

## APPROXIMATION OF TWO-DIMENSIONAL CROSS-SECTIONS OF CONVEX BODIES BY DISKS AND ELLIPSES

V. V. MAKEEV

ABSTRACT. In connection with the well-known Dvoretzky theorem, the following question arises: How close to a disk or to an ellipse can a two-dimensional cross-section through an interior point  $O$  of a convex body  $K \subset \mathbb{R}^n$  be? In the present paper, the attention is focused on a few (close to prime) dimensions  $n$  for which this problem can be solved exactly. Asymptotically, this problem was solved by the author in 1988.

Another problem treated in the paper concerns inscribing a regular polygon in a circle that belongs to a field of circles smoothly embedded into the fibers of the tautological bundle over the Grassmannian manifold  $G_2(\mathbb{R}^n)$ .

Throughout this paper, by a *convex body*  $K \subset \mathbb{R}^n$  we mean a compact convex set with nonempty interior.

As usual, we denote by  $V_k(\mathbb{R}^n)$  the Stiefel manifold of orthonormal  $k$ -frames in  $\mathbb{R}^n$ , and by  $G_k(\mathbb{R}^n)$  (respectively,  $G_k^+(\mathbb{R}^n)$ ) the Grassmannian manifold of nonoriented (respectively, oriented)  $k$ -planes in  $\mathbb{R}^n$  passing through the origin  $O$ . Let  $\gamma_k^{n(+)}: E_k^{(+)}(\mathbb{R}^n) \rightarrow G_k^{(+)}(\mathbb{R}^n)$  be the tautological vector bundle in which the fiber over a  $k$ -plane is the same plane regarded as a  $k$ -dimensional (oriented) subspace of  $\mathbb{R}^n$ .

### §1. INTRODUCTION AND THE BASIC CONJECTURE

The famous Dvoretzky theorem [1] states that each multidimensional convex body admits cross-sections of small dimension that are close to a ball.

We denote by  $\varepsilon_1(n, k)$  (respectively,  $\varepsilon_2(n, k)$ ) the minimal  $\varepsilon > 0$  such that, for each interior point  $O$  of a convex body  $K \subset \mathbb{R}^n$ , there is a  $k$ -plane  $P$  through  $O$  with the property that the cross-section  $P \cap K$  contains a ball (ellipsoid) and is contained in a  $(1 + \varepsilon)$ -homothetic ball (ellipsoid) with the same center.

We also consider the quantities  $\varepsilon_3(n, k)$  and  $\varepsilon_4(n, k)$  defined in the same way as  $\varepsilon_1(n, k)$  and  $\varepsilon_2(n, k)$ , respectively, with the only difference that the centers of the balls (ellipsoids) in question must coincide with the distinguished point  $O$ .

Similar functions  $\varepsilon_i^s(n, k)$  refer to the case where the convex body  $K \subset \mathbb{R}^n$  is symmetric relative to its interior point  $O \in K$ . In this case,  $\varepsilon_1^s(n, k) = \varepsilon_3^s(n, k)$  and  $\varepsilon_2^s(n, k) = \varepsilon_4^s(n, k)$ .

Dvoretzky's theorem [1] states that

$$\lim_{n \rightarrow \infty} \varepsilon_i(n, k) = \lim_{n \rightarrow \infty} \varepsilon_i^s(n, k) = 0, \quad i = 1, 2, 3, 4,$$

for each positive integer  $k$ . In the present paper, we focus our attention on the cases where the values of  $\varepsilon_i(n, k)$  or  $\varepsilon_i^s(n, k)$  can be calculated explicitly.

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In [3], the asymptotics of  $\varepsilon_i(n, 2)$  and  $\varepsilon_i^s(n, 2)$  was calculated for  $i \leq 4$ ; it was proved that if  $i \leq 4$ , then

$$\begin{aligned}\varepsilon_i(n, 2) &\sim \sec \frac{\pi}{n} - 1 \sim \frac{\pi^2}{2n^2}, \\ \varepsilon_i^s(n, 2) &\sim \sec \frac{\pi}{2n} - 1 \sim \frac{\pi^2}{8n^2}.\end{aligned}$$

It was conjectured in [3] that

$$\varepsilon_3(n, 2) = \sec \frac{\pi}{n-1} - 1 \quad \text{and} \quad \varepsilon_3^s(n, 2) = \sec \frac{\pi}{2(n-1)} - 1.$$

This conjecture was proved in [4] in the case where  $n-1$  is an odd prime. In the present paper, we prove the following stronger conjecture for certain dimensions.

