Алгебра и анализ
Том. 16 (2004), вып. 6

St. Petersburg Math. J.
Vol. 16 (2005), No. 6, Pages 1051–1077
S 1061-0022(05)00890-3
Article electronically published on November 22, 2005

GENERATION OF PAIRS OF SHORT ROOT SUBGROUPS IN CHEVALLEY GROUPS

V. NESTEROV

Abstract. On the basis of the Bruhat decomposition, the subgroups generated by pairs of unipotent short root subgroups in Chevalley groups of type B_ℓ, C_ℓ, and F_4 over an arbitrary field are described. Moreover, the orbits of a Chevalley group acting by conjugation on such pairs are classified.

§1. Introduction

Our purpose in this paper is to describe the subgroups of a Chevalley group G of type B_ℓ, C_ℓ, or F_4 that are generated by a pair of unipotent short root subgroups. In fact, we do somewhat more; namely, we classify the orbits of a Chevalley group acting by simultaneous conjugation on pairs of unipotent short root subgroups. Similar problems for a Chevalley group of type G_2 were considered in [N1, N2].

For unipotent long root subgroups such a description is well known. First it appeared in the paper [AS] by Aschbacher and Seitz. Later, it was reproved in [C2, V1]. It turned out that any pair (X_1, X_2) of long root subgroups is simultaneously conjugate to a pair of elementary long root subgroups (X_α, X_β). Essentially, the orbits of G on pairs of root subgroups are determined by the angle between α and β, so that, generically, there are five possible configurations, some of which may give two or three orbits (cf. Theorem A below).

This result played an important role in understanding the irreducible subgroups generated by long root subgroups. Geometry of long root subgroups in Chevalley groups is now a well-established field; see, e.g., McLaughlin [M1, M2], Stark [S1]–[S3], Pollatsek [Po], Wagner [W], Zalesskiĭ and Serezhkin [ZS], Kantor [K1], Cooperstein [C1]–[C4], Brown and Humphries [BH], Shang Zhi Li [L1]–[L3], Timmesfeld [T1]–[T5], Liebeck and Seitz [LS], and Cuypers [Cu].

The irreducible subgroups of the classical groups generated by short root subgroups were classified by Stark [S2] and Shang Zhi Li [L3]. Nevertheless, the geometry of short root subgroups is far from being properly understood. To the best of our knowledge, the orbits of G on pairs of short root subgroups have not been classified even for the classical cases.

In [V1]–[V3], Vavilov calculated the Bruhat decomposition of root unipotent elements. Using these results, he classified the orbits of a Chevalley group on pairs of a long and a short root subgroup (see [V2]). Generically, in this case there are six possible configurations (cf. Theorem B below). This is the starting point of our work.

In the present paper we take the next step, obtaining a similar list for pairs of two short root subgroups. However, our list is considerably longer; for odd characteristics

2000 *Mathematics Subject Classification.* Primary 20G15.

Key words and phrases. Chevalley group, unipotent short root subgroups, Bruhat decomposition.

This work was supported by the Ministry of Education of Russia (grants nos. PD02-1.1-371 and E02-1.0-61).

there are 21 possible configurations (not all of them occur for each type), and some of them split into infinitely many orbits (for an infinite field) parameterized by a continuous parameter. These results are summarized in Theorem 1. For characteristic 2, the short root subgroups behave exactly as the long ones (see Theorem 2).

One of the reasons why the answer is so much more complicated is that a short root subgroup (unlike a long one) does not lie in the center of the unipotent radical U of a Borel subgroup, so that conjugation by elements of U leads to more intricate configurations of roots. In fact, the mutual position of two long root subgroups depends only on two long roots, while that of a pair consisting of a long and a short root subgroups depends on two or three roots (see [V2]). But the mutual position of two short root subgroups depends on two, three, four, and in two cases on five roots.

The present paper is organized as follows. In §2 we recall the known results about pairs of long–long (Theorem A) and long–short (Theorem B) root subgroups and state our principal results about pairs of short–short root subgroups (Theorems 1 and 2). The rest of the paper is devoted to the proofs of these results. Namely, in §3 we use the Bruhat decomposition of short root subgroups to reduce description of orbits to a combinatorial problem about roots. In §4, for each type we draw a list of pairs (X_1, X_2) that contain representatives of all orbits (Tables 5–7). In §5 we identify the corresponding spans $X = \langle X_1, X_2\rangle$ and study long and short root subgroups contained in X. Finally, in §6 (for characteristic not 2) and §7 (for characteristic 2) we prove that the pairs corresponding to different configurations are not conjugate and determine nonconjugate pairs within the same configuration.

