Алгебра и анализ
Том. 16 (2004), вып. 6

St. Petersburg Math. J.
Vol. 16 (2005), No. 6, Pages 923–942
S 1061-0022(05)00884-8
Article electronically published on November 17, 2005

SPECTRAL ANALYSIS OF THE GENERALIZED SURFACE MARYLAND MODEL

F. BENTOSELA, PH. BRIET, AND L. PASTUR

Dedicated to M. S. Birman on the occasion of his 75th birthday

Abstract. The d-dimensional discrete Schrödinger operator whose potential is supported on the subspace $\mathbb{Z}^{d_2}$ of $\mathbb{Z}^d$ is considered: $H = H_a + V_M$, where $H_a = H_0 + V_a$, H_0 is the d-dimensional discrete Laplacian, V_a is a constant "surface" potential, $V_a(\mathbf{x}) = a\delta(x_1)$, $\mathbf{x} = (x_1, x_2)$, $x_1 \in \mathbb{Z}^{d_1}$, $x_2 \in \mathbb{Z}^{d_2}$, $d_1 + d_2 = d$, and $V_M(\mathbf{x}) = g\delta(x_1)\tan\pi(\alpha \cdot x_2 + \omega)$ with $\alpha \in \mathbb{R}^{d_2}$, $\omega \in [0, 1)$. It is proved that if the components of α are rationally independent, i.e., the surface potential is quasiperiodic, then the spectrum of H on the interval $[-d, d]$ (coinciding with the spectrum of the discrete Laplacian) is purely absolutely continuous, and the associated generalized eigenfunctions have the form of the sum of the incident wave and waves reflected by the surface potential and propagating into the bulk of $\mathbb{Z}^d$. If, in addition, α satisfies a certain Diophantine condition, then the remaining part $\mathbb{R} \setminus [-d, d]$ of the spectrum is pure point, dense, and of multiplicity one, and the associated eigenfunctions decay exponentially in both x_1 and x_2 (localized surface states). Also, the case of a rational $\alpha = p/q$ for $d_1 = d_2 = 1$ (i.e., the case of a periodic surface potential) is discussed. In this case the entire spectrum is purely absolutely continuous, and besides the bulk waves there are also surface waves whose amplitude decays exponentially as $|x_1| \to \infty$ but does not decay in x_2. The part of the spectrum corresponding to the surface states consists of q separated bands. For large q, the bands outside of $[-d, d]$ are exponentially small in q, and converge in a natural sense to the pure point spectrum of the quasiperiodic case with Diophantine α's.

§1. Introduction

In this paper we deal with discrete Schrödinger operators acting in $l^2(\mathbb{Z}^d)$, $d \geq 2$, and having a potential whose support is the subspace $\mathbb{Z}^{d_2}$, $1 \leq d_2 < d$. The most known case is that where $d = 3$ and $d_2 = 2$, i.e., where the support of the potential is the plane $\mathbb{Z}^2$. This case has interesting links with wave propagation and surface phenomena (see [16, 20] and the references therein), and this is why the potential is called the surface potential even if $d_1 = d - d_2 > 1$.

In recent years there has been a certain progress in spectral and scattering theory for these operators and their continuous analogs ([2, 3, 5] and [8]–[18]). In all these papers the potential is periodic, almost periodic, or random ergodic in the longitudinal coordinate $x_2 \in \mathbb{Z}^{d_2}$. It was found that in certain settings the spectrum of these operators consists of two parts. The first part coincides with the spectrum of the Laplacian and is absolutely continuous, and the associated generalized eigenfunctions do not decay in any coordinate or decay rather slowly. They can be viewed as analogs of the Sommerfeld solutions in scattering theory for potentials decaying at infinity, because they have the form of the sums of an incident wave and waves reflected by the surface potential and propagating

2000 *Mathematics Subject Classification.* Primary 35J10, 35P25.
Key words and phrases. Discrete Schrödinger operator, Maryland model.

into the bulk of $\mathbb{Z}^d$. Often, this part of the spectrum is called the *bulk spectrum*, and the corresponding generalized eigenfunctions are called the *bulk states* (*bulk waves*).

The second part of the spectrum has a location that depends on the potential. The associated eigenfunctions decay exponentially with respect to the transversal coordinate $x_1 \in \mathbb{Z}^{d_1}$ ($d_1 = d - d_2$). This part of the spectrum can be called the *surface spectrum* and the corresponding generalized eigenfunctions are then the *surface states*. If the potential is periodic, the surface states do not decay in x_2, rather they have a Bloch-type dependence with respect to this coordinate (see, e.g., [2, 9, 17, 18]). However, if the potential is random or quasiperiodic and has a sufficiently big amplitude, then the surface states (or, at least, part of them) decay exponentially also in x_2 (i.e., along the surface), and we can say that the surface states are localized [3, 8, 13, 14, 20]. Here we use the term "localized" in the same sense as in the spectral theory of random operators (see, e.g., [7]), where this term designates the point (and usually dense) spectrum.

In this paper we consider a special example of a surface potential; the "bulk" form of it is well known in localization theory under the name of the "Maryland model" [6, 7, 21]. More precisely, we consider the following selfadjoint operator acting in $l^2(\mathbb{Z}^d)$:

$$H = H_0 + V, \tag{1.1}$$

where

$$(H_0\Psi)(\mathbf{x}) = -\frac{1}{2} \sum_{|\mathbf{x}-\mathbf{y}|=1} \Psi(\mathbf{y}) \tag{1.2}$$

is the discrete Laplacian, and V is the potential of the form

$$V(\mathbf{x}) = \delta(x_1)v(x_2), \tag{1.3}$$

where $\mathbf{x} = (x_1, x_2)$, $x_1 \in \mathbb{Z}^{d_1}$, $x_2 \in \mathbb{Z}^{d_2}$, $d_1 + d_2 = d$, $d_1 \geq 1$, and

$$v(x_2) = a + g \tan \pi(\alpha x_2 + \omega). \tag{1.4}$$

Here $\alpha \in \mathbb{R}^{d_2}$, a and g are real constants, and we shall assume without loss of generality that they are positive:

$$a > 0, \quad g > 0. \tag{1.5}$$

This model case can be analyzed in great detail, thereby providing examples and explicit formulas for spectral and scattering phenomena that are only partly known for general random or almost periodic surface potentials.

The case where $a = 0$ was studied in [2, 12, 15, 20]. In particular, in [2] it was shown that if the components of α are rationally independent, then the spectrum of the respective operator is absolutely continuous on $[-d, d]$, the associated generalized eigenfunctions are quasiperiodic in x_2, and their behavior in x_1 is determined by the Green function of the d_1-dimensional Laplacian on its spectrum. These are bulk waves that combine a Bloch-type dependence on x_2 and a Sommerfeld-type dependence on x_1. If $d = 2$ and α is rational, $\alpha = p/q$, then, in addition to these generalized eigenfunctions corresponding to $[-2, 2]$, there exist surface states that decay exponentially in x_1 and are of Bloch form in x_2. The corresponding surface spectrum consists of q bands that can overlap the bulk spectrum $[-2, 2]$. In the physics paper [20] it was argued that if the "frequency" vector $\alpha \in \mathbb{R}^{d_2}$ in (1.4) satisfies the Diophantine condition

$$|\alpha x_2 - m| \geq C/|x_2|^\beta, \quad x_2 \in \mathbb{Z}^{d_2} \setminus \{0\}, \ m \in \mathbb{Z}, \tag{1.6}$$

for some $C > 0$ and $\beta > 0$, then the spectrum in $\mathbb{R} \setminus [-d, d]$ is pure point, dense, and of multiplicity one. The respective eigenfunctions decay exponentially in all coordinates, although the rate of decay in x_1 and x_2 may be different. It can be said that the surface states corresponding to $\mathbb{R} \setminus [-d, d]$ are localized.

The analysis in [2, 20] was strongly based on the Cayley transformation of the potential and of H_0 and on an explicit form of the Green function of H_0, where H_0 is the discrete Laplacian (1.2). A similar analysis of the case where $a \neq 0$ in (1.4) requires the knowledge of the Green function for the operator

$$H_a = H_0 + V_a, \quad V_a(\mathbf{x}) = a\delta(x_1). \tag{1.7}$$

This Green function has a more complex structure than that of H_0, in accordance with a more complex spectrum structure of the operator H_a. In particular, the spectrum of H_a consists of two parts, the bulk spectrum, $[-d, d]$, and the surface spectrum, $[E_0 - d_2, E_0 + d_2]$, where E_0 is the eigenvalue of the restriction of H_a to $\mathbb{Z}^{d_1}$. Nevertheless, we show that a sufficiently detailed spectral analysis of the operator (1.1)–(1.5) is still possible. If the components of α in (1.4) are rationally independent, we find that $[-d, d]$ belongs to the pure absolutely continuous spectrum, and the corresponding generalized eigenfunctions are superpositions of plane waves with incommensurable phases (§3). If, moreover, α is Diophantine (see (1.6)), then the spectrum in $\mathbb{R} \setminus [-d, d]$ is pure point, dense, and of multiplicity one, and the corresponding eigenfunctions decay exponentially (§4). In particular, we can say that the surface states of H_a are unstable with respect to the addition of the quasiperiodic surface potential

$$V_M(\mathbf{x}) = \delta(x_1) v_M(x_2), \quad v_M(x_2) = g \tan(\alpha x_2 + \omega), \tag{1.8}$$

with an arbitrarily small amplitude g. If $\alpha = p/q$ (periodic surface potential) and $d_1 = d_2 = 1$, we show that the band $[E_0 - 1, E_0 + 1]$ of the surface spectrum of H_a is transformed into q bands of the surface spectrum of H, and we study their dependence on a and g.

§2. Basic formulas

2.1. The resolvent of H. The operator H of (1.1)–(1.5) is selfadjoint as the sum of the multiplication operator V, which is selfadjoint with domain $D(V) = \{\varphi \in l^2(\mathbb{Z}^d); \|V\varphi\| < \infty\}$, and the bounded operator H_0.

Let χ be the indicator of the subspace $\mathbb{Z}^{d_2}$, and let P be the orthogonal projection defined by

$$(P\varphi)(\mathbf{x}) = \chi(\mathbf{x})\varphi(0, x_2), \quad \varphi \in l^2(\mathbb{Z}^d). \tag{2.1}$$

The range of P is then isomorphic to $l^2(\mathbb{Z}^{d_2})$, and we can write the surface potential (1.3) as

$$V = PvP, \tag{2.2}$$

where v acts in $l^2(\mathbb{Z}^{d_2})$ as multiplication by the function (1.4).

