle^{\otimes r}) = F(a)$ for any $a \in \Pi_m X$.*

Proof. For each $t \in \mathbb{N}$, consider the homomorphisms

$$b_t : \langle X\rangle^{\otimes t} \to \langle X\rangle, \quad `x_1\text{'} \otimes \cdots \otimes `x_t\text{'} \mapsto \begin{cases} `x_1\text{'} & \text{if } x_1 = \cdots = x_t, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B_t : (\Psi_m X)^{\otimes t} \to \Psi_m X,$$
$$B_t(w_1 \otimes \cdots \otimes w_t)(`z\text{'}) = b_t(w_1(`z\text{'}) \otimes \cdots \otimes w_t(`z\text{'})), \quad z \in S^m.$$

For $t \in \mathbb{N}$ and $a \in \Pi_m X$, we have $B_t(\langle a\rangle^{\otimes t}) = \langle a\rangle$, because

$$B_t(\langle a\rangle^{\otimes t})(`z\text{'}) = b_t(\langle a\rangle(`z\text{'})^{\otimes t}) = b_t(`a(z)\text{'}^{\otimes t}) = `a(z)\text{'} = \langle a\rangle(`z\text{'})$$

for $z \in S^m$. For each $s = 1, \dots, r$, we introduce the homomorphism

$$G_s : (\Psi_m X)^{\otimes r} \to (\Psi_m X)^{\otimes s},$$
$$w_1 \otimes \cdots \otimes w_r \mapsto w_1 \otimes \cdots \otimes w_{s-1} \otimes B_{r-s+1}(w_s \otimes \cdots \otimes w_r).$$

For $a \in \Pi_m X$, we have

$$G_s(\langle a\rangle^{\otimes r}) = \langle a\rangle^{\otimes(s-1)} \otimes B_{r-s+1}(\langle a\rangle^{\otimes(r-s+1)}) = \langle a\rangle^{\otimes(s-1)} \otimes \langle a\rangle = \langle a\rangle^{\otimes s}.$$

Let

$$P \subset \bigoplus_{s=1}^{r} (\Psi_m X)^{\otimes s}$$

be the subgroup generated by the elements $(\langle a\rangle^{\otimes s})_{s=1}^r$, $a \in \Pi_m X$. Since $\operatorname{Tors} \Psi_m X = 0$ and $J \overset{r}{\mathrel{-\!\!\rhd}} F$, Corollary 2.11 implies the existence of a homomorphism $k : P \to V$ such that $k((\langle a\rangle^{\otimes s})_{s=1}^r) = F(a)$ for any $a \in \Pi_m X$.

If $z \in Q$, then $(G_s(z))_{s=1}^r \in P$ because $(G_s(\langle a\rangle^{\otimes r}))_{s=1}^r = (\langle a\rangle^{\otimes s})_{s=1}^r \in P$ for $a \in \Pi_m X$. Put $l(z) = k((G_s(z))_{s=1}^r)$, $z \in Q$. For $a \in \Pi_m X$, we have

$$l(\langle a\rangle^{\otimes r}) = k((G_s(\langle a\rangle^{\otimes r}))_{s=1}^r) = k((\langle a\rangle^{\otimes s})_{s=1}^r) = F(a).$$

$\square$

14.4. Theorem 1: the proof itself. By Claim 14.2, $J_{m,X} \mathrel{-\!\!\rhd} P_{m,X}$. By Lemma 2.1, for any sufficiently large $r \in \mathbb{N}_0$ we have $J_{m,X} \overset{r}{\mathrel{-\!\!\rhd}} P_{m,X}$. Applying Claim 14.3 concludes the proof.