Further details related to these problems, as well as some applications and many additional references, can be found in the surveys [C3, K, K2, T4, V4, V5, Z] and in the fundamental papers by Cooperstein [C2], Kantor [K1], Liebeck and Seitz [LS] and Timmesfeld [T2, T3, T5]. As a sequel to this paper, we hope to obtain an analogue of one of the main results of [C2] and to classify some important classes of subgroups generated by triples of short root subgroups (cf. also [DV, V3]).

§2. Statement of results

All properties of Chevalley groups used in the sequel can be found in [C] or [St]. First, we fix the notation and recall some notions. Let Φ be a reduced irreducible root system, and let $G = G(\Phi, K)$ be a Chevalley group of type Φ over a field K. Usually, we may (and shall) assume that G is simply connected. For each root $\alpha \in \Phi$ and each element $t \in K$, we denote by $x_\alpha(t)$ the corresponding elementary unipotent root element. Next, $X_\alpha = \{x_\alpha(t) \mid x \in K\}$ is the elementary unipotent root subgroup corresponding to α. For $\alpha \in \Phi$ and $t \in K^*$, we set $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ and $h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}$.

One of the main relations in a Chevalley group (and the main tool in our calculations) is the Chevalley commutator formula

$$[x_\alpha(t), x_\beta(s)] = \prod x_{i\alpha+j\beta}(N_{\alpha\beta ij}t^i s^j),$$

where the product is taken over all roots $i\alpha + j\beta$ (here i and j are positive integers), and the elements $N_{\alpha\beta ij} \in K$ do not depend on t and s (see [C, Chapter 4]).

Here are the cases that we need in the sequel (see [C, Chapter 5]):

$$[x_\alpha(t), x_\beta(s)] = 1, \quad \alpha + \beta \notin \Phi;$$
$$[x_\alpha(t), x_\beta(s)] = x_{\alpha+\beta}(\pm ts), \quad |\alpha| = |\beta| \text{ and } \angle(\alpha, \beta) = 2\pi/3;$$
$$[x_\alpha(t), x_\beta(s)] = x_{\alpha+\beta}(\pm 2ts), \quad \alpha, \beta \text{ are short roots, } \angle(\alpha, \beta) = \pi/2,\ \alpha + \beta \in \Phi;$$
$$[x_\alpha(t), x_\beta(s)] = x_{\alpha+\beta}(\pm ts)x_{\alpha+2\beta}(\pm ts^2), \quad |\alpha| > |\beta| \text{ and } \angle(\alpha, \beta) = 3\pi/4.$$

The signs are defined individually in each case. We need to specify the signs only when carrying out calculations in a Chevalley group of type C_2. Specifically, in this case we fix the choice of structure constants so that the last two relations have the form

$$[x_\alpha(t), x_\beta(s)] = x_{\alpha+\beta}(-2ts),$$
$$[x_\alpha(t), x_\beta(s)] = x_{\alpha+\beta}(ts)x_{\alpha+2\beta}(ts^2),$$

where α is a long root and β is a short one.

Also we use the following relations (see [St, p. 30]):

$$w_\alpha(1)x_\beta(t)w_\alpha^{-1} = x_{w_\alpha\beta}(\pm t),$$
$$h_\alpha(s)x_\beta(t)h_\alpha(s)^{-1} = x_\beta(s^{\langle\beta,\alpha\rangle}t),$$

where $\langle\beta,\alpha\rangle = 2(\beta,\alpha)/(\alpha,\alpha)$ is the Cartan number.

Now we fix an ordering on Φ. Let $B = B(\Phi, K)$ be a Borel subgroup corresponding to this ordering, and let Φ^+, Φ^- be the corresponding sets of positive and negative roots. Then we introduce subgroups U, V, and H of G as follows:

$$U = U(\Phi, K) = \langle x_\alpha(t), \alpha \in \Phi^+, t \in K\rangle,$$

$$V = V(\Phi, K) = \langle x_\alpha(t), \alpha \in \Phi^-, t \in K\rangle,$$

$$H = H(\Phi, K) = \langle h_\alpha(t), \alpha \in \Phi, t \in K^*\rangle.$$

Here, for a subset X of G, $\langle X\rangle$ denotes the subgroup of G generated by X. For a root α we set $H_\alpha = \langle h_\alpha(t),\ t \in K^*\rangle$. The Weyl group of the system Φ is denoted by $W = W(\Phi)$.