If $G(z) = (H - z)^{-1}$, $\operatorname{Im} z \neq 0$, is the resolvent of H, and $G_a(z) = (H_a - z)^{-1}$ is the resolvent of the operator H_a of (1.7), then we can introduce an operator $T(z)$ by the relation [22]

$$G(z) = G_a(z) - G_a(z)T(z)G_a(z), \tag{2.3}$$

and we have

$$T(z) = V_M - T(z)G_a(z)V_M, \tag{2.4}$$

where V_M is defined by (1.8). From (2.2) and (2.4) it follows that

$$T(z) = Pt(z)P, \tag{2.5}$$

where the operator $t(z)$ acts in $l^2(\mathbb{Z}^{d_2})$ and is given formally by

$$t(z) = v_M(1 + v_M\gamma_a(z))^{-1}, \tag{2.6}$$

where v_M is defined in (1.8) and

$$\gamma_a(z) = PG_a(z)P. \tag{2.7}$$

We introduce the following unitary operator u in $l^2(\mathbb{Z}^{d_2})$:

$$(u\psi)(x_2) = e^{-2i\pi\alpha x_2}\psi(x_2), \tag{2.8}$$

and set

$$\sigma = e^{-2i\pi\omega}. \tag{2.9}$$

Then by the Euler formula for the function $\tan : \mathbb{R} \to \mathbb{R}$, we have

$$v_M = \frac{g}{i}\frac{1-\sigma u}{1+\sigma u}. \tag{2.10}$$

Lemma 2.1. *Let $\gamma_a(z)$ be the operator acting in $l^2(\mathbb{Z}^{d_2})$ and defined by* (2.7) *for* $\operatorname{Im} z \neq 0$. *We have*:

(i) $\|\gamma_a(z)\| \leq |\operatorname{Im} z|^{-1}$;
(ii) *if* $\gamma_a(z) = R(z) + iI(z)$, *where* $R(z) = (\gamma_a(z) + \gamma_a(z)^*)/2$ *and* $I(z) = (\gamma_a(z) - \gamma_a(z)^*)/2i$, *then* $R(z)$ *and* $I(z)$ *are bounded and commute, and* $I(z)$ *is positive definite for* $\operatorname{Im} z > 0$;
(iii) $g\gamma_a(z) + i$ *is invertible for* $\operatorname{Im} z > 0$;
(iv) *if*

$$b_a(z) = \frac{g\gamma_a(z) - i}{g\gamma_a(z) + i}, \tag{2.11}$$

then

$$\|b_a(z)\| < 1 \tag{2.12}$$

for $\operatorname{Im} z > 0$.

Proof. Assertions (i)–(ii) follow directly from the definition (2.7) of $\gamma_a(z)$ and the spectral theorem for the resolvent $G_a(z)$ of the selfadjoint operator H_a. Assertion (iii) follows from (ii), because for any $\psi \in l^2(\mathbb{Z}^{d_2})$, $\|\psi\| = 1$, we have

$$\|(g\gamma_a(z) + i)\psi\|^2 = \|g\gamma_a(z)\psi\|^2 + 2g(I(z)\psi, \psi) + 1 \geq 1.$$

Assertion (iv) follows from (ii)–(iii), because for any $\psi \in l^2(\mathbb{Z}^{d_2})$, $\|\psi\| = 1$, we have

$$\|b_a(z)\psi\|^2 = 1 - 4g\left(I(z)(g\gamma_a + i)^{-1}\psi, (g\gamma_a + i)^{-1}\psi\right) < 1. \qquad \square$$

Proposition 2.2. *The resolvent $G(z)$ of H admits the following representations for* $\operatorname{Im} z > 0$:

$$\begin{aligned} G(z) &= G_a(z) - gG_a(z)P(g\gamma_a(z) + i)^{-1}PG_a(z) \\ &\quad + 2igG_a(z)P(g\gamma_a(z) + i)^{-1}\sigma u(1 - \sigma b_a(z)u)^{-1}(g\gamma_a(z) + i)^{-1}PG_a(z), \end{aligned} \tag{2.13}$$

$$G(z) = G_a(z) + \sum_{m=1}^{\infty} G_a(z)Pt_m(z)PG_a(z), \tag{2.14}$$

where

$$t_m(z) = g(g\gamma_a(z) + i)^{-1}\begin{cases} -1 & \text{if } m = 1, \\ 2i\sigma u(\sigma b_a(z)u)^{m-2}(g\gamma_a(z) + i)^{-1} & \text{if } m \geq 2, \end{cases} \tag{2.15}$$

and the series in (2.14) *converges in the operator norm sense.*

Proof. Replace σ of (2.9) in (2.10) by a complex number ζ, $|\zeta| < 1$. Then the operators $v_{\zeta,M}$ and $(v_{\zeta,M})^{-1}$ obtained from (2.10) by the replacement $\sigma \to \zeta$ will be bounded, and we have

$$(v_{\zeta,M})^{-1} + \gamma_a = \frac{i}{g}\frac{1+\zeta u}{1-\zeta u} + \gamma_a = \frac{1}{g}(g\gamma_a + i)(1-\zeta b_a u)(1-\zeta u)^{-1}.$$

Since $|b_a(z)| < 1$ by Lemma 2.1, the operator $1 - \zeta b_a(z)u$ is invertible, and then for the respective operator $t_\zeta(z)$ of (2.6) we get

$$t_\zeta(z) = g(1-\zeta u)(1-\zeta b_a(z)u)^{-1}(g\gamma_a(z)+i)^{-1}.$$

Now a simple algebra yields

$$\begin{aligned} t_\zeta(z) &= g(g\gamma_a(z)+i)^{-1} - 2ig(g\gamma_a(z)+i)^{-1}\zeta u(1-\zeta b_a u)^{-1}(g\gamma_a(z)+i)^{-1} \\ &= -\sum_{m=1}^{\infty} t_{\zeta,m}(z), \end{aligned}$$

where the $t_{\zeta,m}$ are given by formula (2.15) in which σ is replaced by ζ. Let $\mathcal{D}$ denote the set of vectors in $l^2(\mathbf{Z}^{d_2})$ of finite support. $\mathcal{D}$ is dense and $\lim_{\zeta\to\sigma} v_{\zeta,M}\psi = v_M\psi$ for any $\psi \in \mathcal{D}$. Hence, by general principles (see, e.g., [19, Corollary VIII.1.6]), the resolvent of $H_a + Pv_{\zeta,M}P$ converges strongly on $\mathcal{D}$ to the resolvent $G(z)$ of $H_a + Pv_MP$. On the other hand, since $|b_a(z)| < 1$ for $\operatorname{Im} z > 0$ by Lemma 2.1 (iv), the series converges in the operator norm sense uniformly in $|\zeta| \le 1$. Consequently, we can pass to the limit $\zeta \to \sigma$ in the series to obtain the claim of the proposition. □

Our subsequent analysis will be based on formulas (2.13)–(2.14). Namely, (2.14) will be used in §3 to study the absolutely continuous spectrum (we prove that the series for the Green function $G(\mathbf{x},\mathbf{y};z) = (H-z)^{-1}(\mathbf{x},\mathbf{y})$, i.e., the matrix form of (2.14), converges uniformly in $z = E+i\varepsilon$, $|E| \le d-\delta$, $\varepsilon \ge 0$, for every $\delta > 0$ and every $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$), and (2.13) will be used in §4 to study the point spectrum (we diagonalize the operator $1 - \sigma b_a(z)u$ for all z up to $z = E + i0$, $|E| \ge d + \delta$ for every $\delta > 0$). Recall that, for a symmetric operator A, the isometric operator $U = (A-i)(A+i)^{-1}$ is called the Cayley transform of A, and the formula $A = i(1+U)(1-U)$ expresses A via its Cayley transform (see, e.g., [1]). Hence, we can view the Euler formula (2.10) as a particular case of these formulas for $A = -v_M/g$ and $U = -\sigma u$. Similarly, (2.11) can be viewed as a contraction that is the Cayley transform of the dissipative operator $g\gamma_a(z)$, $\operatorname{Im} z > 0$. Thus, the passage from $Pg(H_a - z)^{-1}P$ and $-PV_MP\ /g$ to their Cayley transforms $gb_a(z)$ and $-\sigma u$ and the fact that u is simply the shift by α after the Fourier transform (see formula (3.4) below) provide an "algebraic" basis for the spectral analysis of the operator (1.1)–(1.5), as given in the subsequent sections.

2.2. The operator H_a. To use the formalism presented in the preceding subsection, for the spectral analysis of the operator H we need to know more about the resolvent $G_a(z)$ of the operator H_a of (1.7) corresponding to a constant surface potential.

Denoting by $\mathbb{T}^\nu = [0,1)^\nu$ the ν-dimensional torus, we consider the Fourier transformation defined by the formulas

$$\widehat{\varphi}(\mathbf{k}) = \sum_{\mathbf{x}\in\mathbb{Z}^\nu} e^{-2i\pi\mathbf{x}\mathbf{k}}\varphi(\mathbf{x}),\ \ \mathbf{k}\in\mathbb{T}^\nu, \quad \varphi(\mathbf{x}) = \int_{\mathbb{T}^\nu} d\mathbf{k} e^{2i\pi\mathbf{x}\mathbf{k}}\widehat{\varphi}(\mathbf{k}),\ \ \mathbf{x}\in\mathbb{Z}^\nu, \tag{2.16}$$

for any φ with a finite support in $\mathbb{Z}^\nu$, and extended to $l^2(\mathbb{Z}^\nu)$ by usual arguments. Then the Green function $G_0^{(\nu)}(\mathbf{x}-\mathbf{y};z) = (H_0 - z)^{-1}(\mathbf{x},\mathbf{y})$, $\operatorname{Im} z \ne 0$, of the ν-dimensional

Laplacian (1.2) is

$$
\begin{aligned}
G_0^{(\nu)}(\mathbf{x}-\mathbf{y};z) &= \int_{\mathbb{T}^\nu} d\mathbf{k} \frac{e^{2i\pi\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{E_d(\mathbf{k})-z}, \\
E_\nu(\mathbf{k}) &= -\sum_{i=1}^{\nu} \cos 2\pi k_i, \quad (k_1,\dots,k_\nu)=\mathbf{k}\in\mathbb{T}^\nu.
\end{aligned} \tag{2.17}
$$

Proposition 2.3. *Let H_a be defined in* (1.7). *Then its Green function $G_a(\mathbf{x},\mathbf{y};z) = (H_a - z)^{-1}(\mathbf{x},\mathbf{y})$, $\operatorname{Im} z \neq 0$, admits the following representation*:

$$
\begin{aligned}
&G_a(\mathbf{x},\mathbf{y};z) \\
&\quad = G_0^{(d)}(\mathbf{x}-\mathbf{y};z) \\
&\qquad - a\int_{\mathbb{T}^{d_2}} dk_2 \frac{e^{2i\pi k_2(x_2-y_2)}}{1+a\hat{\gamma}_0(k_2;z)} G_0^{(d_1)}(x_1;z-E_{d_2}(k_2)) G_0^{(d_1)}(y_1;z-E_{d_2}(k_2)),
\end{aligned} \tag{2.18}
$$

where $G_0^{(\nu)}(\mathbf{x};z)$, $nu = d, d_1$, $\mathbf{x} = (x_1,x_2)$, $x_1 \in \mathbb{Z}^{d_1}$, $x_2 \in \mathbb{Z}^{d_2}$ is given by (2.17), *and*

$$
\widehat{\gamma}_0(k_2;z) = G_0^{(d_1)}(0;z-E_{d_2}(k_2)) = \int_{\mathbb{T}^{d_1}} \frac{dk_1}{E_d(\mathbf{k})-z}. \tag{2.19}
$$