§15. Proof of Lemma 9.1

15.1. Lemma. *Suppose $m \in \mathbb{N}$, U and V are finite Abelian groups, and $e : U \to V$ is a homomorphism. If $mU = 0$ and $e/d : U/d \to V/d$ is a monomorphism for every divisor $d \in \mathbb{N}$ of m, then there exists a homomorphism $h : V \to U$ such that $h \circ e = \mathrm{id}_U$.*

Proof. The homomorphism e $(= e/m)$ is a monomorphism. Take an arbitrary $q \in \mathbb{N}$. By [4, Corollary 28.3], it suffices to show that e/q is a monomorphism. Put $d = \text{G.C.D.}(q, m)$. Consider the commutative diagram

$$\begin{array}{ccc} U/q & \xrightarrow{e/q} & V/q \\ {\scriptstyle p}\downarrow & & {\scriptstyle p'}\downarrow \\ U/d & \xrightarrow{e/d} & V/d, \end{array}$$

where p and p' are the homomorphisms of "reduction modulo d". It is easily seen that p is an isomorphism. By assumption, e/d is a monomorphism. Thus, e/q is a monomorphism. □

Systems. Put $\sigma = \{1, 2, \dots, \infty\}$. A *system* of objects and morphisms of a category is a collection

$$(A_s, f_s^t)$$

of objects A_s, $s \in \sigma$, and morphisms

$$f_s^t \colon A_t \to A_s, \quad s, t \in \sigma, \ \ s \leq t,$$

such that

$$f_s^s = \text{id}, \quad s \in \sigma, \quad \text{and} \quad f_r^s \circ f_s^t = f_r^t, \quad r, s, t \in \sigma, \ \ r \leq s \leq t.$$

A system (V_s, l_s^t) of Abelian groups and homomorphisms is *regular* if V_∞ is the projective limit of the groups V_s, $s < \infty$ (more precisely, if for any sequence $(v_s \in V_s)_{s<\infty}$ such that $l_s^t(v_t) = v_s$, $s \leq t < \infty$, there exists a unique element $v_\infty \in V_\infty$ such that $l_s^\infty(v_\infty) = v_s$ for every $s < \infty$).

15.2. Lemma. *Suppose $q \in \mathbb{N}$, and (V_s, l_s^t) is a regular system of Abelian groups and homomorphisms. Suppose the groups V_s, $s < \infty$, are finite. Then the system $(V_s/q, l_s^t/q)$ is also regular.*

This is easy to check by using the fact that the projective limit of a sequence of nonempty finite sets is nonempty.

15.3. Lemma. *Suppose (V_s, l_s^t) is a regular system of finite Abelian groups and homomorphisms. Then for any sufficiently large $s < \infty$ there exists a homomorphism $h \colon V_s \to V_\infty$ such that $h \circ l_s^\infty = \text{id}$.*

Proof. For any $q \in \mathbb{N}$, the system $(V_s/q, l_s^t/q)$ is regular (by Lemma 15.2), and consequently l_s^∞/q is a monomorphism for all sufficiently large $s < \infty$ (because V_∞/q is finite). If $s < \infty$ is sufficiently large, then l_s^∞/d is a monomorphism for all divisors $d \in \mathbb{N}$ of the order of V_∞, and Lemma 15.1 gives the required homomorphism. □

Definition. Suppose $p \in \mathcal{P}$, and U, V are Abelian groups. Let $\hat{U}$ be the p-completion ($= p$-profinite completion) of U, and let $c : U \to \hat{U}$ be the canonical homomorphism. A homomorphism $k : U \to V$ is said to be *p-completing* if there exists an isomorphism $i : \hat{U} \to V$ such that $i \circ c = k$.

15.4. Claim. *Suppose $p \in \mathcal{P}$, U is a finitely generated Abelian group, (V_s, l_s^t) is a regular system of Abelian groups and homomorphisms, the groups V_s, $s < \infty$, are p-special, $k : U \to V_\infty$ is a p-completing homomorphism, W is a p-special Abelian group, and $r : U \to W$ is a homomorphism. Then there exists $s < \infty$ and a homomorphism $g : V_s \to W$ such that $g \circ l_s^\infty \circ k = r$.*