In the root systems of type B_ℓ, C_ℓ, F_4, and G_2, there are two types of roots: long and short ones. We denote by Φ_l the subset of long roots and by Φ_s the subset of short roots. Also, there are two dominant roots in these systems. One of them is the maximal root δ and the other is the dominant short root ρ. A subgroup X is called a *unipotent long root subgroup* if it is conjugate to X_δ, and a *unipotent short root subgroup* if it is conjugate to X_ρ. The elements of X are called *unipotent long* or *short root elements*, respectively. In the sequel we always omit the word "unipotent" and say simply "long/short root subgroup/element".

In the usual realization of root systems (see [B]), we have:

$$\mathrm{B}_\ell = \{\pm e_i, \pm e_i \pm e_j\}, \quad \rho = e_1,\ \ \delta = e_1 + e_2;$$

$$\mathrm{C}_\ell = \{\pm 2e_i, \pm e_i \pm e_j\}, \quad \rho = e_1 + e_2,\ \ \delta = 2e_1;$$

$$\mathrm{F}_4 = \Big\{\pm e_i, \pm e_i \pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\Big\}, \quad \rho = e_1,\ \ \delta = e_1 + e_2.$$

Here $1 \le i \ne j \le \ell$ (with $\ell = 4$ for $\Phi = \mathrm{F}_4$). The $\pm$-signs are chosen independently of each other. In the sequel, when no confusion may arise, we write simply $\pm i$, $\pm i \pm j$, $\pm 1 \pm 2 \pm 3 \pm 4$ to denote the corresponding roots of B_ℓ, C_ℓ, or F_4.

The set of all roots strictly larger than ρ is denoted by Σ. Then for B_ℓ and F_4 we have $\Sigma = \{e_1 + e_2, \dots, e_1 + e_\ell\}$, where, as usual, $\ell = 4$ for $\Phi = \mathrm{F}_4$; for C_ℓ we have $\Sigma = \{2e_1\}$. The elements of Σ will be denoted by $\rho_2, \dots, \rho_m$, where $m = \ell, 4, 2$ for $\Phi = \mathrm{B}_\ell, \mathrm{F}_4, \mathrm{C}_\ell$, respectively. Namely, $\rho_i = e_1 + e_i$, $2 \le i \le \ell$, for $\Phi = \mathrm{B}_\ell$ and F_4, and $\rho_2 = 2e_1$ for $\Phi = \mathrm{C}_\ell$.

Let $\Delta \subset \Phi$ be a subset of roots. We say that Δ generates a root subsystem of type Φ_1 if the intersection of the $\mathbb{Z}$-span of Δ with the root system Φ is a root subsystem of type Φ_1.

If there are two root lengths in Φ and only one root length in Φ_1, then Φ_1 may be embedded in Φ in two essentially different ways: at the long roots and at the short roots. Following Dynkin [D, p. 148], to distinguish these cases we write $\widetilde{\Phi}_1 \subset \Phi$ for the short root embedding and $\Phi_1 \subset \Phi$ for the long root embedding.

Let $\Lambda(G)$ and $\Omega(G)$ be the sets of short and long root subgroups, respectively. For a subgroup $X \leq G(\Phi, K)$ of G, we denote by

$$\Lambda(X) = \{Y \leq X \mid Y \in \Lambda(G)\},$$

$$\Omega(X) = \{Y \leq X \mid Y \in \Omega(G)\}$$

the sets of short and long root subgroups contained in X. As usual, $|\Lambda(X)|$ and $|\Omega(X)|$ are the cardinalities of these sets. We write $K = \mathbb{F}_q$ for the finite field of q elements. Finally, $X \sim Y$ means that the subgroups X and Y are conjugate in G, whereas $X_1 \times X_2$, $X_1 \leftthreetimes X_2$, and $[X_1, X_2]$ denote the direct product, the semidirect product, and the mutual commutator subgroup of the groups X_1 and X_2, respectively.

Before stating our main results, Theorems 1 and 2, we reproduce the known results about pairs of long root subgroups (see [AS, K1, C2] or [V1]) and pairs of a long and a short root subgroup (essentially the latter are contained in [V2], but we state them in a slightly more precise form). These results will be used in the statement of Theorem 1. It is also quite instructive to compare them with Theorems 1 and 2. For brevity, we write Λ, Ω instead of $\Lambda(\langle X_1, X_2\rangle)$, $\Omega(\langle X_1, X_2\rangle)$.

