HOCHSCHILD COHOMOLOGY
OF ALGEBRAS OF DIHEDRAL TYPE,
I: THE FAMILY $D(3K)$ IN CHARACTERISTIC 2

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Abstract. In terms of generators and defining relations, the Hochschild cohomology algebras are described for all algebras of dihedral type in the family $D(3K)$ over an algebraically closed field of characteristic 2. The results are applied to three other families of algebras of dihedral type, namely, $D(3A)_1$, $D(3B)_1$, and $D(3D)_1$. As a corollary, a description is obtained for the Hochschild cohomology algebra for blocks with dihedral defect group and three simple modules; in particular, this applies to principal blocks of the groups $PSL(2,q)$ with odd $q$.

Introduction

Let $R$ be a finite-dimensional algebra over a field $K$, and let $\Lambda = R^e = R \otimes_K R^{op}$ be its enveloping algebra. Then the $n$th Hochschild cohomology group of the algebra $R$ with coefficients in an $R$-bimodule $M$ is defined as follows: $\text{HH}^n(R, M) = \text{Ext}_\Lambda^n(R, M)$. If $M = R$, we use the notation $\text{HH}^n(R) = \text{HH}^n(R, R)$. On the vector space $\text{HH}^*(R) = \bigoplus_{n \geq 0} \text{HH}^n(R) = \bigoplus_{n \geq 0} \text{Ext}_\Lambda^n(R, R)$, we can introduce the $\sim$-product with respect to which it becomes an associative $K$-algebra (see [1, §5], [2, Chapter XI], and [3]); this algebra is called the Hochschild cohomology algebra. It is known that the $\sim$-product coincides with the Yoneda product on the Ext-algebra $\bigoplus_{n \geq 0} \text{Ext}_\Lambda^n(R, R)$ of the $\Lambda$-module $R$ (see, e.g., [4, p. 120]). Moreover, in [3] it was proved that $\text{HH}^*(R)$ is a graded commutative algebra.

Over the last years, progress has been made in the investigation of the multiplicative structure of the Hochschild cohomology algebra in the case of the group algebras of finite groups. In [5, 6] it was proved that

$$\text{HH}^*(K[G]) \simeq H^*(G) \otimes_K K[G]$$

if $G$ is a commutative finite group. A formula proposed in [7] makes it possible to reduce the computation of products in $\text{HH}^*(K[G])$ to manipulations with the usual cohomology of groups, and this technique was employed to obtain a description of the Hochschild cohomology algebra for the symmetric group $S_3$ over the field $F_3$ and for the alternating group $A_4$ and the dihedral 2-groups over the field $F_2$. Furthermore, the subalgebra of even cohomology,

$$\text{HH}^{ev}(B) = \bigoplus_{n \geq 0} \text{HH}^{2n}(B),$$

was described in [8] for the group blocks of finite representation type. We note also that the additive structure of the algebra $\text{HH}^*(B)$ was described in [9] for the group blocks...
having tame representation type and one or three simple modules. The algebra $\text{HH}^*(R)$ has been calculated also for algebras of other classes. In [10], the algebra $\text{HH}^*(R)$ was described in the case where $R$ is a self-injective Nakayama algebra, and in [11], the subalgebra $\text{HH}^*(R)$ generated by the homogeneous elements of degrees divisible by $r$ was identified, where $R$ is the so-called Möbius algebra (here $r$ is a parameter related to the algebra $R$).

In the present paper, we give a description of the Hochschild cohomology algebra for all algebras of dihedral type in the family $D(3K)$ over an algebraically closed field of characteristic 2 (see Theorem 1.2). We recall that algebras of dihedral, semidihedral, and quaternion type appeared in the work of K. Erdmann on classification of group blocks of tame representation type (see [12]). This classification contains dozens of infinite families of algebras; in particular, the algebras of dihedral type with three simple modules form 10 families. By the results of [13], the algebras in three families among these, namely, in the families $D(3A)_1$, $D(3B)_1$, and $D(3D)_1$, are derived equivalent to suitable algebras in the family $D(3K)$ studied in the present paper; hence, in fact we obtain a description of the algebra $\text{HH}^*(R)$ for the four families of algebras of dihedral type mentioned above (see Corollaries 1.3 and 1.4). This allows us to give a description of the Hochschild cohomology algebra for all group blocks with dihedral defect group and three simple modules (see Corollary 1.5), because the results of [12] show that these blocks are contained (up to the Morita equivalence) precisely in these four families. We recall that the principal block (in characteristic 2) of the group $\text{PSL}(2,q)$ with odd $q$ has dihedral defect group; consequently, our description of the algebra $\text{HH}^*(R)$ applies to such blocks. In particular, this description applies to the group $A_4$ and to the principal blocks of the groups $A_5$ and $A_6$; it also applies to the principal block of the group $A_7$.

In order to compute the algebra $\text{HH}^*(R)$ for algebras of the family $D(3K)$, we use a technique similar to that used by the author in the computation of Yoneda algebras for algebras of dihedral or semidihedral type (see [14]–[18]). Namely, on the basis of some empirical observations, a conjecture is stated about the structure of the minimal $\Lambda$-projective resolution of the module $R$, and then this conjecture is proved (see Theorem 1.1 below and its proof in §2). Next, using the resolution obtained, we pick a (finite) set of generators of the algebra $\text{HH}^*(R)$ and find relations satisfied by these generators (see §§3, 4, and 5).

This time, the empirical stage of the investigation was split into two steps. At the first step, I considered only the algebra in $D(3K)$ with the smallest values of the parameters that determine algebras of this family (see the beginning of §1) — this algebra corresponds to the group algebra $K[A_4]$, where $K$ is an algebraically closed field of characteristic 2. Using computer calculation (for this an original computer program was implemented in the Pascal language), I calculated the differentials acting among several initial terms of the minimal projective resolution of the module $R$. Once a “self-reproducing” effect in the initial terms of the resolution was observed, i.e., the resolution began to repeat itself in a regular way, a conjecture was put forward about the construction of the entire resolution; then this conjecture was proved. At the second (purely speculative) step, I managed to formulate a general conjecture concerning the structure of the resolutions for all algebras in the family under consideration.

§1. Statement of the main results

The algebras $R_{n_1,n_2,n_3}$, $n_1,n_2,n_3 \in \mathbb{N}$, of the family $D(3K)$ (over an algebraically closed field $K$ with arbitrary characteristic) are defined by the following quiver with
relations:

\[
Q : \\
\begin{array}{c}
1 \\
\downarrow a_{12} \\
\downarrow a_{13} \\
\downarrow a_{23} \\
\downarrow a_{31} \\
\downarrow a_{32} \\
3
\end{array}
\begin{array}{c}
2
\end{array}
\]

(1.1) \[ 0 = a_{21}a_{13} = a_{31}a_{12} = a_{32}a_{21} = a_{12}a_{23} = a_{13}a_{32} = a_{23}a_{31}, \]

(1.2) \[ (a_{12}a_{21})^{n_3} = (a_{13}a_{31})^{n_2}, \]

\[ (a_{23}a_{32})^{n_1} = (a_{21}a_{12})^{n_3}, \]

\[ (a_{31}a_{13})^{n_2} = (a_{32}a_{23})^{n_1}. \]

(composition is written from right to left). We denote by \( w \in P \) paths. Any module \( w \) corresponds to the vertices of the quiver \( Q \). Then the modules

\[ P_{ij} = \Lambda(e_i \otimes e_j^*), \quad i, j \in \{1, 2, 3\}, \]

form a complete set (of representatives) of the principal indecomposable left \( \Lambda \)-modules, where \( \Lambda \) is also denoted by \( R \). The nonzero images (in \( R \)) of the paths in the quiver \( Q \) are also called paths. Any module \( P_{ij} \) has a basis that consists of elements of the form \( u \otimes v^* \), where \( u \) (respectively, \( v \)) is a (nonzero) path in \( Q \) starting at the vertex \( i \) (respectively, ending at the vertex \( j \)). Such a basis of the module \( P_{ij} \) is said to be standard.

Multiplication on the right by an element \( w \in \Lambda \) induces an endomorphism \( w^* \) of the left \( \Lambda \)-module \( \Lambda \); moreover, if \( w \in \{e_i \otimes e_j^*\} \Lambda(e_k \otimes e_l^*) \), then \( w^* \) induces a homomorphism \( w^*: P_{ij} \rightarrow P_{kl} \). For simplicity, this homomorphism of multiplication (on the right) by \( w \) will also be denoted by \( w \).

In the sequel, we assume that the ground field \( K \) has characteristic 2.

We introduce

\[
(1.3) \quad L_1 = \bigoplus_{i=1}^{3} P_{ii}, \quad L_2' = \bigoplus_{i=1}^{3} P_{i,i+1}, \quad L_2'' = \bigoplus_{i=1}^{3} P_{i+1,i}, \quad L_2 = L_2' \oplus L_2'',
\]

where indices are defined modulo 3. Furthermore, we use abbreviated notation for the following elements of the path algebra of the quiver \( Q \):

(1.4)

\[
\alpha_1 = a_{12}a_{21}, \quad \alpha_2 = a_{23}a_{32}, \quad \alpha_3 = a_{31}a_{13},
\]

\[
\beta_1 = a_{13}a_{31}, \quad \beta_2 = a_{21}a_{12}, \quad \beta_3 = a_{32}a_{23}.
\]

We intend to define several homomorphisms acting between the modules listed in (1.3). These homomorphisms are described by matrices that correspond to the direct decompositions fixed in (1.3). Since these matrices have block structure, first we describe the corresponding auxiliary blocks.

We put

\[
A' = \begin{pmatrix}
0 & a_{31} \otimes e_1' & 0 \\
0 & 0 & 0 \\
e_1 \otimes a_{13}' & 0 & a_{31} \otimes e_1'
\end{pmatrix} : L_2' \rightarrow L_1,
\]

\[
A'' = \begin{pmatrix}
0 & a_{31} \otimes e_1' & 0 \\
e_1 \otimes a_{13}' & 0 & 0 \\
0 & e_3 \otimes a_{32}' & a_{13} \otimes e_3'
\end{pmatrix} : L_2'' \rightarrow L_1,
\]

where indices are defined modulo 3.
\[ B' = \begin{pmatrix} 0 & a_{31} \otimes e'_2 & e_1 \otimes a'_{23} \\ e_2 \otimes a'_3 & 0 & a_{12} \otimes e'_3 \\ a_{23} \otimes e'_1 & e_3 \otimes a'_1 & 0 \end{pmatrix} : L_2' \rightarrow L_2', \]

\[ B'' = \begin{pmatrix} 0 & e_2 \otimes a'_1 & a_{32} \otimes e'_1 \\ a_{13} \otimes e'_2 & 0 & e_3 \otimes a'_2 \\ e_1 \otimes a'_3 & a_{21} \otimes e'_3 & 0 \end{pmatrix} : L_2' \rightarrow L_2'', \]

\[ C' = \begin{pmatrix} e_1 \otimes a'_{21} & a_{21} \otimes e'_2 & 0 \\ a_{13} \otimes e'_1 & 0 & e_3 \otimes a'_{13} \\ 0 & e_2 \otimes a'_3 & a_{32} \otimes e'_3 \end{pmatrix} : L_1 \rightarrow L_2', \]

\[ C'' = \begin{pmatrix} a_{12} \otimes e'_1 & e_2 \otimes a'_1 & 0 \\ e_1 \otimes a'_{31} & a_{23} \otimes e'_2 & 0 \\ 0 & e_3 \otimes a'_3 & a_{31} \otimes e'_3 \end{pmatrix} : L_1 \rightarrow L_2''. \]

\[ D' = \left( \sum_{i=0}^{n-1} \alpha_i^1 \otimes (a_{21}a_1^{n_3-1-i})' \sum_{i=0}^{n-1} \beta_i^{n_3-1-i}a_{21} \otimes (\beta_2')' \right) : L_1 \rightarrow L_2', \]

\[ D'' = \left( \sum_{i=0}^{n-1} \alpha_i^1 \otimes (a_{12}a_1^{n_3-1-i})' \sum_{i=0}^{n-1} \beta_i^{n_3-1-i}a_{12} \otimes (\beta_3')' \right) : L_1 \rightarrow L_2'', \]

\[ E' = \begin{pmatrix} 0 & e_2 \otimes (a_{13}a_3^{n_3-1-i})' & \beta_i^{n_3-1}a_{32} \otimes e'_1 \\ e_1 \otimes (a_{32}a_2^{n_3-1-i})' & 0 & e_3 \otimes (a_{21}a_1^{n_3-1-i})' \\ \beta_i^{n_3-1}a_{13} \otimes e'_2 & \beta_i^{n_3-1}a_{21} \otimes e'_3 & 0 \end{pmatrix} : L_2' \rightarrow L_2'', \]

\[ E'' = \begin{pmatrix} 0 & e_2 \otimes (a_3a_1^{n_3-1-i})' & e_3 \otimes (a_{23}a_3^{n_3-1-i})' \\ a_{21} \otimes e'_2 & 0 & \alpha_i^{n_3-1}a_{12} \otimes e'_3 \\ e_1 \otimes (a_{13}a_3^{n_3-1-i})' & e_3 \otimes (a_{12}a_2^{n_3-1-i})' & 0 \end{pmatrix} : L_2' \rightarrow L_2'', \]

\[ F' = \begin{pmatrix} 0 & e_2 \otimes (a_{23}a_3^{n_3-1-i})' & \alpha_i^{n_3-1}a_{32} \otimes e'_1 \\ e_1 \otimes (a_{12}a_2^{n_3-1-i})' & 0 & \beta_i^{n_3-1}a_{21} \otimes e'_3 \\ \alpha_i^{n_3-1}a_{12} \otimes e'_2 & \alpha_i^{n_3-1}a_{31} \otimes e'_1 & 0 \end{pmatrix} : L_2' \rightarrow L_2', \]

\[ F'' = \begin{pmatrix} 0 & e_2 \otimes (a_{12}a_1^{n_3-1-i})' & \alpha_i^{n_3-1}a_{32} \otimes e'_3 \\ e_1 \otimes (a_{23}a_1^{n_3-1-i})' & 0 & \beta_i^{n_3-1}a_{13} \otimes e'_2 \\ \alpha_i^{n_3-1}a_{12} \otimes e'_1 & \alpha_i^{n_3-1}a_{31} \otimes e'_2 & 0 \end{pmatrix} : L_2' \rightarrow L_2', \]

\[ G' = \begin{pmatrix} 0 & e_2 \otimes (a_{23}a_1^{n_3-1-i})' & \beta_i^{n_3-1}a_{32} \otimes e'_1 \\ e_1 \otimes (a_{12}a_3^{n_3-1-i})' & 0 & \alpha_i^{n_3-1}a_{21} \otimes e'_3 \\ \beta_i^{n_3-1}a_{21} \otimes e'_2 & \beta_i^{n_3-1}a_{13} \otimes e'_1 & 0 \end{pmatrix} : L_1 \rightarrow L_2', \]

\[ G'' = \begin{pmatrix} 0 & e_2 \otimes (a_{23}a_3^{n_3-1-i})' & \alpha_i^{n_3-1}a_{32} \otimes e'_3 \\ e_1 \otimes (a_{12}a_1^{n_3-1-i})' & 0 & \beta_i^{n_3-1}a_{13} \otimes e'_2 \\ \alpha_i^{n_3-1}a_{12} \otimes e'_1 & \alpha_i^{n_3-1}a_{31} \otimes e'_3 & 0 \end{pmatrix} : L_1 \rightarrow L_2''. \]
Now, we use these blocks to define the following homomorphisms:

(1.7) \[ \rho_1 = \begin{pmatrix} A' & A'' \end{pmatrix} : L_2 \to L_1, \quad \rho_2 = \begin{pmatrix} D' \\ D'' \end{pmatrix} : L_1 \to L_2, \]

(1.8) \[ \sigma_1 = \begin{pmatrix} 0 & B'' \\ B' & 0 \end{pmatrix} : L_2 \to L_2, \quad \sigma_2 = C'' \oplus C''' = \begin{pmatrix} C' & 0 \\ 0 & C'' \end{pmatrix} : L_1^2 \to L_2, \]

(1.9) \[ \tau_2 = F' \oplus F'' = \begin{pmatrix} F' & 0 \\ 0 & F'' \end{pmatrix} : L_2 \to L_2, \quad \tau_3 = \begin{pmatrix} 0 & G'' \\ G' & 0 \end{pmatrix} : L_1^2 \to L_2. \]

We build the following bicomplex \( B_{\bullet \bullet} \), lying in the first quadrant of the plane (i.e., its rows and columns are enumerated by 0, 1, 2, ...):

\[
\begin{array}{cccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\sigma_2 & \sigma_1 & \sigma_3 & \sigma_2 & \sigma_1 & \rho_1 \\
L_2 & \tau_1 & L_2 & \tau_3 & L_1 \oplus L_2 & \tau_2 & L_2 & \tau_1 & L_2 & \tau_2 & L_2 & \rho_2 & L_1 \\
\sigma_1 & \sigma_3 & \sigma_2 & \sigma_1 & \rho_1 \\
L_2 & \tau_3 & L_1 \oplus L_2 & \tau_2 & L_2 & \tau_1 & L_2 & \tau_2 & L_1 \\
\sigma_3 & \sigma_3 & \sigma_2 & \sigma_1 & \rho_1 \\
L_2 \oplus L_1 & \tau_2 & L_2 & \tau_1 & L_2 & \tau_2 & L_2 & \rho_2 & L_1 \\
\sigma_2 & \sigma_1 & \rho_1 \\
L_2 & \tau_1 & L_2 & \rho_2 & L_1 \\
\sigma_1 & \rho_1 \\
L_2 & \rho_2 & L_1 \\
\rho_1 & \\
L_1.
\end{array}
\] (1.10)

In detail, all (nonzero) columns of the bicomplex are obtained by the corresponding translation of the 0th column, which has period 3 (after omitting the lower arrow \( \rho_1 \)), and the \( i \)th row of the bicomplex \( (i \geq 0) \) contains \( i \) (nonzero) arrows, the extreme right arrow among those being the arrow \( \rho_2 \), and the other arrows being \( \tau_1, \tau_2, \tau_3 \) repeated successively (from right to left).

**Theorem 1.1.** The total complex \( Q_{\bullet} = \text{Tot}(B_{\bullet \bullet}) \) of the bicomplex (1.10) is the minimal projective resolution of the \( \Lambda \)-module \( R \).

**Remark 1.1.** Let \( m : \Lambda \to R, m(a \otimes b') = ab \), be the canonical map induced by multiplication in \( R \). Then the restriction of \( m \) to \( Q_0 = P_{11} \oplus P_{22} \oplus P_{33} \) can be taken for the role of an augmentation map \( d_{Q_1}^2 : Q_0 \to R \) of the resolution mentioned in Theorem 1.1 because \( \text{Ker} \ d_{Q_1}^2 = \text{Im} \ \rho_1 \) by [19, Proposition 2.1].

Using the resolution \( \Lambda Q_{\bullet} \to \Lambda R \) of Theorem 1.1, we calculate the Hochschild cohomology algebra \( HH^*(R) \) for algebras in the family \( D(3K) \). To state the corresponding result, we describe several graded algebras. Set

(1.11) \[ \mathcal{X}_1 = \{ c_1, c_2, c_3, p_1, p_2, p_3, x_1, x_2, y_1, y_2, y_3, z_1, z_2 \}. \]
On the algebra $K[X_1]$ we introduce a grading such that
\begin{align}
\deg c_i &= \deg p_i = 0 \ (i = 1, 2, 3), \quad \deg x_j = 1 \ (j = 1, 2), \\
\deg y_i &= 2 \ (i = 1, 2, 3), \quad \deg z_j = 3 \ (j = 1, 2).
\end{align}

Then we define a graded $K$-algebra $A_1 = K[X_1]/I_1$, where the ideal $I_1$ of the algebra $K[X_1]$ is generated by the following elements:

— of degree 0:
\begin{align}
c_{1}^{p_3}, c_{2}^{p_1}, c_{3}^{p_2}; \quad p_i p_j, \quad p_i c_j \ (i, j \in \{1, 2, 3\}); \\
c_{i} c_{j} \ (i, j \in \{1, 2, 3\}, i \neq j);
\end{align}

— of degree 1:
\begin{align}p_i x_1 \ (i \in \{1, 2, 3\}), \\
p_i x_2, \ c_i x_2 \ (i \in \{1, 2, 3\});
\end{align}

— of degree 2:
\begin{align}x_1^2, \ x_1 x_2 + n_1 n_2 n_3 \cdot p_1 y_1, \ (p_1 + p_3) y_1; \\
x_2^2, \ (p_1 + p_2) y_1; \\
p_i y_j, \ c_i y_j \ (i \in \{1, 2, 3\}, j \in \{2, 3\});
\end{align}

— of degree 3:
\begin{align}x_1 y_2, \ x_1 y_3 + p_1 z_2; \\
x_2 y_2 + p_1 z_1, \ x_2 y_3 + p_1 z_2; \\
y_2^2, \ y_3^2, \ y_2 y_3 + \kappa_1 \cdot p_1 y_1^2;
\end{align}

— of degree 4:
\begin{align}x_1 z_1 + n_1 n_2 n_3 \cdot y_1 y_2, \ x_1 z_2 + n_1 n_2 n_3 \cdot y_1 y_3 + x_2 z_2; \\
y_2^2, \ y_3^2, \ y_2 y_3 + \kappa_1 \cdot p_1 y_1^2;
\end{align}

with
\begin{align}\kappa_1 = \begin{cases} 1 & \text{if } n_1 = n_2 = n_3 = 1, \\ 0 & \text{otherwise}; \end{cases}
\end{align}

— of degree 5:
\begin{align}y_3 z_1 + \kappa_1 \cdot (x_1 + x_2) y_1^2 + (1 - \kappa_1) \cdot y_2 z_2, \\
y_2 z_2 + \kappa_2 \cdot x_1 y_1^2;
\end{align}

with $\kappa_1$ as in (1.24) and
\begin{align}\kappa_2 = \begin{cases} 1 & \text{if } n_1 = n_2 = n_3 = 1, \\ c_1^{n_1-1} + c_2^{n_2-1} + c_3^{n_3-1} & \text{if } n_i = 1 \text{ for exactly two indices } i \in \{1, 2, 3\}, \\ 0 & \text{in the other cases}; \end{cases}
\end{align}

— of degree 6:
\begin{align}z_1 z_2 + \kappa_3 \cdot y_1^3;
\end{align}

with
\begin{align}\kappa_3 = \begin{cases} 1 & \text{if } n_1 = n_2 = n_3 = 1, \\ n_3 c_1^{n_3-1} + n_1 c_2^{n_1-1} + n_2 c_3^{n_2-1} & \text{if } n_i = 1 \text{ for exactly two indices } i \in \{1, 2, 3\}, \\ 0 & \text{in the other cases}.
\end{cases}
\end{align}
On the algebra \( \mathcal{A}_1 \), we introduce the grading induced by that on \( K[\mathcal{X}_1] \).

**Remark 1.2.** In the case where \( n_3 = 1 \), (1.13) implies that in the algebra \( \mathcal{A}_1 \) we have \( c_1 = 0 \), and in this case we can omit the generator \( c_1 \) (in particular, we put \( c_1^{n_3-1} = 0 \) in (1.20) and (1.28)). A similar remark concerns the elements \( c_2 \) and \( c_3 \) (if \( n_1 = 1 \) or \( n_2 = 1 \), respectively).

Next, we consider the algebra \( \mathcal{A}_1' = K[\mathcal{X}_1']/I_1' \) with

\[
\mathcal{X}_1' = \mathcal{X}_1 \cup \{x_3, x_4\}.
\]

Here the grading on the algebra \( K[\mathcal{X}_1'] \) extends that on \( K[\mathcal{X}_1] \) in such a way that

\[
\deg x_3 = \deg x_4 = 1,
\]

and the ideal \( I_1' \) is generated by the generators of \( I_1 \) listed in (1.13), (1.15), (1.17), (1.18), (1.20), (1.21), (1.23), and by the following homogeneous elements:

\[
(1.29) \quad c_1 x_1, \; c_3 x_1,
\]

\[
(1.30) \quad c_2 x_3, \; c_3 x_3, \; \kappa_4 \cdot x_3 \quad \text{with} \quad \kappa_4 = \begin{cases} c_1^{n_3-1} & \text{if } n_3 > 1, \\ 1 & \text{if } n_3 = 1; \end{cases}
\]

\[
(1.31) \quad x_3 \cdot a \quad \text{with } a \in \mathcal{X}_1' \setminus \{y_1\} \quad \text{and} \quad \deg a > 0;
\]

\[
(1.32) \quad c_1 x_4, \; c_2 x_4, \; \kappa_5 \cdot x_4 \quad \text{with} \quad \kappa_5 = \begin{cases} c_3^{n_2-1} & \text{if } n_2 > 1, \\ 1 & \text{if } n_2 = 1; \end{cases}
\]

\[
(1.33) \quad x_4 \cdot a \quad \text{with } a \in \mathcal{X}_1' \setminus \{y_1\} \quad \text{and} \quad \deg a > 0.
\]

Since the ideal \( I_1' \) is homogeneous, the grading on the algebra \( K[\mathcal{X}_1'] \) induces a grading on \( \mathcal{A}_1' \).

**Remark 1.3.** If \( n_3 = 1 \), then, as in Remark 1.2, we have \( x_3 = 0 \) in the algebra \( \mathcal{A}_1' \), i.e., in this case we can omit the generator \( x_3 \) (and also \( c_1 \)), together with the corresponding relations. The same can be said about the element \( x_4 \) in the case where \( n_2 = 1 \).

Next, we put

\[
\mathcal{X}_2 = (\mathcal{X}_1 \setminus \{x_1\}) \cup \{x_3, x_4, x''_1, x''_3\},
\]

and then introduce the grading on \( K[\mathcal{X}_2] \) that coincides with (1.12) on the elements of \( \mathcal{X}_1 \setminus \{x_1\} \) and additionally satisfies

\[
(1.34) \quad \deg x_3 = \deg x'_1 = \deg x''_1 = 1.
\]

We consider the algebra \( \mathcal{A}_2 = K[\mathcal{X}_2]/I_2 \), where the ideal \( I_2 \) of the algebra \( K[\mathcal{X}_2] \) is generated by the elements listed in (1.13), (1.15), (1.17), (1.18), (1.20), (1.21), (1.23), and (1.27), by the elements in (1.30) and (1.31) with \( a \neq x_1 \), and, moreover, by the elements

\[
(1.35) \quad c_1 x'_1, \; c_3 x'_1, \; c_1 x''_1, \; c_2 x''_1, \; p_i x'_1, \; p_i x''_1 \quad (i \in \{1, 2, 3\});
\]

\[
(1.36) \quad (x'_1 + x''_1) x_2;
\]

\[
(1.37) \quad x'_1 x''_1, \; (x'_1)^2 + \lambda (n_1) \cdot (p_2 + p_3) y_1, \; (x''_1)^2 + \lambda (n_2) \cdot (p_1 + p_3) y_1,
\]

where \( n_1, n_2, p_1, p_2, p_3 \) are the grading parameters as in (1.12).
where for any \( n \in \mathbb{N} \)

\[
\lambda(n) = \begin{cases} 
1 & \text{if } n \text{ is divisible by } 4, \\
0 & \text{otherwise};
\end{cases}
\]

\[
x'_1 y_2, \ y'_1 y_2, \ x'_1 y_3 + p_1 z_2, \ x''_1 y_3 + p_1 z_2,
\]

\[
(x'_1 + x_2) z_2,
\]

\[
x'_1 z_1, \ x''_1 z_1, \ (x'_1 + x''_1) z_2;
\]

\[
y_2 z_2, \ y_3 z_1.
\]

The algebra \( \mathcal{A}_2 \) inherits the grading from the algebra \( K[\mathcal{X}_2] \).

**Remark 1.4.** As in Remark 1.3 if \( n_3 = 1 \), we can omit the generators \( x_3 \) and \( c_1 \) of the algebra \( \mathcal{A}_2 \).

Finally, we introduce the algebra \( \mathcal{A}_3 = K[\mathcal{X}_3]/I_3 \), where

\[ \mathcal{X}_3 = (\mathcal{X}_1 \setminus \{x_1, x_2\}) \cup \{x'_1, x''_1, x'_2, x''_2\}, \]

and the ideal \( I_3 \) is generated by the elements in \((1.36), (1.38), (1.41), (1.43), (1.45), (1.46)\) and by the elements

\[
c_2 x'_2, \ c_3 x'_2, \ c_2 x''_2, \ c_3 x''_2, \ c_1 (x'_2 + x''_2), \ p_1 x'_2, \ p_i x''_2 \ (i \in \{1, 2, 3\});
\]

\[
x'_1 x'_2, \ x''_1 x'_2, \ x'_1 x''_2, \ (x'_2)^2 + (x''_2)^2, (x'_2)^2 + (x''_2)^2 + \lambda(n_3) \cdot (p_1 + p_2) y_1;
\]

\[
x'_2 y_2, \ x''_2 y_3, \ x''_2 y_3 + p_1 z_1, \ x'_2 y_3 + p_1 z_2;
\]

\[
x'_2 z_1, \ x''_2 z_2, \ (x'_1 + x''_1) z_2.
\]

The grading on the algebra \( K[\mathcal{X}_3] \) (and, accordingly, on the algebra \( \mathcal{A}_3 \)) is introduced in such a way that relations \((1.12), (1.34)\) and the relation

\[
\deg x'_2 = \deg x''_2 = 1
\]

are satisfied for the generators in \( \mathcal{X}_3 \).