**Conjecture.** For  $n \geq 3$ , we have

$$\begin{aligned}(1) \quad &\varepsilon_1(n, 2) = \varepsilon_2(n-1, 2) = \sec \frac{\pi}{n} - 1; \\ (2) \quad &\varepsilon_3(n, 2) = \varepsilon_4(n-1, 2) = \sec \frac{\pi}{n-1} - 1; \\ (3) \quad &\varepsilon_3^s(n, 2) = \varepsilon_4^s(n-1, 2) = \sec \frac{\pi}{2(n-1)} - 1.\end{aligned}$$

Thus, the second and third statements of this conjecture were proved partially in [4] in the case where  $n-1$  is an odd prime. It is well known that all statements (1)–(3) of the conjecture are valid for  $n=3$  (see [6]). The second statement means that  $\varepsilon_3(3, 2) = \varepsilon_4(2, 2) = +\infty$ . In [2], it was shown that  $\varepsilon_2(3, 2) = \sqrt{2} - 1$ .

## §2. THE PROOF OF STATEMENTS (2) AND (3) OF THE CONJECTURE IN THE CASE WHERE $n-1$ IS AN ODD PRIME

In the sequel, the following statement will be useful. In some cases, this statement reduces the problem of approximation of a section by an ellipse to the problem of approximation by a disk.

**Lemma 1** (N. Yu. Netsvetaev). For positive integers  $k < n$ , we have

$$\varepsilon_2(n, k) \leq \varepsilon_1(n+1, k), \quad \varepsilon_4^s(n, k) \leq \varepsilon_3^s(n+1, k), \quad \text{and} \quad \varepsilon_4(n, k) \leq \varepsilon_3(n+1, k).$$

*Proof.* We prove the first inequality. It suffices to consider the case where  $\varepsilon(n+1, k) < +\infty$ , which will be assumed in what follows.

Let  $O$  be an interior point of a convex body  $K \subset \mathbb{R}^n$ . Consider a right cylinder  $C \subset \mathbb{R}^{n+1}$  with base  $K$  and with height considerably larger than the diameter of the body  $K$ . We imagine the pair  $(K, O)$  as a midsection equidistant from the bases of  $C$ . Consider a  $k$ -plane  $P$  passing through  $O$  and such that the body  $P \cap C$  contains a ball  $K_1$  and is contained in the ball  $(1 + \varepsilon_1(n+1, k))K_1$  with the same center. If the height of  $C$  is sufficiently large, then the  $k$ -plane  $P$  cannot intersect the base of  $C$ .

Let  $\pi$  be the orthogonal projection of the space  $\mathbb{R}^{n+1}$  onto the hyperplane of the midsection of  $C$ . Then the ellipsoid  $\pi K_1$  is inscribed in the section  $\pi P \cap K$ , and this section lies in the concentric ellipsoid  $(1 + \varepsilon_1(n+1, k))\pi K_1$ .

The remaining inequalities can be proved similarly.  $\square$

*Remark.* Possibly, in many cases the above inequalities are actually equalities. Below, we give several such examples.

Theorem 1 in [4] gives the exact values

$$\varepsilon_3(n, 2) = \sec \frac{\pi}{n-1} \quad \text{and} \quad \varepsilon_1^s(n, 2) = \sec \frac{\pi}{2(n-1)} - 1$$

if  $n-1$  is an odd prime.

The above lemma implies the following.

**Theorem 1.** *If  $n - 1$  is an odd prime, then*

$$\varepsilon_4(n - 1, 2) = \sec \frac{\pi}{n - 1} - 1 \quad \text{and} \quad \varepsilon_2^s(n - 1, 2) = \sec \frac{\pi}{2(n - 1)} - 1.$$

*Proof.* To show that the first estimate is sharp, we consider an  $(n - 1)$ -dimensional simplex  $A_1, \dots, A_n$  the edges  $A_1A_2, A_1A_3, \dots, A_1A_n$  of which are considerably longer than the other edges, and the interior point  $O$  is chosen to be close to  $A_1$ . In this case, the two-dimensional sections intersecting the face  $A_2A_3 \dots A_n$  and passing through  $O$  cannot have good approximations by ellipses with center at  $O$ . The other sections have at most  $n - 1$  sides and also cannot be approximated by ellipses better than stated above.

To show that the second estimate is sharp, consider an  $(n - 1)$ -dimensional parallelepiped the two-dimensional sections of which have at most  $2(n - 2)$  sides.