Proof. Let q be a power of p such that $qW = 0$. By Lemma 15.2, the system $(V_s/q, l_s^t/q)$ is regular. Since V_∞ is isomorphic to the p-completion of a finitely generated Abelian group, V_∞/q is finite (see [7, Chapter VI, 5.2]). By Lemma 15.3, there exist $s < \infty$ and a homomorphism $h\colon V_s/q \to V_\infty/q$ such that $h \circ (l_s^\infty/q) = \mathrm{id}$. Since k is p-completing, there exists a homomorphism $G : V_\infty \to W$ such that $G \circ k = r$. Since $qW = 0$, there exists a homomorphism $G' : V_\infty/q \to W$ such that $G'(X|_q) = G(X)$ for any $X \in V_\infty$. Put $g' = G' \circ h\colon V_s/q \to W$. We define the required homomorphism g by the formula $g(x) = g'(x|_q)$. For $u \in U$, we have

$$\begin{aligned} g(l_s^\infty(k(u))) &= g'(l_s^\infty(k(u))|_q) = G'(h(l_s^\infty(k(u))|_q)) \\ &= G'(h((l_s^\infty/q)(k(u)|_q))) = G'(k(u)|_q) = G(k(u)) = r(u). \end{aligned}$$

□

15.5. Claim. *Suppose $p \in \mathcal{P}$, $m \in \mathbb{N}$, $\mathbf{T}$ is a simply connected simplicial finite pointed set, W is a p-special Abelian group, and $r : \pi_m|\mathbf{T}| \to W$ is a homomorphism. Then there exists a simply connected finite pointed set $\mathbf{T}'$, a homomorphism $r' : \pi_m|\mathbf{T}'| \to W$, and a simplicial bound map $\mathbf{h}\colon \mathbf{T} \to \mathbf{T}'$ such that $r' \circ \pi_m|\mathbf{h}| = r$ and the groups $\pi_q|\mathbf{T}'|$, $q \in \mathbb{N}$, are p-special.*

Proof. We put $R = \mathbb{Z}_p$ and consider the system $(R_s\mathbf{T}, \mathbf{f}_s^t)$ of simplicial pointed sets and simplicial bound maps (see [7, Chapter I, §4]). By construction, for $s < \infty$ the $R_s\mathbf{T}$ are simplicial finite sets. By [7, Chapter I, 6.2 (i)], they are simply connected. Thus, the groups $\pi_q|R_s\mathbf{T}|$, $q \in \mathbb{N}$, are finitely generated by the Serre theorem. By [7, Chapter III, 5.6], they are R-nilpotent. Therefore, they are p-special. Consider the system $(\pi_m|R_s\mathbf{T}|, \pi_m|\mathbf{f}_s^t|)$ of Abelian groups and homomorphisms. Since the groups $\pi_{m+1}|R_s\mathbf{T}|$, $s < \infty$, are finite, the system is regular by [7, Chapter I, 4.3]. Let $\mathbf{c}\colon \mathbf{T} \to R_\infty\mathbf{T}$ be the canonical simplicial bound map (see [7, Chapter I, §4]). Since $\mathbf{T}$ is simply connected and the groups $\pi_q|\mathbf{T}|$, $q \in \mathbb{N}$, are finitely generated (by the Serre theorem), $\pi_m|\mathbf{c}|$ is p-completing (this is implied by [7, Chapter VI, §5]). By Claim 15.4, there exist $s < \infty$ and a homomorphism $g : \pi_m|R_s\mathbf{T}| \to W$ such that $g \circ \pi_m|\mathbf{f}_s^\infty| \circ \pi_m|\mathbf{c}| = r$. It remains to put $\mathbf{T}' = R_s\mathbf{T}$, $r' = g$, and $\mathbf{h} = \mathbf{f}_s^\infty \circ \mathbf{c}$. □