Theorem A ([AS, 12.1], [K1, 3.1], [C2, 2.2], [V1, Theorem 2])**.** *Let X_1, X_2 be two distinct long root subgroups in a universal Chevalley group $G = G(\Phi, K)$. Then*:

1) *X_1 and X_2 generate one of the groups* (L1)–(L5) *listed below*;
2) *pairs (X_1, X_2) of different cases are not conjugate in G, and the number of orbits under conjugation action of G on pairs of the same case is equal to d (we omit $d = 1$).*
   - (L1) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Omega| = q + 1$ *or* ∞, $|\Lambda| = 0$, $d = 2$ *for* A_ℓ, $\Phi \neq \mathrm{C}_\ell$;
   - (L2) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Omega| = 2$, $|\Lambda| = 0$, $d = 3$ *for* D_ℓ, $\Phi \neq \mathrm{C}_\ell$, F_4, G_2;
   - (L3) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Omega| = 2$, $|\Lambda| = q - 1$ *or* ∞, $d = 2$ *for* F_4, $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4;
   - (L4) $\langle X_1, X_2\rangle = X_1X_2Z \simeq U(\mathrm{A}_2, K)$, $Z = [X_1, X_2] \in \Omega$, (X_i, Z) *as in* (L1), $i = 1, 2$, $\Phi \neq \mathrm{C}_\ell$;
   - (L5) $\langle X_1, X_2\rangle = \mathrm{SL}_2(K)$.

Remarks. If (X_1, X_2) and (Y_1, Y_2) are as in (L1), and $\Phi = \mathrm{A}_\ell$, then there exists $g \in G$ such that $gX_1g^{-1} = Y_1$ or Y_2, $gX_2g^{-1} = Y_2$ or Y_1, respectively.

Two orbits for D_ℓ in case (L2) are fused by an outer automorphism.

Theorem B ([V2, Theorem 2])**.** *Let Φ be a root system of type B_ℓ, C_ℓ, or F_4, and let X be a long and Y a short root subgroup in a universal Chevalley group $G(\Phi, K)$. Suppose* $\operatorname{char} K \neq 2$. *Then*:

1) *X and Y generate one of the groups* (LS1)–(LS6) *listed below*;
2) *pairs (X, Y) are conjugate in G if and only if they are of the same case.*
   - (LS1) $\langle X, Y\rangle = X \times Y$, $|\Lambda| = q$ *or* ∞, $|\Omega| = 1$;
   - (LS2) $\langle X, Y\rangle = X \times Y$, $|\Lambda| = 1$, $|\Omega| = 1$;
   - (LS3) $\langle X, Y\rangle = X \times Z$, $Z \in \Omega$, $|\Lambda| = q - 1$ *or* ∞, $|\Omega| = 2$;
   - (LS4) $\langle X, Y\rangle = XYZ$, $Z = [X, Y] \in \Omega$, $|\Lambda(\langle X, Z\rangle)| = 0$, $|\Omega(\langle X, Z\rangle)| = q + 1$ *or* ∞, (Y, Z) *as in* (LS1), $\Phi \neq \mathrm{C}_\ell$;
   - (LS5) $\langle X, Y\rangle = U(\mathrm{C}_2, K)$;
   - (LS6) $\langle X, Y\rangle = \mathrm{SL}_2(K) \times Z$, $Z \in \Omega$;

If $\operatorname{char} K = 2$*, then only* (LS1), (LS2), *and* (LS6) *may occur.*

Our goal in this paper is to obtain analogs of Theorems A and B for pairs of short roots subgroups. Namely, we prove the following results.

Theorem 1. *Let* X_1*,* X_2 *be two distinct short root subgroups in a universal Chevalley group* $G(\Phi, K)$ *of type* B_ℓ, C_ℓ, *or* F_4*. Suppose* $\operatorname{char} K \neq 2$*. Then*:

1) X_1 *and* X_2 *generate one of the groups* (S1)–(S21) *listed below*;
2) *pairs* (X_1, X_2) *of different cases are not conjugate in* G*, and the number of orbits under conjugation action of* G *on pairs of the same case is equal to* d (*we omit* $d = 1$).
   - (S1) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = q + 1$ *or* ∞, $|\Omega| = 0$, $\Phi = \mathrm{C}_\ell$, F_4;
   - (S2) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = q + 1$ *or* ∞, $|\Omega| = 0$, $d = |\{\pm c \in K \mid c^2 - 4 \notin K^{*2}\}|$, $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4;
   - (S3) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = q - 1$ *or* ∞, $|\Omega| = 2$, $d = |\{\pm c \in K \mid c^2 - 4 \in K^{*2}\}|$, $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4;
   - (S4) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = q$ *or* ∞, $|\Omega| = 1$, $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4;
   - (S5) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = 3$, $|\Omega| = 0$, $\Phi = \mathrm{C}_\ell$, F_4;
   - (S6) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = 2$, $|\Omega| = 0$, $d = 1$ *for* $\Phi = \mathrm{B}_\ell$ *and* 2 *for* $\Phi = \mathrm{C}_\ell$, F_4;
   - (S7) $\langle X_1, X_2\rangle = X_1X_2Z$, $Z = [X_1, X_2] \in \Omega$, $|\Lambda| = q^2 + q$ *or* ∞, $|\Omega| = 1$, (X_i, Z) *as in* (LS1), $i = 1, 2$, $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4;
   - (S8) $\langle X_1, X_2\rangle = X_1X_2Z$, $Z = [X_1, X_2] \in \Omega$, $|\Lambda| = 2q$ *or* ∞, $|\Omega| = 1$, $d = (q-1)/2$ *or* ∞, (X_i, Z) *as in* (LS1), $i = 1, 2$, $\Phi = \mathrm{B}_\ell$, F_4;
   - (S9) $\langle X_1, X_2\rangle = X_1X_2Z$, $Z = [X_1, X_2] \in \Omega$, $|\Lambda| = 2q$ *or* ∞, $|\Omega| = 1$, $d = 2$, (X_i, Z) *as in* (LS1), $i = 1, 2$, $\Phi = \mathrm{F}_4$;
   - (S10) $\langle X_1, X_2\rangle = X_1X_2Y \simeq U(\widetilde{\mathrm{A}}_2, K)$, $Y = [X_1, X_2] \in \Lambda$, (X_i, Y) *as in* (S1), $i = 1, 2$, $\Phi = \mathrm{C}_\ell$, F_4;
   - (S11) $\langle X_1, X_2\rangle = Y_1Y_2Z_1Z_2 \simeq U(\mathrm{C}_2, K)$, Y_1, $Y_2 \in \Lambda$, Z_1, $Z_2 \in \Omega$, (Y_1, Y_2) *as in* (S1), (Z_1, Z_2) *as in* (L3), (Y_1, Z_1), (Y_2, Z_i) *as in* (LS1), $i = 1, 2$, $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4;
   - (S12) $\langle X_1, X_2\rangle = X_1X_2YZ \simeq U(\mathrm{C}_2, K)$, $Y \in \Lambda$, $Z \in \Omega$, (X_1, Y) *as in* (S4), (X_2, Y) *as in* (S5), (X_2, Z), (Y, Z) *as in* (LS1), (X_1, Z) *as in* (LS2), $\Phi = \mathrm{C}_\ell$, F_4;
   - (S13) $\langle X_1, X_2\rangle = X_1X_2YZ$, $Y \in \Lambda$, $Z \in \Omega$, (X_2, Z), (Y, Z) *as in* (LS1), (X_1, Z) *as in* (LS3), (X_1, Y) *as in* (S2), (X_2, Y) *as in* (S5), $\Phi = \mathrm{B}_\ell$, F_4;
   - (S14) $\langle X_1, X_2\rangle = U(\mathrm{C}_2, K) \times Z$, $Z \in \Omega$, $\Phi = \mathrm{F}_4$;
   - (S15) $\langle X_1, X_2\rangle = \mathrm{SL}_2(K)$, $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4;
   - (S16) $\langle X_1, X_2\rangle \simeq \mathrm{SL}_2(K) \times Z$, $Z \in \Omega$ *if* $K \neq \mathbb{F}_3$, *and* $\mathrm{SL}_2(K)$ *if* $K = \mathbb{F}_3$, $d = 1$ *for* $\Phi = \mathrm{B}_\ell$ $(\ell \geq 3)$, F_4, *and* d *is equal to the index* $\nu = [K^* : K^{*2} \cup -K^{*2}]$ *for* $\Phi = \mathrm{C}_\ell$;
   - (S17) $\langle X_1, X_2\rangle \simeq \mathrm{SL}_2(K) \times Z_1 \times Z_2$, $Z_1, Z_2 \in \Omega$, (Z_1, Z_2) *as in* (L3) *if* $K \neq \mathbb{F}_3$, *and* $\mathrm{SL}_2(K) \times Z$ *if* $K = \mathbb{F}_3$, Z *is a cyclic group of order* 3, $\Phi = \mathrm{C}_\ell$, F_4;
   - (S18) $\langle X_1, X_2\rangle \simeq \mathrm{SL}_2(K) \leftthreetimes (Z_1 \times Z_2)$, $Z_1, Z_2 \in \Omega$, (Z_1, Z_2) *as in* (L3), $\Phi = \mathrm{F}_4$;
   - (S19) $\langle X_1, X_2\rangle \simeq U(\mathrm{C}_2, K)H_\rho w_\rho X_\rho$, $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4;
   - (S20) $\langle X_1, X_2\rangle \simeq \mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$, $d = |\{\pm c \in K \mid c^2 - 4 \in K^{*2}\}|$, $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4;
   - (S21) $\langle X_1, X_2\rangle \simeq \mathrm{SL}_2(L) \times \mathrm{SL}_2(L)$, L *is a quadratic extension of* K, $d = |\{\pm c \in K \mid c^2 - 4 \notin K^{*2}\}|$, $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4.