Proof. We make the Fourier transformation of the equation for $G_a(\mathbf{x},\mathbf{y};z)$,

$$
-\frac{1}{2}\sum_{|\mathbf{x}-\mathbf{x}'|=1} G_a(\mathbf{x}',\mathbf{y};z) + a\delta(x_1)G_a(\mathbf{x},\mathbf{y};z) = zG_a(\mathbf{x},\mathbf{y};z) + \delta(\mathbf{x}-\mathbf{y}),
$$

with respect to the longitudinal variable x_2. This leads to a similar equation with the d_1-dimensional Laplacian, the potential $a\delta(x_1)$, and the spectral parameter $z - E_{d_2}(k_2)$. The Green function corresponding to this potential (a rank one operator in $l^2(\mathbb{Z}^{d_1})$) can easily be found. This yields (2.18) after the inverse Fourier transformation, because $1+\widehat{\gamma}_0(k_2;z)$ is nonzero for $\operatorname{Im} z \neq 0$ and all $k_2 \in \mathbb{T}^{d_2}$, by (2.19) and (2.17). □

Lemma 2.4. *The operators $\gamma_a(z)$ of* (2.7) *and $b_a(z)$ of* (2.11) *are bounded convolution operators in $l^2(\mathbb{Z}^{d_2})$*:

$$
\gamma_a(z) = \{\gamma_a(x_2-y_2;z)\}_{x_2,y_2\in\mathbb{Z}^{d_2}}, \tag{2.20}
$$

where

$$
\gamma_a(x_2;z) = \int_{\mathbb{T}^{d_2}} dk_2 \widehat{\gamma}_a(k_2;z) e^{ik_2x_2} \tag{2.21}
$$

with

$$
\widehat{\gamma}_a(k_2;z) = \frac{\widehat{\gamma}_0(k_2;z)}{1+a\widehat{\gamma}_0(k_2;z)}, \tag{2.22}
$$

and

$$
b_a(z) = \{b_a(x_2-y_2;z)\}_{x_2,y_2\in\mathbb{Z}^{d_2}}, \tag{2.23}
$$

where

$$
\hat{b}_a(k_2;z) = \frac{g\widehat{\gamma}_a(k_2;z)-i}{g\widehat{\gamma}_a(k_2;z)+i} = \frac{\bar{\mathbf{g}}\widehat{\gamma}_0(k_2;z)-i}{\mathbf{g}\widehat{\gamma}_0(k_2;z)+i} \tag{2.24}
$$

with

$$
\mathbf{g} = g + ia. \tag{2.25}
$$

Proof. Formulas (2.20)–(2.22) follow immediately from (2.17) and (2.18)–(2.19). The fact that $\gamma_a(z)$ is bounded was proved in Lemma 2.1 (and also is a consequence of (2.20)–(2.22)). Formulas (2.23)–(2.25) follow from (2.11) and the fact that the convolution operators form an algebra. Next, $b_a(z)$ is bounded by (2.12). The same can be deduced from the first relation in (2.24), because $\operatorname{Im}\widehat{\gamma}_a > 0$ if $\operatorname{Im} z > 0$. □

Now we describe the spectral structure of the operator H_a. First, we recall the definition of the surface states given in [16] (see [5, 11] for other definitions). Namely, we say that, for a Schrödinger operator H, a generalized eigenfunction $\Psi_E = \{\Psi_E(\mathbf{x})\}_{\mathbf{x}\in\mathbb{Z}^d}$ corresponding to a point E of its spectrum (a polynomially bounded solution of the equation $(H\Psi_E)(\mathbf{x}) = E\Psi_E(\mathbf{x})$) is a *surface state* if for each $\varepsilon > 0$ we have

$$\sup_{x_2\in\mathbb{Z}^{d_2}} (1+|x_2|)^{-(d_2+\varepsilon)} \sum_{x_1\in\mathbb{Z}^{d_1}} |\Psi_E(x_1,x_2)|^2 < \infty. \tag{2.26}$$

Similarly, we say that a generalized eigenfunction Ψ_E is a *bulk state* if

$$\sup_{x\in\mathbb{Z}^d} (1+|\mathbf{x}|)^{-(d+\varepsilon)/2} |\Psi_E(\mathbf{x})| < \infty \tag{2.27}$$

for every $\varepsilon > 0$ but Ψ_E is not in $l^2(\mathbb{Z}^{d_1})$ in x_1 for any $x_2 \in \mathbb{Z}^{d_2}$.

We also need the following result.

Proposition 2.5. *Consider the operator acting on $l^2(\mathbb{Z}^{d_1})$ and defined as the sum of the d_1-dimensional Laplacian* (1.2) *and the potential $a\delta(x_1)$. Then its point spectrum consists of at most a single eigenvalue E_0, $|E_0| > d_1$, which is the solution of the equation*

$$1 + aG_0^{(d_1)}(0,0;E_0) = 0. \tag{2.28}$$

This equation is uniquely soluble if $a > a_c$, where

$$a_c = (G_0^{(d_1)}(0,0;d_1))^{-1}. \tag{2.29}$$

In particular, the solution exists for all $a > 0$ if $d_1 = 1, 2$.

Then we have the following statement.

Theorem 2.6. *Let H_a be the operator defined by* (1.7) *and* (1.2). *Then*

(i) *the spectrum $\sigma(H_a)$ of H_a is $\sigma(H_a) = \Sigma_{a,b} \cup \Sigma_{a,s}$, where*

$$\Sigma_{a,b} = [-d,d] = \{E \in \mathbb{R} : \exists \mathbf{k} \in \mathbb{T}^d, E = E_d(\mathbf{k})\}, \tag{2.30}$$

and

$$\Sigma_{a,s} = \begin{cases} E_0 + [-d_2,d_2] = \{E \in \mathbb{R}; \exists k_2 \in \mathbb{T}^{d_2}, E = E_0 + E_{d_2}(k_2)\} & \text{if } a > a_c, \\ \varnothing & \text{if } a \le a_c; \end{cases} \tag{2.31}$$

here $E_\nu(k)$ ($\nu = d, d_2$), a_c, and E_0 are defined in (2.17), (2.29), *and* (2.28);

(ii) *the spectrum of H_a is purely absolutely continuous*;

(iii) *the generalized eigenfunctions $\Psi^{\pm}_{a,b}(.,\mathbf{k})$ associated with $E = E_d(\mathbf{k}) \in (-d,d)$ for some $\mathbf{k} \in \mathbb{T}^d$ have the form*

$$\Psi^{\pm}_{a,b}(\mathbf{x},\mathbf{k}) = e^{2i\pi\mathbf{kx}} - a\frac{G_0^{(d_1)}(x_1; z - E_{d_2}(k_2))}{1 + a\hat{\gamma}_0(k_2;z)}\bigg|_{z=E_d(\mathbf{k})\mp i0} e^{2i\pi k_2 x_2}; \tag{2.32}$$

(iv) *if $a > a_c$, then the generalized eigenfunctions $\Psi_{a,s}(.,k_2)$ associated with $E = E_0 + E(k_2) \in \Sigma_{a,s}$ for some $k_2 \in \mathbb{T}^{d_2}$ have the form*

$$\Psi_{a,s}(\mathbf{x},k_2) = I^{-1/2} G_0^{(d_1)}(x_1;E_0) e^{2i\pi k_2 x_2}, \tag{2.33}$$

where

$$I = \int_{\mathbb{T}^{d_1}} \frac{dk_1}{(E(k_1) - E_0)^2}. \tag{2.34}$$

The proof of this theorem can be obtained either from formulas (2.18)–(2.19) or directly, by separating the variables in the equation $(H\Psi)(\mathbf{x}) = E\Psi(\mathbf{x})$.

Remarks. 1) It can be shown that the families (2.32) and (2.33) form a complete and orthogonal system of eigenfunctions. This can be written symbolically as follows:

$$\delta(\mathbf{k} - \mathbf{k}') = \sum_{\mathbf{x}\in\mathbb{Z}^d} \Psi^{\pm}_{a,b}(\mathbf{x}, \mathbf{k})\overline{\Psi^{\pm}_{a,b}(\mathbf{x}, \mathbf{k}')}, \quad \delta(k_2 - k_2') = \sum_{\mathbf{x}\in\mathbb{Z}^d} \Psi_{a,s}(\mathbf{x}, k_2)\overline{\Psi_{a,s}(\mathbf{x}, k_2')},$$

$$\sum_{\mathbf{x}\in\mathbb{Z}^d} \Psi^{\pm}_{a,b}(\mathbf{x}, \mathbf{k})\overline{\Psi_{a,s}(\mathbf{x}, k_2')} = 0,$$

and

$$\delta(\mathbf{x} - \mathbf{x}') = \int_{\mathbb{T}^d} d\mathbf{k}\Psi^{\pm}_{a,b}(\mathbf{x}, \mathbf{k})\overline{\Psi^{\pm}_{a,b}(\mathbf{x}', \mathbf{k})} + \int_{\mathbb{T}^{d_2}} dk_2\Psi_{a,s}(\mathbf{x}, k_2)\overline{\Psi_{a,s}(\mathbf{x}', k_2)}.$$

2) By (2.30), the generalized eigenfunctions $\{\Psi^{\pm}_{a,b}(., \mathbf{k}) : E_d(\mathbf{k}) \in (-d, d)\}$ of (2.32) are the bulk states, because for these values of the spectral parameter the Green function $G_0^{(d_1)}$ does not belong to $l^2(\mathbb{Z}^{d_1})$ in x_1. Similarly, the generalized eigenfunctions $\{\Psi_{a,s}(., k_2)\}_{k_2\in\mathbb{T}^{d_2}}$ of (2.33) are surface states, because if the spectral parameter belongs to (2.31), then $G_0^{(d_1)}$ decays exponentially as $x_1 \to \infty$. Note that the band $[E_0 - d_2, E_0 + d_2]$ of the surface states may or may not overlap the band $[-d, d]$ of the bulk states. In particular, if $d_1 = 1$, then $E_0 = \sqrt{1 + a^2}$, and the bands overlap if $E_0 < d + 1$.

3) We do not need Theorem 2.6 in the subsequent analysis of the operator H, although we need the Green function G_a of (2.18)–(2.19). We give this theorem to illustrate the notions of the bulk and the surface spectrum in a simple situation.

§3. Absolutely continuous spectrum

3.1. Almost periodic case. In this subsection, we shall show that if the components of the vector α in (1.8) are rationally independent, then the spectrum of H of (1.1)–(1.5) is purely absolutely continuous on $[-d, d]$, the spectrum of the discrete Laplacian H_0 (see (1.2)). We mention that in our earlier paper [2] we showed that the above is true if $a = 0$, and that the corresponding operator admits an eigenfunction expansion on the interval $[-d, d]$, with the same dispersion law as in the free case, and with the generalized eigenfunctions that are "bulk states". Below we shall show the stability of this property for the operator $H = H_a + V_M$ for all $a > 0$. Our presentation will be mostly of review type, because respective proofs (sometimes rather involved) differ from those in [2] only in certain technical details.

First, note that the spectrum $\sigma(H)$ of H contains the interval $[-d, d] = \sigma(H_0)$ for all $g > 0$, $a > 0$, $\alpha \in \mathbb{R}^{d_2}$, and $\omega \in [0, 1]$. This is a general consequence of the fact that the support of the potential is the "surface" $\mathbb{Z}^{d_2}$, $d_2 < d$. So, we apply the Weyl criterion with the following sequence of test functions:

$$\Psi_n(\mathbf{x}) = \mathbf{1}_n(\mathbf{x})(1 - \delta(x_1))e^{2i\pi\mathbf{kx}}/N_n,$$

$$N_n^2 = \sum_{\mathbf{x}\in\mathbb{Z}^d} |\mathbf{1}_n(\mathbf{x})(1 - \delta(x_1))|^2 = O(n^d), \quad n \to \infty,$$

where $\mathbf{1}_r$ is the indicator of the ball $\{\mathbf{x} \in \mathbb{Z}^d\colon |\mathbf{x}| \le r\}$, and $\mathbf{k} \in \mathbb{T}^d$. Then, clearly, we have $\|(H - E_d(\mathbf{k}))\Psi_n\| = O(n^{-1/2})$ as $n \to \infty$, whence the above statement.