15.6. Lemma 9.1: the proof itself. Since the simplicial set $\mathbf{T}$ is simply connected and the groups $H_q|\mathbf{T}|$, $q \in \mathbb{N}$, are finitely generated (by the Serre theorem), there exists a simplicial finite pointed set $\tilde{\mathbf{T}}$ and an $(m+1)$-connected simplicial map $\mathbf{f} : \tilde{\mathbf{T}} \to \mathbf{T}$ (it is not difficult to construct them by induction on m, applying the relative Hurewicz theorem at each step; cf. [9, Proposition 4C.1]). Put

$$\tilde{r} = r \circ (\pi_m|\mathbf{f}|)^{-1} : \pi_m|\tilde{\mathbf{T}}| \to W.$$

By Claim 15.5, there exists a simply connected simplicial pointed set $\tilde{\mathbf{T}}'$, a homomorphism $\tilde{r}' : \pi_m|\tilde{\mathbf{T}}'| \to W$, and a simplicial bound map $\tilde{\mathbf{h}} : \tilde{\mathbf{T}} \to \tilde{\mathbf{T}}'$ such that $\tilde{r}' \circ \pi_m|\tilde{\mathbf{h}}| = \tilde{r}$ and the groups $\pi_q|\tilde{\mathbf{T}}'|$, $q \in \mathbb{N}$, are p-special. We have a minimal fibrant simplicial pointed set $\mathbf{T}'$ with $\pi_q|\mathbf{T}'| = 0$ for all $q > m$ and an $(m+1)$-connected simplicial bound map $\mathbf{f}'\colon \tilde{\mathbf{T}}' \to \mathbf{T}'$ (see [10, §§8, 9]). Put

$$r' = \tilde{r}' \circ (\pi_m|\mathbf{f}'|)^{-1} : \pi_m|\mathbf{T}'| \to W.$$

Since the simplicial bound map $\mathbf{f}$ is $(m+1)$-connected, $\pi_q|\mathbf{T}'| = 0$ for all $q > m$, and $\mathbf{T}'$ is fibrant, it follows that there exists a simplicial bound map $\mathbf{h} : \mathbf{T} \to \mathbf{T}'$ such that the simplicial bound maps $\mathbf{h} \circ \mathbf{f}, \mathbf{f}' \circ \tilde{\mathbf{h}}\colon \tilde{\mathbf{T}} \to \mathbf{T}'$ are homotopic ("obstruction theory"). It is easy to check that $r' \circ \pi_m|\mathbf{h}| = r$. $\mathbf{T}'$ is a simplicial finite set because $\mathbf{T}'$ is a connected minimal fibrant simplicial set and the groups $\pi_q|\mathbf{T}'|$, $q \in \mathbb{N}$, are finite.

§16. Proof of Lemma 12.1: Auxiliary lemmas

16.0. Fiberwise contraction. Suppose B is a space, (E, E') is a topological pair, $p\colon E \to B$ is a continuous map, and $p' : E' \to B$ is the restriction of p.

Let E'' be the space obtained from $E \sqcup B$ by identifying each point $X \in E'$ with $p'(X)$. Let $p''\colon E'' \to B$ be covered by $p \sqcup \mathrm{id} : E \sqcup B \to B$. We put $(E, p)/(E', p') = (E'', p'')$.

16.1. Lemma. *Suppose B is a Hausdorff space, (E_0, E_1) is a closed Borsuk pair, $p_0\colon E_0 \to B$ and $p_1\colon E_1 \to B$ are Hurewicz fibrations, and p_1 is the restriction of p_0. Put $(E_2, p_2) = (E_0, p_0)/(E_1, p_1)$. Then p_2 is a Serre fibration.*

Proof. Let $i : E_1 \to E_0$ be the inclusion, and let $c : E_0 \to E_2$ be the projection. Consider the map