We consider the following \( 3 \times 3 \)-matrix over \( K \):

\[
C = \begin{pmatrix} 
n_3 & 0 & n_2 \\
n_3 & n_1 & 0 \\
0 & n_1 & n_2
\end{pmatrix}.
\]

The rank \( \text{rk} C \) of the matrix \( C \) does not exceed 2 (recall that char \( K = 2 \)).

**Theorem 1.2.** I) Suppose \( \text{rk} C = 2 \).

a) If, moreover, all numbers \( n_i \ (i = 1, 2, 3) \) are odd, then, as a graded \( K \)-algebra, the Hochschild cohomology algebra \( \text{HH}^*(R) \) is isomorphic to the algebra \( \mathcal{A}_1 \).

b) If \( n_1 \) is even (so that \( n_2, n_3 \) are odd), then \( \text{HH}^*(R) \simeq \mathcal{A}_1 \) as graded \( K \)-algebras.

II) If \( \text{rk} C = 1 \) and \( n_3 \) is odd, then \( \text{HH}^*(R) \simeq \mathcal{A}_2 \) as graded \( K \)-algebras.

III) If \( \text{rk} C = 0 \), then \( \text{HH}^*(R) \simeq \mathcal{A}_3 \) as graded \( K \)-algebras.

**Remark 1.5.** In the case of \( \text{rk} C = 2 \), among the numbers \( n_i \ (i = 1, 2, 3) \) at most one can be even. If such an even number exists, then, by the invariance of the defining relations \((1.1), (1.2)\) of the algebra \( R \) with respect to cyclic permutations of indices, we may assume that \( n_1 \) is even. Similarly, if \( \text{rk} C = 1 \), then exactly one number among the \( n_i \ (i = 1, 2, 3) \) is odd, and we may assume that \( n_3 \) is odd.
We recall that two finite-dimensional $K$-algebras $R_1$ and $R_2$ are said to be derived equivalent if their bounded derived categories $\mathcal{D}^b(R_1\text{-mod})$ and $\mathcal{D}^b(R_2\text{-mod})$ are equivalent as triangulated categories. It is known [20] that if $R_1$ and $R_2$ are derived equivalent, then $\text{HH}^* (R_1) \simeq \text{HH}^* (R_2)$ (as graded algebras). By [13], the algebras of dihedral type in the families $D(3 \mathcal{A}_1), D(3 \mathcal{B}_1)$, and $D(3 \mathcal{D}_1)$ (in the notation of [12]) are derived equivalent to suitable algebras in the family $D(3K)$. More precisely, any algebra in $D(3 \mathcal{D}_1)$ is derived equivalent to an algebra in $D(3K)$ with parameters $n_1, n_2, n_3$. Hence, Theorem 1.2 extends word for word to algebras in the family $D(3 \mathcal{D}_1)$.

Next, the algebras in $D(3 \mathcal{B}_1)$ are defined with the help of two natural parameters $k, s$ ($k > 1$) (see [12, 13]), and any such algebra is derived equivalent to the algebra in $D(3K)$ with the parameters $n_1 = k$, $n_2 = s$, $n_3 = 1$. The algebras in $D(3 \mathcal{A}_1)$ are defined with the help of a single natural parameter $k$, and any such algebra is derived equivalent to the algebra in $D(3K)$ with the parameters $n_1 = k$, $n_2 = 1$, $n_3 = 1$. This implies the following statements.

**Corollary 1.3.** Let $S$ be an algebra in $D(3 \mathcal{B}_1)$ with parameters $n_1, n_2 \in \mathbb{N}$. Then

$$\text{HH}^* (S) \simeq \begin{cases} A_1 & \text{if } n_1 \text{ and } n_2 \text{ are odd,} \\ A_1' & \text{if } n_1 \text{ is even and } n_2 \text{ is odd,} \\ A_2 & \text{if } n_1 \text{ and } n_2 \text{ are even.} \end{cases}$$

**Corollary 1.4.** Let $S$ be an algebra in $D(3 \mathcal{A}_1)$ with parameter $n_1 \in \mathbb{N}$. Then

$$\text{HH}^* (S) \simeq \begin{cases} A_1 & \text{if } n_1 \text{ is odd,} \\ A_1' & \text{if } n_1 \text{ is even.} \end{cases}$$

**Remark 1.6.** When using the description of the algebras $A_1, A_1', A_2$ in Corollary 1.3 we additionally assume that $n_3 = 1$, and similarly in Corollary 1.4 we assume that $n_2 = n_3 = 1$. Moreover, some generators of the algebras $A_1, A_1', A_2$ can be omitted, with the corresponding simplification of the defining relations (see Remarks 1.2–1.4).

Let $G$ be a finite group, and let $B$ be a block of the group algebra $K[G]$ with dihedral defect group of order $2^n$ and with three simple modules. From the results of [21, 22] it follows that $B$ is derived equivalent to an algebra in $D(3K)$ with the parameters $n_1 = 2^{n-2}, n_2 = 1, n_3 = 1$. Thus, Theorem 1.2 implies the following.

**Corollary 1.5.** Let $B$ be a block of the group algebra $K[G]$ with dihedral defect group and three simple modules. Then

$$\text{HH}^* (B) \simeq \begin{cases} A_1 & \text{if } n = 2, \\ A_1' & \text{if } n > 2. \end{cases}$$

We note that, by [23], the last corollary can be applied to principal blocks (over an algebraically closed field $K$ with characteristic 2) of the groups $\text{PSL}(2, q)$, where $q$ is a power of an odd prime number. In particular, since $A_4 \simeq \text{PSL}(2, 3)$, we obtain a description of the Hochschild cohomology algebra $\text{HH}^* (K[A_4])$; in this case we take $n = 2$ in Corollary 1.3 (cf. the description of $\text{HH}^* (\mathbb{F}_2[A_4])$ in [7]).

§2. Resolution

In this section we prove Theorem 1.1. Direct verification shows that diagram (1.10) indeed represents a bicomplex, i.e., its rows and columns are differential sequences and all squares are (anti)commutative.

**Proposition 2.1.** The 0th column $X_\bullet = B_0 \bullet$ of the bicomplex $B_\bullet \bullet$ is exact in all terms except the first two, $X_0 = B_{00} = L_1$ and $X_1 = B_{01} = L_2$. 

Proof. The matrices of the homomorphisms $\sigma_i$ ($i = 1, 2, 3$) show that the complex obtained from $X_*$ by omitting the term $T_0 = L_1$ decomposes into a direct sum of the following complexes:

\[
\begin{align*}
Y_* &: L'_2 \xrightarrow{B'} L''_2 \xrightarrow{C''} L_1 \xleftarrow{A'} L'_3 \xrightarrow{B'} \cdots, \\
Z_* &: L''_2 \xrightarrow{B''} L'_2 \xrightarrow{C'} L_1 \xleftarrow{A''} L''_3 \xrightarrow{B''} \cdots.
\end{align*}
\]

We verify the exactness of the complex $Y_*$ (except for the extreme left term).

Observe that if we reorder the direct summands in $L''_2$ and $L_1$ in the following way (cf. (1.3)):

\[
L''_2 = P_{21} \oplus P_{32} \oplus P_{13}, \quad L_1 = P_{31} \oplus P_{11} \oplus P_{22},
\]

we obtain matrix presentations of the homomorphisms $C''$ and $A'$ that differ only by cyclic permutations of indices from the matrix presentation of the homomorphism $B'$ introduced in (1.5):

\[
\begin{align*}
C'' &= \begin{pmatrix}
0 & a_{12} \otimes c'_1 & e_2 \otimes a'_{12} \\
& e_3 \otimes a'_{23} & 0 & a_{23} \otimes e'_2 \\
& e_{31} \otimes e'_3 & e_1 \otimes e'_{31} & 0
\end{pmatrix}, \\
A' &= \begin{pmatrix}
0 & a_{23} \otimes e'_3 & e_3 \otimes a'_{31} \\
& e_1 \otimes a'_{12} & 0 & a_{31} \otimes e'_1 \\
& e_{12} \otimes e'_2 & e_2 \otimes a'_{23} & 0
\end{pmatrix}
\end{align*}
\]

(it should be noted that one cyclic permutation is used for indices of the first terms of the tensor products, and another cyclic permutation is used for the second terms). Hence, in order to verify the exactness of $Y_*$, it suffices to prove that $\text{Ker } B' \subseteq \text{Im } C''$, and then we can use the symmetry mentioned above.

Suppose $B'(x) = 0$ with $x = (x_{21}, x_{32}, x_{13}) \in L''_2 = P_{21} \oplus P_{32} \oplus P_{13}$. Then

\[
\begin{align*}
x_{32} \cdot (a_{31} \otimes e'_2) + x_{13} \cdot (e_1 \otimes a'_{23}) &= 0, \\
x_{21} \cdot (e_2 \otimes a'_{31}) + x_{13} \cdot (a_{12} \otimes e'_1) &= 0, \\
x_{21} \cdot (a_{23} \otimes e'_1) + x_{32} \cdot (e_3 \otimes a'_{12}) &= 0.
\end{align*}
\]

We claim that

\[
x_{21} = y_{11} \cdot (a_{12} \otimes c'_1) + y_{22} \cdot (e_2 \otimes a'_{12})
\]

for some $y_{11} \in P_{11}, y_{22} \in P_{22}$.

The second relation in (2.2) implies that

\[
x_{21} \in \text{Rad } P_{21} = \Lambda (a_{12} \otimes c'_1, a_{32} \otimes e'_1, e_2 \otimes a'_{12}, e_2 \otimes a'_{13}).
\]

Adding suitable multiples of columns in (2.1),

\[
\lambda' \cdot (a_{12} \otimes e'_1, 0, e_1 \otimes a'_{31}) \quad \text{and} \quad \lambda'' \cdot (e_2 \otimes a'_{12}, a_{23} \otimes e'_2, 0),
\]

to $x = (x_{21}, x_{32}, x_{13})$ (here $\lambda', \lambda'' \in \Lambda$), we may assume that the decomposition of the element $x_{21}$ in the standard basis of the module $P_{21}$ contains no summands of the form $\lambda' \cdot (a_{12} \otimes c'_1)$ and $\lambda'' \cdot (e_2 \otimes a'_{12})$. Then $x_{21}$ is presented in the form

\[
x_{21} = \lambda \cdot (a_{32} \otimes e'_1) + \mu \cdot (e_2 \otimes a'_{13})
\]

for some $\lambda \in P_{31}, \mu \in P_{23}$. Moreover, we may assume that the decomposition of $\mu$ in the standard basis of the module $P_{23}$ contains no summands of the form $\bar{\mu} \cdot (a_{32} \otimes e'_3)$, $\bar{\mu} \in \Lambda$: if this is not the case, then the decomposition of $x_{21}$ contains the summand $\bar{\mu} (a_{32} \otimes e'_3)(e_2 \otimes a'_{13}) = \bar{\mu} (e_3 \otimes a'_{13})(a_{32} \otimes e'_1)$, and we can rewrite (2.3) in the desired form. Next, multiplying the second equation in (2.2) by the element $a_{23} \otimes e'_2$ on the right, we obtain

\[
\lambda \cdot (a_{32} a_{23} \otimes a'_{31}) + \mu \cdot (a_{23} \otimes (a_{31} a_{13})') = 0.
\]
If the second summand in (2.3) is nonzero, then the element \( r = a_{32}(a_{32}a_{23})^{n_{1} - 1} \cdot e' \) does not annihilate this summand under multiplication from the right; on the other hand, \( r \) lies in the right annihilator of the first summand in (2.4), a contradiction. If \( \mu \cdot (a_{23} \otimes (a_{13}a_{13}')) = 0 \), then \( \mu \) is easily seen to be a left multiple of the element \( e_{2} \otimes ((a_{13}a_{31})^{n_{2} - 1}a_{13}') \); hence, \( x_{31} \) contains a summand that is a multiple of \( e_{2} \otimes ((a_{13}a_{31})^{n_{2}})' \). Since the latter opportunity was excluded in the preceding reduction, we necessarily have \( \mu = 0 \) in (2.3). Using the third equation in (2.2), we see similarly that \( \lambda = 0 \) in (2.3).

Consequently, we may assume that \( x \in \text{Ker} B' \) has the form \( x = (0, x_{32}, x_{13}) \). Then the second relation in (2.2) implies that \( x_{13} \in \text{Ker}(a_{12} \otimes e'_3) \), and it is easily seen that \( x_{13} = \nu \cdot (a_{31} \otimes e'_3) \) for some \( \nu \in \Lambda \). Adding to \( x \) the element \( \nu \cdot (0, e_{3} \otimes a'_{23}, a_{31} \otimes e'_{3}) \) (a multiple of the first column of the matrix \( C'' \) in (2.1)), we obtain an element in \( \text{Ker} B' \subset P_{21} \oplus P_{32} \oplus P_{13} \) the first and the third component of which are zero.

Hence, it remains to consider an element of the form \( x = (0, x_{32}, 0) \in \text{Ker} B' \). In this case, \( x_{32} \in \text{Ker}(a_{31} \otimes e'_3) \cap \text{Ker}(e_{3} \otimes a'_{12}) \) by (2.2), whence it easily follows that \( x_{32} = \zeta \cdot (a_{23} \otimes a'_{23}) \) for some \( \zeta \in \Lambda \). It remains to observe that \( x = \zeta \cdot (e_{2} \otimes a'_{23})(e_{2} \otimes a'_{12}, a_{23} \otimes e'_{2}, 0) \in \text{Im} C'', \) and this completes the proof of the embedding \( \text{Ker} B' \subset \text{Im} C'' \).

The proof of the exactness of the complex \( Z_{\bullet} \) is similar.

\[ \square \]

As the first step of the proof of Theorem 1.1 we show that the total complex \( Q_{\bullet} = \text{Tot} B_{\bullet} \) is exact at the term \( Q_{1} \), namely, we prove the following statement.

**Proposition 2.2.** In the notation of (1.10), we have

\[ \text{Ker} \rho_1 = \text{Im} \rho_2 + \text{Im} \sigma_1. \]

**Proof.** We denote by \( v^{(i)} \), \( i = 1, 2, 3 \), the corresponding columns of the matrix \( \rho_2 \) (see (1.7)) and by \( w^{(i)} \), \( 1 \leq j \leq 6 \), the corresponding columns of the matrix \( \sigma_1 \) (see (1.8)). We consider these columns simultaneously as elements of the module

\[ L_2 = (P_{12} \oplus P_{23} \oplus P_{31}) \oplus (P_{21} \oplus P_{32} \oplus P_{13}). \]

Moreover, having in mind the grouping of the direct summands in \( L_2 \), we gather the components of any element of \( L_2 \) in groups of three components each; in particular, a group containing three zero components is denoted briefly by \( O_3 \).

We assume that \( \rho_1(x) = 0 \) for some

\[ x = (x_{12}, x_{23}, x_{31} | x_{21}, x_{32}, x_{13}) \in L_2 \]

with \( x_{ij} \in P_{ij} \). We claim that, since \( \text{Im} \rho_2 + \text{Im} \sigma_1 \subset \text{Ker} \rho_1 \), we can successively add suitable elements of \( \text{Im} \rho_2 + \text{Im} \sigma_1 \) to \( x \) so as to get the zero element, whence we obtain the reverse inclusion.

The relation \( \rho_1(x) = 0 \) is equivalent to the following system of equations:

\[ \begin{align*}
(2.5) \quad & x_{12} \cdot (e_{1} \otimes a'_{12}) + x_{31} \cdot (a_{31} \otimes e'_{1}) + x_{21} \cdot (a_{21} \otimes e'_{1}) + x_{13} \cdot (e_{1} \otimes a'_{13}) = 0, \\
& x_{12} \cdot (a_{2} \otimes e'_{2}) + x_{23} \cdot (e_{2} \otimes a'_{23}) + x_{31} \cdot (e_{2} \otimes a'_{23}) + x_{32} \cdot (a_{23} \otimes e'_{2}) = 0, \\
& x_{23} \cdot (a_{23} \otimes e'_{3}) + x_{31} \cdot (e_{3} \otimes a'_{31}) + x_{32} \cdot (e_{3} \otimes a'_{32}) + x_{13} \cdot (a_{13} \otimes e'_{3}) = 0.
\end{align*} \]

We may assume that \( x_{12} \) contains nonzero summands of the form

\[ \mu \cdot (a_{31} \otimes e'_{2}), \nu \cdot (e_{1} \otimes a'_{23}) \]

with \( \mu, \nu \in \Lambda \).

Indeed, if we subtract from \( x \) suitable multiples of the elements

\[ (a_{31} \otimes e'_{2}, 0, e_{3} \otimes a'_{12} | O_3), \quad (e_{1} \otimes a'_{23}, a_{12} \otimes e'_{3}, 0 | O_3) \in \text{Im} \sigma_1, \]

which coincide with the last two columns of the matrix \( \sigma_1 \), and recall that \( \text{Im} \rho_2 + \text{Im} \sigma_1 \subset \text{Ker} \rho_1 \), we obtain an element in \( \text{Ker} \rho_1 \) with the desired property.
Similarly, we may assume that \( x_{21} \) contains no nonzero summands lying in \( \Lambda \cdot (e_2 \otimes a_{13}') \) or in \( \Lambda \cdot (a_{32} \otimes e_1') \).

Now, we present \( x_{12} \) and \( x_{21} \) in the form

\[
\begin{align*}
\{ & x_{12} = \alpha \cdot (a_{21} \otimes e_2') + \beta \cdot (e_1 \otimes a_{21}'), \\
& x_{21} = \gamma \cdot (e_2 \otimes a_{12}') + \delta \cdot (a_{12} \otimes e_1')
\end{align*}
\]

with \( \alpha, \beta, \gamma, \delta \in \Lambda \). For brevity, the presentation of \( x_{12} \) and \( x_{21} \) as in (2.6) will be called reduced. Substituting relations (2.6) in the first equation in (2.5), we obtain

\[
(\alpha + \gamma)(a_{21} \otimes a_{12}') + \beta(e_1 \otimes (a_{12}a_{21}')) + \delta(a_{12}a_{21} \otimes e_1') + x_{31}(a_{31} \otimes e_1') + x_{13}(e_1 \otimes a_{13}') = 0.
\]

Every summand on the left in (2.7) can be written as a linear combination of the standard basis vectors of \( \Lambda \), and we can talk about “combining like terms” in such presentations.

Suppose \( x_{31} \) contains nonzero summands lying in \( \Lambda \cdot (e_3 \otimes a_{12}') \). Then the nonzero summands of the form

\[
\lambda \cdot (a_{31} \otimes a_{12}'), \quad \lambda \in \Lambda,
\]

occurring in \( x_{31}(a_{31} \otimes e_1') \) may be canceled by like terms that must be in the left-hand side of (2.7), except for the term \( \beta(e_1 \otimes (a_{12}a_{21}')) \) (otherwise \( \beta \), and hence \( x_{12} \), contains a nonzero multiple of the element \( a_{31} \otimes e_1' \)). If there is a nonzero summand of the form (2.8) in \( (\alpha + \gamma)(a_{21} \otimes a_{12}') \) or in \( \delta(a_{12}a_{21} \otimes e_1') \), then, using relations (1.2) that define the algebra \( R \), we deduce that \( \lambda \) in (2.8) must be of the form

\[
\lambda = \tilde{\lambda} \cdot ((a_{13}a_{31})^{n_2-1}a_{13} \otimes e_2'), \quad \tilde{\lambda} \in \Lambda.
\]

Then, adding the element

\[
y = \tilde{\lambda}(e_1 \otimes a_{12}') \cdot v^{(1)} = (y_{12}, 0, \tilde{\lambda}((a_{13}a_{31})^{n_2-1}a_{13} \otimes a_{12}'), y_{21}, 0, 0) \in \text{Im } \rho_2
\]
to \( x \), we obtain an element \( x^{(1)} = x + y \) for which, in the equation

\[
(\alpha^{(1)} + \gamma^{(1)})(a_{21} \otimes a_{12}') + \beta^{(1)}(e_1 \otimes (a_{12}a_{21}')) + \delta^{(1)}(a_{12}a_{21} \otimes e_1') + x^{(1)}_{31}(a_{31} \otimes e_1') + x^{(1)}_{13}(e_1 \otimes a_{13}') = 0
\]

similar to (2.7), the summand \( x^{(1)}_{31}(a_{31} \otimes e_1') \) contains no terms of the form (2.8) that are likes of \((\alpha^{(1)} + \gamma^{(1)})(a_{21} \otimes a_{12}') \) or \((\alpha^{(1)} + \gamma^{(1)})(a_{21} \otimes e_1') \). We observe that, under the replacement of \( x \) by \( x^{(1)} \), we add to the component \( x_{12} \) (respectively, \( x_{21} \)) only summands that are multiples of \( a_{21} \otimes e_2' \) and \( e_1 \otimes a_{12}' \) (respectively, \( a_{12} \otimes e_1' \) and \( e_2 \otimes a_{12}' \)), i.e., the new components \( x^{(1)}_{12} \) and \( x^{(1)}_{21} \) also possess reduced presentation (cf. (2.6)). Moreover, the components \( x_{13}, x_{23}, \) and \( x_{32} \) are not modified.

Finally, if for the summands \( x_{31}(a_{31} \otimes e_1') \) and \( x_{13}(e_1 \otimes a_{13}') \) in (2.7) there are common like terms of the form (2.8), then, for these like terms, the element \( \lambda \) in (2.8) must be of the form

\[
\lambda = \tilde{\lambda} \cdot ((e_3 \otimes (a_{12}a_{21})^{n_3-1} a_{12}))', \quad \tilde{\lambda} \in \Lambda,
\]

and then \( x_{31} \) involves a summand

\[
\tilde{\lambda}(e_3 \otimes ((a_{12}a_{21})^{n_3} a_{12})') = \tilde{\lambda}(e_3 \otimes ((a_{13}a_{31})^{n_2} a_{13})'),
\]

which can be written as a multiple of the element \( e_3 \otimes a_{13}' \). Consequently, we may assume that the component \( x_{31} \) in the initial \( x \) contains no nonzero summands that are multiples of \( e_3 \otimes a_{12}' \).

Similarly, it can be assumed that the component \( x_{13} \) of the element \( x \) contains no nonzero summands that are multiples of \( a_{21} \otimes e_3' \) (we observe that, under the corresponding transformations of the element \( x \) (similar to those used above), the components of
\(P_{12}, P_{23}, \) and \(P_{32}\) are not modified, and the components of \(P_{12}\) and \(P_{21}\) preserve their reduced presentations.

Moreover, adding a suitable multiple of the forth column \(w^{(4)}\) of the matrix \(\sigma_1\) (see (1.8)), we may assume that the component \(x_{21}\) of \(x\) contains no nonzero multiples of the element \(a_{23} \otimes e'_1\) (note that such transformations of \(x\) do not change its components lying in \(P_{12}\) and \(P_{21}\)). Consequently, we may assume that

\[ x_{31} \in \Lambda \cdot (a_{13} \otimes e'_1) + \Lambda \cdot (e_3 \otimes a'_{13}), \]

e.i., \(x_{31}\) has a “reduced presentation” similar to those of the elements \(x_{12}, x_{21}\) in (2.6).

By symmetry, we may also assume that the component \(x_{13}\) has a similar reduced presentation.

Next, replacing the components \(x_{31}\) and \(x_{13}\) in the preceding argument by the components \(x_{23}\) and \(x_{32}\), respectively, and using the second equation in (2.5), we can transform the components \(x_{23}\) and \(x_{32}\) of the element \(x\) so that the new components lying in \(P_{13}\) and \(P_{23}\) contain no nonzero summands that are multiples of \(a_{12} \otimes e'_3\) and \(e_3 \otimes a'_{12}\), respectively. Moreover, the new components lying in \(P_{12}\) and \(P_{21}\) still have reduced presentations, and the components \(x_{31}\) and \(x_{13}\) do not change. Now, writing \(x_{31}\) and \(x_{13}\) in the “reduced form” and using the third equation in (2.5), we obtain a relation similar to (2.7). Analyzing this relation as above, we can, if necessary, rewrite the element \(x\) so that the new component lying in \(P_{13}\) will have no nonzero summands that are multiples of \(e_2 \otimes a'_1\) (here, we use the above argument, with the replacement of \(x_{12}, x_{21}\) by \(x_{31}, x_{13}\), and \(x_{31}, x_{13}\) by \(x_{23}, x_{32}\)). Hence, we may assume that \(x_{23} \in \Lambda(a_{23} \otimes e'_1) + \Lambda(e_2 \otimes a'_3)\), i.e., \(x_{23}\) has a reduced presentation. By symmetry, we may also assume that \(x_{32}\) has a reduced presentation. Moreover, under the corresponding transformations of the element \(x\) all its remaining components preserve their reduced presentations.

Consequently, we may assume that \(x \in \text{Ker} \rho_1\) is such that all its components have reduced presentations. In particular, the decomposition of the element \(x_{12}\) (respectively, \(x_{21}\)) in the standard basis of the module \(P_{12}\) (respectively, \(P_{21}\)) contains only basis elements of the form \(u \otimes v\), where \(u\) and \(v\) are products of the arrows \(a_{12}\) and \(a_{21}\). The sum of lengths of the paths \(u\) and \(v\) is called the degree of this element \(u \otimes v\). Now, we divide the proof into several steps.

Step 1. We prove that \(x_{12}\) and \(x_{21}\) have no summands of degree less than \(2n_3 - 1\).

Step 1a. First, we consider such summands of odd degree. Let

\[ \begin{align*}
\bar{x}_{12} &= \sum_{i=0}^{k} \gamma_i^{(1)} a_{21} \alpha_i^{1} \otimes (\beta_2^{k-i})' + \sum_{i=0}^{k} \delta_i^{(1)} \alpha_i^{1} \otimes (\beta_2^{k-i} a_{21})', \\
\bar{x}_{21} &= \sum_{i=0}^{k} \epsilon_i^{(1)} a_{12} \beta_i^{2} \otimes (\alpha_1^{k-i})' + \sum_{i=0}^{k} \eta_i^{(1)} \beta_i^{2} \otimes (\alpha_1^{k-i} a_{12})'.
\end{align*} \tag{2.9} \]

be the homogeneous summands of degree \(2k + 1, 0 \leq k < n_3 - 1\), included in \(x_{12}\) and \(x_{21}\), respectively, where \(\gamma_i^{(1)}, \delta_i^{(1)}, \epsilon_i^{(1)}, \eta_i^{(1)} \in K\) recall that \(\alpha_1 = a_{12} a_{21}, \beta_2 = a_{21} a_{12}\) (see (1.4)). We substitute (2.9) in the first equation (2.5). Then on the left-hand side of this equation we obtain the following homogeneous summand of degree \(2k + 2\):

\[ \begin{align*}
\sum_{i=0}^{k} \gamma_i^{(1)} a_{21} \alpha_i^{1} \otimes (a_{12} \alpha_1^{k-i})' + \sum_{i=0}^{k} \delta_i^{(1)} \alpha_i^{1} \otimes (\alpha_1^{k-i+1})' \\
+ \sum_{i=0}^{k} \epsilon_i^{(1)} \alpha_1^{i+1} \otimes (\alpha_1^{k-i})' + \sum_{i=0}^{k} \eta_i^{(1)} a_{21} \alpha_i^{1} \otimes (a_{12} \alpha_1^{k-i})',
\end{align*} \]
and since \( k < n_3 - 1 \), this summand must be zero. It follows that
\[
\gamma^{(1)}_i = \eta^{(1)}_i, \quad 0 \leq i \leq k; \quad \varepsilon^{(1)}_{j-1} = \delta^{(1)}_j, \quad 1 \leq j \leq k; \quad \delta^{(1)}_0 = 0 = \varepsilon^{(1)}_k.
\]

In the same way, from the second equation (2.5) we obtain
\[
\delta^{(1)}_i = \varepsilon^{(1)}_i, \quad 0 \leq i \leq k; \quad \gamma^{(1)}_{j-1} = \eta^{(1)}_j, \quad 1 \leq j \leq k; \quad \gamma^{(1)}_k = 0 = \eta^{(1)}_0.
\]

Combining this with (2.10), we see that
\[
\gamma^{(1)}_i = \delta^{(1)}_i = \varepsilon^{(1)}_i = \eta^{(1)}_i = 0 \quad \text{for all } i.
\]

Step 1b. The fact that \( x_{12} \) and \( x_{21} \) have no homogeneous summands of degree 2 with \( k \leq n_3 - 1 \) is proved similarly.

By symmetry, we also conclude that \( x_{23} \) and \( x_{32} \) have no summands of degree less than \( 2n_1 - 1 \) and that \( x_{31} \) and \( x_{13} \) have no summands of degree less than \( 2n_2 - 1 \).