Thus, statements (2) and (3) of the conjecture are valid if  $n - 1$  is an odd prime.  $\square$

§3. PROOF OF STATEMENT (1) OF THE CONJECTURE FOR ODD PRIME DIMENSIONS  $n$

**Theorem 2.** *For any odd prime dimension  $n$ , we have  $\varepsilon_1(n, 2) = \varepsilon_2(n - 1, 2) = \sec \frac{\pi}{n} - 1$ .*

*Proof.* 1) We prove the inequality  $\varepsilon_1(n, 2) \leq \sec \frac{\pi}{n} - 1$ . In the oriented two-dimensional planes passing through the interior point  $O$  of the convex body  $K \subset \mathbb{R}^n$ , we consider ordered collections  $(l_1, \dots, l_n)$  of  $n$  rays with the same origin  $C$  such that the angles between the neighboring rays are  $\frac{2\pi}{n}$ . Let  $M(K)$  denote the set of collections of such rays that intersect the boundary of  $K$  at points  $A_1, \dots, A_n$  with  $\sum_{k=1}^n \overline{CA_k} = \overline{0}$ . For a typical body  $K$  with smooth boundary  $\partial K$ , the set  $M(K)$  is a smooth compact  $(2n - 3)$ -dimensional manifold on which the group  $\mathbb{Z}_n$  acts freely by cyclic permutations of rays.

Let  $CB_i$  be the segment cut by the body  $K$  from the extension of the ray  $CA_i$ . On the set of regular configurations of rays with origin at the interior point  $C$  of the section, we define a continuous mapping  $F_K$  with values in  $\mathbb{R}^{2n}$  by the rule

$$F_K(l_1, \dots, l_n) = (|CA_1|, |CB_1|, |CA_2|, |CB_2|, \dots, |CA_n|, |CB_n|).$$

The cyclic group  $\mathbb{Z}_n$  acts on  $\mathbb{R}^{2n}$  by cyclic permutations of coordinates by an even number of positions. For  $n$  prime, this action is free outside the plane  $P$  given by the equations  $x_1 = x_3 = \dots = x_{2n-1}$  and  $x_2 = x_4 = \dots = x_{2n}$ .

By definition, the mapping  $F_K : M(K) \rightarrow \mathbb{R}^{2n}$  is  $\mathbb{Z}_n$ -equivariant. We prove that the image  $F_K(M(K))$  intersects the plane  $P$ .

Indeed, suppose the contrary. Then we deal with the mapping

$$F_K : M(K) \rightarrow \mathbb{R}^{2n} \setminus P.$$

Let

$$\pi_1 : \mathbb{R}^{2n} - P \rightarrow P^\perp \setminus (0, \dots, 0)$$

be the orthogonal projection onto the orthogonal complement  $P^\perp$  of  $P$ , defined by the formula

$$\pi_1(x_1, \dots, x_{2n}) = (x_1 - s_1, x_2 - s_2, \dots, x_{2n-1} - s_1, x_{2n} - s_2),$$

where

$$s_1 = \frac{x_1 + x_3 + \dots + x_{2n-1}}{n} \quad \text{and} \quad s_2 = \frac{x_2 + x_4 + \dots + x_{2n}}{n}.$$

Let

$$\pi_2 = (P^\perp \setminus (0, \dots, 0)) \rightarrow S^{2n-3} \subset P^\perp, \quad \pi_2(x) = x/|x|,$$

be the projection onto the unit sphere.

By construction, the mappings  $\pi_1$  and  $\pi_2$  are also  $\mathbb{Z}_n$ -equivariant.

Thus, if  $F_K(M(K)) \cap P = \emptyset$ , we obtain a  $\mathbb{Z}_n$ -equivariant continuous mapping

$$G_K = \pi_2 \circ \pi_1 \circ F_K : M(K) \rightarrow S^{2n-3}.$$

We prove that, in general position, when  $M(K)$  is a smooth compact  $(2n-3)$ -dimensional manifold, the integral degree of  $G$  is not divisible by  $n$ .

The ball  $K$  can be deformed smoothly into any other smooth convex body  $K_1$  within the class of smooth convex bodies. Let  $K_t$  be the corresponding deformation, where  $K_0 = K$  and  $0 \leq t \leq 1$ . For a typical deformation, the sets  $M(K_t) \times t$  form a smooth oriented compact  $(2n-2)$ -dimensional manifold  $N$  with boundary  $(M(K) \times 0) \cup (M(K_1) \times 1)$ .