$$f\colon \Gamma B \to B, \quad u \mapsto u(0),$$

which is a Hurewicz fibration. Putting

$$Q_k = \{(X, u) \in E_k \times \Gamma B : p_k(X) = f(u)\}, \quad k = 0, 1, 2,$$

we have $Q_1 \subset Q_0$. We introduce the map

$$d\colon Q_0 \to Q_2, \quad (X, u) \mapsto (c(X), u),$$

and, for $k = 0, 1, 2$, the map

$$h_k\colon \Gamma E_k \to Q_k, \quad U \mapsto (U(0), p_k \circ U).$$

We have a commutative diagram

$$\begin{array}{ccccc}
\Gamma E_1 & \xrightarrow{\Gamma i} & \Gamma E_0 & \xrightarrow{\Gamma c} & \Gamma E_2 \\
{\scriptstyle h_1}\downarrow & & {\scriptstyle h_0}\downarrow & & {\scriptstyle h_2}\downarrow \\
Q_1 & \xrightarrow{j} & Q_0 & \xrightarrow{d} & Q_2,
\end{array}$$

where j is an inclusion. Let

$$g\colon Q_0 \to E_0, \quad (X, u) \mapsto X.$$

(Q_0, Q_1) is a closed Borsuk pair by [3, Lecture 2, Proposition 5], because (E_0, E_1) is a closed Borsuk pair, g is a Hurewicz fibration (induced by the fibration f by means of p_0), and $Q_1 = g^{-1}(E_1)$. Since p_1 is a Hurewicz fibration, there exists a continuous map $s_1\colon Q_1 \to \Gamma E_1$ such that $h_1 \circ s_1 = \mathrm{id}$. Applying the covering homotopy extension theorem (see [3, Lecture 2, Theorem 2]) to the pair (Q_0, Q_1) and the fibration p_0, we get a continuous map $s_0 : Q_0 \to \Gamma E_0$ such that $h_0 \circ s_0 = \mathrm{id}$ and $s_0 \circ j = \Gamma i \circ s_1$. Obviously (?), there exists a unique map $s_2 : Q_2 \to \Gamma E_2$ such that $s_2 \circ d = \Gamma c \circ s_0$. It is not difficult to check that s_2 is weakly continuous. Clearly, $h_2 \circ s_2 = \mathrm{id}$. This implies that p_2 is a Serre fibration. □

16.2. Lemma. *Suppose X is a convenient space. Let $D \subset X^{\times 2}$ be the diagonal, let*

$$A = X^{\times 2} \times \{0\} \cup D \times I \subset X^{\times 2} \times I,$$

and let $p : X^{\times 2} \times I \to X$ be the first projection. Then there exists a retraction $R : X^{\times 2} \times I \to A$ such that $p|_A \circ R = p$.

Proof. By the Borsuk theorem, there exists a retraction $r\colon X^{\times 2} \times I \to A$. Let

$$f, g\colon X^{\times 2} \times I \to X, \quad k\colon X^{\times 2} \times I \to I$$

be the maps for which $r(Z) = (f(Z), g(Z), k(Z))$, $Z \in X^{\times 2} \times I$. We construct continuous maps

$$G\colon X^{\times 2} \times I \to X \quad \text{and} \quad K\colon X^{\times 2} \times I \to I.$$

For $Z = (x, y, t) \in X^{\times 2} \times I$, let

$$G(Z) = \begin{cases} g(x, y, 3k(Z)) & \text{if } 3k(Z) \le t, \\ f(x, y, 2t - 3k(Z)) & \text{if } t \le 3k(Z) \le 2t \text{ and } f(Z) = g(Z), \\ x & \text{if } 3k(Z) \ge 2t \text{ and } f(Z) = g(Z), \end{cases}$$

$$K(Z) = m(3k(Z) - 2t),$$

where $m : \mathbb{R} \to I$ is the increasing retraction. We put $R(Z) = (p(Z), G(Z), K(Z))$, $Z \in X^{\times 2} \times I$. □

16.3. Lemma. *Suppose $n \in \mathbb{N}$, (B, A) is an n-connected convenient pointed pair of simply connected spaces, F is a simply connected convenient pointed space, and $g\colon B \to F$ is a convenient bound map. Let $G \subset B \times F$ be the graph of g. Then the pair $(B \times F, (A \times F) \cup G)$ is $(n+2)$-connected.*

Proof. All the spaces considered below are simply connected. Therefore, all the homotopy sets (including the relative ones) are Abelian groups. We take an arbitrary $q \in \mathbb{N}$ and put $F' = \{*\} \times F \subset B \times F$. Let

$$k\colon F' \to (B \times F, G), \quad j\colon F' \to (A \times F, A \times F \cap G)$$

be the inclusions.