Proposition 3.1. *Suppose that* $\operatorname{Im} z > 0$. *Then the Green function* $G(\mathbf{x},\mathbf{y};z) = (H-z)^{-1}(\mathbf{x},\mathbf{y})$ *of* (1.1)–(1.5) *admits the representation*

$$\begin{aligned} G(\mathbf{x},\mathbf{y};z) &= G_0^{(d)}(\mathbf{x}-\mathbf{y};z) \\ &\quad + \sum_{m=0}^{\infty}\int_{\mathbb{T}^{d_2}} dk_2\, e^{2i\pi k_2(x_2-y_2)} t_{a,m}(k_2;z) G_0^{(d_1)}(x_1; z-E_{d_2}(k_2)) \\ &\qquad\qquad \times G_0^{(d_1)}(y_1; z-E_{d_2}(k_2+m\alpha)) e^{-2i\pi m\alpha y_2}, \end{aligned} \tag{3.1}$$

where

$$t_{a,m}(k_2;z) = \frac{1}{\mathbf{g}\widehat{\gamma}_0(k_2;z)+i} \begin{cases} -\mathbf{g} & \text{if } m=0, \\ 2ig\sigma(\mathbf{g}\widehat{\gamma}_0(k_2+\alpha;z)+i)^{-1} & \text{if } m=1, \\ 2ig\sigma^m(\mathbf{g}\widehat{\gamma}_0(k_2+m\alpha;z)+i)^{-1}P_{m-1}(k_2;z) & \text{if } m\geq 2, \end{cases} \tag{3.2}$$

and

$$P_m(k_2;z) = \prod_{l=1}^{m} \hat{b}_a(k_2+l\alpha;z), \quad k_2 \in \mathbb{T}^{d_2}. \tag{3.3}$$

Proof. This follows easily from Propositions 2.2, 2.3 and the observation that the operator u of (2.8) acts in $L^2(\mathbb{T}^{d_2})$ as the shift by $\alpha \in \mathbb{R}^{d_2}$, i.e., for any $\varphi \in L^2(\mathbb{T}^{d_2})$ we have

$$(u\varphi)(k_2) = \varphi(k_2+\alpha). \tag{3.4}$$

□

Theorem 3.2. *Let* H *be the selfadjoint operator defined in* (1.1)–(1.5), *where the vector* $\alpha \in \mathbb{R}^{d_2}$ *has rationally independent components. Then the spectrum of* H *on the interval* $[-d,d]$ *is purely absolutely continuous.*

Proof. The proof of this theorem is based on the same method as that developed in [2] for the case of $a=0$. In fact, it yields a stronger result saying that $G(\mathbf{x},\mathbf{y};E+i0)$ exists and is bounded for every $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$ and every real $E \in (-d,d)$. We show that the series in (3.1), which converges for $\operatorname{Im} z > 0$ (because $\|b_a(z)\| < 1$ in this case; see (2.12)), also converges if $z = E+i0$, $|E| < d$.

Given $\gamma \in (0,1)$, we define

$$K_\gamma(E) = \{k_2 \in \mathbb{T}^{d_2} : E - E_{d_2}(k_2) \in [-d_1+\gamma, d_1-\gamma]\}. \tag{3.5}$$

Then $K_\gamma(E)$ is a closed subset of $\mathbb{T}^{d_2}$, and if $|E| < d$, then there exists γ such that $|K_\gamma(E)| > 0$, where $|K_\gamma(E)|$ is the Lebesgue measure of $K_\gamma(E)$. If $k_2 \in K_\gamma(E)$, then $\operatorname{Im}\hat{\gamma}_0(k_2, E+i0) > 0$, and $|\hat{b}_a(k_2, E+i0)| < 1$ by (2.24). Hence, by continuity, there exists $\delta > 0$ such that

$$|\hat{b}_a(k_2, E+i0)| \leq 1-\delta, \;\; k_2 \in K_\gamma(E). \tag{3.6}$$

Now we recall the assumption that the vector α has rationally independent components. This implies the following limit relation, valid uniformly in $k_2 \in \mathbb{T}^{d_2}$ (see [4]):

$$\lim_{m\to\infty} \sharp\{l \in \mathbb{N} : k_2 + l\alpha \in K_\gamma(E), 1 \leq l \leq m\} m^{-1} = |K_\gamma(E)| > 0.$$

Then there exists an integer $m_0 > 0$ such that

$$\sharp\{l \in \mathbb{N} : k_2 + l\alpha \in K_\gamma(E), 1 \leq l \leq m\} \geq \frac{m}{2}|K_\gamma(E)|, \quad m \geq m_0.$$

This and (3.6) imply that the product P_m on the right in (3.2) admits the bound

$$\left|\prod_{l=0}^{m-1} \hat{b}_a(k_2+l\alpha; E+i0)\right| \leq (1-\delta)^{m|K_\gamma(E)|/2}, \quad m \geq m_0. \tag{3.7}$$

Moreover, since $\operatorname{Im}\hat{\gamma}_0(k_2;z)\geq 0$ for $\operatorname{Im} z\geq 0$ and $k_2\in\mathbb{T}^{d_2}$, we have $\hat{\gamma}_0(k_2;z)\mathbf{g}+i\neq 0$. Then a standard argument shows that for $m\geq 1$ the factors

$$2ig\sigma^m\frac{G_0^{(d_1)}(x_1;z-E_{d_2}(k_2))}{\mathbf{g}\widehat{\gamma}_0(k_2;z)+i}\cdot\frac{G_0^{(d_1)}(y_1;z-E_{d_2}(k_2+m\alpha))}{\mathbf{g}\widehat{\gamma}_0(k_2+m\alpha;z)+i}\Big|_{z=E+i\varepsilon}$$

of the integrands in (3.1) are bounded uniformly in $k_2\in\mathbb{T}^{d_2}$ and $\varepsilon\geq 0$. Clearly, the same is true for $m=0$. This fact and the bound (3.7) show that the terms of the series on the right-hand side in (3.1) decay exponentially as $m\to\infty$ uniformly in $k_2\in\mathbb{T}^{d_2}$ and $\varepsilon\geq 0$. Thus, the series converges uniformly in $k_2\in\mathbb{T}^{d_2}$ and $\varepsilon\geq 0$, and the theorem follows. □

Our next result concerns the eigenfunction expansion of the operator H in the interval $[-d,d]$.

Theorem 3.3. *Let $\dot{\mathbb{T}}^d$ be defined as*

$$\dot{\mathbb{T}}^{d_2}=\mathbb{T}^{d_2}\setminus\{\overbrace{(0,0,\dots,0)}^{d_2\text{ times}},\overbrace{(\pi,\pi,\dots,\pi)}^{d_2\text{ times}}\};\quad \dot{\mathbb{T}}^d=\mathbb{T}^{d_1}\times\dot{\mathbb{T}}^{d_2}. \tag{3.8}$$

Then, under the assumptions of Theorem 3.2, the functions

$$\begin{aligned}&\Psi_\pm(\mathbf{x},\mathbf{k})\\&\quad=e^{2i\pi\mathbf{k}\mathbf{x}}\\&\qquad+\sum_{m=0}^{\infty}t_{a,m}(k_2-m\alpha;z)G_0^{(d_1)}(x_1;z-E_{d_2}(k_2-m\alpha))e^{2i\pi(k_2-m\alpha)x_2}\Big|_{z=E_d(\mathbf{k})\mp i0},\end{aligned} \tag{3.9}$$

where the coefficients $t_{a,m}$, $m\geq 0$, are given by (3.2), are bounded in $\mathbf{x}\in\mathbb{Z}^d$ for every $\mathbf{k}\in\dot{\mathbb{T}}^d$, are continuous in $\mathbf{k}$ varying in any compact subset of $\dot{\mathbb{T}}^d$, and satisfy the Schrödinger equation in $\mathbf{x}$:

$$(H\Psi_\pm)(\mathbf{x},\mathbf{k})=E_d(\mathbf{k})\Psi_\pm(\mathbf{x},\mathbf{k}). \tag{3.10}$$

The families $\{\Psi_\pm(\cdot,\mathbf{k});\mathbf{k}\in\dot{\mathbb{T}}^d\}$ are complete families of generalized eigenfunctions for the operator H on $(-d,d)$, i.e.,

(i) *for any $\varphi\in l^2(\mathbb{Z}^d)$, the series*

$$\Phi_\pm(\mathbf{k})=\sum_{\mathbf{x}\in\mathbb{Z}^d}\overline{\Psi_\pm(\mathbf{x},\mathbf{k})}\varphi(\mathbf{x}) \tag{3.11}$$

converges in $l^2(\mathbb{Z}^d)$;

(ii) *if $\mathcal{E}_H(\Delta)$ is the spectral projection of H corresponding to the closed interval $\Delta=[a,b]\subset(-d,d)$, then*

$$\|\mathcal{E}_H(\Delta)\varphi\|^2=\int_{\{\mathbf{k}\in\mathbb{T}^d:E_d(\mathbf{k})\in\Delta\}}|\Phi_\pm(\mathbf{k})|^2\,d\mathbf{k}, \tag{3.12}$$

where $E_d(\mathbf{k})$ is defined in (2.17), and

$$\|H\mathcal{E}_H(\Delta)\varphi\|^2=\int_{\{\mathbf{k}\in\mathbb{T}^d:E_d(\mathbf{k})\in\Delta\}}|E_d(\mathbf{k})\Phi_\pm(\mathbf{k})|^2\,d\mathbf{k}; \tag{3.13}$$

(iii) *the families $\{\Psi_\pm(\cdot,\mathbf{k})\}_{\mathbf{k}\in\dot{\mathbb{T}}^d}$ are orthonormal, i.e., for any continuous functions $\widehat{\Phi}^{(1)}$ and $\widehat{\Phi}^{(2)}$ with compact support in $\dot{\mathbb{T}}^d$ we have*

$$\sum_{\mathbf{x}\in\mathbb{Z}^d}\varphi_\pm^{(1)}(\mathbf{x})\overline{\varphi_\pm^{(2)}(\mathbf{x})}=\int_{\mathbb{T}^d}\widehat{\Phi}^{(1)}(\mathbf{k})\overline{\widehat{\Phi}^{(2)}(\mathbf{k})}\,d\mathbf{k}, \tag{3.14}$$

where

$$\varphi_{\pm}^{(i)}(\mathbf{x}) = \int_{\mathbb{T}^d} \Psi_{\pm}(\mathbf{x},\mathbf{k})\widehat{\Phi}^{(i)}(\mathbf{k})\,d\mathbf{k}, \quad i = 1, 2. \tag{3.15}$$

We omit the rather involved proof of this theorem, because it differs from those of Theorems 3.3 and 3.4 in [2] only by technical details.