Step 2. Now we simultaneously consider the summands of degree \( 2n_3 - 1 \) in \( x_{12} \) and \( x_{21} \), the summands of degree \( 2n_1 - 1 \) in \( x_{23} \) and \( x_{32} \), and the summands of degree \( 2n_2 - 1 \) in \( x_{31} \) and \( x_{13} \). We write the sum of such summands occurring in \( x_{12} \) and \( x_{21} \) in the form
\[
\gamma^{(j)}_i, \delta^{(j)}_i, \varepsilon^{(j)}_i, \eta^{(j)}_i \in K.
\]

Using relations (1.2), from the first equation in (2.5) we deduce the following:
\[
\gamma^{(1)}_i = \eta^{(1)}_i, \quad 0 \leq i \leq n_3 - 1; \quad \varepsilon^{(1)}_{j-1} = \delta^{(1)}_j, \quad 1 \leq j \leq n_3 - 1;
\]
\[
\gamma^{(3)}_i = \varepsilon^{(3)}_i, \quad 0 \leq i \leq n_2 - 1; \quad \gamma^{(3)}_{j-1} = \eta^{(3)}_j, \quad 1 \leq j \leq n_2 - 1;
\]
\[
\varepsilon^{(3)}_{n_3-1} = \gamma^{(3)}_{n_2-1}; \quad \delta^{(3)}_0 = \delta^{(3)}_0 = \eta^{(3)}_0.
\]

The second equation in (2.5) implies that
\[
\varepsilon^{(1)}_i = \delta^{(1)}_i, \quad 0 \leq i \leq n_3 - 1; \quad \gamma^{(1)}_{j-1} = \eta^{(1)}_j, \quad 1 \leq j \leq n_3 - 1;
\]
\[
\gamma^{(2)}_i = \varepsilon^{(2)}_i, \quad 0 \leq i \leq n_1 - 1; \quad \gamma^{(2)}_{j-1} = \eta^{(2)}_j, \quad 1 \leq j \leq n_1 - 1;
\]
\[
\gamma^{(2)}_{n_3-1} = \varepsilon^{(2)}_{n_1-1}; \quad \eta^{(2)}_0 = \delta^{(2)}_0 = \delta^{(2)}_0 = \varepsilon^{(2)}_0.
\]

The identities in (2.11) and (2.12) that relate \{\gamma^{(1)}_i\}_i and \{\eta^{(1)}_i\}_j show that
\[
\gamma^{(1)}_0 = \gamma^{(1)}_1 = \cdots = \gamma^{(1)}_{n_3-1} = \eta^{(1)}_0 = \eta^{(1)}_1 = \cdots = \eta^{(1)}_{n_3-1} = a.
\]

Similarly, we obtain
\[
\delta^{(1)}_0 = \delta^{(1)}_1 = \cdots = \delta^{(1)}_{n_3-1} = \varepsilon^{(1)}_0 = \varepsilon^{(1)}_1 = \cdots = \varepsilon^{(1)}_{n_3-1} = b.
\]

By symmetry, this implies the following groups of identities:
\[
\gamma^{(2)}_0 = \gamma^{(2)}_1 = \cdots = \gamma^{(2)}_{n_1-1} = \eta^{(2)}_0 = \eta^{(2)}_1 = \cdots = \eta^{(2)}_{n_1-1} = c,
\]
\[
\delta^{(2)}_0 = \delta^{(2)}_1 = \cdots = \delta^{(2)}_{n_1-1} = \varepsilon^{(2)}_0 = \varepsilon^{(2)}_1 = \cdots = \varepsilon^{(2)}_{n_1-1}.
\]
moreover, from the last identity in (2.12) it follows that \( \delta_0^{(3)} = a \). Also, we obtain
\[
\begin{align*}
\gamma_0^{(3)} &= \gamma_1^{(3)} = \cdots = \gamma_{n_2-1}^{(3)} = \eta_0^{(3)} = \eta_1^{(3)} = \cdots = \eta_{n_2-1}^{(3)}, \\
\delta_0^{(3)} &= \delta_1^{(3)} = \cdots = \delta_{n_2-1}^{(3)} = \epsilon_0^{(3)} = \epsilon_1^{(3)} = \cdots = \epsilon_{n_2-1}^{(3)},
\end{align*}
\]
and, moreover, \( \gamma_0^{(3)} = b, \ \delta_0^{(3)} = c \). Consequently, adding the element
\[
b \cdot v^{(1)} + a \cdot v^{(2)} + c \cdot v^{(3)} \in \text{Im} \rho_2
\]
to \( x \), we obtain a new element in Ker \( \rho_1 \) for which the components lying in \( P_{12} \) and \( P_{21} \) have no summands of degree \( 2n_3 - 1 \), the components lying in \( P_{23} \) and \( P_{32} \) have no summands of degree \( 2n_1 - 1 \), and the components lying in \( P_{31} \) and \( P_{13} \) have no summands of degree \( 2n_2 - 1 \).

Step 3. Now we consider the homogeneous components in \( x_{12} \) and \( x_{21} \) of degree \( r \), the components in \( x_{23} \) and \( x_{32} \) of degree \( s \), and the components in \( x_{31} \) and \( x_{13} \) of degree \( t \), where \( r > 2n_3 - 1, s > 2n_1 - 1, t > 2n_2 - 1 \).

Step 3a. First, we consider the case where \( r \) is odd: \( r = 2k + 1 \) with \( k \geq n_3 \); moreover, we additionally assume that \( k < 2n_3 - 1 \). Put \( \overline{k} = k - n_3 + 1 \). Like (2.9), we write the corresponding homogeneous components of \( x_{12} \) and \( x_{21} \) in the form
\[
\begin{align*}
\overline{x}_{12} &= \sum_{i=k-1}^{n_3-1} \gamma_i^{(1)} a_{21} \alpha_i^{(1)} \otimes (\beta_2^{k-i})' + \sum_{i=k}^{n_3} \delta_i^{(1)} \alpha_i^{(1)} \otimes (\beta_2^{k-i} a_{21})', \\
\overline{x}_{21} &= \sum_{i=k-1}^{n_3-1} \epsilon_i^{(1)} a_{12} \beta_i^{(1)} \otimes (\alpha_1^{k-i})' + \sum_{i=k}^{n_3} \eta_i^{(1)} \beta_i^{(1)} \otimes (\alpha_1^{k-i} a_{12})',
\end{align*}
\]
with \( \gamma_i^{(1)}, \delta_i^{(1)}, \epsilon_i^{(1)}, \eta_i^{(1)} \in K \). Substituting (2.13) and similar presentations of the homogeneous components of \( x_{31} \) and \( x_{13} \) in the first equation (2.10), we do not obtain any common “like terms” (in the decompositions in the standard basis of \( P_{11} \)) in the summands \( \overline{x}_{12}(e_1 \otimes a'_{12}) + \overline{x}_{21}(a_{21} \otimes e_1') \) and in the summands \( x_{31}(a_{31} \otimes e_1') + x_{13}(e_1 \otimes a'_{13}) \), because \( \overline{k} < n_3 \). It follows that
\[
\gamma_i^{(1)} = \eta_i^{(1)}, \ \overline{k} \leq i \leq n_3 - 1; \quad \epsilon_{j-1}^{(1)} = \delta_j^{(1)}, \ \overline{k} \leq j \leq n_3.
\]
Similarly, from the second equation in (2.10) we deduce that
\[
\delta_j^{(1)} = \epsilon_j^{(1)}, \ \overline{k} \leq j \leq n_3 - 1; \quad \gamma_{j-1}^{(1)} = \eta_j^{(1)}, \ \overline{k} \leq j \leq n_3.
\]
Relations (2.14) and (2.15) imply
\[
\begin{align*}
\gamma_0^{(1)} = \cdots = \gamma_{n_3-1}^{(1)} = \eta_0^{(1)} = \cdots = \eta_{n_3-1}^{(1)} = a, \\
\delta_0^{(1)} = \cdots = \delta_{n_3}^{(1)} = \epsilon_0^{(1)} = \cdots = \epsilon_{n_3}^{(1)} = b.
\end{align*}
\]
Adding the element
\[
\left( b \cdot a_{21} \otimes (a_{12} \beta_1^{k-1})' + a \cdot a_{12} \otimes (a_{21} \alpha_1^{k-1})' \right) \cdot v^{(1)} \in \text{Im} \rho_2
\]
to \( x \), we obtain a new element in Ker \( \rho_1 \) for which the components lying in \( P_{12} \) and \( P_{21} \) have no summands of degree \( r \). Observe that, under this transformation, the remaining components of the element \( x \) do not change.

If \( s < 4n_1 - 1, t < 4n_2 - 1, \) and \( s, t \) are odd, then, by symmetry, we can modify \( x \) by using an element also lying in \( \text{Im} \rho_2 \), which yields an element of Ker \( \rho_1 \) such that the components lying in \( P_{23} \) and \( P_{32} \) have no summands of degree \( s \) and the components lying in \( P_{31} \) and \( P_{13} \) have no summands of degree \( t \).
In the case where \( r = 4n_3 - 1 \) but \( s < 4n_1 - 1 \), we can argue as above to show that \( \varepsilon^{(1)}_{n_3 - 1} = \delta^{(1)}_{n_3} \) (cf. (2.14)). Modifying \( x \) with the help of a suitable element in \( \text{Im} \rho_2 \), we may assume that in \( \tilde{x}_{12} \) and \( \tilde{x}_{21} \) we have \( \varepsilon^{(1)}_{n_3 - 1} = 0 = \delta^{(1)}_{n_3} \) from the outset. A similar argument applies to the remaining cases where one of the following three identities:

\[
(2.16) \quad r = 4n_3 - 1, \quad s = 4n_1 - 1, \quad t = 4n_2 - 1,
\]

is true, but all three of them are not fulfilled simultaneously.

Finally, assume that all three identities in (2.16) are fulfilled. We write the corresponding homogeneous summands of the components \( x_{31}, x_{13}, x_{23}, x_{32} \) in a form similar to (2.13):

(2.17)

\[
\begin{align*}
\tilde{x}_{23} &= \gamma^{(2)}_{n_3 - 1} a_{32} \alpha^{n_1 - 1}_2 \otimes (\beta^n_3)^{r'} + \delta^{(2)}_{n_1} \alpha^{n_1 - 1}_2 \otimes (\beta^n_3)^{r'} a_{32}, \\
\tilde{x}_{32} &= \gamma^{(2)}_{n_1 - 1} a_{31} \alpha^{n_1 - 1}_2 \otimes (\alpha^n_2)^{r'} + \eta^{(2)}_{n_1} \alpha^{n_1 - 1}_2 \otimes (\alpha^n_2)^{r'} a_{31}, \\
\tilde{x}_{31} &= \gamma^{(3)}_{n_3 - 1} a_{13} \alpha^{n_2 - 1}_3 \otimes (\beta^n_2)^{r'} + \delta^{(3)}_{n_2} \alpha^{n_2 - 1}_3 \otimes (\beta^n_2)^{r'} a_{13}, \\
\tilde{x}_{13} &= \gamma^{(3)}_{n_1 - 1} a_{31} \alpha^{n_2 - 1}_3 \otimes (\alpha^n_2)^{r'} + \eta^{(3)}_{n_2} \alpha^{n_2 - 1}_3 \otimes (\alpha^n_2)^{r'} a_{31},
\end{align*}
\]

with \( \gamma^{(j)}_i, \delta^{(j)}_i, \varepsilon^{(j)}_i, \eta^{(j)}_i \in K \). Substituting (2.13) and (2.17) in the first equation in (2.5), we see that

\[
\varepsilon^{(1)}_{n_3 - 1} + \delta^{(1)}_{n_3} + \gamma^{(3)}_{n_2 - 1} + \eta^{(3)}_{n_2} = 0.
\]

With the help of the remaining two equations (2.5), we obtain in the same way that

(2.18)

\[
\begin{align*}
\varepsilon^{(2)}_{n_3 - 1} + \delta^{(2)}_{n_3} + \gamma^{(1)}_{n_1 - 1} + \eta^{(1)}_{n_1} &= 0, \\
\varepsilon^{(3)}_{n_3 - 1} + \delta^{(3)}_{n_3} + \gamma^{(2)}_{n_1 - 1} + \eta^{(2)}_{n_1} &= 0.
\end{align*}
\]

We observe that

\[
\tilde{x} := \begin{pmatrix}
\delta^{(1)}_{n_3} (a_{12} a_{21})^{n_3} \otimes ((a_{21} a_{12})^{n_2 - 1} a_{21})^{r'} \\
\gamma^{(3)}_{n_3 - 1} a_{13} (a_{31} a_{13})^{n_2 - 1} \otimes ((a_{31} a_{13})^{n_2} a_{13})^{r'} \\
\varepsilon^{(1)}_{n_3 - 1} a_{12} (a_{21} a_{12})^{n_2 - 1} \otimes ((a_{21} a_{12})^{n_3} a_{21})^{r'} \\
\eta^{(3)}_{n_2} a_{31} (a_{31} a_{31})^{n_2 - 1} \otimes ((a_{31} a_{31})^{n_2 - 1} a_{31})^{r'}
\end{pmatrix}
\]

\[
= \delta^{(1)}_{n_3} (a_{13} (a_{31} a_{13})^{n_2 - 1} \otimes (a_{31} (a_{31} a_{13})^{n_2 - 1} a_{21})^{r'}) \cdot u^{(5)}
\]

\[
+ \varepsilon^{(1)}_{n_3 - 1} (a_{21} (a_{21} a_{12})^{n_2 - 1} \otimes (a_{21} (a_{12} a_{21})^{n_3} a_{21})^{r'}) \cdot u^{(2)}
\]

\[
+ (\delta^{(3)}_{n_3} + \gamma^{(3)}_{n_3 - 1}) \cdot v^{(3)}
\in \text{Im} a_1 + \text{Im} \rho_2.
\]

Replacing \( x \) by \( x + \tilde{x} \), we may assume that for the initial \( x \) we have

\[
\varepsilon^{(1)}_{n_3 - 1} = \delta^{(1)}_{n_3} = \gamma^{(3)}_{n_2 - 1} = \eta^{(3)}_{n_2} = 0.
\]

Similar application of (2.18) allows us to replace \( x \) by an element for which the components lying in \( P_{12} \) and \( P_{21} \) (respectively, in \( P_{23} \) and \( P_{32} \), or in \( P_{31} \) and \( P_{13} \)) have no summands of the maximal odd degree \( 4n_3 - 1 \) (respectively, \( 4n_1 - 1 \), or \( 4n_2 - 1 \)).

Step 3b. Now we consider the case where \( r \) is even: \( r = 2k \) with \( k \geq n_3 \); we additionally assume that \( k < 2n_3 \). Again, put \( \bar{k} = k - n_3 + 1 \). The homogeneous components in \( x_{12} \)
and \( x_{21} \) of degree \( r \) are written in the following form:

\[
\begin{align*}
\bar{x}_{12} &= \sum_{i=k-1}^{n_3} \gamma_i^{(1)} \alpha_1^i \otimes (\beta_2^k)^i' + \sum_{i=k-1}^{n_3-1} \delta_i^{(1)} a_{21} \alpha_1^i \otimes (\beta_2^{k-1-i} a_{21})', \\
\bar{x}_{21} &= \sum_{i=k-1}^{n_3} \varepsilon_i^{(1)} \beta_2^i \otimes (\alpha_1^k)^i' + \sum_{i=k-1}^{n_3-1} \eta_i^{(1)} a_{12} \beta_2^i \otimes (\alpha_1^{k-1-i} a_{12})'
\end{align*}
\]

with \( \gamma_j^{(1)}, \delta_i^{(1)}, \varepsilon_i^{(1)}, \eta_i^{(1)} \in K \). As in the case where \( r \) is odd, we can use the first and the second equations (2.20) to prove that

\[
\gamma^{(1)}_{\mathcal{F}-1} = \gamma^{(1)}_{\mathcal{F}} = \cdots = \gamma^{(1)}_{n_3} = \eta^{(1)}_{\mathcal{F}-1} = \cdots = \eta^{(1)}_{n_3-1} = a.
\]

Observe that if \( \mathcal{F} > 1 \), then

\[
\bar{x} := \left( \sum_{i=k-1}^{n_3} \alpha_1^i \otimes (\beta_2^k)^i', 0, 0 \right| \sum_{i=k-1}^{n_3-1} a_{12} \beta_2^i \otimes (\alpha_1^{k-1-i} a_{12})', 0, 0 \right) = ((a_{12} a_{21})^{-1} \otimes a_{12}^t) \cdot v^{(1)} \in \text{Im } \rho_2,
\]

and if \( \mathcal{F} = 1 \), then for the same element \( \bar{x} \) we have

\[
\bar{x} = (e_1 \otimes a_{12}^t) \cdot v^{(1)} + ((a_{13} a_{31})^{-1} a_{13} \otimes e_2^t) \cdot w^{(5)} \in \text{Im } \rho_2 + \text{Im } \sigma_1.
\]

Hence, replacing \( x \) by \( x + a \bar{x} \), we may assume that \( a \) in (2.20) is equal to zero. Similarly, we may assume that all \( \delta_i^{(1)}, \varepsilon_i^{(1)} \) in (2.19) are zero, i.e., \( x_{12} \) and \( x_{21} \) have no nonzero summands of degree \( r = 2k \). By symmetry, we may also assume that \( x_{23}, x_{32} \) (respectively, \( x_{31}, x_{13} \)) have no nonzero summands of even degree \( s < 4n_1 \) (respectively, of even degree \( t < 4n_2 \)).

Finally, the homogeneous components in \( x_{12} \) and \( x_{21} \) of degree \( r = 4n_3 \) (i.e., \( k = 2n_3 \)) have the form (cf. (2.19))

\[
\bar{x}_{12} = \gamma_{n_3}^{(1)} \alpha_1^{n_3} \otimes (\beta_2^{n_3})', \quad \bar{x}_{21} = \varepsilon_{n_3}^{(1)} \beta_2^{n_3} \otimes (\alpha_1^{n_3})'.
\]

In the above argument, we replace \( \bar{x} \) by

\[
(\alpha^{n_3} \otimes (\beta^{n_3})', 0, 0 | O_3) = ((a_{12} a_{21})^{n_3} \otimes a_{12}^t) \cdot v^{(1)} \in \text{Im } \rho_2.
\]

Continuing in the same way as above, we complete the proof of the embedding \( \ker \rho_1 \subset \text{Im } \rho_2 + \text{Im } \sigma_1 \). \( \square \)

**Proposition 2.3.** In the notation of (1.19), we have \( \ker \rho_2 \subset \text{Im } \rho_1 \).
Proof. We denote by \( u^{(j)} \) (\( 1 \leq j \leq 6 \)) the columns of the matrix of the homomorphism \( \rho_1 \) (see (1.7)). Let \( x = (x_1, x_2, x_3) \in \text{Ker} \rho_2 \) with \( x_i \in P_{ii} \) (\( i = 1, 2, 3 \)). Then we have

\[
\begin{align*}
&x_1 \cdot \sum_{i=0}^{n_3-1} \alpha^i_1 \otimes (a_{21} \alpha_1^{n_3-1-i})' + x_2 \cdot \sum_{i=0}^{n_3-1} \beta^i_2 a_{21} \otimes (\beta_2^{n_3-1-i})' = 0, \\
&x_1 \cdot \sum_{i=0}^{n_1-1} \alpha^i_2 \otimes (a_{32} \alpha_2^{n_1-1-i})' + x_3 \cdot \sum_{i=0}^{n_1-1} \beta^i_3 a_{32} \otimes (\beta_3^{n_1-1-i})' = 0, \\
&x_1 \cdot \sum_{i=0}^{n_2-1} \beta^i_1 a_{13} \otimes (\beta_1^{n_2-1-i})' + x_3 \cdot \sum_{i=0}^{n_2-1} \alpha^i_3 \otimes (\alpha_3 a_2^{n_2-1-i})' = 0, \\
&x_1 \cdot \sum_{i=0}^{n_3-1} \alpha^i_1 a_{12} \otimes (\alpha_1^{n_3-1-i})' + x_2 \cdot \sum_{i=0}^{n_3-1} \beta^i_2 a_{12} \otimes (\beta_2^{n_3-1-i})' = 0, \\
&x_2 \cdot \sum_{i=0}^{n_1-1} \alpha^i_2 a_{23} \otimes (\alpha_2^{n_1-1-i})' + x_3 \cdot \sum_{i=0}^{n_1-1} \beta^i_3 a_{23} \otimes (\beta_3^{n_1-1-i})' = 0, \\
&x_1 \cdot \sum_{i=0}^{n_2-1} \beta^i_1 a_{31} \otimes (\beta_1^{n_2-1-i})' + x_3 \cdot \sum_{i=0}^{n_2-1} \alpha^i_3 a_{31} \otimes (\alpha_3 a_2^{n_2-1-i})' = 0.
\end{align*}
\]

(2.21)

As in the proof of Proposition 2.2, we shall modify \( x \) with the help of elements of \( \text{Ker} \rho_2 \cap \text{Im} \rho_1 \) so as to get the zero element after several steps; this will imply the desired inclusion.

Clearly, \( x_i \in \text{Rad} P_{ii} \) (\( i = 1, 2, 3 \)). We consider the decompositions of the components \( x_i \) in the standard bases of the corresponding modules \( P_{ii} \).

Step 1. Suppose that such a decomposition of the component \( x_1 \) has a summand of the form \( \zeta \cdot p \otimes q' \), where \( \zeta \in K \setminus \{0\} \), \( p \) is a path of nonzero length containing only the arrows \( a_{13} \) and \( a_{31} \), and \( q' \) is a path containing only the arrows \( a_{12} \) and \( a_{21} \) (perhaps, \( q = e_1 \)). We have \( p = p_1 a_{31} \) for a subpath \( p_1 \) of \( p \). Hence, on the left-hand side of the first equation (2.21) there is a summand of the form

\[
\zeta (p_1 \otimes q')(a_{31} \otimes (a_{21} a_1^{n_3-1-i})').
\]

(2.22)

If this summand is nonzero, then we can use (1.7) to show that \( p_1 \) must be a (left) multiple of the element \( p_1^{n_2-1} a_{13} \), and then \( p \), and hence \( q' \), is represented as a product of the arrows \( a_{12} \) and \( a_{21} \): \( p = p_1^{n_2} = \alpha_1^{n_3} \).

If the summand (2.22) is zero, then we necessarily have \( q = a_{12} a_{21} \cdot q_1 \), where \( q_1 \) is a path. In this case, on the left-hand side of the third equation (2.21) there is a summand of the form

\[
\zeta (p_1 \otimes q_1')(a_{31} \otimes (a_1^{n_3-1} a_1')).
\]

(2.23)

and if it is nonzero, then \( q_1 = a_1^{n_3-1} \) by (1.2). Thus, in this case \( q \) and \( p \) are represented as products of the arrows \( a_{13} \) and \( a_{31} \): \( q = a_1^{n_3} = \beta_1^{n_2} \).

Finally, if the summand (2.23) is zero, then \( p_1 = p_2 a_{31} \), where \( p_2 \) is a subpath of \( p_1 \), whence

\[
\zeta \cdot p \otimes q' = \zeta (p_2 \otimes q')(a_{13} a_{31} \otimes (a_{12} a_{21})').
\]

We observe that

\[
\bar{x} := \begin{pmatrix}
\beta_1 & \alpha_1' \\
0 & 0
\end{pmatrix} = (a_{13} \otimes a_1') \cdot u^{(3)} \in \text{Im} \rho_1 \cap \text{Ker} \rho_2.
\]
Consequently, replacing \( x \) by \( x + \zeta_2 \otimes q'_1 \cdot \bar{x} \), we may assume that, in the initial decomposition of \( x_1 \) in the standard basis of the module \( P_{11} \), the coefficient of the basis element \( p \otimes q' \) is zero.

Using the forth and the sixth equation (2.21) in a similar way, we can restrict our attention to an element \( x = (x_1, x_2, x_3) \in \text{Ker} p_2 \) for which the decomposition of the component \( x_1 \) in the standard basis of the module \( P_{11} \) has nonzero coefficients only at the basis elements of the form \( p \otimes q' \), where \( p \) and \( q \) are products of the same arrows (either \( a_{12} \) and \( a_{21} \), or \( a_{13} \) and \( a_{31} \)).

By symmetry, we may assume that the components \( x_2 \) and \( x_3 \) of \( x \) have the same property.

Next, in the decompositions of \( x_1 \) and \( x_2 \) in the standard bases of the modules \( P_{11} \) and \( P_{22} \) respectively, we single out the “homogeneous” summands \( \bar{x}_1 \) and \( \bar{x}_2 \) of degree \( r \) that involve only products of the arrows \( a_{12} \) and \( a_{21} \).

Step 2. First, we consider the case where \( r \) is even: \( r = 2s, s \in \mathbb{N} \). Then

\[
\begin{align*}
\bar{x}_1 &= \sum_{j=0}^{s} \gamma_j \alpha_1^j \otimes (\alpha_1^{s-j})' + \sum_{j=0}^{s-1} \delta_j a_{21} \alpha_1^j \otimes (\alpha_1^{s-j} a_{12})', \\
\bar{x}_2 &= \sum_{j=0}^{s} \epsilon_j \beta_2^j \otimes (\beta_2^{s-j})' + \sum_{j=0}^{s-1} \eta_j a_{12} \beta_2^j \otimes (\beta_2^{s-j} a_{21})'
\end{align*}
\]

(2.24)

with \( \gamma_j, \delta_j, \epsilon_j, \eta_j \in K \).

Step 2a. Assume additionally that \( s < n_3 \). Substituting (2.24) in the first equation in (2.21), we obtain

\[
\begin{align*}
\sum_{k=s}^{n_3-1} \sum_{j=0}^{s} \left( \sum_{j=0}^{s} \gamma_j \right) \alpha_1^k \otimes (a_{21} \alpha_1^{n_3+s-1-k})' + \left( \sum_{j=1}^{s} \gamma_j \right) \alpha_1^{n_3} \otimes (a_{21} \alpha_1^{s-1})' \\
+ \sum_{k=s-1}^{n_3-1} \left( \sum_{j=0}^{s-1} \delta_j \right) a_{21} \alpha_1^k \otimes (\alpha_1^{n_3+s-1-k})' \\
+ \left( \sum_{j=0}^{s-1} \epsilon_j \right) \beta_2^{s-1} a_{21} \otimes (\beta_2^{n_3})' + \sum_{k=s}^{n_3-1} \left( \sum_{j=0}^{s} \epsilon_j \right) \beta_2^k a_{21} \otimes (\beta_2^{n_3+s-1-k})' \\
+ \sum_{k=s}^{n_3-1} \left( \sum_{j=0}^{s-1} \eta_j \right) \alpha_1^{k+1} \otimes (\beta_2^{n_3+s-2-k} a_{21})' = 0,
\end{align*}
\]

(2.25)

because the substitution of the remaining summands of \( x_1 \) and \( x_2 \) in the first equation in (2.21) leads to no “like” terms for the summands in (2.25). Consequently,

\[
\begin{align*}
\sum_{j=0}^{s} \gamma_j &= \sum_{j=1}^{s} \gamma_j = \sum_{j=0}^{s-1} \eta_j, \\
\sum_{j=0}^{s-1} \delta_j &= \sum_{j=0}^{s-1} \epsilon_j = \sum_{j=0}^{s} \epsilon_j;
\end{align*}
\]

(2.26)

in particular, \( \gamma_0 = 0 = \epsilon_0 \). Using the fourth equation in (2.21), we similarly obtain \( \gamma_s = 0 = \epsilon_0 \). We need the following auxiliary statement.
Lemma 2.4. For any \( t \geq 0 \), we have:

\[
\sum_{i=0}^{t} \left( \sum_{j=0}^{i} \gamma_{j} + \sum_{j=0}^{i-1} \eta_{j} \right) \alpha_{i} \otimes (a_{21} \alpha_{1}^{t-i})' \cdot u^{(1)} + \sum_{i=0}^{t} \sum_{j=0}^{i} \gamma_{j} \eta_{j} a_{12} \beta_{2}^{j} \otimes (\alpha_{1}^{r-i})' \cdot u^{(4)} \right) = \left( \sum_{i=0}^{t} \gamma_{i} \alpha_{i} \otimes (\alpha_{1}^{s-i})' + \sum_{j=0}^{t} (\gamma_{j} + \eta_{j}) \alpha_{1}^{t+1} \otimes \epsilon_{1}' \right) \sum_{i=0}^{t} \eta_{i} a_{12} \beta_{2}^{i} \otimes (a_{21} \alpha_{1}^{t-i})' \cdot u^{(1)} \right); \tag{2.27}
\]

b)

\[
\sum_{i=0}^{t} \left( \sum_{j=0}^{i} \delta_{j} + \sum_{j=0}^{i-1} \epsilon_{j} \right) \alpha_{i} \otimes (\beta_{2}^{r-i})' \cdot u^{(1)} + \sum_{i=0}^{t} \sum_{j=0}^{i} \delta_{j} \epsilon_{j} a_{12} \beta_{2}^{j} \otimes (\alpha_{1}^{s-i} a_{12})' \cdot u^{(4)} \left) = \left( \sum_{i=0}^{t} \delta_{i} \alpha_{i} \otimes (\alpha_{1}^{s-i})' + \left( \sum_{j=0}^{t} \delta_{j} + \sum_{j=0}^{t-1} \epsilon_{j} \right) \alpha_{1} \otimes \beta_{2}^{12} \right) \right) = \left( \sum_{i=0}^{t} \epsilon_{i} a_{12} \beta_{2}^{i} \otimes (\beta_{2}^{s-i})' + \left( \sum_{j=0}^{t} \delta_{j} + \sum_{j=0}^{t-1} \epsilon_{j} \right) a_{12} \beta_{2}^{i} \otimes \beta_{2}^{12} \right) = \left( \sum_{i=0}^{t} \eta_{i} a_{12} \beta_{2}^{i} \otimes (\alpha_{1}^{s-i})' \right) \sum_{i=0}^{t} \gamma_{i} \alpha_{i} \otimes (\beta_{2}^{r-i})' \cdot u^{(1)} \right); \tag{2.28}
\]

Both relations are proved by direct inspection.