We show that $\pi_q k$ and $\pi_q j$ are isomorphisms. Let $p : B \times F \to B$ be the projection. Then $p|_G$ is a homeomorphism and $(p|_G)^{-1} \circ p\colon B \times F \to G$ is a retraction. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_q G & \longrightarrow & \pi_q(B \times F) & \longrightarrow & \pi_q(B \times F, G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_q(p|_G)} & & \| & & \uparrow{\scriptstyle \pi_q k} & & \\ 0 & \longleftarrow & \pi_q B & \xleftarrow{\pi_q p} & \pi_q(B \times F) & \longleftarrow & \pi_q F' & \longleftarrow & 0 \end{array}$$

where the unlabeled arrows are induced by inclusions. The rows are exact. Since $\pi_q(p|_G)$ is an isomorphism, so is $\pi_q k$. Similarly, $\pi_q j$ is an isomorphism.

Consider the commutative diagram

$$\begin{array}{ccccc} \pi_q F' & & \xrightarrow{\text{id}} & & \pi_q F' \\ \downarrow{\scriptstyle \pi_q j} & & & & \downarrow{\scriptstyle \pi_q k} \\ \pi_q(A \times F, (A \times F) \cap G) & \xrightarrow{\pi_q e} & \pi_q((A \times F) \cup G, G) & \xrightarrow{\pi_q i} & \pi_q(B \times F, G), \end{array}$$

where e and i are inclusions. The pair $(A \times F, (A \times F) \cap G)$ is simply connected. The pair $(G, (A \times F) \cap G)$ is homeomorphic to the pair (B, A) and, consequently, is n-connected. By the homotopy excision theorem, $\pi_q e$ is an epimorphism if $q \le n+1$. By the above diagram, $\pi_q i$ is an epimorphism for any q and an isomorphism if $q \le n+1$. Comparing the homotopy sequences of the pairs $(A \times F \cup G, G)$ and $(B \times F, G)$, we see that the pair $(B \times F, A \times F \cup G)$ is $(n+2)$-connected. □

§17. The proof itself of Lemma 12.1

Let $V_0 = \Delta^r \times X^{\times(r+1)}$, and let $f_0 : V_0 \to X$ be the last projection. Put

$$V_1 = \bigcup_{s=0}^{r} \{((z_1, \dots, z_r), x_1, \dots, x_{r+1}) \in V_0 : z_s = z_{s+1}, x_s = x_{s+1}\} \subset V_0,$$

where $z_0 = 0$, $z_{r+1} = 1$, and $x_0 = *$. We denote

$$f_1 := f_0|\colon V_1 \to X.$$

Below we show that f_1 is a Hurewicz fibration. Put $(V,f)=(V_0,f_0)/(V_1,f_1)$. By Lemma 16.1, f is a Serre fibration. Let

$$V_0^*=f_0^{-1}(*),\quad V_1^*=f_1^{-1}(*),\quad V^*=V_0^*/V_1^*,$$

and let $i: V^*\to V$ be the embedding covered by the inclusion $V_0^*\to V_0$.

We put

$$P=\{u\in\Gamma X: u(0)=*\},\quad W_0=\Delta^r\times P,$$

and introduce the map

$$g_0\colon W_0\to X,\quad (z,u)\mapsto u(1).$$

If

$$W_1=\partial\Delta^r\times P\subset W_0,\quad g_1:=g_0|: W_1\to X,$$

then, clearly, g_0 and g_1 are Hurewicz fibrations. Put $(W,g)=(W_0,g_0)/(W_1,g_1)$. By Lemma 16.1, g is a Serre fibration. Let $W_0^*=g_0^{-1}(*)$ and $W_1^*=g_1^{-1}(*)$, and let $j: W^*\to W$ be the embedding covered by the inclusion $W_0^*\to W_0$.