Let

$$\pi = \bigcup_{0 \leq t \leq 1} K_t \times t \rightarrow K_1$$

be a continuous projection of the cylinder onto the upper base. This projection is assumed to be a diffeomorphism on each fiber  $K_t \times t$ . By  $\pi$ , we can continuously project  $N$  onto the ‘‘upper base’’ with preservation of the directions of the rays  $l_i$ ; namely, we project them from the origin by  $\pi$  and then, applying  $\pi_1 \circ F_{K_1}$ , we map  $N$  continuously to  $P^\perp$ .

If the image of  $N$  intersects  $P$ , then the corresponding regular collection of rays belongs to  $M(K_1)$  automatically, and  $F_{K_1}(M(K_1)) \cap P \neq \emptyset$ . Otherwise, we obtain a continuous  $\mathbb{Z}_n$ -equivariant mapping  $h : N \rightarrow S^{2n-3}$ .

Observe that, for a disk (and for a set  $K$  that is  $C^1$ -close to a disk), we have  $M(K) \cong V_2(\mathbb{R}^n)$  because, for the regular collections of rays  $(l_1, \dots, l_n)$  as above, the condition  $\sum_{k=1}^n \overline{CA}_k = \overline{0}$  is fulfilled for a disk  $K$  only if  $C$  is the center of  $K$ . In the case in question, where  $M(K) \cong V_2(\mathbb{R}^n)$ , the mapping  $G_K : V_2(\mathbb{R}^n) \rightarrow S^{2n-3}$  has integral degree not divisible by  $n$  if  $n$  is an odd prime (see [3]).

Since the degree of the restriction  $h| : \partial N \rightarrow S^{2n-3}$  is zero, we have

$$\deg(h|_{M(K) \times 0}) = \deg(h|_{M(K_1) \times 1});$$

therefore,  $\deg(G_{K_1} : M(K_1) \rightarrow S^{2n-3})$  is not divisible by  $n$ .

Consequently, the mapping  $G_{K_1}$  is surjective, and its image contains the points of  $S^{2n-3}$  with the coordinates

$$\left( \pm \frac{1}{\sqrt{n^2-n}}, 0 \pm \frac{1}{\sqrt{n^2-n}}, 0, \dots, \pm \frac{1}{\sqrt{n^2-n}}, 0, \mp \sqrt{\frac{n-1}{n}}, 0 \right).$$

Thus, for the corresponding regular collections of rays  $(l_1, \dots, l_n) \in M_{K_1}$ , we obtain

$$|CA_1| = |CA_2| = \dots = |CA_{n-1}| \quad \text{and} \quad |CB_1| = |CB_2| = \dots = |CB_n|.$$

Since  $\sum_{k=1}^n \overline{CA}_k = \overline{0}$ , we have

$$|CA_1| = |CA_2| = \dots = |CA_{n-1}| = |CA_n|,$$

whence  $F_{K_1}(l_1, \dots, l_n) \in P$ , which contradicts our assumption.

Thus, we have proved that a semiregular convex equilateral  $2n$ -gon  $A_1B_1 \dots A_nB_n$  such that

$$\angle A_1 = \angle A_2 = \dots = \angle A_n, \quad \angle B_1 = \angle B_2 = \dots = \angle B_n$$

is inscribed in some cross-section of  $K \subset \mathbb{R}^n$  passing through the interior point  $O$  of  $K$ . This cross-section lies in the  $2n$ -gonal star bounded by the extensions of the sides of the  $2n$ -gon. It is easily seen that the ratio of the radius of the disk inscribed in the  $2n$ -gon to the radius of the disk circumscribed about the star is equal to  $\cos \pi/n$ , which proves the inequality  $\varepsilon_1(n, 2) \leq \sec \frac{\pi}{n} - 1$ .

2) The inequality  $\varepsilon_2(n-1, 2) \leq \varepsilon_1(n, 2)$  follows from Lemma 1.

3) It remains to observe that  $\varepsilon_2(n-1, 2) \geq \sec \frac{\pi}{n} - 1$ , which is shown by the example of an  $(n-1)$ -dimensional simplex  $K$  with an arbitrary distinguished interior point  $O$ . In this case, the two-dimensional cross-sections through  $O$  are polygons with at most

$n$  sides. These polygons cannot be approximated by ellipses better than stated above. Only for an affine-regular  $n$ -gon can the ratio of the area of the ellipse containing the polygon to the area of the ellipse contained in the polygon be equal to  $\sec^2 \pi/n$ .  $\square$

§4. ON POLYGONS INSCRIBED INTO FIELDS OF SMOOTH JORDAN CURVES  
IN THE BUNDLE  $\gamma_2^n$

The results of the preceding sections are based on the existence of a polygon having a certain symmetry group and inscribed in some two-dimensional cross-section of a multidimensional convex body.