Remark. Since the functions (3.9) do not decay in x_2 (being the sum of the incident plane wave and a "quasi-Bloch" function) and decay slowly or do not decay at all in x_1 (recall that $G_0^{(d_1)}(x_1, E + i0) = O(|x_1|^{-(d_1-1)/2})$ as $|x_1| \to \infty$ if $|E| < d_1$), they are bulk states (see (2.27)). Hence, the surface states of the operator H_a, which exist on $[-d, d]$ for $E_0 < d+d_2$, disappear after addition to H_a of the surface potential V_M of (1.8), whatever small is the coupling constant g. However, since the expression $E_d(\mathbf{k}) - E_{d_2}(k_2 - m\alpha)$ takes values inside the interval $(-d_1, d_1)$ as well as outside this interval as m varies, the Green function $G_0^{(d_1)}(x_1; E_d(\mathbf{k}) - E_{d_2}(k_2 - m\alpha))$ involved in (3.9) may be exponentially decaying or slowly decaying in x_1 (i.e., as $1/|x_1|^{(d_1-1)/2}$). Thus, despite the fact that in the quasiperiodic case (which we are considering now) the surface states are absent on the part $[-d, d]$ of the spectrum [15], the bulk states (3.9) contain both terms slowly decaying (or even only oscillating for $d_1 = 1$) in $|x_1|$ and terms exponentially decaying in $|x_1|$, which correspond to waves scattered by the surface potential and propagating inside the bulk $\mathbb{Z}^d$ and along the subspace $\mathbb{Z}^{d_2}$, i.e., the support of the quasiperiodic perturbation (strongly corrugated quasiperiodic surface in the case where $d_1 = 1$). In other words, we can write

$$\Psi(\mathbf{x},\mathbf{k}) = e^{2i\pi\mathbf{kx}} + \Psi_{\text{bulk}}(\mathbf{x},\mathbf{k}) + \Psi_{\text{surf}}(\mathbf{x},\mathbf{k}), \tag{3.16}$$

where $e^{2i\pi\mathbf{kx}} + \Psi_{\text{bulk}}$ satisfies (2.27) and Ψ_{surf} satisfies (2.26), but they are not solutions of the Schrödinger equation (3.10), i.e., are not bulk states and surface states, respectively.

3.2. Periodic case. In this section we consider the operator H of (1.1)–(1.5) for $d_1 = d_2 = 1$ and V_M of (1.8) in which α is a rational number: $\alpha = p/q$, where p and $q > p$ are positive integers and q is not a divisor of p. In this case the potential is periodic with period q. As in the case of $a = 0$ (see [2]) the entire spectrum of H is absolutely continuous, but there are two types of generalized eigenfunctions. The first type corresponds to energies in the interval $(-2, 2)$, where the generalized eigenfunctions have the Bloch–Floquet behavior in the x_2-direction and are finite sums of plane waves in x_1, so that they do not decay in the x_1-direction. The second type corresponds to energies which may be outside or inside the interval $(-2, 2)$. Here the generalized eigenfunctions have the Bloch–Floquet behavior in the x_2-direction again, but decay exponentially in the x_1-direction. The corresponding energies fill q separated intervals, which can be called the *surface bands*.

Again, here we use the same strategy as in [2], basing our analysis on the following formula for the Green function $G(\mathbf{x}, \mathbf{y}; z)$ of H:

$$\begin{aligned} G(\mathbf{x},\mathbf{y};z) &= G_a^{(2)}(\mathbf{x}-\mathbf{y};z) \\ &\quad + \sum_{m=0}^{q} \int_{\mathbb{T}} dk_2\, e^{2i\pi k_2(x_2-y_2)} t_{a,m}(k_2;z) G_0^{(1)}(x_1; z + \cos 2\pi k_2) \\ &\qquad \times G_0^{(1)}(y_1; z + \cos 2\pi(k_2 + m\alpha)) e^{-2i\pi m\alpha y_2}, \end{aligned} \tag{3.17}$$

where now

$$t_{a,m}(k_2;z) = \frac{1}{g\widehat{\gamma}_a(k_2;z)+i} \begin{cases} -\mathbf{g} & \text{if } m=0, \\ \frac{1}{1-P_q(k_2;z)} \frac{2i\sigma}{\mathbf{g}\hat{\gamma}_a(k_2+\alpha;z)+i} & \text{if } m=1, \\ \frac{1}{1-P_q(k_2;z)} \frac{2i\sigma}{\mathbf{g}\hat{\gamma}_a(k_2+m\alpha;z)+i} P_{m-1}(k_2;z) & \text{if } m \geq 2, \end{cases} \tag{3.18}$$

$G_0^{(1)}(.;z)$ is the Green function (2.17) for $\nu = 1$, $\hat{\gamma}_0(.;z)$, and $\hat{\gamma}_a(.;z)$ are defined in (2.19) and (2.22), and $P_q(k_2, E)$ is given by (3.3) for $m = q$.

This formula can be obtained from (3.1) if we take into account that $b_a(k_2 + lp/q; z)$ is q-periodic in l, so that $P_m = P_q^\mu \cdot P_\nu$ for $m = \mu q + \nu$, $\mu \in \mathbb{N}$, $0 \leq \nu < q-1$, and the series in (3.1) can be summed with respect to μ, which leads to the denominator $1 - P_q$ in (3.18). This denominator gives rise to singularities of the integrand, thereby determining the surface energy bands $E_j(k_2)$ as solutions of the equation

$$1 - P_q(k_2; E + i0) = 0. \tag{3.19}$$

Relation (3.3) and the inequality $|\hat{b}_a(k_2; E + i0)| \leq 1$, valid for every $k_2 \in \mathbb{T}^{d_2}$ and every $E \in \mathbb{R}$, show that equation (3.19) can only be fulfilled if the absolute values of all the $\hat{b}_a$ are equal to 1. In this case we write

$$\hat{b}_a(k_2, E) = \exp\{2\pi i \varphi(k_2, E)\},$$

where the phase φ is given by

$$\varphi(k_2, E) = (1/\pi) \tan^{-1}\left((1/g)(\sqrt{(E + \cos 2\pi k_2)^2 - 1} - a)\right). \tag{3.20}$$

Thus, equation (3.19) is equivalent to

$$\Phi_q(k_2, E) \equiv q\omega \pmod 1, \tag{3.21}$$

where

$$\Phi_q(k_2, E) = \sum_{l=0}^{q-1} \varphi(k_2 + l\alpha, E). \tag{3.22}$$

Observe that the phases are only defined if $E > 1 - \cos 2\pi(k_2 + lp/q)$.

It is easily seen that, for every $k_2 \in \mathbb{T}^1$, $\Phi_q(k_2, E)$ is a monotone increasing function of E of total variation q. Therefore, the total number of solutions $E_j(k_2)$ of (3.21) is exactly q.

Our analysis in [2] remains true because it is based on the analyticity of φ with respect to E. In particular, we have an analog of Proposition 4.1 in [2] on the separation of the surface bands:

(i) for every $j = 1, \ldots, q$ there exists a finite subset $\mathcal{D}'_j$ of the domain $\mathcal{D}_j \subset \mathbb{T}^1$ of the band function $E_j(k_2)$ such that for all $\mathbf{k} \in \mathbb{T}^1 \times (\mathcal{D}_j \setminus \mathcal{D}'_j)$ we have

$$|E_j(k_2) - E_2(\mathbf{k})| > 0, \tag{3.23}$$

where $E_2(\mathbf{k}) = -(\cos 2\pi k_1 + \cos 2\pi k_2)$;

(ii) there exists a positive constant $\eta_q > 0$ such that for any $j \neq j'$ we have

$$\inf_{k_2 \in \mathcal{D}_j \cap \mathcal{D}_{j'}} |E_j(k_2) - E_{j'}(k_2)| \geq \eta_q > 0.$$

For a fixed q, now we can examine the location of the surface bands as a increases. First, note that, by [2], for $a = 0$ the number of positive and negative surface bands coincide up to 1. When a is positive and increases, the surface bands are shifting to the right, and for a certain $a > 0$ all the bands are on the positive semiaxis. The center of this cluster of bands is at $E_0 = \sqrt{1 + a^2}$, which is the center of the surface band

$[E_0-1, E_0+1]$ for $g=0$ (see (2.31)), i.e., for the operator H_a of (1.7). For a fixed $g>0$, the behavior of all the surface bands with respect to the parameter a is continuous.

In the case where a is fixed and g varies, first we see that the geometric structure of the spectrum is discontinuous in g at $g=0$: we have a unique surface band at $g=0$ and q bands whatever small is $g>0$. If g is small, the terms of (3.22) vary rapidly in E from $-1/2$ to $1/2$, and their sum $\Phi_q(k_2,E)$ looks like a stair, with q steps of height 1. The width of the stair is close to 2, the first band is close to E_0-1, and the qth band is close to E_0+1.

From (3.21) it also follows that if $q\to\infty$, then the width of the bands decreases exponentially, the distance between bands is $O(1/q)$ in the center of the cluster and is $O(1)$ for the highest bands, and the number of bands in any fixed interval of the spectral axis tends to infinity. This is in agreement with our results in the next section, where we show that for a Diophantine α in (1.8), i.e., in the "limit" $q=\infty$, the spectrum is pure point and dense outside the interval $[-2,2]$.

§4. Pure point spectrum of H

In this section we prove that if the frequency vector α in (1.4) satisfies the Diophantine condition (1.6), then the spectrum of H is pure point in $\mathbb{R}\setminus[-d,d]$.

Theorem 4.1. *Let H be the operator defined by* (1.1)–(1.5), *where the vector α in* (1.4) *satisfies* (1.6). *Then*:

(i) *the spectrum of H in $\mathbb{R}\setminus[-d,d]$ is pure point, dense, and of multiplicity* 1;

(ii) *the eigenvalues of H in $\mathbb{R}\setminus[-d,d]$ are indexed by the points of $\mathbb{Z}^{d_2}$, i.e., the set of eigenvalues is of the form*

$$\{E_{s_2}\}_{s_2\in\mathbb{Z}^{d_2}}, \tag{4.1}$$

and E_{s_2} solves the equation

$$f(E)\equiv\alpha s_2+\omega \pmod 1, \tag{4.2}$$

where

$$f(E)=-\frac{1}{\pi}\int_{\mathbb{T}^{d_2}}\tan^{-1}\frac{1+a\widehat{\gamma}_0(k_2;E)}{g\widehat{\gamma}_0(k_2;E)}\,dk_2,\quad |E|>d, \tag{4.3}$$

f is monotone increasing, varying from $f(-\infty)=-1/2$ to $f(-d-0)<0$, and from $f(d+0)>0$ to $f(\infty)=1/2$;

(iii) *the eigenfunction $\Psi_{s_2}(\mathbf{x})$ corresponding to the eigenvalue E_{s_2} is of the form*

$$\Psi_{s_2}(\mathbf{x})=\chi(x_1,x_2-s_2;E_{s_2}), \tag{4.4}$$

where $\chi(x_1,x_2;E)$ decays exponentially in x_1 and x_2 and is real analytic in E, $|E|>d$.

Proof. Given $\delta>0$, consider the set

$$\Sigma_\delta=\{E\in\mathbb{R}:|E|\ge d+\delta\}. \tag{4.5}$$

We show that for any $\delta>0$ there exists $\delta_1>\delta$ such that $\delta_1\to 0$ as $\delta\to 0$, and that the spectrum on Σ_{δ_1} possesses the property described in the theorem.