For the map

$$h_0\colon W_0\to V_0,\quad h_0(z,u):=(z,u(z_1),\dots,u(z_r),u(1)),$$
$$z=(z_1,\dots,z_r)\in\Delta^r,\quad u\in P,$$

we have $f_0\circ h_0=g_0$ and $h_0(W_1)\subset V_1$. Next, let $h\colon W\to V$ be the map covered by h_0, and let $h_0^*:=h_0|: W_0^*\to V_0^*$. Next, let $h^*: W^*\to V^*$ be the map covered by h_0^*.

We put

$$K=\Delta^r\times\{*\}\subset V_0,\quad \tilde V=V_0/(V_1\cup K),\quad \tilde V^*=V_0^*/(V_1^*\cup K).$$

Let $c\colon V\to\tilde V$ and $c^*: V^*\to\tilde V^*$ be the maps covered by $\mathrm{id}: V_0\to V_0$ and $\mathrm{id}: V_0^*\to V_0^*$ (respectively).

If

$$L=\Delta^r\times\{*\}\subset W_0,\quad \tilde W=W_0/(W_1\cup L),\quad \tilde W^*=W_0^*/(W_1^*\cup L),$$

then $\tilde W$ is homeomorphic to S^rP and, consequently, contractible. Let $d\colon W\to\tilde W$ and $d^*: W^*\to\tilde W^*$ be the maps covered by $\mathrm{id}: W_0\to W_0$ and $\mathrm{id}: W_0^*\to W_0^*$ (respectively).

We have $h_0(W_1\cup L)\subset V_1\cup K$. Let $\tilde h\colon\tilde W\to\tilde V$ be the map covered by h_0. We have $h_0^*(W_1^*\cup L)\subset V_1^*\cup K$. Let $\tilde h^*: \tilde W^*\to\tilde V^*$ be the map covered by h_0^*.

We put $Y_0=X^{\times(r+1)}$,

$$Y_1=\bigcup_{s=0}^{r}\{(x_1,\dots,x_{r+1})\in Y_0: x_s=x_{s+1}\}\subset Y_0,$$

where $x_0=*$. Let $Z=Y_0/Y_1$, and let $p_0\colon V_0\to Y_0$ be the projection. Clearly, p_0 is an equivalence. Put $p_1:=p_0|\colon V_1\cup K\to Y_1$. The preimage of each point under the map p_1 is contractible, because it is either the simplex Δ^r, or the union of at most r $(r-1)$-dimensional faces of that simplex. Therefore, p_1 is also an equivalence.

Let $q\colon\tilde V\to Z$ be the map covered by p_0. We have a commutative diagram

$$\begin{array}{ccccc}
V_1\cup K & \longrightarrow & V_0 & \longrightarrow & \tilde V\\
{\scriptstyle p_1}\big\downarrow & & {\scriptstyle p_0}\big\downarrow & & {\scriptstyle q}\big\downarrow\\
Y_1 & \longrightarrow & Y_0 & \longrightarrow & Z
\end{array}$$

with cofibration rows. Since p_0 and p_1 are equivalences, so is q. Put

$$Y_0^*=X^{\times r}\times\{*\}\subset Y_0,\quad Y_1^*=Y_0^*\cap Y_1,\quad Z^*=Y_0^*/Y_1^*,$$
$$p_0^*:=p_0|\colon V_0^*\to Y_0^*,\quad p_1^*:=p_0^*|\colon V_1^*\to Y_1^*,$$

and let $q^*: \tilde{V}^* \to Z^*$ be the map covered by p_0^*. Like p_0, p_1, and q, the maps p_0^*, p_1^*, and q^* are equivalences.