When $\operatorname{char} K = 2$, the short root subgroups behave exactly as the long root subgroups. In this case we have the following result.

Theorem 2. *Under the same assumptions as in Theorem* 1, *suppose* $\operatorname{char} K = 2$. *Then the following cases are possible*:

(T1) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = q+1$ *or* ∞, $|\Omega| = 0$, $\Phi = \mathrm{C}_\ell$, F_4;
(T2) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = 2$, $|\Omega| = 0$, $d = 1$ *for* $\Phi = \mathrm{B}_\ell$, F_4 *and* 2 *for* C_ℓ $(\ell \geq 4)$;
(T3) $\langle X_1, X_2\rangle = U(\widetilde{\mathrm{A}}_2, K)$, $\Phi = \mathrm{C}_\ell$, F_4;
(T4) $\langle X_1, X_2\rangle = \mathrm{SL}_2(K)$, $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4.

Remarks. The index ν of case (S16) is equal to 1 if K is really or quadratically closed, or if $K = \mathbb{F}_q$ where $q \equiv 3 \pmod 4$, and ν is equal to 2 if $K = \mathbb{F}_q$ where $q \equiv 1 \pmod 4$.

In cases (S2), (S3), (S8), (S20), and (S21), each orbit corresponds to an element of the field up to the sign.

For an algebraically closed field K, the pairs of cases (S1)–(S3), (S8), (S20), and (S21) are conjugate in the appropriate $\mathrm{GL}(n, K)$.

No automorphism identifies two orbits of cases (S6) and (S9).

For a finite field $\mathbb{F}_q$ with $q \neq 2^n$, there exist $6 + (3q+1)/2$ orbits for a root system of type B_ℓ, $10 + \nu + q$ for C_ℓ, and $16 + (3q+1)/2$ for F_4.

The length of the list in Theorem 1 suggests that it is much harder to work with short root subgroups than with long ones. The presence of continuous parameters and arithmetic conditions in the description of orbits shows that, unlike the geometry of long root subgroups, the geometry of short root subgroups depends essentially on the arithmetic nature of the ground field. This might explain why they have not attracted as much attention as the long root subgroups.

The rest of the paper is devoted to the proof of Theorems 1 and 2.

§3. Possible configurations of roots

The starting point of our work is the calculation of the Bruhat decomposition of a short root element (see [V2, Theorem 1]).

Lemma 1. *Let* $\Phi = \mathrm{B}_\ell$, C_ℓ, *or* F_4, *and let* $gX_\rho g^{-1}$ *be a short root subgroup. Then one of the following statements is true*:

a) *there exists* $u \in U$, *a short root* γ, *and* $t \in K^*$ *such that*

$$gx_\rho(\epsilon)g^{-1} = ux_\gamma(t\epsilon)u^{-1} \quad \text{for } \epsilon \in K;$$

b) $\operatorname{char} K \neq 2$, *and there exists* $u \in U$, *orthogonal long roots* α *and* β *for which* $(\alpha - \beta)/2$ *is also a root, and* r, $s \in K^*$ *with* $s/r \in -K^{*2}$ *such that*

$$gx_\rho(\epsilon)g^{-1} = ux_\alpha(s\epsilon)x_\beta(r\epsilon)u^{-1} \quad \text{for } \epsilon \in K.$$

This result implies the following key lemma.

Lemma 2. *Let* X_1, X_2 *be short root subgroups in* $G(\Phi, K)$, *where* $\Phi = \mathrm{B}_\ell$, C_ℓ, *or* F_4 *and* $\operatorname{char} K \neq 2$. *Then there exists* $g \in G(\Phi, K)$ *such that*

$$gX_1g^{-1} = \langle x_\rho(t)x_{\rho_2}(c_2t)\cdots x_{\rho_m}(c_mt),\ t \in K\rangle,$$

and one of the following statements is true:

$$\text{a)}\quad gX_2g^{-1} = X_\gamma;$$
$$\text{b)}\quad gX_2g^{-1} = X_{\alpha,\beta} = \langle x_\alpha(s)x_\beta(-s),\ s \in K\rangle,$$

where γ, α, β *have the same meaning as in Lemma* 1; $c_2, \dots, c_m \in K$ *depend only on* X_1 *and* X_2; *and* $m = \ell, 2, 4$ *for* $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4, *respectively.*

If $\operatorname{char} K = 2$, *we have* $gX_1g^{-1} = X_\rho$, $gX_2g^{-1} = X_\gamma$.