It will be convenient to present our principal formula (2.13) in the form

$$\begin{aligned}G(z)&=G_0(z)-\mathbf{g}G_0(z)P\Gamma_a(z)PG_0(z)\\&\quad+2igG_0(z)P\Gamma_a(z)\sigma u(1-\sigma b_a u)^{-1}\Gamma_a(z)PG_0(z),\end{aligned} \tag{4.6}$$

where $\Gamma_a(z)=(\mathbf{g}\gamma_0(z)+i)^{-1}$ and $G_0(z)$ is the resolvent of H_0. Since $\gamma_0(z)$ is a convolution operator in $l^2(\mathbb{Z}^{d_2})$ the symbol $\widehat{\gamma}_0(k_2;z)$ of which is given by (2.19), $\Gamma_a(z)$ is also a

convolution, and the symbol of it is the inverse of $\mathbf{g}\widehat{\gamma}_0(k_2;z)+i$. Note that $\mathbf{g}\widehat{\gamma}_0(k_2;z)+i = -g/a$ if $1+a\widehat{\gamma}_0(k_2;z)=0$ and that, by (2.22), we have

$$\begin{aligned}|\mathbf{g}\widehat{\gamma}_0(k_2;z)+i| &= |1+a\widehat{\gamma}_0(k_2;z)|\,|g\widehat{\gamma}_a(k_2;z)+i| \\ &\geq |1+a\widehat{\gamma}_0(k_2;z)|\,|\operatorname{Im} g\widehat{\gamma}_a(k_2;z)+1| \geq |1+a\widehat{\gamma}_0(k_2;z)| > 0\end{aligned}$$

if $1+a\widehat{\gamma}_0(k_2;z)\neq 0$ (because $\operatorname{Im}\widehat{\gamma}_a(k_2;z)>0$ for $\operatorname{Im} z>0$ and $g>0$). We conclude that

$$\min_{k_2\in\mathbb{T}^{d_2}} |\mathbf{g}\widehat{\gamma}_0(k_2;z)+i| > 0, \quad z\in\mathcal{N}(\delta),\ \ \delta>0, \tag{4.7}$$

where

$$\mathcal{N}(\delta) = \mathbb{C}\setminus\{z=E+i\epsilon : |E|\leq d+\delta,\ \ 0\leq\epsilon\leq\delta\} \tag{4.8}$$

is the exterior of the complex δ-neighborhood of $[-d,d]$. Therefore, the operator $\Gamma_a(z)$ is bounded for every $z\in\mathcal{N}(\delta)$, $\delta>0$, and formula (4.6) is well defined.

The idea of the proof of the theorem is to "diagonalize" the operator $(1-\sigma b_a u)^{-1}$ in (4.6), the only operator that may have poles. Thus, we suppose that there exists a bounded convolution operator $c(z)$ and a bounded diagonal operator $d(z)$ in $l^2(\mathbb{Z}^{d_2})$ such that

$$b_a(z)u = c(z)d(z)c^{-1}(z). \tag{4.9}$$

If u is the diagonal operator defined by (2.8), then $uc(z)=c_\alpha(z)u$ by (3.4). Here $c_\alpha(z)$ is the convolution operator with symbol $\hat{c}(k_2+\alpha;z)$, and $\hat{c}(k_2;z)$ denotes the symbol of $c(z)$. Hence, assuming that (4.9) is valid, we can rewrite the above relation as follows:

$$c^{-1}(z)b_a(z)c_\alpha(z) = d(z)u^{-1}. \tag{4.10}$$

Since the left-hand side of (4.10) is a convolution operator and the right-hand side is a diagonal operator, they are both equal to a scalar operator $\kappa(z)\cdot\mathbf{1}$:

$$c^{-1}(z)b_a(z)c_\alpha(z) = \kappa(z)\cdot\mathbf{1}, \tag{4.11}$$

$$d(z) = \kappa(z)u. \tag{4.12}$$

Thus, we must find $c(z)$ and $\kappa(z)$ from (4.11). Then $d(z)$ will be given by (4.12). □

Lemma 4.2. *For any $\delta>0$, there exists $\delta_1>\delta$ such that $\delta_1\to 0$ as $\delta\to 0$ and equation (4.11) is solvable for every $z\in\mathcal{N}(\delta_1)$, where $\mathcal{N}(\delta)$ is as in (4.8). More precisely, the following is true for any $z\in\mathcal{N}(\delta_1)$:*

(i) *there exists a bounded and invertible convolution operator*

$$c(z) = \{c(x_2-y_2;z)\}_{x_2,y_2\in\mathbb{Z}^{d_2}}$$

acting in $l^2(\mathbb{Z}^{d_2})$ and such that $c(x_2;z)$ is analytic in $z\in\mathcal{N}(\delta_1)$ for every $x_2\in\mathbb{Z}^{d_2}$, and

$$|c(x_2;z)| \leq Ce^{-\lambda|x_2|}, \tag{4.13}$$

$$\Big|\frac{\partial}{\partial z}c(x_2;z)\Big| \leq C'e^{-\lambda|x_2|}, \tag{4.14}$$

where $C, C', \lambda>0$ depend only on δ_1;

(ii) *there exists a function f analytic in $\mathcal{N}(\delta_1)$ and such that*

$$\kappa(z) = e^{2\pi i f(z)}, \tag{4.15}$$

and f is given by (4.3) for $z=E$, $|E|\geq d+\delta_1$.

The proof of the lemma will be given at the end of the section. Now, using the lemma and relations (4.9), (4.12), (2.9), and (4.15), we can write

$$(1-\sigma b_a(z)u)^{-1} = c(z)(1-e^{2\pi i(f(z)-\omega)}u)^{-1}c^{-1}(z),$$

or, in the matrix form,

$$\begin{aligned}&((1-\sigma b_a(z)u)^{-1})(x_2,y_2)\\ &\quad= \sum_{s_2\in\mathbb{Z}^{d_2}} c(x_2-s_2;z)(1-e^{2\pi i(f(z)-\alpha s_2-\omega)})^{-1}(c^{-1})(s_2-y_2;z).\end{aligned}$$

We see that the poles of the operator on the left are the solutions of equation (4.2).

It is easily seen that if $\alpha\in\mathbb{R}$ satisfies the Diophantine condition (1.6), then for every $s_2\in\mathbb{Z}^{d_2}$ there exists a unique solution E_{s_2}, $|E_{s_2}|>d$, of equation (4.2).

We introduce the following notation:

$$\beta_1(\mathbf{x},s_2;z) = \big(G_0(z)P\Gamma_a(z)c_\alpha(z)\big)(\mathbf{x},s_2;z), \tag{4.16}$$

$$\beta_2(s_2,\mathbf{x};z) = \big(c^{-1}(z)\Gamma_a(z)PG_0(z)\big)(\mathbf{x},s_2;z). \tag{4.17}$$

Then

$$\begin{aligned}&\beta_1(\mathbf{x},s_2;z)\\ &\quad= \int_{\mathbb{T}^{d_2}} dk_2 e^{ik_2(x_2-s_2)}G_0^{(d_1)}(x_1,z-E_{d_2}(k_2))\Gamma_a(k_2;z)c(k_2+\alpha;z).\end{aligned} \tag{4.18}$$

By (2.17), for every $k_2\in\mathbb{T}^{d_2}$ and any $\delta>0$ the function $\widehat{\gamma}_0(k_2;z)$ is analytic in z outside the complex δ-neighborhood (4.8) of the cut $[-d,d]$. Parametrizing the torus $\mathbb{T}^{d_2}$ by $\zeta=(\zeta_1,\dots,\zeta_{d_2})\in\mathbb{C}^{d_2}$ so that $\mathbb{T}^{d_2}=\{\zeta\in\mathbb{C}^{d_2}:|\zeta_1|=1,\dots,|\zeta_{d_2}|=1\}$, we see that $E_{d_2}(k_2)=\mathcal{E}_{d_2}(e^{2\pi ik_{21}},\dots,e^{2\pi ik_{2d_2}})$, where $k_2=(k_{21},\dots,k_{2d_2})\in\mathbb{T}^{d_2}$, and the function

$$\mathcal{E}_{d_2}(\zeta) = -\frac{1}{2}\sum_{j=1}^{d_2}(\zeta_j+\zeta_j^{-1})$$

is analytic in $\mathbb{C}^{d_2}$ for $0<|\zeta_j|<\infty$, $j=1,\dots,d_2$. Since, moreover, $|E_{d_2}(k_2)|\le d_2$, $k_2\in\mathbb{T}^{d_2}$, we conclude that for every $z\in\mathcal{N}(\delta_1)$ with $\delta_1>\delta$ (say, for $\delta_1=2\delta$) there exists $\delta_2>0$ such that $\widehat{\gamma}_0(k_2;z)$ admits analytic continuation from $\mathbb{T}^{d_2}$ to the domain

$$\mathcal{T}(\delta_2) = \{\zeta\in\mathbb{C}^{d_2}:\big||\zeta_1|-1\big|\le\delta_2,\dots,\big||\zeta_{d_2}|-1\big|\le\delta_2\}. \tag{4.19}$$

The same property of $\hat{c}(k_2;z)$ is established in the proof of Lemma 4.2 below. Furthermore, in accordance with (4.7), $\mathbf{g}\widehat{\gamma}_0(k_2,z)+i$ is not zero in $\mathbb{T}^{d_2}\times\mathcal{N}(\delta)$ for any $\delta>0$. Thus, the integrand in (4.18) can also be continued to the domain (4.19). This implies that $\beta_1(\mathbf{x},s_2;z)$ decays exponentially as $s_2\to\infty$ for every $\mathbf{x}\in\mathbb{Z}^d$ and $z\in\mathcal{N}(\delta_1)$. Indeed, it suffices to rewrite (4.18) as the d_2-fold contour integral in $\zeta_1=e^{2\pi ik_{21}},\dots,\zeta_{d_2}=e^{2\pi ik_{2d_2}}$ over the product of d_2 unit circles $|\zeta_1|=\dots=|\zeta_{d_2}|=1$, and then to deform every unit circle into the circle $|\zeta_j|=1-\delta_2'$, $0<\delta_2'<\delta_2$. This yields the bound

$$|\beta_1(\mathbf{x},s_2;z)|\le C_\beta e^{-\lambda_\beta|s_2|}, \tag{4.20}$$

which is valid for every $\mathbf{x}\in\mathbb{Z}^d$ and $z\in\mathcal{N}(\delta_1)$, with positive C_β and λ_β independent of on $\mathbf{x}$ and z.