Without any changes, all theorems proved above can be generalized to “quasisections”, i.e., to the case of continuous fields of convex figures in fibers of the tautological bundle  $\gamma_2^n$ .

In this section, we lift the convexity restriction and consider continuous fields of smooth Jordan curves in the fibers of the tautological bundle  $\gamma_2^n$ .

There is a well-known Shnirel’man theorem [8] about the possibility to inscribe a square in a regular Jordan plane curve  $\gamma$ . The theorems that follow are analogs of Shnirel’man’s theorem for a field  $\gamma$  of smooth Jordan curves in the fibers of the tautological bundle  $\gamma_2^n$ .

The following conjecture seems quite plausible.

**Conjecture.** *Each continuous field of “smooth” Jordan curves in  $\gamma_2^n$  contains a curve with an inscribed regular  $2n$ -gon.*

For  $n = 2$ , this is Shnirel’man’s theorem. The author does not know whether this conjecture is true for  $n \geq 3$ .

**Theorem 3.** *If  $2n - 1$  is a prime, then each continuous field  $\gamma$  of circles that are  $C^2$ -smoothly embedded into the fibers of  $\gamma_2^n$  and have curvatures depending continuously on a fiber, contains a circle with an inscribed regular  $(2n - 1)$ -gon.*

*Proof.* It suffices to prove Theorem 3 for smooth fields of smooth curves in fibers, i.e., in the case where the union of the curves of the field is a smooth submanifold transversal to the fibers of the total space  $E_2(\mathbb{R}^n)$ . In the other cases, the theorem is obtained by passing to the limit and taking into account the fact that the limiting regular polygon cannot degenerate because the curvature of the curves of the field is uniformly continuous. In what follows, we assume that the field of curves  $\gamma$  in  $\gamma_2^n$  is smooth.

A field of curves in  $\gamma_2^n$  induces a field of curves in the bundle  $(\gamma_2^n)^+$ ; these curves are assumed to be oriented counterclockwise. We consider the set  $M_0$  of collections of points  $(A_1, \dots, A_{2n-1})$  arranged counterclockwise on the curves in the fibers of the bundle  $(\gamma_2^n)^+$  and such that

$$|A_1A_2| = |A_2A_3| = \dots = |A_{2n-2}A_{2n-1}| = |A_{2n-1}A_1|.$$

The set  $M_0$  is compact, because no two points  $A_i$  can become arbitrarily close; this follows from the uniform continuity of the curvature of the curves in the family.

For a typical field of curves  $\gamma$ , the set  $M_0$  is a compact  $(2n - 3)$ -dimensional smooth submanifold of the total space of the bundle  $(\gamma_2^n)^+ \oplus \dots \oplus (\gamma_2^n)^+$  ( $n$  terms).

The cyclic group  $\mathbb{Z}_{2n-1}$  acts on  $M$  by cyclic permutations of the points  $(A_1, \dots, A_{2n-1})$  and, on the space  $\mathbb{R}^{2n-1}$ , by cyclic permutations of the coordinates.

We consider the continuous mapping

$$F_0 : M \rightarrow \mathbb{R}^{2n-1}, \quad F_0(A_1, \dots, A_{2n-1}) = (\angle A_1, \angle A_2, \dots, \angle A_{2n-1}),$$

where  $\angle A_1$  is the angle at the vertex  $A_1$  of the oriented broken line  $A_1, \dots, A_{2n-1}$  (we mean the angle located to the left of this broken line). By definition, the mapping  $F$  is  $\mathbb{Z}_{2n-1}$ -equivariant.

Let  $\gamma_t$  be a smooth deformation of the field of curves  $\gamma = \gamma_0$  to the field of circles  $\gamma_1$  in the fibers of  $\gamma_2^n$ . The one-parameter family of mappings  $F_t : M_t \rightarrow \mathbb{R}^{2n-1}$  yields a  $\mathbb{Z}_{2n-1}$ -equivariant continuous mapping  $\bigcup_t M_t \rightarrow \mathbb{R}^{2n-1}$ . For a typical deformation  $\gamma_t$  of the field of curves, the set  $\bigcup_{0 \leq t \leq 1} M_t = N$  is a compact oriented  $(2n-2)$ -dimensional manifold with boundary  $\partial N = M_0 \cup M_1$ , where  $M_1 \cong V_2(\mathbb{R}^n)$ ; this property is assumed in the sequel.