We continue the proof of Proposition 2.3. Lemma 2.4 a) (with \( t = s - 1 \)) and relations (2.26) show that

\[
(2.28) \quad \tilde{x} := \left( \sum_{j=0}^{s-1} \gamma_{j} \alpha_{j} \otimes (\alpha_{1}^{s-j})' + \sum_{j=0}^{s-1} \eta_{j} a_{12} \beta_{2}^{j} \otimes (\beta_{2}^{s-1-j} a_{21})' \right) \in \text{Im} \rho_{1}.
\]

Substitution of \( \tilde{x} \) in place of \( x \) in (2.21) gives \( \tilde{x} \in \text{Ker} \rho_{2} \) (we have used the identities \( \gamma_{0} = \gamma_{s} = 0 \)). Hence, replacing \( x \) by \( x + \tilde{x} \), we may assume that for the initial \( x \) in (2.24) we have \( \gamma_{i} = 0 = \eta_{i} \) for all \( i \). If in the above argument we interchange the roles of the first and fourth equations (2.21), then in the same way we see that all \( \delta_{i} \) and \( \epsilon_{i} \) in (2.24) can be assumed to be zero.

Step 2b. If \( s > n_{3} \) in (2.24), then, setting \( t = s - 1 \) in (2.27), we again obtain the inclusion (2.28) for any \( \gamma_{i}, \eta_{i} \). It is clear that \( \tilde{x} \) defined by (2.28) lies also in \( \text{Ker} \rho_{2} \). Repeating the above argument, we may assume that \( x_{1} \) and \( x_{2} \) have no nonzero summands of even degree \( r = 2s \neq 2n_{3} \) that are products of the arrows \( a_{12} \) and \( a_{21} \).

Step 2c. Put \( s = n_{3} \) in (2.24). Then, on the left-hand side of the first equation (2.21) we obtain the summand

\[
(2.29) \quad \tilde{x}_{1} \cdot \sum_{i=0}^{n_{3}-1} \alpha_{i} \otimes (a_{21} \alpha_{1}^{n_{3}-1-i})' + \tilde{x}_{2} \cdot \sum_{i=0}^{n_{3}-1} \beta_{2}^{i} a_{21} \otimes (\beta_{2}^{n_{3}-1-i})' = \left( \sum_{j=1}^{n_{3}} \gamma_{j} \alpha_{1}^{n_{3}} \otimes (a_{21} \alpha_{1}^{n_{3}-1})' + \sum_{j=0}^{n_{3}-1} \delta_{j} a_{21} \alpha_{1}^{n_{3}-1} \otimes (a_{21} \alpha_{1}^{n_{3}-1})' \right) \right) + \left( \sum_{j=0}^{n_{3}-1} \eta_{j} a_{12} \beta_{2}^{j} \otimes (\alpha_{1}^{n_{3}}) \right) \sum_{i=0}^{t} \gamma_{i} \alpha_{i} \otimes (\beta_{2}^{r-i})' \cdot u^{(1)} \right); \tag{2.29}
\]
Using relations \( \text{(2.32)} \) that define the algebra \( R \), we may assume that \( x_1 \) (respectively, \( x_2 \)) has no summands of the form \( \lambda (\beta_1^{n_2} \otimes e_1') \) with \( \lambda \in \Lambda \) (respectively, of the form \( \mu (e_2 \otimes (\alpha_2^{n_2})') \) with \( \mu \in \Lambda \)). Consequently, the left-hand side of the first equation in \( \text{(2.21)} \) involves no terms similar to summands occurring on the right in \( \text{(2.29)} \), whence we see that

\[
\text{(2.30)} \quad \sum_{j=1}^{n_3} \gamma_j + \sum_{j=0}^{n_3-1} \eta_j = 0, \quad \sum_{j=0}^{n_3-1} \delta_j + \sum_{j=0}^{n_3-1} \varepsilon_j = 0.
\]

Using the fourth equation in \( \text{(2.21)} \), in a similar way we obtain

\[
\text{(2.31)} \quad \sum_{j=0}^{n_3-1} \gamma_j + \sum_{j=0}^{n_3-1} \eta_j = 0, \quad \sum_{j=0}^{n_3-1} \delta_j + \sum_{j=1}^{n_3-1} \varepsilon_j = 0.
\]

In particular, relations \( \text{(2.30)} \) and \( \text{(2.31)} \) imply that

\[
\gamma_0 = \gamma_{n_3}, \quad \varepsilon_0 = \varepsilon_{n_3}.
\]

Observe that

\[
\tilde{y} := \begin{pmatrix}
\alpha_1^{n_3} \otimes e_1' + e_1 \otimes (\alpha_2^{n_3})' \\
\alpha_1 \otimes (\beta_2^{n_3-1} \otimes a_{21})' \\
\alpha_1 \alpha_3 a_2 \otimes (\alpha_4^{n_3} \\ 1)
\end{pmatrix} = (e_1 \otimes (\beta_2^{n_3-1} \otimes a_{21})') \cdot u^{(1)} + (\alpha_1 \alpha_3 \otimes e_1') \cdot u^{(3)} \in \text{Im} \rho_1.
\]

Moreover, the replacement of \( x \) by \( \tilde{y} \) in \( \text{(2.21)} \) immediately shows that \( \tilde{y} \in \text{Ker} \rho_2 \). Hence, replacing \( x \) by \( x + \gamma_0 \cdot \tilde{y} \), we may assume that \( \gamma_0 = \gamma_{n_3} = 0 \) in the initial \( x \). A similar argument allows us to assume that \( \varepsilon_0 = \varepsilon_{n_3} = 0 \) in \( x \). Now we may apply the argument of Step 2a and assume that the components \( x_1 \) and \( x_2 \) of \( x \) have no nonzero summands that have even degree \( r = 2s \) and consist of products of the arrows \( a_{12} \) and \( a_{21} \).

By symmetry, we can restrict our consideration to \( x = (x_1, x_2, x_3) \in \text{Ker} \rho_2 \) such that \( x_2 \) and \( x_3 \) have no nonzero summands of even degree that consist of products of the arrows \( a_{23} \) and \( a_{32} \), assuming that \( x_3 \) and \( x_1 \) have no nonzero summands of even degree that consist of products of the arrows \( \beta_3 \) and \( a_{13} \).

Step 3. Now we consider the case where \( r = 2s + 1 \) is odd \( (s \geq 0) \) and put

\[
\text{(2.32)} \quad \begin{cases}
\tilde{x}_1 = \sum_{j=0}^{s} \gamma_j a_{21} \alpha_1^j \otimes (\alpha_1^{-s-j})' + \sum_{j=0}^{s} \delta_j a_1 \otimes (\alpha_1^{-s-j} a_{12})', \\
\tilde{x}_2 = \sum_{j=0}^{s} \varepsilon_j a_{12} \beta_2^j \otimes (\beta_2^{-s-j})' + \sum_{j=0}^{s} \eta_j \beta_2' \otimes (\beta_2^{-s-j} a_{21})',
\end{cases}
\]

where \( \gamma_j, \delta_j, \varepsilon_j, \eta_j \in K \).

Step 3a. Assume that \( s \leq n_3 - 1 \). Substituting \( \text{(2.32)} \) in the first three equations \( \text{(2.21)} \) and arguing as in Step 1a, we obtain

\[
\text{(2.33)} \quad \sum_{j=0}^{s} \gamma_j = \sum_{j=0}^{s} \eta_j, \quad \sum_{j=0}^{s} \delta_j = \sum_{j=1}^{s} \varepsilon_j, \quad \delta_0 = 0 = \varepsilon_s.
\]

(Note that if \( s < n_3 - 1 \), these relations can be obtained by using only the first equation \( \text{(2.21)} \).) Then Lemma \( \text{(2.33b)} \) and relations \( \text{(2.33)} \) imply that

\[
\text{(2.34)} \quad \begin{pmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
0
\end{pmatrix} \in \text{Im} \rho_1 \cap \text{Ker} \rho_2.
\]

Arguing as in Step 2a, we reduce the situation to the case where all \( \gamma_j, \delta_j, \varepsilon_j, \eta_j \) in \( \text{(2.32)} \) are zero.
Step 3b. Suppose \( s > n_3 \) in (2.32); using Lemma 2.4b) once again, we obtain the inclusion (2.33). Then, as in Step 2a (cf. also Step 2b), we reduce the situation to the case where all \( \gamma_j, \delta_j, \varepsilon_j, \eta_j \) in (2.32) are zero.

Step 3c. Let \( s = n_3 \). When analyzing (2.32) with \( s = n_3 \), we may additionally assume that \( \delta_0 = 0 = \varepsilon_3 \). Moreover, we suppose that the summands in \( x_1 \) of the form \( \lambda \cdot (\beta^{n_2} \otimes e^i) = \lambda \cdot (\alpha^{n_3} \otimes e^i) \) and \( \mu \cdot (\beta^{n_2})' = \mu \cdot (\alpha^{n_3})' \) with \( \lambda, \mu \in \Lambda \) occur in \( \bar{x} \). Similarly, the summands in \( x_2 \) of the form \( \lambda \cdot (\alpha^{n_3} \otimes e_2) = \lambda \cdot (\beta^{n_2} \otimes e_2) \) and \( \mu \cdot (\alpha^{n_3})' = \mu \cdot (\beta^{n_2})' \) (\( \lambda, \mu \in \Lambda \)) are assumed to occur in \( \bar{x}_2 \). Then the substitution of (2.32) in the first equation (2.21) yields \( \sum_{j=1}^{n_3} \delta_j = \sum_{j=0}^{n_3} - \varepsilon_j \). Using Lemma 2.4b), we see that

\[
\bar{y} := \left( \sum_{j=1}^{n_3} \delta_j \alpha^{n_3-j} \otimes (\alpha^{n_3-j} \otimes a_{12})' \right) \in \text{Im } \rho_1 \cap \text{Ker } \rho_2.
\]

Replacing \( x \) by \( x + \bar{y} \), we may assume that all \( \delta_j \) and \( \varepsilon_j \) in (2.32) are zero.

Consequently, we have reduced the matter to the case where in the initial \( x = (x_1, x_2, x_3) \in \text{Ker } \rho_2 \) the components admit no summands of odd degree that consist of products of the arrows \( a_{12} \) and \( a_{21} \).

By symmetry, it can be assumed that \( x_2 \) and \( x_3 \) have no nonzero summands of odd degree that consist of products of the arrows \( a_{23} \) and \( a_{32} \), and that \( x_3 \) and \( x_1 \) have no nonzero summands of odd degree that consist of products of the arrows \( a_{31} \) and \( a_{13} \).

Therefore, any element \( x \in \text{Ker } \rho_2 \) can be transformed into zero with the help of elements belonging to \( \text{Im } \rho_1 \cap \text{Ker } \rho_2 \); consequently, \( \text{Ker } \rho_2 \subset \text{Im } \rho_1 \). \( \square \)

**Proposition 2.5.** In the notation of (1.10), we have

\[
\text{Im } \rho_2 \cap \text{Im } \sigma_1 = \text{Im } (\rho_2 \rho_1).
\]

**Proof.** The inclusion \( \text{Im } \rho_2 \rho_1 \subset \text{Im } \rho_2 \cap \text{Im } \sigma_1 \) follows from the fact that \( \rho_2 \rho_1 = \sigma_1 \tau_1 \).

We prove the reverse inclusion. First, we observe that, with respect to the fixed direct decomposition of the module \( L_2 \) (see (1.3)), we have

\[
\rho_2 \rho_1 = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix},
\]

where

\[
X = \begin{pmatrix}
\alpha_1 \otimes (\beta_2) & + & \alpha_1 \otimes e_2' & \beta_2 \otimes (\beta_3) & \alpha_1 \otimes (\alpha_3) & \beta_2 \otimes (\alpha_3)\\
\alpha_2 \otimes (\alpha_3) & \alpha_2 \otimes (\beta_3) & \alpha_2 \otimes (\beta_3) & \alpha_2 \otimes (\beta_3) & \alpha_2 \otimes (\beta_3) & \alpha_2 \otimes (\beta_3)
\end{pmatrix},
\]

\[
Y = \begin{pmatrix}
\beta_3 & \beta_3 & \beta_3 & \beta_3 & \beta_3 & \beta_3
\end{pmatrix}.
\]

We denote by \( z(i), 1 \leq i \leq 6 \), the columns of the matrix representing the homomorphism \( \rho_2 \rho_1 \). Let

\[
x = (x_1, x_2, x_3) \in L_1, \quad y = (y_{12}, y_{23}, y_{31} | y_{21}, y_{32}, y_{13}) \in L_2,
\]

where \( x_i \in P_{ii}, y_{ij} \in P_{ij} \) (\( i, j \in \{1, 2, 3\}, i \neq j \)), be elements such that

\[
\rho_2(x) = \sigma_1(y).
\]

(2.35)
Comparing the first three components on the left and on the right in (2.35), which lie in $L'_2 = P_{12} \oplus P_{23} \oplus P_{31}$, we obtain

$$
\begin{aligned}
& x_1 \cdot \sum_{i=0}^{n_3-1} \alpha^i_3 \otimes (a_3a_1\alpha^{n_3-1-i}_3)' + x_2 \cdot \sum_{i=0}^{n_3-1} \beta^i_2a_{21} \otimes (\beta^2_2\alpha^{n_3-1-i}_3)' \\
& \quad = y_{32} \cdot (a_3 \otimes a'_2) + y_{13} \cdot (a_1 \otimes a'_2), \\
& x_2 \cdot \sum_{i=0}^{n_3-1} \alpha^i_3 \otimes (a_3a_2\alpha^{n_3-1-i}_2)' + x_3 \cdot \sum_{i=0}^{n_3-1} \beta^i_2a_{32} \otimes (\beta^2_2\alpha^{n_3-1-i}_2)' \\
& \quad = y_{21} \cdot (a_2 \otimes a'_1) + y_{13} \cdot (a_1 \otimes a'_2), \\
& x_1 \cdot \sum_{i=0}^{n_2-1} \beta^i_1a_{13} \otimes (\beta^1_1\alpha^{n_2-1-i}_1)' + x_3 \cdot \sum_{i=0}^{n_2-1} \alpha^i_3 \otimes (a_1\alpha^{n_2-1-i}_1)' \\
& \quad = y_{21} \cdot (a_3 \otimes a'_1) + y_{32} \cdot (a_3 \otimes a'_2).
\end{aligned}
$$

(2.36)

Step 1. We may assume that $y_{32}$ has no nonzero summands of the form $\lambda \cdot (e_3 \otimes a'_2)$ with $\lambda \in \Lambda$, because otherwise we can simultaneously replace $y_{32}$ by $y_{32} + \lambda(e_3 \otimes a'_2)$ and $y_{13}$ by $y_{13} + \lambda(a_3 \otimes e'_1)$ without altering the right-hand sides of all equations (2.36).

Similarly, we may assume that $y_{13}$ has no nonzero summands of the form $\mu \cdot (a_3 \otimes e'_2)$ with $\mu \in \Lambda$. By symmetry, we may also assume that $y_{21}$ has no nonzero multiples of the element $a_1 \otimes e'_1$ and $y_{13}$ has no nonzero multiples of the element $e_1 \otimes a'_3$, and that $y_{21}$ has no nonzero multiples of the element $e_2 \otimes a'_1$ and $y_{32}$ has no nonzero multiples of the element $a_2 \otimes e'_2$.

Step 2. Assume that $y_{32}(a_3 \otimes e'_2) \neq 0$ on the right-hand side of the first equation in (2.36).

Step 2a. If $y_{32}$ has no nonzero summands of the form $\lambda \cdot (a_1 \otimes e'_1)$, $\lambda \in \Lambda$, the left-hand side of the first equation in (2.36) necessarily contains a nonzero summand of the form $\mu \cdot (a_3 \otimes (a_2a_1\alpha^{n_3-1}_3)'')$. To both sides of (2.36), we add the element $\mu \cdot z^{(3)} = \rho_2\rho_1(h_3) = \sigma_1\tau_1(h_3)$ with $h_3 = (0, 0, e_3 \otimes e'_1) \in O_3 \subset L_2$. Then, replacing $y$ by $x + \mu\rho_1(h_3)$ and $y$ by $y + \mu\tau_1(h_3)$, we may assume that the initial $y_{32}$ has no nonzero summands of the form $\mu \cdot (e_3 \otimes (a_2a_1\alpha^{n_3-1}_3)'')$, but has a nonzero summand of the form $\lambda \cdot (\beta^2_2\alpha^{n_2-1}_1a_1 \otimes e'_2)$, $\lambda \in \Lambda$. Similarly, we may assume that $y_{13}$ has no nonzero summands of the form $\mu \cdot (\beta^1_1\alpha^{n_2-1}_1a_1 \otimes e'_2)$, $\mu \in \Lambda$.

Step 2b. We assume that, in the decomposition of the element $y_{32}$ in the standard basis of the module $P_{32}$, the coefficient $\zeta_0$ of the basis element $\beta^2_2\alpha^{n_2-1}_1a_1 \otimes e'_2$ is nonzero. Let $\xi_0$ denote the coefficient of $e_1 \otimes (a_2a_1\alpha^{n_2-1}_3)'$ in a similar decomposition of the element $y_{13}$ in the standard basis. We consider the homogeneous summands in $x_1$ and $x_2$ that have degree 1 and consist of the arrows $a_{12}$ and $a_{21}$ (cf. (2.32)):

$$
\begin{aligned}
& \bar{x}_1 = \gamma a_{21} \otimes e'_1 + \delta e_1 \otimes a'_2, \quad \bar{x}_2 = \varepsilon a_{12} \otimes e'_2 + \eta e_2 \otimes a'_2
\end{aligned}
$$

(2.37)

with $\gamma, \delta, \varepsilon, \eta \in K$. As in Step 3a of the proof of Proposition 2.3, from the first equation (2.36) we obtain $\gamma = \eta = 0, \delta = \varepsilon = \zeta_0 = \xi_0$. Adding the element $\zeta_0 \cdot z^{(1)}$ to both sides of (2.36), we obtain a new element $x$ that has no degree 1 summands of the form (2.37). Hence, replacing the corresponding $y$ by $y + \zeta_0\tau_1(h_1)$, where $h_1 = (e_1 \otimes e'_2, 0, 0) \in O_3 \subset L_2$, we may assume that in the initial $y_{32}$ and $y_{13}$ the coefficients of $\beta^2_2\alpha^{n_2-1}_1a_1 \otimes e'_2$ and $\xi_0$ are zero.

Step 2c. Similarly, if in the decomposition of $y_{32}$ in the standard basis the coefficient $\zeta_s$ of $\beta^1_1\alpha^{n_2-1}_1a_1 \otimes (\beta^2_2)'$, where $1 \leq s < n_2$, is nonzero, then to both sides of (2.36) we add
the element
\[ z_2^{(1)} = \zeta_s(e_1 \otimes (\alpha_1^s))^t \cdot \zeta_s(e_1 \otimes (\alpha_1^s))^t = \zeta_s(\beta_1^{n_2} \otimes (\alpha_1^s))^t, \]
and after that we replace \( y \) by \( y + \zeta_s(e_1 \otimes (\alpha_1^s))^t \cdot \tau_1 \). As a result, in the decomposition of the right-hand side of the first equation \((2.36)\) in the standard basis, the coefficient of \( \beta_1^{n_2-1} a_{13} \otimes (\alpha_2^s)^t \) becomes zero.

Moreover, such replacement does not change \( y_{13} \), because the last component of the element \( (e_1 \otimes (\alpha_1^s))^t \cdot \tau_1 \) is a multiple of the first column of the matrix representing the homomorphism \( \tau_1 \) (see \((1.3)\)), is equal to zero. Consequently, we may assume that in the initial \( y \) we have \( z_2 = 0 \) for the component \( y_{13} \).

Step 2d. In the same way, we reduce the situation to the case where in the decomposition of \( y_{32} \) the coefficient of any basis element of the form \( \beta_1^{n_2-1} a_{13} \otimes (\beta_3^s)^t \), \( 0 \leq s < n_2 \), is equal to zero, i.e., we may assume that \( y_{32}(a_{31} \otimes e_2^r) = 0 \) on the right-hand side of the first equation \((2.36)\). Finally, since
\[ y_{32} \in \text{Rad } P_{32} = \Lambda \langle e_3 \otimes a_{21}', e_3 \otimes a_{23}', a_{13} \otimes e_2', a_{23} \otimes e_2' \rangle, \]
the conditions imposed on \( y_{32} \) imply that \( y_{32} = 0 \). Similarly, we may assume that \( y_{13} = 0 \).

Step 3. Suppose that on the right-hand side of the third equation \((2.36)\), we have \( y_{21}(a_{23} \otimes e_1') \neq 0 \). As in Step 1, we need to consider two possibilities: \( y_{21}(a_{23} \otimes e_1') \) contains a nonzero summand either of the form \( \lambda \cdot (a_{23} \otimes (\alpha_3^{n_2-1})') \) with \( \lambda \in \Lambda \), or of the form \( \mu \cdot (\beta_3^{n_2} \otimes e_1') \) with \( \mu \in \Lambda \).

Step 3a. In the first case, \( y_{21} \) has a nonzero summand of the form \( \lambda \cdot (e_2 \otimes (\alpha_3^{n_2-1})'). \)

Hence, the right-hand side of the second equation \((2.36)\) contains the summand \( \lambda \cdot (e_2 \otimes (\alpha_3^{n_2-1})'). \) If we assume that \( \lambda \in \Lambda \setminus \{0\} \), then, arguing as in Step 2b, we deduce that \( y_{32}(a_{12} \otimes e_1') \) has a nonzero summand \( \lambda \cdot (\alpha_3^{n_2} \otimes e_1') \), but this is impossible (see Step 1). Consequently, \( \lambda \in \text{Rad } \Lambda \), and we may assume that \( \lambda \) is a multiple of \( a_{32} \otimes e_3' \). Adding the element
\[ \lambda \cdot z_2^{(2)} = \left( 0, \lambda(e_2 \otimes (\beta_3^{n_2})'), \lambda(a_{23} \otimes (\alpha_3^{n_2-1})') \right) \]
to both sides of \((2.36)\), we can transform \( x \) and \( y \) into new elements for which the relation similar to the third equation \((2.36)\) contains no nonzero summands of the form \( \lambda \cdot (a_{32} \otimes (\alpha_3^{n_2-1})'). \)

Step 3b. Now, suppose that \( y_{21} \) has a nonzero summand of the form \( \lambda \cdot (\beta_3^{n_2-1} a_{32} \otimes e_1') \), \( \lambda \in \Lambda \). If \( \lambda \in \Lambda \setminus \{0\} \), then we can argue as in Step 2b and use the second equation \((2.36)\) to show that \( y_{32} \cdot (e_3 \otimes a_{12}') \) contains a nonzero summand \( \lambda \cdot (e_3 \otimes (\beta_3^{n_2})') \), but this is impossible. Consequently, \( \lambda \in \text{Rad } \Lambda \), and we may assume that \( \lambda \) is a multiple of \( e_3 \otimes a_{12}' \). Adding the element \( \lambda \cdot z_2^{(3)} \) to both sides of \((2.36)\), we reduce the situation to the case where the right-hand side of the third equation \((2.36)\) contains no nonzero summands of the form \( \lambda \cdot (\beta_3^{n_2} \otimes e_1') \); hence, we necessarily have \( y_{21} = 0 \).

Comparing the last three components of the left-hand and the right-hand side of \((2.35)\) that lie in \( L''_2 = P_{21} \oplus P_{32} \oplus P_{13} \), we obtain relations similar to the equations occurring in \((2.36)\). Then the obvious symmetry (compare the submatrices \( X \) and \( Y \) of the matrix representing the homomorphism \( \rho_{2 \rho_1} \)) allows us to assume that the first three components of \( y \) in \((2.35)\) are also equal to zero.

Consequently, adding to \((2.35)\) some elements that lie in \( \text{Im } \rho_{2 \rho_1} \), we reduce the situation to the case where we have zero elements on both sides of the resulting relation. This completes the proof of the inclusion \( \text{Im } \rho_{2 \rho_1} \cap \text{Im } \sigma_1 \subset \text{Im } (\rho_{2 \rho_1}) \).
End of the proof of Theorem \ref{thm:main}. We consider the spectral sequence of the bicomplex \( B_{\bullet} \) (see \eqref{eq:bicomplex}):

\begin{equation}
E_{pq}^2 = H_p^h H_q^v(B_{\bullet}) \implies H_{p+q}(\text{Tot}(B_{\bullet})).
\end{equation}

We claim that the second page of this spectral sequence degenerates. By Proposition \ref{prop:degeneration}, in every row of the first page of the spectral sequence \eqref{eq: spectral sequence} (except the 0th row) there are exactly two nonzero terms, and the homomorphism induced by the homomorphism \( \rho_2 \) acts between them:

\[
\ker \rho_1 / \text{Im} \, \sigma_1 \xrightarrow{\varphi_2} \text{Coker} \, \rho_1.
\]

The homomorphism \( \varphi_2 \) is surjective by Proposition \ref{prop:surjectivity}. Next, if \( \pi \in \ker \varphi_2 \), where \( \pi \) is the image of an element \( x \in L_1 \) under the canonical homomorphism \( L_1 \to \text{Coker} \, \rho_1 \), then \( \rho_2(x) = \sigma_1(y) \) for some \( y \in L_2 \). By Proposition \ref{prop:degeneration}, this implies that \( \rho_2(x) = \rho_2 \rho_1(h) \) with \( h \in L_2 \). Proposition \ref{prop:degeneration} shows that \( x + \rho_1(h) \in \ker \rho_2 \subset \text{Im} \, \rho_1 \), whence, \( \pi = 0 \).

Thus, we have proved that \( Q_{\bullet} = \text{Tot} \, B_{\bullet} \) is a \( \Lambda \)-projective resolution of the module \( R \). Its minimality follows from the fact that, by the construction of the bicomplex \( B_{\bullet} \), we have \( \text{Im} \, \partial_n^Q \subset \text{Rad} \, Q_n \) for all \( n \geq 0 \).

\section{Additive structure of the cohomology ring}

In this section, we use the minimal projective resolution \( Q_{\bullet} = (Q_n, d_n^Q) \) of the \( \Lambda \)-module \( R \) (see Theorem \ref{thm:resolution}) to determine the cohomology groups \( \text{HH}^i(R) \), \( i \geq 0 \). For this, we investigate the complex

\[
\text{Hom}_\Lambda(Q_{\bullet}, R) = \left( \text{Hom}_\Lambda(Q_n, R), \Delta_n := (d_n^Q)^* \right)
\]

and calculate its cohomology:

\[
\text{HH}^n(\text{Hom}_\Lambda(Q_{\bullet}, R)) = \text{Ext}_\Lambda^n(R, R) = \text{HH}^n(R).
\]

\begin{proposition}
Let \( X_{\bullet} = B_{\bullet} \) be the complex occurring in Proposition \ref{prop:degeneration}. Then we have the following short exact sequence of complexes:

\begin{equation}
0 \to X_{\bullet} \xrightarrow{\iota} Q_{\bullet} \xrightarrow{\pi} Q_{\bullet}[-2] \to 0.
\end{equation}

\end{proposition}

\begin{proof}
By the construction of the bicomplex \( B_{\bullet} \) (see \eqref{eq:bicomplex}), for any \( n \geq 0 \) we have a split sequence of \( \Lambda \)-modules:

\begin{equation}
0 \to X_n \xrightarrow{\iota_n} Q_n \xrightarrow{\pi_n} Q_{n-2} \to 0,
\end{equation}

where \( \iota_n \) is the embedding of \( X_n \) as a direct summand, and \( \pi_n \) is the projection onto a direct summand (we put \( Q_n = 0 \) for \( n < 0 \)). It is easily seen that \( \iota = (\iota_n)_{n \geq 0} \) and \( \pi = (\pi_n)_{n \geq 0} \) are chain mappings.