The same is true for $\beta_2(s_2, \mathbf{x}; z)$ of (4.17). Consequently, the matrix form of (4.6) will be

$$\begin{aligned} G(\mathbf{x}, \mathbf{y}; z) &= G_0^{(d)}(\mathbf{x} - \mathbf{y}; z) \\ &\quad - g \int_{T^{d_2}} dk_2 e^{ik_2(x_2 - y_2)} G_0^{(d_1)}(x_1; \zeta) G_0^{(d_1)}(y_1; \zeta) \Gamma_a(k_2; z)|_{\zeta = z - E_{d_2}(k_2)} \\ &\quad + 2ig \sum_{s_2 \in \mathbb{Z}^{d_2}} \beta_1(\mathbf{x}, s_2; z) \sigma u(s_2) h^{-1}(s_2; z) \beta_2(s_2, \mathbf{y}; z), \end{aligned} \tag{4.21}$$

where

$$h(s_2; z) = 1 - e^{2\pi i(f(z) - \alpha s_2 - \omega)}, \quad u(s_2) = e^{-2\pi i \alpha s_2}, \tag{4.22}$$

and the series converges uniformly in $z \in \mathcal{N}(\delta_1)$. This implies that, if E_{s_2}, $s_2 \in \mathbb{Z}^{d_2}$, is a solution of (4.2) such that $|E_{s_2}| \geq d + \delta_1$ and $z \in \mathcal{N}(\delta_1)$, then

$$\lim_{z \to E_{s_2}} (E_{s_2} - z) G(\mathbf{x}, \mathbf{y}; z) = P_{s_2}(\mathbf{x}, \mathbf{y}),$$

where

$$P_{s_2}(\mathbf{x}, \mathbf{y}) = \frac{g}{\pi f'(E_{s_2})} \beta_1(\mathbf{x}, s_2; E_{s_2}) \sigma u(s_2) \beta_2(s_2, \mathbf{y}; E_{s_2}).$$

By the general principles of spectral theory, $\{P_{s_2}(\mathbf{x}, \mathbf{y})\}_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d}$ is the matrix of the orthogonal projection to the subspace of eigenfunctions that correspond to the eigenvalue E_{s_2}. An easy calculation based on the above formulas shows that

$$\sum_{\mathbf{x} \in \mathbb{Z}^d} P_{s_2}(\mathbf{x}, \mathbf{x}) = 1.$$

We have shown that the operator H of (1.1)–(1.5) has a dense set $\{E_{s_2}\}_{s_2 \in \mathbb{Z}^{d_2}}$ of eigenvalues of multiplicity 1 defined by (4.1)–(4.2). We shall prove that (4.1) is the entire spectrum of H outside $[-d, d]$. For this, it suffices to verify that for any C^1-function $\varphi : \mathbb{R} \to \mathbb{R}$ with support outside of $[-d, d]$ we have the limit relation

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{R}} \varphi(E) \operatorname{Im} G(\mathbf{x}, \mathbf{y}; E + i\varepsilon)\, dE = \sum_{s_2 \in \mathbb{Z}^{d_2}} \varphi(E_{s_2}) \Psi_{s_2}(\mathbf{x}) \Psi_{s_2}(\mathbf{y}),$$

where the functions $\{\Psi_{s_2}\}_{s_2 \in \mathbb{Z}^{d_2}}$ are given by (4.4).

By using the resolvent identity for $G(z)$, we can write

$$\frac{1}{\pi} \operatorname{Im} G(\mathbf{x}, \mathbf{y}; E + i\varepsilon) = \frac{\varepsilon}{\pi} \sum_{\mathbf{t} \in \mathbb{Z}^d} G(\mathbf{x}, \mathbf{t}; E + i\varepsilon) G(\mathbf{y}, \mathbf{t}; E - i\varepsilon). \tag{4.23}$$

The right-hand side of this formula contains $G(\mathbf{y}, \mathbf{t}; E - i\varepsilon)$, while our basic formula (4.6) is valid for $z = E + i\varepsilon$, $\varepsilon > 0$. However, applying to $G(\mathbf{y}, \mathbf{t}; E - i\varepsilon)$ the arguments that led to (4.6), we obtain the Hermitian conjugate version of (4.6) for $\overline{z} = E - i\varepsilon$. Moreover, for $G(\mathbf{y}, \mathbf{t}; E - i\varepsilon)$ we have an analog of the factorization Lemma 4.2 in which the operator $b_a^*(z)$ is diagonalized, hence an analog of (4.21). We must plug these formulas in (4.23), integrate with $\varepsilon\varphi/\pi$, and pass to the limit $\lim_{\varepsilon \to 0^+}$. Denoting the sum of the first two terms in (4.21) by G_1, the third term by G_2, and the respective terms of $G(\overline{z})$ by G_1^* and G_2^*, we first see that the contribution of the product $G_1 G_1^*$ vanishes after application of the operation

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi} \int \cdots \varphi(E) dE. \tag{4.24}$$

Indeed, the term $G_1(\mathbf{x},\mathbf{y};z)$ can be written as

$$\int_{\mathbb{T}^{d_2}} dk_2 e^{ik_2(x_2-y_2)}\Big(G_0^{(d_1)}(x_1-y_1;\zeta)-\frac{G_0^{(d_1)}(x_1;\zeta)G_0^{(d_1)}(y_1;\zeta)}{\mathbf{g}G_0^{(d_1)}(0;\zeta)+i}\Big)\Big|_{\zeta=z-E_{d_2}(k_2)}.$$

Since $G_0^{(d_1)}(x_1;\zeta)$ decays exponentially in x_1 and is analytic in ζ outside $[-d_1,d_1]$, the integrand can be analytically continued in k_2 to the domain $\mathcal{T}_{\delta_2}$ (see (4.19)) if $z\in\mathcal{N}(\delta_1)$. Consequently, $G_1(\mathbf{x},\mathbf{y};z)$ decays exponentially in $|\mathbf{x}-\mathbf{y}|$ for all z up to $z=E+i0\in\mathcal{N}(\delta_1)$, and

$$\lim_{\varepsilon\to 0^+}\sum_{\mathbf{x}'\in\mathbb{Z}^d} G_1(\mathbf{x},\mathbf{x}';E+i\varepsilon)G_1^*(\mathbf{x}',\mathbf{y};E-i\varepsilon)\le\infty.$$

We conclude that the operation (4.24) converts this expression into zero.

A similar but more involved argument shows that the contribution of the products $G_1G_2^*$ and $G_2G_1^*$ also vanishes. We shall not give details of the proof of this fact, because they will be clear from the analysis of the contribution of $G_2G_2^*$ below.

By using relation (4.21) and its analog for $G(\mathbf{x},\mathbf{y};\overline{z})$, the term $G_2G_2^*$ can be written as

$$-4g^2\sum_{s_2,t_2\in\mathbb{Z}^{d_2}}\beta_1(\mathbf{x},s_2;z)\frac{\sigma u(s_2)}{h(s_2)}\beta_3(s_2,t_2;z)\overline{\frac{\sigma u(t_2)}{h(t_2)}\beta_1(\mathbf{y},t_2;z)},\tag{4.25}$$

where

$$\beta_3(s_2,t_2;z)=\sum_{\mathbf{x}'\in\mathbb{Z}^d}\beta_2(s_2,\mathbf{x}';z)\overline{\beta_2(t_2,\mathbf{x}';z)},\tag{4.26}$$

and β_1 and β_2 are defined in (4.16)–(4.17). In accordance with the above, β_1 and β_2 and their conjugates have finite limits for $z=E+i0$, $|E|\ge d+\delta_1$, and these limits decay exponentially in $|x_1|$ and $|x_2-s_2|$. It follows that β_3 is well defined up to $z=E+i0$, $|E|\ge d+\delta_1$, and the double series in (4.25) converges uniformly in $\varepsilon>0$. Moreover, denoting

$$e(s_2;z)=(E_{s_2}-z)/h(s_2;z),$$

we see that for every $s_2\in\mathbb{Z}^{d_2}$ the function $e(s_2;E+i\varepsilon)$ is of class C^1 in E, $|E|\ge d+\delta_1$, uniformly in $\varepsilon\ge 0$, and that

$$\lim_{z\to E+i0} e(s_2;z)=(-2\pi i f'(E_{s_2}))^{-1}.$$

Therefore, the problem reduces to the calculation of the limit

$$\lim_{\varepsilon\to 0^+}\frac{\varepsilon}{\pi}\int_{-\infty}^{\infty}dE\varphi(E)\sum_{s_2,t_2\in\mathbb{Z}^{d_2}}A_1(s_2,E,\varepsilon)\frac{A_2(s_2,t_2,E,\varepsilon)}{E_{s_2}-(E+i\varepsilon)}\,\frac{A_3(t_2,E,\varepsilon)}{E_{t_2}-(E-i\varepsilon)}\tag{4.27}$$

where the functions A_1,A_2,A_3 are of class C^1 in E uniformly in $\varepsilon\ge 0$, admit the bounds

$$|A_{1,2}(s_2,E,\varepsilon)|\le Ce^{-\lambda|s_2|},\quad |A_3(s_2,t_2,E,\varepsilon)|\le Ce^{-\lambda|s_2-t_2|},$$

and their derivatives in E admit the same bounds, where $C>0$ and $\lambda>0$ depend only on δ_1.

We split the double series in (4.27) into two parts, over $s_2\ne t_2$ and $s_2=t_2$. In the first sum we use the identity

$$\frac{\varepsilon}{(E_{s_2}-z)(E_{t_2}-\overline{z})}=\Big(\frac{1}{E_{s_2}-z}-\frac{1}{E_{t_2}-\overline{z}}\Big)\frac{\varepsilon}{E_{s_2}-E_{t_2}+2i\varepsilon},\quad z=E+i\varepsilon.\tag{4.28}$$

After integration with respect to E with a C^1-function whose support is in $|E|>d+\delta_1$, the contribution of the terms in parentheses in (4.28) is bounded by

$$Ce^{-\lambda(|s_2|+|t_2|+|s_2-t_2|)}\tag{4.29}$$

uniformly in $\varepsilon \geq 0$, where C and λ depend only on δ_1. Next, using (1.6) and (4.2), we see that $|E_{t_2} - E_{s_2}| \geq C/|s_2 - t_2|^\beta$, $s_2 \neq t_2$. Thus, we have

$$\Big|\frac{\varepsilon}{E_{s_2} - E_{t_2} + 2i\varepsilon}\Big| \leq \frac{\varepsilon|s_2 - t_2|^\beta}{C}, \quad s_2 \neq t_2.$$

This inequality and (4.29) imply that the contribution to (4.27) of the "nondiagonal" part $s_2 \neq t_2$ of the double series in (4.27) vanishes as $\varepsilon \to 0$.