Let $l = 2r + 1$. Using Lemma 16.3 and induction on r, we see that the pair (Y_0, Y_1) is l-connected. Therefore, Z is l-connected. Consequently, $\tilde{V}$ is l-connected. Since $\tilde{W}$ is contractible, $\tilde{h}$ is l-connected. We have the commutative diagram

$$\begin{array}{ccccc} S^r \vee X & \longrightarrow & W & \xrightarrow{d} & \tilde{W} \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \tilde{h}}\downarrow \\ S^r \vee X & \longrightarrow & V & \xrightarrow{c} & \tilde{V} \end{array}$$

with cofibration rows. Thus, h is l-connected (because V and W are simply connected). We have the commutative diagram

$$\begin{array}{ccccc} W^* & \xrightarrow{j} & W & \xrightarrow{g} & X \\ {\scriptstyle h^*}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ V^* & \xrightarrow{i} & V & \xrightarrow{f} & X \end{array}$$

with fibration rows. Thus, h^* is l-connected. We have the commutative diagram

$$\begin{array}{ccccc} S^r & \longrightarrow & W^* & \xrightarrow{d^*} & \tilde{W}^* \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle h^*}\downarrow & & {\scriptstyle \tilde{h}^*}\downarrow \\ S^r & \longrightarrow & V^* & \xrightarrow{c^*} & \tilde{V}^* \end{array}$$

with cofibration rows. Thus, $\tilde{h}^*$ is l-connected.

Consider the map

$$E: I^r \times \Omega X \to W_0^*, \quad (t, u) \mapsto (t^{\triangle}, u).$$

Let $e: S^r \Omega X \to \tilde{W}^*$ be the map covered by E. It is easily seen that e is a homeomorphism. We have a commutative diagram

$$\begin{array}{ccccc} S^r \Omega X & & \xrightarrow{k} & & M^r X \\ {\scriptstyle e}\downarrow & & & & \downarrow \\ \tilde{W}^* & \xrightarrow{\tilde{h}^*} & \tilde{V}^* & \xrightarrow{q^*} & Z^*, \end{array}$$

where the second vertical arrow is the evident homeomorphism. Since q^* is an equivalence, k is l-connected.

Why is f_1 *a Hurewicz fibration*? Let $D \subset X^{\times 2}$ be the diagonal, let

$$A = X^{\times 2} \times \{0\} \cup D \times I \subset X^{\times 2} \times I,$$

and let $p: X^{\times 2} \times I \to X$ be the first projection. By Lemma 16.2, there exists a retraction $R: X^{\times 2} \times I \to A$ such that $p|_A \circ R = p$. For $x \in X$, we put

$$A_x := X \times \{0\} \cup \{x\} \times I \subset X \times I$$

and introduce the retraction

$$R_x: X \times I \to A_x$$

defined by the condition

$$R(x, y, t) = (x, R_x(y, t)), \quad y \in X, \ \ t \in I.$$

Given any point $B = (z^0, x_1^0, \ldots, x_{r+1}^0) \in V_1$, where $z^0 = (z_1^0, \ldots, z_r^0) \in \Delta^r$, we put $b = f_1(B) = x_{r+1}^0$. Suppose $u \in \Gamma X$ is a path with $u(0) = b$. We construct a path $U \in \Gamma V_1$ such that $U(0) = B$ and $f_1 \circ U = u$. We define a map $g: A_b \to X$ by putting

$g(y,0) = y$ for $y \in X$ and $g(b,t) = u(t)$ for $t \in I$. Let $h = g \circ R_b \colon X \times I \to X$. For $t \in I$, put $U(t) = (z^0, x_1, \dots, x_{r+1})$, where $x_s = h(x_s^0, tz_s^0)$, $s = 1, \dots, r+1$, and $z_{r+1}^0 = 1$. The path U depends on b and u continuously.

Acknowledgements. The statement of the problem was taken from a conversation with M. N. Gusarov; the proof roughly follows an argument in his manuscript "Axiomatic theory of cubic spaces". It was a pleasure to discuss various questions with N. E. Mnëv, N. Yu. Netsvetaev, I. A. Panin, O. Ya. Viro, and M. Yu. Zvagel′skiĭ.

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Received 1/FEB/2003

Translated by THE AUTHOR