We put \( N = n_1 + n_2 + n_3 \).

\begin{proposition}
\( \dim_K \text{HH}^0(R) = N + 1 \).
\end{proposition}

This result is contained in the tables of the book \cite{12}.

\begin{remark}
Consider the following elements of the algebra \( R \):

\[
c_1 = a_{12}a_{21} + a_{21}a_{12}, \quad c_2 = a_{23}a_{32} + a_{32}a_{23}, \quad c_3 = a_{31}a_{13} + a_{13}a_{31},
\]

\[
p_1 = (a_{12}a_{21})^{n_1}, \quad p_2 = (a_{23}a_{32})^{n_1}, \quad p_3 = (a_{31}a_{13})^{n_2}.
\]

It is easily seen that the set

\begin{equation}
\{ c_i^{n_i} \}_{i=1}^{n_1-1} \cup \{ c_i^{n_i} \}_{i=1}^{n_1-1} \cup \{ c_i^{n_i} \}_{i=1}^{n_2-1} \cup \{ 1, p_1, p_2, p_3 \}
\end{equation}

is a \( K \)-basis of the center \( Z(R) \) of the algebra \( R \). Since \( \text{HH}^0(R) \cong Z(R) \), after the corresponding identification the set \eqref{eq:basis} can be viewed as a basis of \( \text{HH}^0(R) \).

\end{remark}
Proposition 3.3. If \( \{i, j, k\} = \{1, 2, 3\} \), then

a) \( \dim_K \text{Hom}_\Lambda(P_{ij}, R) = n_k \);

b) \( \dim_K \text{Hom}_\Lambda(P_{ii}, R) = n_j + n_k \).

Proof. It suffices to consider the case where \( i = 1, j = 2 \); the remaining cases follow by symmetry.

a) Any \( \Lambda \)-homomorphism \( f : P_{12} \to R \) is determined by its value on the element \( e_1 \otimes e'_2 \); moreover, we have \( f(e_1 \otimes e'_2) = e_1 \cdot f(e_1 \otimes e'_2) \cdot e_2 \). Therefore, the element \( f(e_1 \otimes e'_2) \) must be a linear combination of (nonzero) paths that start at the vertex 2 and end at the vertex 1. Consequently, \( f \) is a linear combination of \( \Lambda \)-homomorphisms \( \varphi_i : P_{12} \to R \) such that

\[
\varphi_i(e_1 \otimes e'_2) = a^i_1, \quad 0 \leq i \leq n_3 - 1.
\]

It is readily verified that the set \( \{\varphi_i\}_{i=0}^{n_3-1} \) is linearly independent over \( K \), so that it is a basis of the vector space \( \text{Hom}_\Lambda(P_{12}, R) \).

b) In a similar way, we can prove that the \( \Lambda \)-homomorphisms \( \chi_i, \psi_j : P_{11} \to R \) such that

\[
\chi_i(e_1 \otimes e'_1) = a^i_1, \quad 0 \leq i \leq n_3; \quad \psi_j(e_1 \otimes e'_1) = b^j_1, \quad 1 \leq j \leq n_2 - 1,
\]

form a \( K \)-basis of the vector space \( \text{Hom}_\Lambda(P_{11}, R) \).

\[ \square \]

Corollary 3.4. For any \( n \geq 0 \), we have

\[
\dim_K \text{Hom}_\Lambda(Q_n, R) = \begin{cases} 
(2k+1)2N & \text{if } n = 3k \text{ or } n = 3k+1, \\
(2k+2)2N & \text{if } n = 3k+2,
\end{cases}
\]

where \( k \) is a nonnegative integer.

Proof. If \( n \leq 1 \), the above formulas are verified directly. Then we complete the proof by induction on \( n \) with the use the fact that, for \( n > 1 \), by (3.2) we have

\[
Q_n = Q_{n-2} \oplus X_n, \quad \text{where } X_n = \begin{cases} 
L^1_n & \text{if } n = 3k, \\
L_2 & \text{otherwise.}
\end{cases}
\]

To describe the homomorphisms in \( \text{Hom}_\Lambda(Q_n, R) \), we fix the decomposition

\[
Q_n = \sum_{i+j=n} B_{ij}
\]

in the order of increasing second index \( j \). Moreover, replacing \( B_{ij} \) in (3.6) by the decompositions of the modules \( L_1 \) and \( L_2 \) fixed earlier (see (1.3)), we obtain a presentation of \( Q_n \) as a direct sum of the modules \( P_{ij} (i, j \in \{1, 2, 3\}) \); by the way, the latter turn out to be gathered in triples (sometimes we consider also a suitable unions of several consecutive triples). Such a decomposition of the module \( Q_n \) is said to be \textit{canonical}.

Since \( P_{ij} = \Lambda \cdot (e_i \otimes e'_j) \), any \( \Lambda \)-homomorphism \( f : Q_n \to R \) is determined by the set of its values on the corresponding generators \( e_i \otimes e'_j \). In the sequel, we identify \( f \) with this set of values, and we group these values in accordance with the grouping of the modules \( P_{ij} \) in the canonical decomposition of \( Q_n \). Moreover, we put this set of values in square brackets and often write \( f = [f(e_i \otimes e'_j)]_{i,j} \); such a presentation of a homomorphism is said to be \textit{standard}. Sometimes we use similar notation for homomorphisms defined only on direct summands included in the canonical decomposition of the module \( Q_n \). For example, the homomorphisms in (3.3) can be written as \( \varphi_i = [a^i_1 a_{12}] \).

Proposition 3.5. The homomorphisms \( \sigma_1 : L_2 \to L_2 \) and \( \tau_1 : L_2 \to L_2 \) induce zero maps \( \sigma^*_1 : \text{Hom}_\Lambda(L_2, R) \to \text{Hom}_\Lambda(L_2, R), \quad \tau^*_1 : \text{Hom}_\Lambda(L_2, R) \to \text{Hom}_\Lambda(L_2, R) \).
Proposition 3.6. Let C be the matrix defined in (1.8). Then we have:

a) \( \dim_K \mathrm{Ker} \Delta^1 = 2N - \mathrm{rk} C \);

b) \( \dim_K \mathrm{Ker} \Delta^2 = N + 3 \);

c) \( \dim_K \mathrm{Ker} \Delta^3 = 4N + 2 - \mathrm{rk} C \).

Proof. a) We recall that \( Q_1 = L_2 = (P_{12} \oplus P_{23} \oplus P_{31}) \oplus (P_{21} \oplus P_{32} \oplus P_{13}) \). We consider a homomorphism \( f \in \mathrm{Hom}_A(Q_1, R) \) determined by the collection

\[
\begin{align*}
&\{\zeta_{12}a_{12}, \zeta_{23}a_{23}, \zeta_{31}a_{31} \mid \zeta_{21}a_{21}, \zeta_{32}a_{32}, \zeta_{13}a_{13}\}
\end{align*}
\]

with \( \zeta_{ij} \in K \). We claim that the condition \( \Delta^1(f) = 0 \) is equivalent to the following system of linear equations (over \( K \)):

\[
\begin{align*}
&n_3\zeta_{12} + n_2\zeta_{31} + n_3\zeta_{21} + n_2\zeta_{13} = 0, \\
n_3\zeta_{12} + n_1\zeta_{31} + n_3\zeta_{21} + n_1\zeta_{32} = 0, \\
n_1\zeta_{23} + n_2\zeta_{31} + n_1\zeta_{32} + n_2\zeta_{13} = 0.
\end{align*}
\]

(We observe that the matrix of this system has the form \((C \mid C)\), and hence, at least one among the equations \((3.8)\) can be omitted; we avoid doing this for the sake of symmetry.) Using Proposition 3.5, we obtain

\[
\Delta^1(f) = f \circ d_1 = (\rho^*_2(f) \mid \sigma^*_1(f)) = (\rho^*_2(f) \mid O_6).
\]

Next, we have

\[
\rho^*_2(f) = f \circ \rho_2 = (g_1, g_2, g_3).
\]
where

\[ g_1 = f_{12} \circ \left( \sum_{i=0}^{n_2-1} \alpha_i^1 \otimes (a_{21} \alpha_1^{n_3-1-i})' \right)^* + f_{31} \circ \left( \sum_{i=0}^{n_2-1} \beta_i^{n_2-1-i} a_{13} \otimes (\beta_1^i)' \right)^* + f_{21} \circ \left( \sum_{i=0}^{n_2-1} \alpha_i^1 \otimes (a_{21} \alpha_1^{n_3-1-i})' \right)^* + f_{13} \circ \left( \sum_{i=0}^{n_2-1} \beta_i^{n_2-1-i} a_{13} \otimes (\beta_1^i)' \right)^* \]

\[ g_2 = f_{12} \circ \left( \sum_{i=0}^{n_2-1} \beta_i^2 \otimes (a_{12} \alpha_2^{n_3-1-i})' \right)^* + f_{23} \circ \left( \sum_{i=0}^{n_2-1} \alpha_i^2 \otimes (a_{12} \alpha_2^{n_3-1-i})' \right)^* + f_{32} \circ \left( \sum_{i=0}^{n_2-1} \alpha_i^2 \otimes (a_{12} \alpha_2^{n_3-1-i})' \right)^* \]

\[ g_3 = f_{23} \circ \left( \sum_{i=0}^{n_2-1} \beta_i^3 \otimes (a_{13} \alpha_3^{n_3-1-i})' \right)^* + f_{31} \circ \left( \sum_{i=0}^{n_2-1} \alpha_i^3 \otimes (a_{13} \alpha_3^{n_3-1-i})' \right)^* + f_{32} \circ \left( \sum_{i=0}^{n_2-1} \alpha_i^3 \otimes (a_{13} \alpha_3^{n_3-1-i})' \right)^* \]

Moreover,

\[ f_{12} \circ \left( \sum_{i=0}^{n_2-1} \alpha_i^1 \otimes (a_{21} \alpha_1^{n_3-1-i})' \right)^* (e_1 \otimes e_1') = n_2 \zeta_{12} \cdot \alpha_1^{n_3}, \]

\[ f_{31} \circ \left( \sum_{i=0}^{n_2-1} \beta_i^1 \otimes (\beta_1^i)' \right)^* (e_1 \otimes e_1') = n_2 \zeta_{31} \cdot \beta_1^{n_2}, \]

\[ f_{21} \circ \left( \sum_{i=0}^{n_2-1} \alpha_i^1 \otimes (a_{12} \alpha_1^{n_3-1-i})' \right)^* (e_1 \otimes e_1') = n_3 \zeta_{21} \cdot \alpha_1^{n_3}, \]

\[ f_{13} \circ \left( \sum_{i=0}^{n_2-1} \beta_i^1 \otimes (a_{31} \beta_1^{n_2-1-i})' \right)^* (e_1 \otimes e_1') = n_2 \zeta_{13} \cdot \beta_1^{n_2}. \]

Consequently, the condition \( g_1 = 0 \) is equivalent to the first equation \( (3.8) \). Similarly, we see that \( g_2 = 0 \) (respectively, \( g_3 = 0 \)) if and only if the second (respectively, the third) equation \( (3.8) \) is fulfilled.

By a similar argument, we deduce that the homomorphisms belonging to \( \text{Hom}_\Lambda(Q_1, R) \) and determined by the collections of values

\[ [\alpha_i^1 a_{12}, 0, 0 \mid O_3], \quad [O_3 \mid a_{21} \alpha_1^i, 0, 0], \quad 1 \leq i \leq n_3 - 1; \]

\[ [0, a_j^i a_{23}, 0 \mid O_3], \quad [O_3 \mid 0, a_{32} \alpha_2^j, 0], \quad 1 \leq j \leq n_1 - 1; \]

\[ [0, 0, \alpha_k^a a_{31} \mid O_3], \quad [O_3 \mid 0, 0, a_{13} \alpha_3^k], \quad 1 \leq k \leq n_2 - 1, \]

lie in \( \text{Ker} \Delta^1 \). It is clear that the set \( (3.9) \) is linearly independent. Moreover, choosing a basis of the vector space of solutions of the system \( (3.8) \), we obtain \( 6 - \text{rk} C \) linearly independent homomorphisms that correspond to the vectors of this basis. It is easily seen that, adding these homomorphisms to the set \( (3.9) \), we obtain a basis of the vector space \( \text{Ker} \Delta^1 \). Consequently,

\[ \text{dim}_K \text{Ker} \Delta^1 = 2 \left( (n_1 - 1) + (n_2 - 1) + (n_3 - 1) \right) + (6 - \text{rk} C) = 2N - \text{rk} C. \]
b) Since $Q_2 = L_1 \oplus L_2$, an arbitrary homomorphism $f \in \text{Hom}_\Lambda(Q_2, R)$ can be written in the form $f = (f', f'')$, where

$$f' = (f_1, f_2, f_3) \in \text{Hom}_\Lambda(L_1, R), \quad f_i \in \text{Hom}_\Lambda(P_{ii}, R) \quad (i \in \{1, 2, 3\});$$

$$f'' = (f_{12}, f_{23}, f_{31}, f_{32}, f_{13}) \in \text{Hom}_\Lambda(L_2, R), \quad f_{ij} \in \text{Hom}_\Lambda(P_{ij}, R) \quad (i, j \in \{1, 2, 3\}, i \neq j).$$

Using Proposition 3.3 we obtain $\Delta^2(f) = (\rho_1^*(f'), \sigma_2^*(f''))$, whence it follows that

$$\text{Ker } \Delta^2 \simeq \text{Ker } \rho_1^* \oplus \text{Ker } \sigma_2^*.$$

By (1.7), the condition $\rho_1^*(f') = 0$ means that $(A')^*(f') = 0 = (A'')^*(f')$. The condition $(A')^*(f') = 0$ is equivalent to the following system of equations:

$$f_1 \circ (e_1 \otimes a_{12})^* + f_2 \circ (a_{12} \otimes e_2)^* = 0,$$

$$f_2 \circ (e_2 \otimes a_{23})^* + f_3 \circ (a_{23} \otimes e_3)^* = 0,$$

$$f_1 \circ (a_{31} \otimes e_1)^* + f_3 \circ (e_3 \otimes a_{31})^* = 0.$$

Using formulas (3.9) and Proposition 3.3(b), we represent $f_i \ (i = 1, 2, 3)$ in the form

$$f_1 = \gamma(1)[e_1] + \sum_{k=1}^{n_1-1} \delta_k^{(1)}[a_1^k] + \sum_{l=1}^{n_2-1} \varepsilon_l^{(1)}[\beta_1^l] + \eta(1)[p_1],$$

$$f_2 = \gamma(2)[e_2] + \sum_{k=1}^{n_1-1} \delta_k^{(2)}[a_2^k] + \sum_{l=1}^{n_2-1} \varepsilon_l^{(2)}[\beta_2^l] + \eta(2)[p_2],$$

$$f_3 = \gamma(3)[e_3] + \sum_{k=1}^{n_1-1} \delta_k^{(3)}[a_3^k] + \sum_{l=1}^{n_2-1} \varepsilon_l^{(3)}[\beta_3^l] + \eta(3)[p_3]$$

with $\gamma(i), \delta_k^{(i)}, \varepsilon_l^{(i)}, \eta(i) \in K$. Substituting (3.11) in (3.10), we obtain

$$(\gamma(1) + \gamma(2))a_{12} + \sum_{k=1}^{n_1-1} \delta_k^{(1)}a_1^ka_{12} + \sum_{l=1}^{n_2-1} \varepsilon_l^{(2)}a_{12}\beta_2^l = 0,$$

$$(\gamma(2) + \gamma(3))a_{23} + \sum_{k=1}^{n_1-1} \delta_k^{(2)}a_2^ka_{23} + \sum_{l=1}^{n_2-1} \varepsilon_l^{(3)}a_{23}\beta_3^l = 0,$$

$$(\gamma(3) + \gamma(1))a_{31} + \sum_{k=1}^{n_1-1} \delta_k^{(3)}a_3^ka_{31} + \sum_{l=1}^{n_2-1} \varepsilon_l^{(1)}a_{31}\beta_1^l = 0.$$

The last system breaks up into the following conditions:

$$\delta_k^{(1)} + \varepsilon_k^{(2)} = 0, \quad 1 \leq k \leq n_3 - 1; \quad \delta_l^{(2)} + \varepsilon_l^{(3)} = 0, \quad 1 \leq l \leq n_1 - 1;$$

$$\varepsilon_m^{(1)} + \delta_m^{(3)} = 0, \quad 1 \leq m \leq n_2 - 1.$$

Similar arguments show that the equation $(A'')^*(f') = 0$ leads to the same conditions (3.12) and (3.13).

Since $\text{char } K = 2$, the system (3.12) is equivalent to the conditions $\gamma(1) = \gamma(2) = \gamma(3)$. In particular, the $\Lambda$-homomorphism $f': L_1 \to R$ determined by the collection $[e_1, e_2, e_3]$ lies in $\text{Ker } \rho_1^1$. Moreover, relations (3.13) yield the following linearly independent set of homomorphisms in $\text{Hom}_\Lambda(L_1, R)$:

$$[\alpha_i^j, \beta_i^j, 0], \quad 1 \leq i \leq n_3 - 1; \quad [0, \alpha_j^j, \beta_3^j], \quad 1 \leq j \leq n_1 - 1;$$

$$[\beta_1^k, 0, \alpha_3^k], \quad 1 \leq k \leq n_2 - 1.$$
It is clear that, together with the homomorphisms 

\[ e_1, e_2, e_3, [p_1, 0, 0], [0, p_2, 0], [0, 0, p_3], \]

the set \((3.14)\) forms a basis of the vector space \(\text{Ker} \rho^*_1\), whence we obtain \(\dim_K \text{Ker} \rho^*_1 = N + 1\).

Next, since \(\sigma_2 = C' \oplus C''\) (see \((3.8)\)), it suffices to consider

\[ \text{Ker} \left( (C')^* : \text{Hom}_\Lambda(L_2', R) \to \text{Hom}_\Lambda(L_1, R) \right), \]

because the corresponding result for \(C''\) will follow by symmetry.

Let \(\varphi = (f_{12}, f_{23}, f_{31}) \in \text{Ker}(C')^*\), where the homomorphisms \(f_{ij}\) are represented in the form

\[ f_{12} = \sum_{k=0}^{n_1-1} \gamma_k^{(1)} a_1^{k+1}, \quad f_{23} = \sum_{l=0}^{n_2-1} \gamma_l^{(2)} a_2^{l+1}, \quad f_{31} = \sum_{m=0}^{n_3-1} \gamma_m^{(3)} a_3^{m+1} \]

(see the proof of Proposition \((3.3a)\)). The condition \((C')^*(\varphi) = 0\) means that

\[
\begin{align*}
\gamma_k^{(1)} a_1^{k+1} + \gamma_l^{(2)} a_2^{l+1} + \gamma_m^{(3)} a_3^{m+1} &= 0, \\
\sum_{k=0}^{n_1-1} \gamma_k^{(1)} a_1^{k+1} + \sum_{l=0}^{n_2-1} \gamma_l^{(2)} a_2^{l+1} + \sum_{m=0}^{n_3-1} \gamma_m^{(3)} a_3^{m+1} &= 0,
\end{align*}
\]

Substituting formulas \((3.16)\) in \((3.17)\), we obtain

\[ \sum_{k=0}^{n_1-1} \gamma_k^{(1)} a_1^{k+1} = 0, \quad \sum_{l=0}^{n_2-1} \gamma_l^{(2)} a_2^{l+1} = 0, \quad \sum_{m=0}^{n_3-1} \gamma_m^{(3)} a_3^{m+1} = 0.\]

Consequently,

\[ \gamma_k^{(1)} = 0 \quad \text{if} \quad k < n_1 - 1, \]

\[ \gamma_l^{(2)} = 0 \quad \text{if} \quad l < n_2 - 1, \]

\[ \gamma_m^{(3)} = 0 \quad \text{if} \quad m < n_3 - 1, \]

and also \(\gamma_{n_1-1}^{(1)} = \gamma_{n_2-1}^{(2)} = \gamma_{n_3-1}^{(3)}\). Therefore, \(\text{Ker}(C')^*\) is generated by the homomorphism \([a_1^{n_1-1} a_1, a_2^{n_2-1} a_2, a_3^{n_3-1} a_3]\). Similarly, we obtain

\[ \text{Ker}(C'')^* = \mathbb{K}([\beta_2^{n_2-1} a_2, \beta_3^{n_3-1} a_3, \beta_1^{n_1-1} a_1]). \]

Finally, we have

\[ \dim_K \text{Ker} \Delta^2 = \dim_K \text{Ker} \rho^*_1 + \dim_K \text{Ker} \sigma^*_2 = N + 3. \]

c) Since \(Q_3 = L_2 \oplus L_2^*\), any homomorphism \(f \in \text{Hom}_\Lambda(Q_3, R)\) is represented in the form \(f = (f', f'')\), where \(f' \in \text{Hom}_\Lambda(L_2, R)\), \(f'' \in \text{Hom}_\Lambda(L_2^*, R)\). Moreover, by Proposition \((3.6)\) we obtain

\[ \Delta^3(f) = \left( \rho^*_2(f'), \tau^*_2(f''), \sigma^*_3(f'') \right). \]

We prove the following auxiliary statement.

**Lemma 3.7.** \(\text{Ker} \sigma^*_3 \subset \text{Ker} \tau^*_2\).
Proof. Since
\[
\sigma_3 = \begin{pmatrix} 0 & A'' \\ A' & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} F' & 0 \\ 0 & F'' \end{pmatrix},
\]
we must prove that \(\ker(A')^\ast \subset \ker(F'')^\ast\), \(\ker(A'')^\ast \subset \ker(F')^\ast\). If we represent a homomorphism \(\varphi = (f_1, f_2, f_3) \in \text{Hom}_\Lambda(L_1, R)\) in the form (3.11), then, as in part b) of the proof of Proposition 3.6, we deduce that each of the conditions \((F')^\ast(\varphi) = 0\) and \((F'')^\ast(\varphi) = 0\) is equivalent to system (3.12). Consequently,
\[
\ker(A')^\ast = \ker(A'')^\ast \subset \ker(F')^\ast = \ker(F'')^\ast.
\]
We continue the proof of Proposition 3.6. Using Lemma 3.17 and (3.18), we see that
\[
\ker \Delta^3 \simeq \ker \rho_2^3 \oplus \ker \sigma_3^3.
\]
Moreover, in the proof of part a) we obtained the formula \(\dim_K \ker \rho_2^3 = 2N - \text{rk} C\), and from the proof of part b) it follows that \(\dim_K \ker \sigma_3^3 = 2 \dim_K (A')^\ast = 2(N + 1)\). Finally, we arrive at
\[
\dim_K \ker \Delta^3 = (2N - \text{rk} C) + 2(N + 1) = 4N + 2 - \text{rk} C.
\]
Computations in the proof of Propositions 3.6 and 3.3 yield the following statement.

Corollary 3.8.

a) \(\dim_K \ker \sigma_2^3 = 2\), \(\dim_K \im \sigma_2^3 = 2N - 2\);
b) \(\dim_K \ker \sigma_3^3 = 2N + 2\), \(\dim_K \im \sigma_3^3 = 2N - 2\), \(\dim_K \text{coker} \sigma_3^3 = 2\).

Corollary 3.9.

a) \(\dim_K \HH^1(R) = N + 1 - \text{rk} C\);
b) \(\dim_K \HH^2(R) = N + 3 - \text{rk} C\);
c) \(\dim_K \HH^3(R) = N + 5 - \text{rk} C\).

Proof. By Proposition 3.2 and Corollary 3.3, we have \(\dim_K \im \Delta^0 = 2N - (N + 1) = N - 1\). Similarly, from Proposition 3.6 and Corollary 3.3 it follows that \(\dim_K \im \Delta^1 = \text{rk} C\), \(\dim_K \im \Delta^2 = 3N - 3\). From this and Proposition 3.6 we obtain the desired statements.

Proposition 3.10. a) If \(\text{rk} C = 2\), then a \(K\)-basis for the vector space \(\HH^1(R)\) is formed by the set of cohomology classes of the following homomorphisms:
\[
\begin{align*}
  x_1 & = [n_1n_2a_{12}, n_2n_3a_{23}, n_1n_3a_{31} | O_3]; \\
  x_2 & = [a_{12}, 0, 0 | a_{21}, 0, 0]; \\
  x_{12}^{(i)} & = [a_{12}, 0, 0 | O_3], \quad 1 \leq i \leq n_3 - 1; \\
  x_{23}^{(j)} & = [0, a_{23}, 0 | O_3], \quad 1 \leq j \leq n_1 - 1; \\
  x_{31}^{(k)} & = [0, 0, a_{31} | O_3], \quad 1 \leq k \leq n_2 - 1.
\end{align*}
\]

b) Let \(\text{rk} C = 1\), and assume additionally that \(n_3\) is odd (then \(n_1 = n_2 = 0\) in \(K\)). In order to get a \(K\)-basis of \(\HH^1(R)\), it suffices to replace the class of \(x_1\) in (3.19) by the cohomology classes of the following homomorphisms:
\[
\begin{align*}
  x_1' & = [0, a_{23}, 0 | O_3], \\
  x''_1 & = [0, 0, a_{31} | O_3].
\end{align*}
\]

c) If \(\text{rk} C = 0\), then in order to get a \(K\)-basis of \(\HH^1(R)\), we replace the classes of \(x_1, x_2\) by the cohomology classes of the homomorphisms \(x_1', x_1''\) (see (3.20)), and also of the following homomorphisms:
\[
x_1' = [a_{12}, 0, 0 | O_3], \quad x_2'' = [O_3 | a_{21}, 0, 0].
\]
Remark 3.2. If, for example, \( n_3 = 1 \), we agree that the collection \( \{ x_{12}^{(i)} \}_i \) is empty. A similar remark concerns the cases where \( n_1 = 1 \) and/or \( n_2 = 1 \). In the sequel, such a convention is adopted in the other similar cases.

Proof of Proposition 3.10. We observe that, for \( f = (f_1, f_2, f_3) \in \text{Hom}_A(L_1, R) \), we have

\[
\Delta^0(f) = \left( f_1 \circ (e_1 \otimes a_1^{12})^* + f_2 \circ (a_1^{12} \otimes e_2')^*, \right.
\left. f_2 \circ (e_2 \otimes a_2^{23})^* + f_3 \circ (a_2^{23} \otimes e_3')^*, \right.
\left. f_1 \circ (a_3 \otimes e_1')^* + f_3 \circ (e_3 \otimes a_3')^*, \right.
\left. f_1 \circ (a_21 \otimes e_1')^* + f_2 \circ (e_2 \otimes a_2^{12})^*, \right.
\left. f_2 \circ (a_3 \otimes e_2')^* + f_3 \circ (e_3 \otimes a_3')^*, \right.
\left. f_1 \circ (e_1 \otimes a_1^{13})^* + f_3 \circ (a_13 \otimes e_3')^* \right).
\]

(3.21)

For the role of \( f \) we take the following basis elements of the vector space \( \text{Hom}_A(L_1, R) \):

\[
[a_1^i, 0, 0], \quad 0 \leq i \leq n_3 - 1;
0, a_2^j, 0], \quad 0 < j \leq n_1 - 1;
0, 0, a_3^k], \quad 0 \leq k \leq n_2 - 1.
\]

This yields the following set of homomorphisms in \( \text{Im} \Delta^0 \):

\[
[a_{12}, 0, a_{31} | a_{21}, 0, a_{13}],
[a_{12}, a_{23}, 0 | a_{21}, a_{32}, 0], \quad [0, a_{23}, a_{31} | 0, a_{32}, a_{13}];
(3.21)
\[
[a_1^i a_{12}, 0, 0 | a_{21} a_1^i, 0, 0], \quad 1 \leq i \leq n_3 - 1;
0, a_2^j a_{23}, 0 | 0, a_{32} a_2^j, 0], \quad 1 \leq j \leq n_1 - 1;
0, 0, a_3^k a_{31} | 0, 0, a_{13} a_3^k], \quad 1 \leq k \leq n_2 - 1.
\]

If \( f \) runs through the remaining basis elements of \( \text{Hom}_A(L_1, R) \):

\[
[a_1^i, 0, 0], \quad 1 \leq i \leq n_2 - 1;
0, a_2^j, 0], \quad 1 \leq j \leq n_3 - 1;
0, 0, a_3^k], \quad 1 \leq k \leq n_1 - 1;
\]

then, again, the values of \( \Delta^0(f) \) belong to the set (3.21). Hence, this set generates \( \text{Im} \Delta^0 \).

(We observe that the first three elements in (3.21) are linearly dependent, and if we omit one of them, the remaining set is linearly independent.) Therefore, the 1-cocycles

\[
[a_1^i a_{12}, 0, 0 | O_3] \quad \text{and} \quad [O_3 | a_{21} a_1^i, 0, 0] \quad (1 \leq i \leq n_3 - 1)
\]

in (3.9) (respectively,

\[
[0, a_2^j a_{23}, 0 | O_3] \quad \text{and} \quad [O_3 | 0, a_{32} a_2^j, 0] \quad (1 \leq j \leq n_1 - 1),
\]

or

\[
[0, 0, a_3^k a_{31} | O_3] \quad \text{and} \quad [O_3 | 0, 0, a_{13} a_3^k] \quad (1 \leq k \leq n_2 - 1)
\]

are cohomological to each other.