The diagonal part of the series in (4.27) is of the form

$$\sum_{s_2 \in \mathbb{Z}^{d_2}} \int_{\mathbb{R}} \frac{\varepsilon}{\pi[(E_{s_2} - E)^2 + \varepsilon^2]} A_4(s_2, E, \varepsilon)\, dE,$$

where A_4 has the same properties as A_1, A_2, A_3 in (4.27), i.e., it is of class C^1 in E uniformly in s_2 and ε, and decays exponentially in s_2 together with its derivative with respect to E, with rate uniformly positive in E, $|E| \geq \delta_1$, and in $\varepsilon \geq 0$. Thus, passing to the limit as $\varepsilon \to 0$ in the above expression, we obtain

$$\sum_{s_2 \in \mathbb{Z}^{d_2}} A_4(s_2, E_{s_2}, 0).$$

Returning to the matrices β_1, β_2, and β_3 of (4.16), (4.17), and (4.26), we can now write the contribution of the diagonal part of the series in (4.27) in the form

$$\sum_{s_2 \in \mathbb{Z}^{d_2}} \varphi(E_{s_2})\beta_1(x, s_2, E_{s_2})\beta_3(s_2, s_2, E_{s_2})\overline{\beta_1(y, s_2, E_{s_2})}.$$

This shows that the eigenfunction of H corresponding to the eigenvalue E_{s_2} is

$$\psi_{s_2}(x) = \beta_3^{1/2}(s_2, s_2, E_{s_2})\beta_1(x, s_2, E_{s_2}).$$

By using (4.16), (4.17), and (4.26), it can be shown that $\beta_3(s_2, s_2, E_{s_2})$ is independent of s_2, that β_1 depends on x and s_2 via x_1 and $x_2 - s_2$, and decays exponentially in these variables. This proves (4.4). It can also be shown that $P_{s_2}(x, y) = \psi_{s_2}(x)\psi_{s_2}(y)$. The theorem is proved. □

Proof of Lemma 4.2. Since the operators $b_a(z)$ and $c(z)$ are bounded convolutions in $l^2(\mathbb{Z}^{d_2})$, equation (4.11) will be a functional equation after the Fourier transformation. Namely, denoting the symbol of $c(z)$ by $\widehat{c}(k_2; z)$ and using (3.4) and Lemma 2.4, from (4.11) we obtain

$$(\widehat{c}(k_2; z))^{-1}\hat{b}_a(k_2; z)\widehat{c}(k_2 + \alpha; z) = \kappa(z).$$

Setting

$$\widehat{c}(k_2; z) = e^{-2\pi i \hat{l}(k_2; z)}, \tag{4.30}$$

and using (4.15), we can write

$$\hat{l}(k_2; z) - \hat{l}(k_2 + \alpha; z) = f(z) - (2\pi i)^{-1} \log \hat{b}_a(k_2; z), \tag{4.31}$$

where we have used the branch of the logarithm such that $\log(-1 \pm i) = \pm i\pi$.

We have seen above (see the text after (4.18)) that $\widehat{\gamma}_0(k_2; z)$ admits analytic continuation to the domain

$$\mathcal{T}(\delta_2) \times \mathcal{N}(\delta_1), \tag{4.32}$$

where $\mathcal{T}(\delta)$ and $\mathcal{N}(\delta)$ are defined in (4.19) and (4.8). Moreover, $\mathbf{g}\widehat{\gamma}_0(k_2, z) + i$ is not zero in $\mathbb{T}^{d_2} \times \mathcal{N}(\delta)$ for any $\delta > 0$ (see (4.7)). Hence, there exists a function B analytic in (4.32) and such that

$$B(e^{2\pi i k_{21}}, \ldots, e^{2\pi i k_{2d_2}}; z) = \hat{b}_a(k_2, z), \quad k_2 = (k_{21}, \ldots, k_{2d_2}) \in \mathbb{T}^{d_2}.$$

Furthermore, $\widehat{\gamma}_0(k_2;z)$ is real and nonzero for $z = E \in \mathbb{R}$, $|E| \geq d+\delta$, for any $\delta > 0$. Consequently, in this case we have $|\hat{b}_a(k_2,E)| = 1$, and by continuity B is not zero in (4.32) if δ_1 and δ_2 are sufficiently small. We conclude that $\log B(\zeta;z)$ is also analytic in $\zeta \in \mathcal{T}(\delta_2)$ for every $z \in \mathcal{N}(\delta_1)$. Now we can apply the arguments used in the proof of (4.20) to show that the Fourier coefficients

$$\psi(x_2;z) = \int_{\mathbb{T}^{d_2}} e^{2\pi i k_2 x_2} \log \hat{b}_a(k_2;z)\, dk_2 \tag{4.33}$$

decay exponentially in x_2:

$$|\psi(x_2;z)| \leq C' e^{-\lambda'|x_2|}, \quad x_2 \to \infty, \tag{4.34}$$

where C' and λ' are strictly positive for $\delta_2 > 0$. Now we can use the Fourier transform (4.31), obtaining

$$l(x_2;z)\left(1 - e^{2\pi i \alpha x_2}\right) = \psi(x_2;z), \quad x_2 \neq 0, \tag{4.35}$$

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{T}^{d_2}} \log \hat{b}_a(k_2,z)\, dk_2, \tag{4.36}$$

where $l(x_2;z)$ is the Fourier coefficient of $\hat{l}(k_2;z)$. From (4.35) we deduce that

$$l(x_2;z) = \frac{\psi(x_2;z)}{1 - e^{2\pi i \alpha x_2}}, \quad x_2 \neq 0, \tag{4.37}$$

and (4.34) and the Diophantine condition (1.6) imply that $l(x_2;z)$ decays exponentially in x_2 for every $z \in \mathcal{N}(\delta_1)$, with a rate $0 < \lambda'' < \lambda'$. Thus, the series

$$\hat{l}(k_2;z) = \sum_{x_2 \in \mathbb{Z}^{d_2}\setminus\{0\}} l(x_2;z) e^{-2\pi i k_2 x_2} \tag{4.38}$$

converges for every $z \in \mathcal{N}(\delta_1)$ and $k_2 \in \mathbb{T}^{d_2}$; moreover, it admits analytic continuation in k_2 to $\mathcal{T}(\delta')$ for some $\delta' > 0$. By (4.30), $\widehat{c}(k_2;z)$ and $(\widehat{c}(k_2;z))^{-1}$ possess the same property, so that their Fourier coefficients are analytic in $\mathcal{N}(\delta_1)$, and $c(x_2;z)$ and $(c^{-1}(z))(x_2)$ satisfy (4.13).

To prove (4.14), we note that, because of (4.33)–(4.38), the derivative of $c(x_2;z)$ with respect to z will be expressed eventually via

$$\frac{\partial}{\partial z}\log \hat{b}_a(k_2;z) = -\frac{2i}{(g\gamma_0(k_2;z))^2 + (1 + a\gamma_0(k_2;z))^2}\frac{\partial}{\partial z}\gamma_0(k_2;z). \tag{4.39}$$

The denominator of the right-hand side is analytic in the domain (4.32) and is not zero for $z = E + i0$, $|E| \geq d+\delta$, $\delta > 0$, because $\widehat{\gamma}_0(k_2;z)$ is real and is nonzero for all such z. Hence, the right-hand side of (4.39) can be continued into $\mathcal{T}(\delta')$ for some $\delta' > 0$. Now we can repeat the above arguments that lead from (4.33) to (4.13), and obtain (4.14). □

References

[1] N. I. Akhiezer and I. M. Glazman, *The theory of linear operators in Hilbert space*, GITTL, Moscow–Leningrad, 1950; English transl., Vol. 2, F. Ungar Publ. Co., New York, 1963. MR0044034 (13:3586); MR0264421 (41:9015b)

[2] F. Bentosela, Ph. Briet, and L. Pastur, *On the spectral and wave propagation properties of the surface Maryland model*, J. Math. Phys. **44** (2003), 1–35. MR1946689 (2003k:81049)

[3] A. Boutet de Monvel and A. Surkova, *Localisation des états de surface pour une classe d'opérateurs de Schrödinger discrets à potentiels de surface quasi-périodiques*, Helv. Phys. Acta **71** (1998), 459–490. MR1651042 (2000a:47070a)

[4] I. P. Kornfel′d, Ya. G. Sinaĭ, and S. V. Fomin, *Ergodic theory*, "Nauka", Moscow, 1980; English transl., Grundlehren Math. Wiss., vol. 245, Springer-Verlag, New York, 1982. MR0610981 (83a:28017); MR0832433 (87f:28019)

[5] E. B. Davies and B. Simon, *Scattering theory for systems with different spatial asymptotics on the left and right*, Comm. Math. Phys. **63** (1978), 277–301. MR0513906 (80c:81110)

[6] A. L. Figotin and L. A. Pastur, *An exactly solvable model of a multidimensional incommensurate structure*, Comm. Math. Phys. **95** (1984), 401–425. MR0767188 (86f:81027)

[7] ———, *Spectra of random and almost-periodic operators*, Grundlehren Math. Wiss., vol. 297, Springer-Verlag, Berlin, 1992. MR1223779 (94h:47068)

[8] V. Grinshpun, *Localization for random potentials supported on a subspace*, Lett. Math. Phys. **34** (1995), 103–117. MR1335579 (96e:82052)

[9] A. Grossmann, R. Høegh-Krohn, and M. Mebkhout, *The one particle theory of periodic point interactions. Polymers, monomolecular layers, and crystals*, Comm. Math. Phys. **77** (1980), 87–110. MR0588688 (82i:81022)

[10] V. Jakšić and Y. Last, *Corrugated surfaces and a.c. spectrum*, Rev. Math. Phys. **12** (2000), 1465–1503. MR1809458 (2001m:47143)

[11] ———, *Surface states and spectra*, Comm. Math. Phys. **218** (2001), 459–477. MR1828849 (2002g:81030)

[12] V. Jakšić and S. Molchanov, *On the spectrum of the surface Maryland model*, Lett. Math. Phys. **45** (1998), 189–193. MR1641176 (99h:82046)

[13] ———, *On the surface spectrum in dimension two,* Helv. Phys. Acta **71** (1998), 629–657. MR1669046 (2000b:81031)

[14] ———, *Localization of surface spectra*, Comm. Math. Phys. **208** (1999), 153–172. MR1729882 (2000k:47048)

[15] ———, *Wave operators for the surface Maryland model*, J. Math. Phys. **41** (2000), 4452–4463. MR1765613 (2002a:81052)

[16] V. Jakšić, S. Molchanov, and L. Pastur, *On the propagation properties of surface waves,* Wave Propagation in Complex Media (Minneapolis, MN, 1994), IMA Vol. Math. Appl., vol. 96, Springer, New York, 1998, pp. 143–154. MR1489748

[17] Yu. E. Karpeshina, *The spectrum and eigenfunctions of the Schrödinger operator in a three-dimensional space with point-like potential of the homogeneous two-dimensional lattice type*, Teoret. Mat. Fiz. **57** (1983), no. 3, 414–423; English transl., Theoret. and Math. Phys. **57** (1983), no. 3, 1231–1238. MR0735399 (85f:81011)

[18] ———, *An eigenfunction expansion theorem for the Schrödinger operator with a homogeneous simple two-dimensional lattice of potentials of zero radius in a three-dimensional space*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **1984**, vyp. 1, 11–17; English transl. in Vestnik Leningrad Univ. Math. **17**. MR0743579 (85h:35158)

[19] T. Kato, *Perturbation theory for linear operators*, Grundlehren Math. Wiss., vol. 132, Springer-Verlag, New York, 1966. MR0203473 (34:3324)

[20] B. Khoruzhenko and L. Pastur, *Localization of surface states: an explicitly solvable model,* Phys. Rep. **288** (1997), 109–125.

[21] B. Simon, *Almost periodic Schrödinger operators.* IV. *The Maryland model*, Ann. Physics **159** (1985), 157–183. MR0776654 (86m:81038)

[22] D. Yafaev, *Mathematical scattering theory. General theory*, S.-Peterburg. Gos. Univ., St. Petersburg, 1994; English transl., Amer. Math. Soc., Providence, RI, 1999. MR1784870 (2001e:47015)

Centre de Physique Théorique, Luminy, Case 907, Marseille 13288, France
E-mail address: `Francois.Bentosela@cpt.univ-mrs.fr`

U. F. R. de Mathématiques, Université Paris 7, 2, Pl. Jussieu, Paris 75251, France
E-mail address: `briet@cpt.univ-mrs.fr`

Institute for Low Temperature Physics, Kharkiv, Ukraine
E-mail address: `pastur@math.jussieu.fr`

Received 17/MAR/2004
Originally published in English