Proof. There exists $z \in G$ such that $zX_1z^{-1} = X_\rho$. By Lemma 1, we can pick $u \in U$ such that $u^{-1}zX_2z^{-1}u$ coincides with X_γ or with $\{x_\alpha(st)x_\beta(rt) \mid t \in K\}$. In the first case we set $g = u^{-1}z$; in the second we set $g = h_\beta(k)u^{-1}z$, where $-k^2 = s/r$. Then gX_2g^{-1} will have the required form and $gX_1g^{-1} = u^{-1}X_\rho u$. Since $u \in U$ is a product of $x_\tau(t)$, $\tau \in \Phi^+$, the conjugate $u^{-1}x_\rho(r)u$ equals $x_\rho(r) \cdot \prod x_\sigma(*)$ with $\sigma \in \Sigma$.

This means that the elements of gX_1g^{-1} have the following form:

$$x_\rho(t)x_{\rho_2}(c_2t)\cdots x_{\rho_m}(c_mt),$$

where $m = \ell, 2, 4$ for $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4, respectively. Now from the Chevalley commutator relations it follows that all c_i are divisible by 2, and so if $\operatorname{char} K = 2$, then they all are equal to zero. This proves the lemma. □

Note that since x_ρ and x_{ρ_i} commute, there is no need to worry about their ordering.

Let R_J denote the set of $\rho_i \in \Sigma$ that appear in the decomposition of elements of gX_1g^{-1} with nonzero coefficients c_i, and let J be the set of the corresponding indices i. We put

$$X_{\rho,J} = \Big\langle x_\rho(t)\prod_i x_{\rho_i}(c_it),\ i \in J,\ c_i \neq 0,\ t \in K\Big\rangle.$$

If $J = \varnothing$, then $X_{\rho,J} = X_\rho$, and if $J = \{i\}$, we write $X_{\rho,c}$, where $c = c_i$.

Recall that the angle between short roots can take the values 0, $\pi/3$, $\pi/2$, $2\pi/3$, π for root systems of type C_ℓ and F_4 and the values 0, $\pi/2$, π for root systems of type B_ℓ. The angle between a short and a long root can be $\pi/4$, $\pi/2$, or $3\pi/4$ (see [B, p. 148]).

Case 1. $gX_1g^{-1} = X_{\rho,J}$, $gX_2g^{-1} = X_\gamma$. First, suppose that $R_J = \varnothing$, i.e., $gX_1g^{-1} = X_\rho$. This is the only situation that can occur for the field of characteristic 2.

Denote by θ the angle between ρ and γ. The possibilities for γ are listed in Table 1. In the following tables we always assume that the indices i, j, k are pairwise distinct within the same variant and are strictly greater than 1 in the cases of B_ℓ, F_4 or strictly greater than 2 in the case of C_ℓ.

Table 1

	(θ)	γ	Φ
1	0	e_1	$\mathrm{B}_\ell, \mathrm{F}_4$
		$e_1 + e_2$	C_ℓ
2	$\pi/3$	$\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$	F_4
		$e_{1,2} \pm e_j$	C_ℓ
3	$\pi/2$	$\pm e_i$	$\mathrm{B}_\ell, \mathrm{F}_4$
		$\pm(e_1 - e_2)$	C_ℓ
4	$\pi/2$	$\pm e_j \pm e_k$	C_ℓ
5	$2\pi/3$	$\frac{1}{2}(-e_1 \pm e_2 \pm e_3 \pm e_4)$	F_4
		$-e_{1,2} \pm e_j$	C_ℓ
6	π	$-e_1$	$\mathrm{B}_\ell, \mathrm{F}_4$
		$-e_1 - e_2$	C_ℓ

Observe that the angle $\pi/2$ corresponds to two variants for C_ℓ: when ρ and γ generate a subsystem of type C_2 or $2\widetilde{\mathrm{A}}_1$.