Leaving the mapping  $F_t$  unchanged for small  $t$ , we perturb it preserving the action of  $\mathbb{Z}_{2n-1}$  so that the entire family of mappings  $F_t : N \rightarrow \mathbb{R}^{2n-1}$  be transversal to the line  $l : x_1 = x_2 = \dots = x_{2n-1}$  and the images of  $F_t$  for  $t$  close to 1 be disjoint with  $l$ .

If the image  $F_0(M_0)$  does not touch  $l$ , then the theorem is proved.

Let  $p : (\mathbb{R}^{2n-1} \setminus l) \rightarrow S^{2n-3}$  be a  $\mathbb{Z}_{2n-1}$ -equivariant retraction onto the sphere given by the equations  $x_1 + \dots + x_{2n-1} = 0$  and  $x_1^2 + \dots + x_{2n-1}^2 = 1$ ; such a restriction is obtained by projecting  $\mathbb{R}^{2n-1} \setminus l$  orthogonally onto the plane  $x_1 + \dots + x_{2n-1} = 0$  in accordance with the formula

$$\text{pr}(x_1, \dots, x_{2n-1}) = (x_1 - s, \dots, x_{2n-1} - s), \quad \text{where } s = (x_1 + \dots + x_{2n-1})/2n - 1,$$

with subsequent projection onto the sphere  $S^{2n-3}$  along the rays outgoing from the origin.

Since the mapping  $N \rightarrow \mathbb{R}^{2n-1}$  is transversal to the line  $l$ , its image consists of finitely many orbits of the action of  $\mathbb{Z}_{2n-1}$ . We surround the points of these orbits by a  $\mathbb{Z}_{2n-1}$ -invariant set of small disjoint balls  $B_i$ . The degree of the restriction of the mapping in question to the boundary of  $N \setminus \bigcup_i \text{int}(B_i)$  is zero. By  $\mathbb{Z}_{2n-1}$ -invariance, the degree of the restriction to the boundary of the balls belonging to one orbit is divisible by  $2n-1$ , which proves the claim.  $\square$

In [3] it was proved that the degree of the  $\mathbb{Z}_{2n-1}$ -equivariant mapping  $p \circ F_1 : V_2(\mathbb{R}^n) \rightarrow S^{2n-3}$  is not divisible by  $2n-1$ . Consequently,  $\deg(p \circ F : M_0 \rightarrow S^{2n-3})$  is also not divisible by  $2n-1$ . Thus, the latter mapping is surjective, so that the image of  $F$  contains a point whose first  $2n-2$  coordinates are equal; on  $S^{2n-3}$ , these points are

$$\left( \pm \frac{1}{\sqrt{(2n-2)(2n-1)}}, \dots, \pm \frac{1}{\sqrt{(2n-2)(2n-1)}}, \mp \frac{2n-2}{\sqrt{(2n-2)(2n-1)}} \right).$$

We have proved that an  $(2n-1)$ -gon  $A_1 \dots A_{2n-1}$  with equal sides and equal  $2n-2$  angles,  $\angle A_1 = \dots = \angle A_{2n-2}$ , is inscribed in some curve belonging to the field. It follows that the vertices  $A_i$  lie on the same circle circumscribed about  $\Delta A_1 A_2 A_3$ . Now, since the sides of  $A_1, \dots, A_{2n-1}$  are equal, we see that  $A_1, \dots, A_{2n-1}$  are the vertices of a regular  $(2n-1)$ -gon. Theorem 3 is proved.  $\square$

The following statement can be proved similarly.

**Theorem 4.** *For an odd prime  $n$  and an arbitrary continuous field of  $C^2$ -smoothly embedded circles in  $\gamma_2^n$  the curvatures of which smoothly depend on a fiber, we can find a circle with an inscribed regular (possibly, nonconvex)  $2n$ -gon  $A_1 \dots A_{2n}$  such that*

$$\angle A_1 = \angle A_3 = \dots = \angle A_{2n-1}, \quad \text{and} \quad \angle A_2 = \angle A_4 = \dots = \angle A_{2n-2}.$$

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DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, UNIVERSITETSKIĬ PROSPEKT 28, STARYĬ PETERHOF, ST. PETERSBURG 198904, RUSSIA

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