Again, we consider the system (3.8) (recall that the indeterminates have been ordered as follows: \( (\zeta_{12}, \zeta_{23}, \zeta_{31}, \zeta_{21}, \zeta_{32}, \zeta_{13}) \)). Depending on the value of \( \text{rk} C \), we can choose the following bases of the vector space of solutions of this system (if \( \text{rk} C = 1 \), we use the assumption that \( n_3 \neq 0 \) in \( K \)):

\[
\text{rk} C = 2: \quad \{ n_1 n_2, n_2 n_3, n_3 n_4, 0, 0, 0 \}, \quad \{ 1, 0, 0, 1, 0, 0 \}, \quad \{ 0, 1, 0, 0, 1, 0 \}, \quad \{ 0, 0, 1, 0, 0, 1 \};
\]

\[
\text{rk} C = 1: \quad \{ 0, 1, 0, 0, 0, 0 \}, \quad \{ 0, 0, 1, 0, 0, 0 \}, \quad \{ 1, 0, 0, 1, 0, 0 \}, \quad \{ 0, 0, 0, 0, 0, 1 \};
\]

\[
\text{rk} C = 0: \quad \{ 1, 0, 0, 0, 0, 0 \}, \quad \{ 0, 1, 0, 0, 0, 0 \}, \quad \{ 0, 0, 1, 0, 0, 0 \}, \quad \{ 0, 0, 0, 0, 0, 1 \}.
\]
The corresponding homomorphisms in $\mathrm{Ker} \Delta^1$ look like this:

- **rk** $C = 2$: $x_1 = [n_1 n_2 a_{12}, n_2 n_3 a_{23}, n_1 n_3 a_{31} | O_3]$, $x_2 = [a_{12}, 0, 0 | a_{21}, 0, 0]$, $x_3 = [0, a_{23}, 0 | 0, a_{32}, 0]$, $x_4 = [0, 0, a_{31} | 0, 0, a_{13}]$.

- **rk** $C = 1$: $x'_1 = [0, a_{23}, 0 | O_3]$, $x'_2 = [0, 0, a_{31} | O_3]$, $x_2 = [a_{12}, 0, 0 | a_{21}, 0, 0]$, $x_5 = [O_3 | 0, a_{32}, 0 | O_3]$, $x_6 = [O_3 | 0, 0, a_{13}]$.

- **rk** $C = 0$: $x'_1 = [0, a_{23}, 0 | O_3]$, $x'_2 = [0, 0, a_{31} | O_3]$, $x'_2 = [O_3 | a_{21}, 0, 0]$, $x_5 = [O_3 | 0, a_{32}, 0 | O_3]$, $x_6 = [O_3 | 0, 0, a_{13}]$.

It is clear that $x_3$ and $x_4$ are cohomological to $x_2$. Hence, if **rk** $C = 2$, then $\mathrm{HH}^1(R)$ is generated (over $K$) by the classes of homomorphisms

$$x_1, x_2, x_2^{(i)} (1 \leq i \leq n_3 - 1), x_2^{(j)} (1 \leq j \leq n_1 - 1), x_2^{(k)} (1 \leq k \leq n_2 - 1).$$

Since the cardinality of the set of these homomorphisms is equal to $\dim_K \mathrm{HH}^1(R) = N + 1 - \mathrm{rk} C = N - 1$, their classes form a basis of $\mathrm{HH}^1(R)$.

If **rk** $C = 1$, then $x_5$ is cohomological to $x'_1 + x_2$ and $x_6$ is cohomological to $x''_1 + x_2$. Consequently, the classes of the homomorphisms

$$x'_1, x''_1, x_2, x_2^{(i)} (1 \leq i \leq n_3 - 1), x_2^{(j)} (1 \leq j \leq n_1 - 1), x_2^{(k)} (1 \leq k \leq n_2 - 1)$$

generate $\mathrm{HH}^1(R)$, and again the cardinality of this set is equal to $\dim_K \mathrm{HH}^1(R)$. The case where **rk** $C = 0$ is treated similarly.

**Remark 3.3.** In the sequel, if $x \in \mathrm{Ker} \Delta^3$ is an $n$-cocycle ($n \geq 1$), its cohomology class is often denoted also by $x$.

**Proposition 3.11.** a) If **rk** $C = 2$, then a $K$-basis of the vector space $\mathrm{HH}^2(R)$ is formed by the set of cohomology classes of the following homomorphisms:

- $y_1 = [e_1, e_2, e_3 | O_6]$;
- $y_2 = [O_3 | a_1^{n_1-1} a_{21}, a_2^{n_2-1} a_{23}, a_3^{n_3-1} a_{31} | O_3]$;
- $y_3 = [O_6 | a_{21} a_1^{n_1-1}, a_{32} a_2^{n_2-1}, a_{13} a_3^{n_3-1}]$;
- $y_4 = [p_1, 0, 0 | O_6]$;
- $y_1^{(i)} = [\alpha_i, 0, 0 | O_6], 1 \leq i \leq n_3 - 1$;
- $y_2^{(j)} = [0, \alpha_j, 0 | O_6], 1 \leq j \leq n_1 - 1$;
- $y_3^{(k)} = [\beta_k, 0, 0 | O_6], 1 \leq k \leq n_2 - 1$.

\begin{equation}
\text{(3.22)}
\end{equation}

b) Let **rk** $C = 1$, and let $n_3$ be odd (then $n_1 = n_2 = 0$ in $K$). To get a $K$-basis of $\mathrm{HH}^2(R)$, we need to supplement the set (3.22) with the class of the homomorphism

\begin{equation}
\text{(3.23)}
y_5 = [0, 0, p_3 | O_6].
\end{equation}

c) If **rk** $C = 0$, then, to get a $K$-basis of $\mathrm{HH}^2(R)$, we add to the set (3.22) the cohomology classes of $y_5$ (see (3.23)) and of

$$y_6 = [0, p_2, 0 | O_6].$$

**Proof.** We note that, in fact, a basis of $K^2 = \mathrm{Ker} \rho^*_1 \oplus \mathrm{Ker} \sigma^*_2$ was constructed in the proof of Proposition 3.6(b): to obtain a standard presentation of its elements, it suffices to complete the basis vectors of $\mathrm{Ker} \rho^*_1$ by six zero components and preface the basis vectors of $\mathrm{Ker} \sigma^*_2$ with three zero components.
Next, as in the proof of Proposition 3.10, we show that $\text{Im} \Delta^1$ is generated by the homomorphisms

$$
[n_3 p_1, n_3 p_2, 0 \mid O_6], \quad [0, n_1 p_2, n_1 p_3 \mid O_6], \quad [n_2 p_1, 0, n_2 p_3 \mid O_6].
$$

We assume that $\text{rk} C = 2$. Then at least two numbers among $n_1, n_2, n_3$ are odd, and it is easily seen that each of the 2-cocycles $[0, p_2, 0 \mid O_6]$ and $[0, 0, p_3 \mid O_6]$ (see (3.15)) is cohomological to $y_1 = [p_1, 0, 0 \mid O_6]$. Consequently, the classes of the 1-cocycles:

$$
y_m (1 \leq m \leq 4), \quad y_{12}^{(i)} (1 \leq i \leq n_3 - 1), \quad y_{23}^{(j)} (1 \leq j \leq n_1 - 1), \quad y_{31}^{(k)} (1 \leq k \leq n_2 - 1)
$$

generate $\text{HH}^2(R)$. Since the number of them is equal to $\dim_K \text{HH}^2(R) = N + 3 - \text{rk} C = N + 1$, these classes form a basis of $\text{HH}^2(R)$.

If $\text{rk} C = 1$ and $n_3$ is odd, then $\text{Im} \Delta^1$ is generated by one 2-coboundary $[p_1, p_2, 0 \mid O_6]$, and we can eliminate the 2-cocycle $[0, p_2, 0 \mid O_6]$ from the basis of $\text{Ker} \Delta^2$, in order to obtain a basis of $\text{HH}^2(R)$ after passage to the corresponding cohomology classes.

Finally, in the case where $\text{rk} C = 0$, we have $\text{Im} \Delta^1 = 0$, whence $\text{HH}^2(R) \simeq \text{Ker} \Delta^2$. □

**Proposition 3.12.** a) If $\text{rk} C = 2$, then a $K$-basis of the vector space $\text{HH}^3(R)$ is formed by the set of cohomology classes of the following homomorphisms:

$$
z_1 = [O_6 \mid e_1, e_2, e_3 \mid O_3], \quad z_2 = [O_3 \mid e_1, e_2, e_3], \quad z_3 = [n_1 n_2 a_{12}, n_2 n_3 a_{23}, n_1 n_3 a_{31} \mid O_9], \quad z_4 = [a_{12}, 0, 0 \mid O_6];
$$

$$
z_5 = [O_6 \mid p_1, 0, 0 \mid O_3]; \quad z_6 = [O_9 \mid p_1, 0, 0];
$$

(3.24)

\[ z_{12}^{(i)} = [a_1 a_{12}, 0, 0 \mid O_9], \quad 1 \leq i \leq n_3 - 1; \]

\[ z_{23}^{(j)} = [0, a_2 a_{23}, 0 \mid O_9], \quad 1 \leq j \leq n_1 - 1; \]

\[ z_{31}^{(k)} = [0, 0, a_3 a_{31} \mid O_6], \quad 1 \leq k \leq n_2 - 1. \]

b) Suppose $\text{rk} C = 1$ and $n_3$ is odd (then $n_1 = n_2 = 0$ in $K$). In order to get a $K$-basis of $\text{HH}^3(R)$ we must replace the class of $z_3$ in (3.24) by the cohomology classes of the following homomorphisms:

(3.25)

\[ z_3' = [0, a_{23}, 0 \mid O_9], \quad z_3'' = [0, 0, a_{31} \mid O_9]. \]

c) If $\text{rk} C = 0$, in order to get a $K$-basis of the vector space $\text{HH}^3(R)$, we must replace the classes of $z_3, z_4$ in (3.24) by the cohomology classes of the homomorphisms $z_3', z_3''$ (see 3.25) and also of the following homomorphisms:

$$
z_4' = [a_{12}, 0, 0 \mid O_9], \quad z_4'' = [O_9 \mid a_{21}, 0, 0 \mid O_6].
$$

**Proof.** Again, much as in the proof of Proposition 3.10, we easily verify that the following set of homomorphisms is a basis in $\text{Im} \Delta^2$:

$$
[a_{12}, 0, a_{31} \mid a_{21}, 0, a_{13} \mid O_6];
$$

$$
[0, a_{23}, a_{31} \mid 0, a_{32}, a_{13} \mid O_6];
$$

$$
[a_1 a_{12}, 0, 0 \mid a_{21} a_1, 0, 0 \mid O_6], \quad 1 \leq i \leq n_3 - 1;
$$

$$
[0, a_2 a_{23}, 0 \mid 0, a_{32} a_2, 0 \mid O_6], \quad 1 \leq j \leq n_1 - 1;
$$

$$
[0, 0, a_3 a_{31} \mid 0, 0, a_{13} a_3 \mid O_6], \quad 1 \leq k \leq n_2 - 1;
$$

$$
[O_6 \mid a_1^2, \beta_2, 0 \mid O_9], \quad [O_9 \mid a_1^2, \beta_2, 0], \quad 1 \leq i \leq n_3;
$$

$$
[O_6 \mid 0, a_2^2, \beta_2 \mid O_3], \quad [O_9 \mid 0, a_2^2, \beta_2], \quad 1 \leq j \leq n_1;
$$

$$
[O_6 \mid \beta_1, 0, a_3^2 \mid O_3], \quad [O_9 \mid \beta_1, 0, a_3^2], \quad 1 \leq k \leq n_2 - 1.
(It should be noted that the number of such elements is equal to $2 + (n - 3) + 2(N - 1) = 3N - 3$, and this agrees with the calculation of $\dim K \operatorname{Im} \Delta^2$ done earlier.)

Next, in fact, a basis of $\ker \Delta^3$ was constructed in the proof of Proposition 3.6c. The remaining details are similar to the proof of Proposition 3.10. We leave this to the reader. □

Proposition 3.13. a) If $\operatorname{rk} C = 2$, then a $K$-basis of the vector space $\text{HH}^4(R)$ is formed by the set of cohomology classes of the following homomorphisms:

$$
t_1 = [e_1, e_2, e_3] \mid O_{12};
$$

$$
t_2 = [p_1, 0, 0] \mid O_{12};
$$

$$
t_{i i} = [\alpha_i^1, \beta_{i 2}, 0] \mid O_{12}, \quad 1 \leq i \leq n_3 - 1;
$$

$$
t_{i j} = [0, \alpha_j^1, \beta_{i 2}] \mid O_{12}, \quad 1 \leq j \leq n_1 - 1;
$$

$$
t_{i j k} = [\beta_i^k, 0, \alpha_j^k] \mid O_{12}, \quad 1 \leq k \leq n_2 - 1;
$$

$$
t_3 = [O_3 \mid \alpha_3^{n_3 - 1} a_{12}, \alpha_2^{n_3 - 1} a_{23}, \alpha_3^{n_3 - 1} a_{31}] \mid O_9];
$$

$$
t_4 = [O_3 \mid \alpha_3^{n_3 - 1} a_{12}, 0, 0 \mid O_6 \mid a_{21}, 0, 0];
$$

$$
t_5 = [O_6 \mid a_{21} a_3^{n_3 - 1}, a_{32} a_2^{n_2 - 1}, a_{13} a_3^{n_2 - 1}] \mid O_6];
$$

$$
t_6 = [O_6 \mid a_{21} a_3^{n_3 - 1}, 0, 0 \mid a_{12}, 0, 0 \mid O_3].
$$

b) Suppose $\operatorname{rk} C = 1$ and $n_3$ is odd (then $n_1 = n_2 = 0$ in $K$). In order to get a $K$-basis of the vector space $\text{HH}^4(R)$, we must add to the set (3.26) the cohomology class of the homomorphism

$$
t_7 = [0, p_2, 0] \mid O_{12}.
$$

c) If $\operatorname{rk} C = 0$, in order to get a $K$-basis of $\text{HH}^4(R)$, we add to the set (3.26) the cohomology classes of $t_7$ (see (3.27)) and of the homomorphism

$$
t_8 = [0, 0, p_3] \mid O_{12}.
$$

Proof. Since the matrix of the differential $d^Q_4$ has the form

$$
\begin{pmatrix}
\rho_1 & 0 & 0 \\
\tau_1 & \sigma_2 & 0 \\
0 & \tau_3 & \sigma_1
\end{pmatrix},
$$

by Proposition 3.5 we obtain

$$
\ker \Delta^4 \simeq \ker \rho_1^* \oplus \ker \left( \frac{\sigma_2}{\tau_3} \right)^*;
$$

where $(\frac{\sigma_2}{\tau_3})^* : \text{Hom}_A(L_2 \oplus L_2, R) \to \text{Hom}_A(L_2^2, R)$. The condition $(f, g) \in \ker (\frac{\sigma_2}{\tau_3})^*$, where $f, g \in \text{Hom}_A(L_2, R)$, is equivalent to the equation

$$
f \cdot \sigma_2 + g \cdot \tau_3 = 0.
$$

Put

$$f = (f_{12}, f_{23}, f_{31}, f_{21}, f_{32}, f_{13}), \quad g = (g_{12}, g_{23}, g_{31}, g_{21}, g_{32}, g_{13})$$

with $f_{ij}, g_{ij} \in \text{Hom}_A(P_{ij}, R)$. Then, using the first columns of the matrices $\sigma_2$ and $\tau_3$ (see (1.8) and (1.9)), from (3.29) we deduce that

$$f_{12} \circ (e_1 \otimes a_{21}^1)^* + f_{31} \circ (a_{31} \otimes e_1')^* + g_{21} \circ (a_{31}^{n_3 - 1} a_{12} \otimes e_1')^* + g_{13} \circ (e_1 \otimes (\alpha_3^{n_2 - 1} a_{31})')^* = 0.
Substituting in (3.30) the following expansions in bases:

\[ f_{12} = \sum_{i=0}^{n_3-1} \gamma_i^{(1)} [\alpha_1^i]_{a12}, \quad f_{23} = \sum_{j=0}^{n_1-1} \gamma_j^{(2)} [\alpha_2^j]_{a23}, \quad f_{31} = \sum_{k=0}^{n_2-1} \gamma_k^{(3)} [\alpha_3^k]_{a31}, \]

\[ g_{21} = \sum_{i=0}^{n_3-1} \delta_i^{(1)} [\omega_{21}^i], \quad g_{32} = \sum_{j=0}^{n_1-1} \delta_j^{(2)} [\omega_{32}^j], \quad g_{13} = \sum_{k=0}^{n_2-1} \delta_k^{(3)} [\omega_{13}^k] \]

with \( \gamma_i^{(j)}, \delta_i^{(j)} \in K \), we obtain the relation

\[ \sum_{i=0}^{n_3-1} \gamma_i^{(1)} \alpha_1^{i+1} + \sum_{k=0}^{n_2-1} \gamma_k^{(3)} \beta_1^{k+1} + \delta_0^{(1)} \alpha_1^{n_3} + \delta_0^{(3)} \beta_1^{n_2} = 0. \]

Hence, we have

\[ (3.31) \quad \gamma_i^{(1)} = 0 \quad \text{if} \quad i < n_3 - 1; \]

\[ (3.32) \quad \gamma_k^{(3)} = 0 \quad \text{if} \quad k < n_2 - 1; \]

\[ (3.33) \quad \gamma_{n_3-1}^{(1)} + \gamma_{n_2-1}^{(3)} + \delta_0^{(1)} + \delta_0^{(3)} = 0. \]

Similarly, from (3.30) we deduce the equation

\[ (3.33) \quad \gamma_j^{(2)} = 0 \quad \text{if} \quad j < n_1 - 1, \]

and two more equations that are similar to (3.32). Together with (3.33), they form a system of homogeneous linear equations (with respect to the corresponding \( \gamma_i^{(j)}, \delta_i^{(j)} \)) with the matrix

\[
\begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1
\end{pmatrix}
\]

As a basis of the vector space of solutions of this system, we take the following vectors:

\( (1,1,1 \mid O_3), \quad (1,0,0 \mid 1,0,0), \quad (0,1,0 \mid 0,1,0), \quad (0,0,1 \mid 0,0,1). \)

The corresponding elements included in a basis of Ker \( \Delta^4 \) (here, we use the isomorphism (3.28)) are \( t_3, t_4 \) in (3.26) and also

\[ [O_3 \mid 0, \alpha_1^{n_2-1}a_{23} \mid 0, \alpha_3^{n_2-1} \mid 0, \omega_{32} \mid 0, a_{13}^k]. \]

Moreover, relations (3.31) and (3.33) imply that we can include the following elements in a basis of Ker \( \Delta^4 \):

\[ [O_{12} \mid \alpha_{21}\alpha_1^i, 0, 0], \quad 1 \leq i \leq n_3 - 1; \]

\[ [O_{12} \mid 0, \alpha_{32}\alpha_2^j, 0], \quad 1 \leq j \leq n_1 - 1; \]

\[ [O_{12} \mid 0, 0, \alpha_{13}\alpha_3^k], \quad 1 \leq k \leq n_2 - 1. \]

Similarly, using the last three columns of the matrices \( \sigma_2 \) and \( \tau_3 \), by (3.29) we see that we can include the following elements in a basis of Ker \( \Delta^4 \): \( t_5, t_6 \) in (3.26) and

\[ [O_6 \mid 0, \omega_{32} \alpha_2^{n_2-1}, 0, 0, a_{23} \mid 0, 0, \alpha_3^{n_2-1} \mid 0, 0, a_{31} \mid O_3]; \]

\[ [O_9 \mid \alpha_1^i a_{12}, 0, 0 \mid O_3], \quad 1 \leq i \leq n_3 - 1; \]

\[ [O_9 \mid 0, \alpha_2^j a_{23} \mid 0, 0, \alpha_3^{n_2-1} \mid 0, 0, a_{31} \mid O_3]; \]

\[ [O_9 \mid 0, 0, \alpha_3^k \omega_{13} \mid O_3], \quad 1 \leq k \leq n_2 - 1. \]

Finally, in order to get a basis of Ker \( \Delta^4 \), it suffices to supplement the union of the sets indicated above with the elements that correspond to the basis elements of Ker \( \rho_1^* \) in (3.14) and (3.15).
Now, the required statement easily follows from the fact that, as in the proof of Proposition 3.14, \( \text{Im} \Delta^3 \) is generated by the following elements:

\[
\begin{align*}
&[n_3 p_1, n_3 p_2, 0 \mid O_{12}], \quad [0, n_1 p_2, n_1 p_3, \mid O_{12}], \quad [n_2 p_1, 0, n_2 p_3, \mid O_{12}], \\
&[O_3 \mid \alpha_1^{n_3 - 1} a_{12}, 0, a_2^{n_3 - 1} a_{23}, \mid O_6 \mid a_{21}, 0, a_{13}], \quad [O_6 \mid a_2 \alpha_1^{n_3 - 1}, 0, a_1 \alpha_2^{n_3 - 1} \mid a_{12}, 0, a_{13} \mid O_3], \\
&[O_3 \mid \alpha_1^{n_3 - 1} a_{12}, \alpha_2^{n_3 - 1} a_{23}, 0 \mid O_6 \mid a_{21}, a_{13}, 0], \quad [O_6 \mid a_2 \alpha_1^{n_3 - 1}, a_3 \alpha_2^{n_3 - 1}, 0 \mid a_{12}, a_{23}, 0 \mid O_3]; \\
&[O_{12} \mid a_2 \alpha_1^i, 0, 0], \quad [O_9 \mid a_1^i a_{12}, 0, 0 \mid O_3], \quad 1 \leq i \leq n_3 - 1; \\
&[O_{12} \mid 0, a_3 \alpha_2^i, 0], \quad [O_9 \mid 0, a_1^i a_{23}, 0 \mid O_3], \quad 1 \leq j \leq n_1 - 1; \\
&[O_{12} \mid 0, 0, a_3 \alpha_2^k \mid O_3], \quad 1 \leq k \leq n_2 - 1.
\end{align*}
\]

\[\square\]

**Proposition 3.14.** For any \( n \geq 3 \), we have

\[
(3.34) \quad \text{HH}^n(R) \simeq \text{HH}^{n-2}(R) \oplus W_n,
\]

where

\[
W_n = \left\{ \begin{array}{ll}
\text{Ker} \sigma_3^* / \text{Im} \sigma_2^* & \text{if } n \equiv 0 \pmod{3}, \\
\text{Coker} \sigma_3^* & \text{if } n \equiv 1 \pmod{3}, \\
\text{Ker} \sigma_2^* & \text{if } n \equiv 2 \pmod{3}.
\end{array} \right.
\]

**Proof.** Since the sequence of complexes \([3.1]\) splits in each degree, applying to it the functor \( \text{Hom}_A(-, R) \) we obtain a short exact sequence of complexes

\[
0 \to \text{Hom}_A(Q_\bullet[-2], R) \xrightarrow{\pi^*} \text{Hom}_A(Q_\bullet, R) \xrightarrow{\iota^*} \text{Hom}_A(X_\bullet, R) \to 0,
\]

which in turn yields the following long exact sequence:

\[
(3.35) \quad \cdots \to \text{HH}^{n-3}(R) \xrightarrow{\pi^*} \text{HH}^{n-2}(R) \xrightarrow{\iota^*} \text{H}^n(\text{Hom}_A(X_\bullet, R)) \xrightarrow{\delta^n} \text{HH}^{n-1}(R) \xrightarrow{\pi^*} \cdots.
\]

We claim that the connecting homomorphisms \( \delta^n \) in the sequence \((3.35)\) are zero if \( n \geq 2 \). Then the required statement will follow from the exactness of this sequence and from the construction of the complex \( X_\bullet \).

**Case 1:** assume that \( n = 3k, k \in \mathbb{N} \). We consider a 3k-cocycle \( f \in \text{Hom}_A(X_{3k}, R) \), i.e., we have \( \sigma_3^*(f) = 0 \).

\[
0 \to \text{Hom}_A(Q_{3k-2}, R) \xrightarrow{\pi^*} \text{Hom}_A(Q_{3k}, R) \xrightarrow{\iota^*} \text{Hom}_A(X_{3k}, R) \to 0
\]

\[
\Delta^{3k-2} \downarrow \quad \Delta^{3k} \downarrow \quad \text{Ker} \sigma_3^* \quad \text{Im} \sigma_2^* \quad \text{Coker} \sigma_3^* \quad \text{Ker} \sigma_2^*
\]

\[
0 \to \text{Hom}_A(Q_{3k-1}, R) \xrightarrow{\pi^*} \text{Hom}_A(Q_{3k+1}, R) \xrightarrow{\iota^*} \text{Hom}_A(X_{3k+1}, R) \to 0.
\]

We define \( \tilde{f} = f \circ \nu \), where \( \nu: Q_{3k} \to X_{3k} = L_3^2 \) is the projection onto the corresponding direct summand; in other words, the map \( \tilde{f} \) is obtained from \( f \) by extension by zero to the direct summand complementary to \( L_3^2 \) in \( Q_{3k} \). Hence, we have \( \iota^*(\tilde{f}) = \tilde{f} \circ \nu = \tilde{f} = f \circ \nu = f \).

By the construction of the connecting homomorphism, we have \( \delta^{3k}(\text{cl} f) = \text{cl} g \), where \( g: Q_{3k-1} \to R \) is a cocycle for which \( \pi^*(g) = \Delta^{3k}(\tilde{f}) \). Here and below, \( \text{cl} h \) denotes the cohomology class represented by a cocycle \( h \). Since

\[
\Delta^{3k}(\tilde{f}) = (0, \ldots, 0, \tau_2^*(f), \sigma_3^*(f))
\]

(this presentation corresponds to the direct decomposition of \( \text{Hom}_A(Q_{3k+1}, R) \) that is induced by the canonical decomposition of the module \( Q_{3k+1} \)) and \( \text{Ker} \sigma_3^* \subset \text{Ker} \tau_2^* \) (see Lemma 3.7), we obtain \( \Delta^{3k}(\tilde{f}) = 0 \), whence \( g = 0 \).

**Case 2:** we assume that \( n = 3k + 1, k \in \mathbb{N} \). Let \( f \in \text{Hom}_A(X_{3k+1}, R) \) (recall that \( \sigma_1^* = 0 \) by Proposition 3.5). If \( \tilde{f} \) is obtained from \( f \) by extension by zero to the direct
summand complementary to \(X_{3k+1} = L_2\) in \(Q_{3k+1}\), then \(\iota^*(\tilde{f}) = f\). Moreover, we have
\[
\Delta^{3k+1}(\tilde{f}) = (0, \ldots, 0, \tau_3^*(f), \sigma_1^*(f)) = (0, \ldots, 0, \tau_3^*(f), 0).
\]

We now need the following auxiliary statement.

**Lemma 3.15.** \(\text{Im} \, \tau_3^* \subset \text{Im} \, \sigma_2^*\).

**Proof.** As in the proof of Proposition 3.6 we show that \(\text{Im} \, \sigma_2^*\) is generated by the following homomorphisms:
\[
\begin{align*}
&[\alpha_1^i, \beta_2^i, 0 \mid O_3], \ [O_3 \mid \alpha_1^i, \beta_2^i, 0], \ 1 \leq i \leq n_3; \\
&[0, \alpha_2^j, \beta_3^j \mid O_3], \ [O_3 \mid 0, \alpha_2^j, \beta_3^j], \ 1 \leq j \leq n_1; \\
&[\beta_1^k, 0, \alpha_3^k] \mid O_3], \ [O_3 \mid \beta_1^k, 0, \alpha_3^k], \ 1 \leq k \leq n_2;
\end{align*}
\]

furthermore, \(\text{Im} \, \tau_3^*\) is generated by the homomorphisms in (3.36) that correspond to \(i = n_3, j = n_1\) and \(k = n_2\).

We continue the proof of Proposition 3.14. Put \(g = (0, \ldots, 0, \tau_3^*(f)) \in \text{Hom}_\Lambda(X_{3k}, R)\). It is clear that \(\pi^*(g) = \Delta^{3k+1}(f)\). By Lemma 3.14 there exists a homomorphism \(\varphi \in \text{Hom}_\Lambda(L_2, R)\) such that \(\tau_3^*(f) = \sigma_2^*(\varphi)\). Using Proposition 3.3 we obtain
\[
\Delta^{3k+1}(0, \ldots, 0, \varphi) = (0, \ldots, 0, \tau_1^*(\varphi), \sigma_2^*(\varphi)) = g.
\]

Consequently, we have \(\delta^{3k+1}(\text{cl} \, f) = \text{cl} \, g = 0\).

**Remark 3.4.** Propositions 3.12 and 3.13 can be derived from relation (3.34) with \(n = 3\) and \(n = 4\), respectively. But for us it is more convenient to directly produce representatives of basis classes in \(\text{HH}^1(R)\) and in \(\text{HH}^4(R)\), as was done in the propositions mentioned above.

**Corollary 3.16.** For any \(n > 0\), we have
\[
\dim_K \text{HH}^n(R) = \begin{cases} 
N - \text{rk} \, C + 4\left[\frac{n}{3}\right] + 1 & \text{if } n \equiv 0 \text{ or } 1 \pmod{3}, \\
N - \text{rk} \, C + 4\left[\frac{n}{3}\right] + 3 & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

**Proof.** If \(n \leq 3\), this statement is contained in Corollary 3.9. Induction on \(n\) with the use of (3.34) and Corollary 3.8 finishes the proof.

**Remark 3.5.** Corollary 3.10 extends the corresponding result proved in [9] for blocks with a dihedral defect group and three simple modules to all algebras in the families \(D(3K), D(3A)_1, D(3B)_1, \) and \(D(3D)_1\).

### §4. Generators

We propose an interpretation of the Yoneda product in
\[
\text{HH}^*(R) = \text{Ext}^*_\Lambda(R, R) = \bigoplus_{m \geq 0} \text{Ext}^m_\Lambda(R, R).
\]
that is convenient for our calculations. Let \( Q_\bullet \to R \) be the minimal \( \Lambda \)-projective resolution. We denote by \( \Omega^n(R) \) the \( n \)th syzygy of the module \( R \), that is, \( \Omega^n(R) = \text{Im}(d_{n-1}^Q: Q_n \to Q_{n-1}) \). Let

\[
\text{Hom}_\Lambda(Q_\bullet, R) = \left( \text{Hom}_\Lambda(Q_n, R), \Delta^n \right)
\]

be the complex investigated in \( \S 3 \). Since \( \text{Ker} \Delta^n \simeq \text{Hom}_\Lambda(\Omega^n(R), R) \), for any cocycle \( f: Q_n \to R \) there is a unique \( \Lambda \)-homomorphism \( \hat{f}: \Omega^n(R) \to R \), and we have \( f = \hat{f} \circ \pi_n \), where \( \pi_n: Q_n \to \Omega^n(R) \) is the projective cover. The sequence

\[
\cdots \xrightarrow{d_{n+1}^Q} Q_{n+1} \xrightarrow{d_n^Q} Q_n \xrightarrow{\pi_n} \Omega^n(R) \to 0
\]

is a minimal \( \Lambda \)-projective resolution of the module \( \Omega^n(R) \). Therefore, the homomorphism \( \hat{f}: \Omega^n(R) \to R \) is lifted (uniquely up to homotopy) to a chain map of complexes \( \{ \varphi_i: Q_{n+i} \to Q_i \}_{i \geq 0} \). The homomorphism \( \varphi_i \) is called the \( i \)th translate of the homomorphism \( \hat{f} \) or of the corresponding cocycle \( f \), and it will be denoted by \( T^i(\hat{f}) \) or \( T^i(f) \). Moreover, the homomorphism \( \varphi_i \) induces the maps \( \Omega^k(\hat{f}): \Omega^{n+k}(R) \to \Omega^k(R), k \geq 0 \).

Now, for cocycles \( f \in \text{Ker} \Delta^n \) and \( g \in \text{Ker} \Delta^t \) we have \( \text{cl} g \cdot \text{cl} f = \text{cl}(\hat{g} \circ \Omega^t(\hat{f})) \).

On the other hand, from the commutative diagram

\[
\begin{array}{ccc}
Q_{n+t} & \xrightarrow{\pi_{n+t}} & \Omega^{n+t}(R) \\
\downarrow{T^i(\hat{f})} & & \downarrow{\Omega^t(\hat{f})} \\
Q_t & \xrightarrow{\pi_t} & \Omega^t(R) \\
\downarrow{T^0(\hat{g})} & & \downarrow{\hat{g}} \\
Q_0 & \xrightarrow{d_{-1}^Q} & R
\end{array}
\]

we obtain \( (\hat{g} \circ \Omega^t(\hat{f})) \circ \pi_{n+t} = d_{-1}^Q \circ (T^0(\hat{g}) \circ T^t(\hat{f})) \); consequently, for us it suffices to know the composition of the translates \( T^0(\hat{g}) \) and \( T^t(\hat{f}) \). In the sequel, using direct decompositions (in particular, canonical decompositions) of the modules \( Q_i \), we shall describe translates of cocycles and their products with the help of matrices that correspond to such decompositions.

**Proposition 4.1.** Put \( y_1 = [e_1, e_2, e_3 | Q_0] \in \text{HH}^2(R) \). Then, for any \( i \geq 0 \), the projection onto the direct summand \( \pi_{i+2}: Q_{i+2} = Q_i \oplus X_{i+2} \to Q_i \) is the \( i \)th translate \( T^i(y_1) \) of the cocycle \( y_1 \).

**Proof.** We know already that \( \pi = \{ \pi_{i+2} \}_{i \geq 0} \) is a chain map (see Proposition 3.1). Further, it is clear that \( y_1 = d_{-1}^Q \circ \pi_2 \). Hence, we can take \( \pi_{i+2} \) as \( T^i(y_1) \) \((i \geq 0)\). \( \square \)

Before stating the next proposition, we introduce additional notation. Let

\[
\begin{align*}
\text{id}_{L_1} &= \begin{pmatrix} c_1 \otimes e_1' & 0 & 0 \\ 0 & e_2 \otimes e_2' & 0 \\ 0 & 0 & e_3 \otimes e_3' \end{pmatrix} : L_1 \to L_1, \\
\text{id}_{L_2'} &= \begin{pmatrix} e_1 \otimes e_2' & 0 & 0 \\ 0 & e_2 \otimes e_3' & 0 \\ 0 & 0 & e_3 \otimes e_1' \end{pmatrix} : L_2' \to L_2', \\
\text{id}_{L_2''} &= \begin{pmatrix} e_2 \otimes e_1' & 0 & 0 \\ 0 & e_3 \otimes e_2' & 0 \\ 0 & 0 & e_1 \otimes e_3' \end{pmatrix} : L_2'' \to L_2''
\end{align*}
\]
be the identity maps of the corresponding modules, and also their matrices with respect to the decompositions of these modules fixed earlier (see (1.3)), let

$$\Delta_{L_1} = \begin{pmatrix} \text{id}_{L_1} \\ \text{id}_{L_1} \end{pmatrix} : L_1 \to L_1^2, \quad \nabla_{L_1} = \begin{pmatrix} \text{id}_{L_1} & \text{id}_{L_1} \end{pmatrix} : L_1^2 \to L_1$$

be the diagonal and the codiagonal map of the module $L_1$, respectively, and let $M = (M_{11}, M_{12})$ be a block matrix in which $M_{ij}$ are the following diagonal $(3 \times 3)$-matrices:

\begin{equation}
(4.1) \quad M_{11} = \text{diag} \left( \sum_{i=0}^{n_1-1} \alpha_1^i \otimes (\beta_2^{n_3-1-i})', \sum_{i=0}^{n_1-1} \alpha_2^i \otimes (\beta_3^{n_2-1-i})', \sum_{i=0}^{n_2-1} \alpha_3^i \otimes (\beta_4^{n_1-1-i})' \right),
\end{equation}

\begin{equation}
M_{12} = \text{diag} \left( \sum_{i=1}^{n_3-1} a_{21} \alpha_1^{i-1} \otimes (\beta_2^{n_3-1-i})', \sum_{i=1}^{n_3-1} a_{22} \alpha_2^{i-1} \otimes (\beta_3^{n_2-1-i})', \sum_{i=1}^{n_2-1} \alpha_3^{i-1} \otimes (\beta_4^{n_1-1-i} a_{32})' \right),
\end{equation}

\begin{equation}
M_{21} = \text{diag} \left( \sum_{i=1}^{n_3-1} \alpha_1^{i-1} a_{12} \otimes (a_{12} \beta_2^{n_3-1-i})', \sum_{i=1}^{n_3-1} \alpha_2^{i-1} a_{23} \otimes (a_{23} \beta_3^{n_2-1-i})', \sum_{i=1}^{n_2-1} \alpha_3^{i-1} a_{31} \otimes (a_{31} \beta_4^{n_1-1-i})' \right),
\end{equation}

\begin{equation}
(4.2) \quad M_{22} = \text{diag} \left( \sum_{i=0}^{n_1-1} \beta_1^{i-1} \otimes (\alpha_2^{n_3-1-i})', \sum_{i=0}^{n_1-1} \beta_2^{i} \otimes (\alpha_3^{n_2-1-i})', \sum_{i=0}^{n_2-1} \beta_3^{i} \otimes (\alpha_4^{n_1-1-i})' \right).
\end{equation}

It should be noted that if, for example, $n_3 = 1$, then the $(1,1)$-position in the matrix $M_{11}$ is occupied by the element $e_1 \otimes e_2$, and the same position in the matrix $M_{12}$ is occupied by zero.

Also, let $N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$, $U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$, $S = \begin{pmatrix} S' & 0 \\ 0 & S' \end{pmatrix}$, $Y = \begin{pmatrix} S' & 0 \\ 0 & S' \end{pmatrix}$ be block matrices, where

\begin{equation}
(4.3) \quad N_1 = \text{diag} \left( \beta_1^{n_2-1} \otimes (\alpha_2^{n_3-1-i})', \beta_2^{n_3-1} \otimes (\alpha_3^{n_2-1-i})', \beta_3^{n_1-1} \otimes (\alpha_4^{n_1-1-i})' \right),
\end{equation}

\begin{equation}
(4.4) \quad N_2 = \text{diag} \left( \alpha_2^{n_3-1} \otimes (\beta_1^{n_2-1})', \alpha_3^{n_2-1} \otimes (\beta_2^{n_3-1})', \alpha_1^{n_1-1} \otimes (\beta_3^{n_1-1})' \right),
\end{equation}

\begin{equation}
(4.5) \quad S' = \text{diag} \left( 0, \sum_{i=0}^{n_1-1} \alpha_2^{i} \otimes (\alpha_2^{n_1-1-i})', \sum_{i=0}^{n_1-1} \beta_3^{i} \otimes (\beta_3^{n_1-1-i})' \right),
\end{equation}

and let $W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$, $V = \begin{pmatrix} V_{11} & 0 \\ W_{21} & W_{22} \end{pmatrix}$ be the matrices in which

\begin{equation}
(4.6) \quad W_{11} = \text{diag} \left( e_1 \otimes (\alpha_2^{n_1-1-i})', \sum_{i=0}^{n_1-1} \alpha_2^{i} \otimes (\beta_3^{n_1-1-i})', \beta_3^{n_1-1} \otimes e_1' \right),
\end{equation}

\begin{equation}
W_{12} = \text{diag} \left( 0, \sum_{i=1}^{n_1-1} a_{32} \alpha_2^{i-1} \otimes (\beta_3^{n_1-1-i} a_{32})', 0 \right),
\end{equation}

\begin{equation}
W_{21} = \text{diag} \left( 0, \sum_{i=1}^{n_1-1} a_{23} \alpha_2^{i-1} a_{23} \otimes (a_{23} \beta_3^{n_1-1-i})', 0 \right),
\end{equation}

\begin{equation}
W_{22} = \text{diag} \left( \alpha_2^{n_1-1} \otimes e_1', \sum_{i=0}^{n_1-1} \beta_3^{i} \otimes (\alpha_2^{n_1-1-i})', e_1 \otimes (\beta_3^{n_1-1})' \right).
\end{equation}
We denote by \( O_{m,n} \) the \((m \times n)\)-matrix consisting of zero maps, and we omit the indication of its size if this is clear from the context. We recall also that \( X_\bullet \) is the complex occurring in Proposition 2.1.

**Proposition 4.2.** In the notation of (3.24), we put \( z := z_1 + z_2 \in \text{HH}^3(R) \). We assume additionally that

\[
(4.6) \quad n_1 \geq n_2 \geq n_3.
\]

a) If \( i = 0, 1, 2 \), then the translates \( T^i(z) : Q_{i+3} \to Q_i \) can be defined by the following block matrices:

\[
T^0(z) = \begin{pmatrix} O_{3,6} & \nabla L_1 \\ \nabla L_1 & M & O_{5,2} \\ O_{5,2} & \text{id}_{L_2} & O \end{pmatrix}, \quad T^1(z) = \begin{pmatrix} O_{6,3} & M & \text{id}_{L_2} \\ N & \text{id}_{L_1} & O \end{pmatrix}, \quad T^2(z) = \begin{pmatrix} O & \nabla L_1 & O \\ N & \text{id}_{L_1} & \text{id}_{L_2} \end{pmatrix}.
\]

b) If \( n_3 > 1 \), then for \( i \geq 3 \) we have \( T^i(z) = T^{i-2}(z) \oplus \text{id}_{X_3} \).

c) If \( n_2 > 1 \) and \( n_3 = 1 \), then

\[
T^3(z) = \begin{pmatrix} O & M & \text{id}_{L_2} & O \\ U & \text{id}_{L_1} & \text{id}_{L_2} & O \end{pmatrix},
\]

and for \( i \geq 4 \) we have \( T^i(z) = T^{i-2}(z) \oplus \text{id}_{X_3} \).

d) If \( n_1 > 1 \) and \( n_2 = 1 \), then

\[
T^4(z) = \begin{pmatrix} O & M & \text{id}_{L_2} & O \\ S & \text{id}_{L_1} & \text{id}_{L_2} & O \end{pmatrix}, \quad T^5(z) = \begin{pmatrix} Z^{(i)}(O_{3,6}) & 0 \\ \text{id}_{L_1} & \text{id}_{L_2} \end{pmatrix},
\]

where \( Z^{(i)} = (w_{0,12}) \), and for \( i \geq 5 \) we have

\[
T^i(z) = \begin{pmatrix} T^{i-2}(z) & 0 \\ Z^{(i)} & \text{id}_{X_3} \end{pmatrix},
\]

where

\[
Z^{(i)} = \begin{cases} O_{6,3i-12} & \text{if } i \equiv 0 \pmod{3}, \\
0_{6,3i-12} & \text{otherwise.}
\end{cases}
\]

e) If \( n_1 = 1 \), then

\[
T^3(z) = \begin{pmatrix} O & \text{id}_{L_2} & \text{id}_{L_2} & O \\ \Delta_{L_1} & O & \text{id}_{L_1} & \text{id}_{L_2} \end{pmatrix},
\]

and for \( i \geq 4 \) we have

\[
T^i(z) = \begin{pmatrix} T^{i-2}(z) & 0 \\ Z^{(i)} & \text{id}_{X_3} \end{pmatrix}, \quad \text{where } Z^{(i)} = \begin{pmatrix} O_{6,3i-12} & \text{id}_{X_{i-3}} & O_{6,12} \end{pmatrix}.
\]

**Remark 4.1.** We imposed the additional condition (4.6) in order to simplify the presentation of the translates \( T^i(z) \). In the sequel, citing Proposition 4.2, we use additionally the following symmetry: the defining relations (1.1), (1.2) of the algebras under consideration are invariant under cyclic permutations of indices.

The proof of Proposition 4.2 is a direct verification of the relations

\[
d_{i-1}^Q d_{0}^{Q_0} = z, \quad d_{i-1}^{Q_0} T_i(z) = d_{i-1}^{Q_0} T^i(z) d_{i+2}^{Q_0} (i > 0)
\]

by induction on \( i \). In particular, in the proof of part d) the following facts are used:

\[
\sigma_1 V = V \sigma_1, \quad \tau_1 V = V \tau_1, \quad \sigma_2 Y = V \sigma_2, \quad \tau_2 V = Y \tau_2, \quad \sigma_3 Y = V \sigma_3, \quad \tau_3 Y = V \tau_3.
\]

The details of calculations are left to the reader. \( \square \)
Proposition 4.3. In the notation of Propositions 3.10, 3.13 and Remark 3.1 we have:

\[
\begin{align*}
(4.7) & \quad t_1 = y_1^2, \quad t_3 = y_2 y_1, \quad t_5 = y_3 y_1, \\
(4.8) & \quad t_4 = \begin{cases} 
    x_2 z_1 & \text{if } \text{rk} \, C \geq 1, \\
    x_2'' z_1 & \text{if } \text{rk} \, C = 0,
\end{cases} \quad t_6 = \begin{cases} 
    x_2 z_2 & \text{if } \text{rk} \, C \geq 1, \\
    x_2'' z_2 & \text{if } \text{rk} \, C = 0,
\end{cases} \\
& \quad t_2 = p_1 y_1, \quad t_7 = p_3 y_1, \quad t_8 = p_2 y_1^2, \\
(4.9) & \quad t_{12}^{(i)} = c_1^i y_1^2 \quad (1 \leq i < n_3), \\
& \quad t_{23}^{(j)} = c_2^j y_1^2 \quad (1 \leq j < n_1), \\
& \quad t_{31}^{(k)} = c_3^k y_1^2 \quad (1 \leq k < n_2).
\end{align*}
\]

**Proof.** We prove only the relation \( t_3 = y_3 y_1 \) and also the relation \( t_4 = x_2 z_1 \) in the case where \( \text{rk} \, C \geq 1 \). The remaining relations (4.7) and (4.8) are verified in the same way, and the relations (4.9) are obvious.

As the translate \( T^0(y_2) : Q_2 \to Q_0 \), we can take the map that is presented by the following matrix with respect to the decomposition \( Q_2 = L_1 \oplus L_2' \oplus L_2'' \):

\[
(4.10) \quad T^0(y_2) = \begin{pmatrix} O_{3,3} & \gamma & O_{3,3} \end{pmatrix},
\]

where \( \gamma = \text{diag} \left( e_1 \otimes (a_1^{n_3-1} a_{12})', e_2 \otimes (a_2^{n_1-1} a_{23})', e_3 \otimes (a_3^{n_2-1} a_{31})' \right) \).

By Proposition 4.3 with respect to the same decomposition the translate \( T^2(y_1) \) is presented by the matrix

\[
T^2(y_1) = \begin{pmatrix}
    \text{id}_{L_1} & O & O & O \\
    O & \text{id}_{L_2'} & O & O \\
    O & O & \text{id}_{L_2''} & O
\end{pmatrix}.
\]

Then \( T^0(y_2) T^2(y_1) = \begin{pmatrix} O_{3,3} & \gamma & O_{3,3} \end{pmatrix} \), and, clearly, \( \delta_{1}^{Q}(T^0(y_2) T^2(y_1)) = t_3 \). Consequently, \( y_3 y_1 = t_3 \).

Direct verification shows that, as the translates \( T^0(x_2), T^0(z_1), \) and \( T^1(z_1) \), we can take the maps

\[
(4.11) \quad T^0(x_2) = \begin{pmatrix} e_1 \otimes a_{12}' & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & e_2 \otimes a_{21}' & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
(4.12) \quad T^0(z_1) = \begin{pmatrix} O_{3,6} & \text{id}_{L_1} & O_{3,3} \end{pmatrix}, \quad T^1(z_1) = \begin{pmatrix} O_{3,3} & M_{11} & O_{3,6} & O_{3,3} \\
    O_{3,3} & M_{21} & O_{3,6} & \text{id}_{L_2''} \end{pmatrix},
\]

where \( M_{11}, M_{21} \) are the matrices in (4.1), (4.2). Hence, we have

\[
T^0(x_2) T^1(z_1) = \begin{pmatrix} O_{3,3} & \phi_1 & O_{3,6} & \phi_2 \end{pmatrix},
\]

where

\[
\phi_1 = \begin{pmatrix}
    \sum_{i=0}^{n_3-1} a_1^i \otimes (a_1^{n_3-1-i})' & 0 & 0 & 0 \\
    \sum_{i=1}^{n_3-1} a_1^{i-1} a_{12} \otimes (b_2^{n_3-1-i})' & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}, \quad \phi_2 = \begin{pmatrix} e_2 \otimes a_{21}' & 0 & 0 & 0 \end{pmatrix}.
\]

We consider the restriction of the composition \( d_{1}^{Q} T^0(x_2) T^1(z_1) \) to the first occurrence of the module \( L_2 \) as a direct summand in the decomposition \( Q_4 = L_1 \oplus L_2 \oplus L_2'. \) The
values of the map $d^Q_1 T^0(x_2) T^1(z_1)$ at all generators $e_1 \otimes e'_2$, $e_2 \otimes e'_3$, $e_3 \otimes e'_1$, $e_1 \otimes e'_3$, $e_2 \otimes e'_1$, $e_3 \otimes e'_2$, $e_1 \otimes e'_2$, $e_2 \otimes e'_3$, of this module, except the first, are zero, and, furthermore,

$$d^Q_1 T^0(x_2) T^1(z_1)(e_1 \otimes e'_2) = \sum_{i=0}^{n+1} a_1^i a_2^{n_3-1-i} + \sum_{i=1}^{n_1-1} a_1^{-i} a_2^{n_3-i} = n_3 a_1^{n_3-1} a_2 + (n_3-1) a_1^{-1} a_2 = a_1^{n_3-1} a_2.$$

In the same way, we calculate the values of the map $d^Q_1 T^0(x_2) T^1(z_1)$ at the other direct summands of the module $Q_4$; finally, we obtain $d^Q_1 T^0(x_2) T^1(z_1) = t_4$, whence $x_2 z_1 = t_4$.

**Proposition 4.4.** The ring $\text{HH}^*(R)$ is generated as $K$-algebra by finitely many elements with degrees less than or equal to 3.

**Proof.** Let $\mathcal{H}$ be a $K$-subalgebra of $\text{HH}^*(R)$ generated by the set $\bigoplus_{i=0}^3 \text{HH}^i(R)$. Since $\text{HH}^i(R)$ is finite-dimensional, the $K$-algebra $\mathcal{H}$ is finitely generated. We recall also that in [6.3] and in Propositions [6.10] [6.12] we presented bases for $\text{HH}^i(R)$, $i \leq 3$. Moreover, Proposition [4.3] implies that $\text{HH}^i(R) \subset \mathcal{H}$.

Consider an arbitrary element $f \in \text{HH}^m(R)$, $m > 3$. We use induction on $m$ to prove that $f \in \mathcal{H}$. Consider the corresponding cocycle, which is also denoted by $f$, $f \in \text{Hom}_\Lambda(Q_m, R)$. Using the canonical decomposition (see [4.6])

$$Q_m = \sum_{i+j=m} B_{ij},$$

we can confine ourselves to consideration of $f$ with $f(B_{i,m-i}) \neq 0$ for a unique $i$. Using the decomposition $Q_m = Q_{m-2} \oplus X_m$, we may analyze only two possibilities: either $B_{i,m-i}$ is contained in $Q_{m-2}$ or $B_{i,m-i} = X_m$ (i.e., $i = 0$).

a) If $f(B_{i,m-i}) \neq 0$ and $i > 0$, then, by Proposition [4.1] there exists a homomorphism $g: Q_{m-2} \to R$ such that $g T^{m-2}(y_1) = f$. Furthermore, we have $g d_{m-2} Q^{m-1}(y_1) = 0$, and by the surjectivity of $T^{m-1}(y_1)$ we obtain $g \in \ker \Delta^{m-2}$. Consequently, $f = g \cdot y_1$ in $\text{HH}^*(R)$. By the inductive hypothesis, we have $g \in \mathcal{H}$, whence $f \in \mathcal{H}$.

b) Assume now that $Q_{m-2} \subset \ker f$. Since the decomposition $Q_m$ is ordered by increasing of the second index $j$, we can present the homomorphism $f$ in the form $f = (0, \ldots, 0, f_1)$ with $f_1: X_m \to R$. Using the fact that $X_m = X_{m-3}$, we define a homomorphism $g = (0, \ldots, 0, f_1): Q_{m-3} \to R$. Since $f d_m = 0$, we have $g d_{m-3} = 0$ because if $i > 4$, then the nonzero elements in the bottom row of the matrix of the differential $d_{m-3}$ are the same as those in the matrix of the differential $d^Q$. Proposition [4.3] implies that the matrix of the map $T^{m-3}(z)$ has the following triangular block form: $T^{m-3}(z) = \left( \begin{array}{cc} A & 0 \\ B & \id_{X_{m-3}} \end{array} \right)$; therefore, the cocycle $\tilde{f} := f - g \cdot z$ annihilates the last summand in the canonical decomposition of the module $Q_m$. Applying the argument of part a) to $\tilde{f}$, we see that $\tilde{f} \in \mathcal{H}$. Moreover, by the inductive hypothesis we have $g \in \mathcal{H}$, and hence $f = \tilde{f} + g \cdot z \in \mathcal{H}$.

**Corollary 4.5.** Put

$$\mathcal{X} = \{ e_i, p_i \mid i = 1, 2, 3 \} \cup \{ x_1, x_2, y_1, y_2, y_3, z_1, z_2 \}.$$

1) Let $\text{rk} C = 2$. If all $n_i$, $i = 1, 2, 3$, are odd, then $\text{HH}^*(R)$ (as a $K$-algebra) is generated by the set $\mathcal{X}$. If $n_3$ is even (hence, $n_2$ and $n_3$ are odd), then $\text{HH}^*(R)$ is generated by the set $\mathcal{X} \cup \{ x_{12}^{(1)}, x_{31}^{(1)} \}$. 
2) Let \( \text{rk} \, C = 1 \) and assume additionally that \( n_3 \) is odd (hence, \( n_1 \) and \( n_2 \) are even). Then \( \text{HH}^* (R) \) is generated by the set
\[
(\mathcal{X} \setminus \{x_1\}) \cup \{x'_1, x''_1, x^{(1)}_{12}\}.
\]

3) If \( \text{rk} \, C = 0 \), then \( \text{HH}^* (R) \) is generated by the set
\[
(\mathcal{X} \setminus \{x_1, x_2\}) \cup \{x'_1, x''_1, x'_2, x''_2\}.
\]

Proof. 1) It is clear that, if \( n_2 \) and \( n_3 \) are odd, then
\[
x^{(1)}_{23} = c_2 x_1, \quad y_4 = p_1 y_1, \quad x_1 y_1 = z_3, \quad x_2 y_1 = z_4, \quad z_5 = p_1 z_1, \quad z_6 = p_1 z_2,
\]
\[
x^{(i)}_{12} = c_1^{i-1} x^{(1)}_{12} \quad (2 \leq i \leq n_3 - 1),
\]
\[
x^{(j)}_{23} = c_2^{j-1} x^{(1)}_{23} \quad (2 \leq j \leq n_1 - 1),
\]
\[
x^{(k)}_{31} = c_3^{k-1} x^{(1)}_{31} \quad (2 \leq k \leq n_2 - 1),
\]
\[
y^{(i)}_{12} = c_1^i y_1, \quad x^{(i)}_{12} = x^{(1)}_{12} y_1 \quad (1 \leq i \leq n_3 - 1),
\]
\[
y^{(j)}_{23} = c_2^j y_1, \quad x^{(j)}_{23} = x^{(1)}_{23} y_1 \quad (1 \leq j \leq n_1 - 1),
\]
\[
y^{(k)}_{31} = c_3^k y_1, \quad x^{(k)}_{31} = x^{(1)}_{31} y_1 \quad (1 \leq k \leq n_2 - 1).
\]
Moreover, if \( n_1 \) is also odd, then \( x^{(1)}_{12} = c_1 x_1, \) \( x^{(1)}_{31} = c_3 x_1 \). Consequently, the required statements follow from Proposition [4.4] by using Propositions [3.10 3.12] and Remark [3.1].

2) Let \( \text{rk} \, C = 1 \), and assume that \( n_3 \) is odd. The relations mentioned in the proof of part 1) show that, in order to establish the required statement, it suffices to observe that
\[
x^{(1)}_{31} = c_3 x''_1, \quad z'_3 = x'_1 y_1, \quad z''_3 = x''_1 y_1, \quad y_5 = p_3 y_1.
\]

3) Finally, if \( \text{rk} \, C = 0 \), then the relations mentioned above show that it remains to observe that
\[
x^{(1)}_{12} = c_1 x'_2, \quad z'_4 = x'_2 y_1, \quad z''_4 = x''_2 y_1, \quad y_6 = p_2 y_1\] \[\square\]

Remark 4.2. If \( n_3 = 1 \), then \( x^{(1)}_{12} \) and \( c_1 \) can be omitted in the sets of generators mentioned above. A similar remark concerns the elements \( x^{(1)}_{31} \) and \( c_3 \) (if \( n_2 = 1 \)) and also \( c_2 \) (if \( n_1 = 1 \)); cf. Remark [3.2].

\[\text{§5. Relations}\]

**Proposition 5.1.** 1) Let \( \text{rk} \, C = 2 \). Then the elements of the set \( \mathcal{X} \) (see [4.14]) satisfy the relations
\[
c_1^{n_3} = c_2^{n_2} = c_3^{n_2} = 0, \quad c_i c_j = 0 \quad (i, j \in \{1, 2, 3\}, i \neq j),
\]
\[
p_1 p_3 = p_1 c_j = 0 \quad (i, j \in \{1, 2, 3\}),
\]
\[
p_i x_j = c_i x_j = p_i y_j + 1 = c_i y_j + 1 = c_i z_j = 0 \quad (i \in \{1, 2, 3\}, j \in \{1, 2\}),
\]
\[
p_1 z_1 = p_2 z_2 = p_3 z_1, \quad p_1 z_2 = p_2 z_2 = p_3 z_2;
\]
\[
\]
\[
p_1 y_1 = p_2 y_1 = p_3 y_1;
\]
\[
x^i_1 = x^i_2 = y^i_2 = y^i_3 = 0, \quad x_1 x_2 = n_1 n_2 n_3 \cdot p_1 y_1,
\]
\[
x_1 y_2 = 0, \quad x_1 y_1 = x_2 y_1 = p_1 z_2, \quad x_2 y_2 = p_1 z_2, \quad x_1 z_1 = n_1 n_2 n_3 \cdot y_1 y_2, \quad x_1 z_2 = n_1 n_2 n_3 \cdot y_1 y_2 + x_2 z_2;
\]
\[
\]
\[
y_2 y_3 = x_1 \cdot p_1 y_1^2, \quad y_3 z_1 = x_1 \cdot (x_1 y_1^2 + x_2 y_2^2 + (1 - x_1) \cdot y_2 z_2)
\]
\[
\quad \text{with } \kappa_1 \text{ as in [1.24]};
\]
\[
y_2 z_2 = x_2 \cdot x_1 y_1^2 \quad \text{with } \kappa_2 \text{ as in [1.26]};
\]
\[
z_1 z_2 = x_3 \cdot y_1^3 \quad \text{with } \kappa_3 \text{ as in [1.28]}.
\]
If \( n_1 \) is even (and \( n_2, n_3 \) are odd), then additionally the following relations are satisfied:

\[
\begin{align*}
  c_1 x_1 &= c_3 x_1 = c_2 x_{12}^{(1)} = c_3 x_{12}^{(1)} = c_1 x_{31}^{(1)} = c_2 x_{31}^{(1)} = c_1^{n_1-1} x_{12}^{(1)} = c_3^{n_2-1} x_{31}^{(1)} = 0, \\
  x_{12} x_{31}^{(1)} &= (x_{12}^{(1)})^2 = (x_{31}^{(1)})^2 = x_{12}^{(1)} x_2 = x_{31}^{(1)} x_2 = 0,
\end{align*}
\]

and also

\[
(5.7) \quad x_{12}^{(1)} a = x_{31}^{(1)} a = 0 \text{ for any } a \in X \setminus \{y_1\} \text{ with } \deg a > 0.
\]

2) Let \( \text{rk } R = 1 \), and assume additionally that \( n_3 \) is odd. Then the generators of the algebra \( \text{HH}^*(R) \) in the set \((4.11)\) satisfy all relations \((5.1)\) and \((5.3)-(5.4)\) that do not involve \( x_1 \) and \( x_{31}^{(1)} \), and also the relations

\[
\begin{align*}
  c_1 x_1' &= c_3 x_1' = c_1 x_1'' = c_2 x_1'' = x_1' x_1'' = 0, \\
  p_1 x_1' &= p_1 x_1'' = 0 \text{ for } i \in \{1, 2, 3\}; \\
  p_1 y_1 &= p_2 y_1; \\
  x_1' x_2 &= x_1'' x_2, \quad (x_1')^2 = \lambda(n_1) \cdot (p_2 + p_3) y_1, \quad (x_1'')^2 = \lambda(n_2) \cdot (p_1 + p_3) y_1,
\end{align*}
\]

where \( \lambda(n), n \in \mathbb{N} \), is defined by formula \((4.38)\):

\[
\begin{align*}
  x_1' y_2 &= x_1'' y_3 = 0, \quad x_1' y_3 = x_1'' y_3 = p_1 z_2, \\
  x_1' z_1 &= x_1'' z_1 = 0, \quad x_1' z_2 = x_1'' z_2 = 0.
\end{align*}
\]

3) Let \( \text{rk } R = 0 \). Then the generators of the algebra \( \text{HH}^*(R) \) belonging to the set \((4.11)\) satisfy all relations \((5.1), (5.3), (5.5), (5.6), (5.8), (5.10)-(5.12)\) that do not involve \( x_1 \) and \( x_2 \), and also the relations

\[
\begin{align*}
  c_2 x_2' &= c_3 x_2' = c_2 x_2'' = c_3 x_2'' = 0; \\
  c_1 x_2' &= c_2 x_2', \quad p_2 x_2' = p_2 x_2'' = 0 \text{ for } i \in \{1, 2, 3\}; \\
  x_1' x_2' &= x_1' x_2'' = x_1' x_2'' = 0, \quad (x_2')^2 = (x_2'')^2 = x_2'' x_2'' = \lambda(n_3) \cdot (p_1 + p_2) y_1,
\end{align*}
\]

where \( \lambda(n), n \in \mathbb{N} \), is defined in \((4.38)\):

\[
\begin{align*}
  x_2' y_2 &= x_2'' y_3 = 0, \quad x_2' y_3 = x_2'' y_3 = p_1 z_1, \quad x_2' z_1 = x_2' z_2 = p_1 z_2, \\
  x_2' z_1 &= x_2'' z_2 = 0, \quad x_2' z_2 = x_2'' z_2.
\end{align*}
\]

Proof. Let \( \text{rk } R = 2 \). Relations \((5.1)\) and \((5.2)\) are verified directly. In order to compute the product of any pair of elements of the set \( X \) that have positive degree, beforehand we need to compute translates of these elements. By the commutativity of the algebra \( \text{HH}^*(R) \), for an element of degree \( i \) it suffices to know its translates up to the \( i \)th order inclusive. The translates of \( y_1 \) are already known (see Proposition \(4.1\)). Below we present the translates of the remaining generators. The verification of the corresponding commutativity relations (cf. the proof of Proposition \((1.2)\) is left to the reader.

To simplify the description of some translates, we assume additionally that inequalities \((4.10)\) are fulfilled.

We define the following elements of the algebra \( \Lambda \):

\[
\begin{align*}
  \mu_1 &= n_1 n_2 \sum_{i=1}^{n_3-1} i a_1^{n_1-i-1} \otimes (a_2 i_1)'; \\
  \nu_1 &= n_1 n_2 \sum_{i=1}^{n_3-1} i a_1^{n_2-i-1} \otimes (a_1 i_1)', \\
  \mu_2 &= n_2 n_3 \sum_{i=1}^{n_1-1} i a_2^{n_1-i-1} \otimes (a_2 i_2)'; \\
  \nu_2 &= n_2 n_3 \sum_{i=1}^{n_1-1} i a_2^{n_2-i-1} \otimes (a_1 i_2)', \\
  \mu_3 &= n_1 n_3 \sum_{i=1}^{n_2-1} i a_3^{n_2-i-1} \otimes (a_3 i_3)', \\
  \nu_3 &= n_1 n_3 \sum_{i=1}^{n_2-1} i a_3^{n_3-i-1} \otimes (a_3 i_3)'.
\end{align*}
\]
We observe that $\mu_2$ and $\mu_3$ (respectively, $\nu_2$ and $\nu_3$) are obtained from $\mu_1$ (respectively, from $\nu_1$) with the help of cyclic permutations (modulo 3) of indices in the corresponding formulas. Then we consider the matrix

$$X_1 = \begin{pmatrix} \mu_1 & \nu_1 & 0 \\ 0 & \mu_2 & \nu_2 \\ \nu_3 & 0 & \mu_3 \end{pmatrix}.$$ 

The following matrix is constructed similarly:

$$X_2 = \begin{pmatrix} \zeta_1 & \theta_1 & 0 \\ 0 & \zeta_2 & \theta_2 \\ \theta_3 & 0 & \zeta_3 \end{pmatrix},$$

where

$$\zeta_1 = n_1 n_2 \sum_{i=1}^{n_3-1} i a_1^{n_3-i-1} a_{12} \otimes (a_1^i)', \quad \theta_1 = n_1 n_2 \sum_{i=1}^{n_3} i a_2^{n_3-i} \otimes (a_{12} a_{1}^{i-1})',$$

and $\zeta_2$, $\zeta_3$, $\theta_2$, and $\theta_3$ are obtained from $\zeta_1$ and $\theta_1$ with the help of suitable cyclic permutations of indices in the formulas. Furthermore, we consider the matrices

$$X_0 = \text{diag} \left( n_1 n_2 e_1 \otimes a_{12}', n_2 n_3 e_2 \otimes a_{23}', n_1 n_3 e_3 \otimes a_{31}' \right),$$

$$X_3 = \begin{pmatrix} 0 & 0 & n_2 n_3 e_1 \otimes a_{23}' \\ n_1 n_3 e_2 \otimes a_{31}' & 0 & 0 \\ 0 & n_1 n_2 e_3 \otimes a_{12}' & 0 \end{pmatrix}.$$ 

Then

$$T^0(x_1) = \begin{pmatrix} X_0 & O_{3,3} \end{pmatrix}, \quad T^1(x_1) = \begin{pmatrix} X_1 & O_{3,3} \\ X_2 & X_3 \end{pmatrix}. $$

Next, $T^0(x_2)$ is described in [4,11] and, moreover, we have

$$T^1(x_2) = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

Here the matrix of the map $T^1(x_2): Q_2 \to Q_1$ contains four nonzero entries (they are denoted by asterisk), and the latter look like this:

$$T^1(x_2)_{1,1} = \sum_{i=0}^{n_3-1} a_1^{n_3-i-1} \otimes (a_{21} a_1^i)', \quad T^1(x_2)_{3,8} = e_3 \otimes a_{12}',$$

$$T^1(x_2)_{4,2} = \sum_{i=0}^{n_3-1} b_2^{n_3-i-1} \otimes (a_1^i a_{12})', \quad T^1(x_2)_{5,6} = e_3 \otimes a_{21}'.
The translate $T^0(y_2)$ has also been calculated; see (4.10). To describe the remaining translates of the maps $y_2$ and $y_3$ of the order $i \leq 2$, we consider the following matrices:

\[
Y_1^{(0)} = \begin{pmatrix}
0 & 0 & e_1 \otimes (a_{13}a_{32}^{-1})' \\
0 & 0 & 0 \\
e_3 \otimes (a_{32}a_{21}^{-1})' & 0 & 0
\end{pmatrix},
\]

\[
Y_1^{(1)} = \begin{pmatrix}
\alpha_2^{n_1-1}a_{23} \otimes (\beta_1^{n_2-1})' & 0 \\
0 & 0 & 0 \\
\alpha_2^{n_1-1}a_{23} \otimes (\beta_1^{n_2-1})' & 0 & 0
\end{pmatrix},
\]

\[
Y_2^{(1)} = \begin{pmatrix}
e_2 \otimes (a_{13}a_{32}^{-1})' & 0 & 0 \\
e_3 \otimes (a_{32}a_{21}^{-1})' & 0 & 0 \\
e_3 \otimes (a_{32}a_{21}^{-1})' & 0 & 0
\end{pmatrix},
\]

\[
Y_3^{(1)} = \begin{pmatrix}
g_{n_1-1} \otimes (a_{13}a_{32}^{-1})' & 0 \\
g_{n_1-1} \otimes (a_{13}a_{32}^{-1})' & 0 & 0 \\
\beta_3^{n_1-1} \otimes (a_{21}a_{13}^{-1})' & 0 & 0
\end{pmatrix},
\]

\[
Y_4^{(1)} = \text{diag} \left( e_1 \otimes (a_{21}a_{13}^{-1})', e_2 \otimes (a_{32}a_{21}^{-1})', e_3 \otimes (a_{13}a_{32}^{-1})' \right),
\]

\[
Y_1^{(2)} = \begin{pmatrix}
\alpha_1^{n_2-1}a_{12} \otimes e_2' & 0 & 0 \\
0 & 0 & 0 \\
\alpha_2^{n_1-1}a_{23} \otimes e_3' & 0 & 0
\end{pmatrix},
\]

\[
Y_2^{(2)} = \begin{pmatrix}
e_2 \otimes (a_{32}a_{21}^{-1})' & 0 & 0 \\
e_3 \otimes (a_{13}a_{21}^{-1})' & 0 & 0 \\
e_3 \otimes (a_{13}a_{21}^{-1})' & 0 & 0
\end{pmatrix},
\]

\[
Y_3^{(2)} = \begin{cases}
diag(a_{12} \otimes e_1', a_{23} \otimes e_2', a_{31} \otimes e_3') & \text{if } n_1 = 1, \\
\beta_3^{n_1-1} \otimes (a_{13}a_{32}^{-1})' & \text{if } n_1 > 1, n_2 = 1, \\
\beta_3^{n_1-1} \otimes (a_{13}a_{32}^{-1})' & \text{if } n_1 > 1, n_3 = 1, \\
\beta_3^{n_1-1} \otimes (a_{13}a_{32}^{-1})' & \text{if } n_3 > 1,
\end{cases}
\]

\[
Y_4^{(2)} = \begin{pmatrix}
e_2 \otimes (a_{21}a_{13}^{-1})' & 0 & 0 \\
e_3 \otimes (a_{13}a_{32}^{-1})' & 0 & 0 \\
e_3 \otimes (a_{13}a_{32}^{-1})' & 0 & 0
\end{pmatrix},
\]

\[
Y_5^{(2)} = \begin{pmatrix}
e_1 \otimes (a_{13}a_{32}^{-1})' & 0 \\
e_2 \otimes (a_{32}a_{21}^{-1})' & 0 & 0 \\
e_3 \otimes (a_{13}a_{32}^{-1})' & 0
\end{pmatrix},
\]

\[
Y_6^{(2)} = \begin{cases}
diag(e_1 \otimes a_{21}', e_2 \otimes a_{32}', e_3 \otimes a_1') & \text{if } n_1 = 1, \\
diag(e_2 \otimes a_{32}' \otimes (a_{13}a_{32}^{-1})', e_3 \otimes a_1') & \text{if } n_1 > 1, n_2 = 1, \\
diag(e_2 \otimes a_{32}' \otimes (a_{13}a_{32}^{-1})', e_3 \otimes a_1') & \text{if } n_2 > 1, n_3 = 1, \\
diag(0, 0, 0) & \text{if } n_3 > 1.
\end{cases}
\]
Now we put
\[
T^1(y_2) = \begin{pmatrix} O_{3,3} & Y_1^{(1)} & O_{3,3} & O_{3,3} \\ O_{3,3} & Y_2^{(1)} & O_{3,3} & O_{3,3} \end{pmatrix}, \quad T^2(y_2) = \begin{pmatrix} O_{3,3} & Y_1^{(2)} & O_{3,6} & O_{3,3} \\ O_{3,3} & O_{3,3} & O_{3,3} & O_{3,3} \\ Y_3^{(2)} & O_{3,3} & O_{3,3} & O_{3,3} \end{pmatrix},
\]
\[
T^0(y_3) = \begin{pmatrix} O_{3,6} \\ O_{3,6} \end{pmatrix}, \quad T^1(y_3) = \begin{pmatrix} O_{3,3} & Y_3^{(1)} & O_{3,6} & O_{3,3} \\ O_{3,3} & O_{3,3} & O_{3,3} & O_{3,3} \end{pmatrix},
\]
\[
T^2(y_3) = \begin{pmatrix} O_{3,3} & O_{3,3} & Y_4^{(2)} & O_{3,3} \\ Y_6^{(2)} & O_{3,3} & O_{3,3} & O_{3,3} \\ O_{3,3} & O_{3,3} & O_{3,3} & Y_5^{(2)} \end{pmatrix}.
\]
It only remains to compute the translates of the map $z_1$, because the translates of $z_2 = z - z_1$ can be obtained by using Proposition 4.2. We recall that $T^0(z_1)$ and $T^1(z_1)$ were described in (4.12). Finally, we have
\[
T^2(z_1) = \begin{pmatrix} O_{3,3} & O_{3,3} & id_{L_1} & O_{3,3} & O_{3,3} \\ O_{3,3} & O_{3,3} & O_{3,3} & id_{L_2} & O_{3,3} \\ O_{3,3} & N_2 & O_{3,3} & O_{3,3} & O_{3,3} \end{pmatrix},
\]
\[
T^3(z_1) = \begin{pmatrix} O_{3,3} & M_{11} & O_{3,6} & O_{3,3} & O_{3,3} \\ O_{3,3} & M_{21} & id_{L_2'} & O_{3,3} & O_{3,3} \\ O_{3,3} & O_{3,3} & O_{3,6} & id_{L_1} & O_{3,3} \\ Z_3^{(3)} & O_{3,3} & O_{3,6} & O_{3,3} & O_{3,3} \end{pmatrix},
\]
where the submatrices $M_{11}, M_{21}, N_2$ are defined in (4.11), (4.22), and (4.3), respectively; moreover,
\[
Z_3^{(3)} = \begin{cases} 0 & \text{if } n_3 > 1, \\ U_2 & \text{if } n_2 > 1, n_3 = 1, \\ S' & \text{if } n_1 > 1, n_2 = 1, \\ \text{id}_{L_1} & \text{if } n_1 = 1, \end{cases}
\]
and $U_2$ and $S'$ are defined in (4.4) and (4.5).

Now, we verify relations (5.5) and the second relation (5.4). The verification of the remaining relations (including the case where $n_1$ is even) is similar, and we leave this to the reader. First, we consider the case where inequalities (4.6) are true.

We have
\[
T^0(y_3) \circ T^2(z_1) = \begin{pmatrix} O_{3,3} & Y_2^{(0)} \cdot N_2 & O_{3,12} \end{pmatrix},
\]
where
\[
Y_2^{(0)} \cdot N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ if } n_3 > 1,
\]
\[
= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ if } n_2 > 1, n_3 = 1,
\]
\[
= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ if } n_1 > 1, n_2 = 1,
\]
\[
= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ if } n_1 = 1.
\]

Consequently,
\[
\alpha_{21}^3 \circ T^0(y_3) \circ T^2(z_1) = \begin{cases} \begin{pmatrix} O_3 | 0, a_{32} a_{21}^{n_1-1} | 0, O_{12} \end{pmatrix} & \text{if } n_1 > 1, n_2 = 1, \\ \begin{pmatrix} O_3 | a_{21}, a_{32}, a_{13} | O_{12} \end{pmatrix} & \text{if } n_1 = 1. \end{cases}
\]
Similar calculations show that
\[
d_{-1}^Q \circ T^0(y_2) \circ T^2(z_2) = \begin{cases} 
0 & \text{if } n_2 > 1, \\
[0, a_2^{-1} a_{23}, 0] | O_{15} & \text{if } n_1 > 1, n_2 = 1, \\
[a_{12}, a_{23}, a_{31}] | O_{15} & \text{if } n_1 = 1.
\end{cases}
\]

Furthermore, using Proposition 4.1, we easily see that \(x_1 y_1^2\) and \(x_2 y_1^2\) are represented by the cocycles
\[
[n_1 n_2 a_{12}, n_2 n_3 a_{23}, n_1 n_3 a_{31}] | O_{15} \quad \text{and} \quad [a_{12}, 0, 0] | a_{21}, 0, 0 | O_{12},
\]
respectively. From this, we immediately derive the relation \(y_2 z_2 = \varphi_2 \cdot x_1 y_1^2\) in (5.5), and also the relation \(y_2 z_2 = y_3 z_1 = 0\) in the case where \(n_2 > 1\). If \(n_1 > 1\) and \(n_2 = 1\), then \(y_2 z_2 + y_3 z_1\) is represented by the map \([0, a_2^{-1} a_{23}, 0] | 0, a_{32} a_2^{-1}, 0 | O_{12}\) lying in \(\text{Im} \Delta^4\) (cf. (3.21)) in the proof of Proposition 3.10, and again we obtain \(y_2 z_2 = y_3 z_1\). If \(n_1 = 1\), we show in the same way that the 5-cocycles
\[
[O_3 | a_{21}, a_{32}, a_{13}] | O_{12} \quad \text{and} \quad [0, a_{23}, a_{31} | a_{21}, 0, 0 | O_{12}]
\]
representing \(y_3 z_1\) and \(x_1 y_1^2 + x_2 y_1^2\), respectively, are mutually cohomological (again, cf. (3.21)), and hence in this case we obtain \(y_3 z_1 = x_1 y_1^2 + x_2 y_1^2\).

Next, since \(T^0(z_2) \circ T^3(z_1) = \left( Z^{(3)}_1 O_{3,18} \right)\), we see that
\[
d_{-1}^Q \circ T^0(z_2) \circ T^2(z_1) = \begin{cases} 
0 & \text{if } n_2 > 1, \\
[0, n_1 a_2^{-1}, n_1 a_3^{-1}] | O_{18} & \text{if } n_1 > 1, n_2 = 1, \\
[e_1, e_2, e_3] | O_{18} & \text{if } n_1 = 1.
\end{cases}
\]

Using Proposition 4.1, we obtain \(y_3^3 = [e_1, e_2, e_3] | O_{18}\). Consequently, \(z_3 z_1 = \varphi_3 \cdot y_3^3\).

Finally, in the case where condition (4.6) is not fulfilled, the desired relations follow from the above by symmetry.

If \(\text{rk} C < 2\), the corresponding relations are established much as before, by using the translates (calculated beforehand) for the generators of the algebra \(\text{HH}^*(R)\). We leave this to the reader. \(\square\)

Suppose that \(\text{rk} C = 2\) and all \(n_i\), \(i = 1, 2, 3\), are odd. Corollary 4.5 and Proposition 5.1 imply that there exists a surjective homomorphism of graded \(K\)-algebras \(\varphi: A_1 \rightarrow \text{HH}^*(R)\) that takes the generators in the set \(X_1\) (see (1.11)) to the corresponding generators in \(X\) (see (1.14)). Note that, not hesitating about ambiguity, we use the same letters to denote elements of both sets that correspond to each other. If \(\text{rk} C = 2\) and some \(n_i\) is even (hence, the remaining two parameters \(n_j, j \neq i\), are odd), we can assume by symmetry that \(n_1\) is even. Then, as above, we construct a surjective homomorphism of graded algebras \(\varphi: A'_1 \rightarrow \text{HH}^*(R)\) for which we have \(\varphi(x_3) = x_{12}^{(1)}\), \(\varphi(x_1) = x_{31}^{(1)}\). In the same way, we construct surjective homomorphisms of graded algebras \(A_2 \rightarrow \text{HH}^*(R)\) if \(\text{rk} C = 1\) (in this case, it is convenient to assume that \(n_3\) is odd), respectively, \(A_3 \rightarrow \text{HH}^*(R)\) if \(\text{rk} C = 0\). Now, Theorem 1.2 is derived from the following statement.

**Proposition 5.2.** Let \(\varphi: A \rightarrow \text{HH}^*(R)\) be a homomorphism among those constructed above, where the algebra \(A\) is either \(A_1\), or \(A'_1\), or \(A_2\), or \(A_3\), and let \(A = \bigoplus_{m \geq 0} A^m\) be a direct decomposition into homogeneous direct summands. Then
\[
(5.17) \quad \dim_K A^m = \dim_K \text{HH}^m(R).
\]

**Proof.** We prove (5.17) only in the case where \(\text{rk} C = 2\) and all \(n_i\) are odd. The remaining cases are treated similarly.
The (nonzero) images of the monomials in $K[x_1]$ under the canonical epimorphism $K[x_1] \to A$ are also called monomials. Any element $a \in A$ is represented as a linear combination of monomials (with coefficients in $K$).

By definition, the reduction of a monomial $m$ in $A$ is the process, and also the result, of replacement of some submonomials of $m$ by other elements of $A$ by the following rules $(a \mapsto b$ means the replacement of every occurrence of the monomial $a$ by the element $b)$:

\[
\begin{align*}
x_1 x_2 & \mapsto p_1 y_1, \\
x_1 y_3 & \mapsto x_2 y_3 \mapsto p_1 z_2, \\
p_3 z_1 & \mapsto p_2 z_1 \mapsto p_1 z_1, \\
y_2 y_3 & \mapsto p_1 y_1^2 \quad \text{(if $y_2 y_3 \not= 0$ (in $A$))}, \\
x_1 z_2 & \mapsto y_1 y_3 + x_2 z_2, \\
y_3 z_1 & \mapsto \begin{cases} y_2 z_2, & \text{if } y_2 z_2 \not= 0, \\
x_1 y_1^2 + x_1 y_1^2, & \text{if } y_2 z_2 = 0, \end{cases}
\end{align*}
\]

Any replacement in the above list is called an elementary step of reduction.

Let $\chi : K[x_1] \to \mathbb{N}_0$ be a function such that, for any elementary step of reduction $a \mapsto b$, we have $\chi(a) \geq \chi(b)$ (more precisely, here $a$ and $b$ are the corresponding elements in $K[x_1]$). For example, it is easily seen that these properties are fulfilled for the degree function if we introduce a new grading $\chi$ on $K[x_1]$ such that

\[
\begin{align*}
\chi(c_i) & = 0 \quad (i = 1, 2, 3), \\
\chi(p_1) & = 0, \\
\chi(p_2) & = 1, \\
\chi(p_3) & = 2, \\
\chi(x_1) & = 7, \\
\chi(y_1) & = 1, \\
\chi(y_2) & = 3, \\
\chi(y_3) & = 5, \\
\chi(z_1) & = 6, \\
\chi(z_2) & = 7.
\end{align*}
\]

Since the values of $\chi$ strictly decrease under reduction, after a finite number of steps we obtain monomials or their linear combinations such that no elementary step of reduction can be applied to them. We say that a presentation of an element $a \in A$ as a linear combination of monomials has normal form if reduction cannot be applied to any of these monomials.

Put $d_k = \dim_K A^k$. We denote by $\bar{d}_k$ the number of monomials in $A^k$ presented in the normal form. It is clear that $\bar{d}_k \geq d_k$. Since there is an epimorphism $A^k \to \HH^k(R)$, we have $d_k \geq \dim_K \HH^k(R)$, and hence it suffices to show that

\[(5.18) \quad \bar{d}_k = \dim_K \HH^k(R).\]

We prove this inequality by induction on $k$. If $k \leq 2$, the following monomials have normal form:

- $k = 0$: $1$, $c_1'$ ($1 \leq i \leq n_3 - 1$), $c_2'$ ($1 \leq j \leq n_1 - 1$), $c_3'$ ($1 \leq l \leq n_2 - 1$), $p_1, p_2, p_3$;
- $k = 1$: $x_1, x_2, c_1 x_1$ ($1 \leq i \leq n_3 - 1$), $c_2 x_1$ ($1 \leq j \leq n_1 - 1$), $c_3 x_1$ ($1 \leq l \leq n_2 - 1$);
- $k = 2$: $y_1, y_2, y_3, c_1 y_1$ ($1 \leq i \leq n_3 - 1$), $c_2 y_1$ ($1 \leq j \leq n_1 - 1$), $c_3 y_1$ ($1 \leq l \leq n_2 - 1$), $p_1 y_1$.

Consequently, relation \((5.18)\) follows from Proposition \(5.2\) and Corollary \(5.9\). Next, we assume that $k > 2$. We claim that

\[(5.19) \quad \bar{d}_k - \bar{d}_{k-2} = \begin{cases} 4 & \text{if } k \equiv 0 \pmod{3}, \\
2 & \text{otherwise.} \end{cases}\]

Let $M^k$ denote the set of all (nonzero) monomials in $A^k$ presented in the normal form. The elements in $M^k$ the normal form of which contains $y_1$ are in one-to-one correspondence with all elements in $M^{k-2}$; hence, $\bar{d}_k - \bar{d}_{k-2}$ is the number of elements in $M^k$ the normal form of which contains no factor equal to $y_1$. We fix an element $a \in M^k$ presented in the normal form and assume that this normal form does not contain $y_1$.

Next, we consider several cases successively.
1) Assume additionally that the monomial \( a \) contains a factor \( x_1 \). Then \( a \) cannot contain \( x_2, y_2, y_3, z_1, z_2 \), whence we see that \( \deg a = 1 \), but this is impossible.

2) Now, assume that \( a \) contains \( x_2 \). Then (the normal form of) \( a \) does not contain \( p_i, c_i \, (i = 1, 2, 3), \, y_2, y_3 \). Moreover, \( z_1 \) and \( z_2 \) cannot be involved in \( a \) simultaneously. Consequently, \( a = x_2 z_1^j \) or \( a = x_2 z_2^j \) with \( j \in \mathbb{N} \). We have \( k = 3j + 1 \) in this case.

3) Assume that \( x_1 \) and \( x_2 \) are not contained in \( a \), and \( y_2 \) is contained in \( a \). Then \( a \) does not contain \( p_i, c_i \, (i = 1, 2, 3), \, y_3 \), and also \( z_2 \), whence \( a = y_2 z_1^j, \, j \in \mathbb{N} \); moreover, we have \( k = 3j + 2 \).

4) We assume that \( x_1, x_2, \, y_2 \) are not contained in \( a \) and \( y_3 \) is contained in \( a \). As in the preceding case, we obtain \( a = y_2 z_1^{j+1}, \, j \in \mathbb{N} \), and also \( k = 3j + 2 \).

5) Finally, we assume that \( a \) does not contain \( x_1, \, x_2, \, y_2, \, y_3 \). As above, we obtain \( a \in \{ z_1^j, z_2^j, p_1 z_1^j, p_1 z_2^j \mid j \in \mathbb{N} \} \) and \( k = 3j \).

Relation \((5.19)\) follows immediately from this. Now, using Corollary \(5.16\) we obtain

\[
\bar{d}_k - \bar{d}_{k-2} = \dim_K \text{HH}^k(R) - \dim_K \text{HH}^{k-2}(R).
\]

By the inductive hypothesis, we have \( \bar{d}_{k-2} = \dim_K \text{HH}^{k-2}(R) \), whence \((5.15)\) follows. \(\Box\)

**Remark 5.1.** The proof of Proposition \(5.2\) implies the uniqueness of a normal form of the elements of the algebra \( A \).

**References**


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