Now suppose $R_J \neq \varnothing$. We introduce the notation (θ, ψ), where $\theta = \angle(\rho, \gamma)$ is the angle between ρ and γ, and $\psi = \max\{\angle(\rho_i, \gamma)\}$, where i runs over J. Since $\angle(\rho, \rho_i) = \pi/4$ for any $\rho_i \in \Sigma$, the configurations $(0, \pi/2)$, $(0, 3\pi/4)$, $(\pi/3, 3\pi/4)$, $(2\pi/3, \pi/4)$, $(\pi, \pi/4)$, $(\pi, \pi/2)$ are impossible and we have nine variants for C_ℓ, F_4 and five variants for B_ℓ. For each pair (θ, ψ) we fix an index i such that $\psi = \angle(\rho_i, \gamma)$. The corresponding roots γ are listed in Table 2.

Table 2

	(θ, ψ)	γ	Φ
7	$(0, \pi/4)$	e_1	$\mathrm{B}_\ell, \mathrm{F}_4$
		$e_1 + e_2$	C_ℓ
8	$(\pi/3, \pi/4)$	$\frac{1}{2}(e_1 + e_i \pm e_j \pm e_k)$	F_4
		$e_1 \pm e_j$	C_ℓ
9	$(\pi/3, \pi/2)$	$\frac{1}{2}(e_1 - e_i \pm e_j \pm e_k)$	F_4
		$e_2 \pm e_j$	C_ℓ
10	$(\pi/2, \pi/4)$	e_i	$\mathrm{B}_\ell, \mathrm{F}_4$
		$e_1 - e_2$	C_ℓ
11	$(\pi/2, \pi/2)$	$\pm e_j$	$\mathrm{B}_\ell, \mathrm{F}_4$
		$\pm e_j \pm e_k$	C_ℓ
12	$(\pi/2, 3\pi/4)$	$-e_i$	$\mathrm{B}_\ell, \mathrm{F}_4$
		$-e_1 + e_2$	C_ℓ
13	$(2\pi/3, \pi/2)$	$\frac{1}{2}(-e_1 + e_i \pm e_j \pm e_k)$	F_4
		$-e_2 \pm e_j$	C_ℓ
14	$(2\pi/3, 3\pi/4)$	$\frac{1}{2}(-e_1 - e_i \pm e_j \pm e_k)$	F_4
		$-e_1 \pm e_j$	C_ℓ
15	$(\pi, 3\pi/4)$	$-e_1$	$\mathrm{B}_\ell, \mathrm{F}_4$
		$-e_1 - e_2$	C_ℓ

Case 2. $gX_1g^{-1} = X_{\rho,J}$, $gX_2g^{-1} = X_{\alpha,\beta}$. The restrictions imposed upon the roots α, β imply that (α, β) is one of the following pairs: $(e_i + e_j, e_i - e_j)$ or $(-e_i + e_j, -e_i - e_j)$ with $i \neq j$ for B_ℓ; $(\pm 2e_i, \pm 2e_j)$ with $i \neq j$ for C_ℓ; and $(e_i + e_j, e_i - e_j)$, $(-e_i + e_j, -e_i - e_j)$ or $(\pm e_1 \pm e_i, \pm e_j \pm e_k)$ with $i \neq j \neq k \neq i$ for F_4. We set

$$(\theta_1, \theta_2) = (\angle(\rho, \alpha), \angle(\rho, \beta)),$$
$$(\psi_1, \psi_2) = (\max\{\angle(\rho_i, \alpha)\}, \max\{\angle(\rho_i, \beta)\},\ i \in J) \quad \text{if } J \neq \varnothing.$$

Interchanging the roots α and β, we may assume that $\theta_1 \leq \theta_2$ and $\psi_1 \leq \psi_2$.

Suppose that $R_J = \varnothing$. Then the possible configurations of roots are determined by the angles (θ_1, θ_2). In Tables 3 and 4 it is assumed that $i \neq j \neq k \neq i$ and j, $k > 1$. The $\pm$-signs are chosen independently.

TABLE 3

	(θ_1, θ_2)	(α, β)	Φ
16	$(\pi/4, \pi/4)$	$(e_1 + e_j, e_1 - e_j)$ up to ordering	$\mathrm{B}_\ell, \mathrm{F}_4$
		$(2e_1, 2e_2)$ up to ordering	C_ℓ
17	$(\pi/4, \pi/2)$	$(e_1 \pm e_i, \pm e_j \pm e_k)$	F_4
		$(2e_1, \pm 2e_j)$ or $(2e_2, \pm 2e_j)$	C_ℓ
18	$(\pi/2, \pi/2)$	$(\pm(e_j + e_k), \pm(e_j - e_k))$ up to ordering	$\mathrm{B}_\ell, \mathrm{F}_4$
		$(\pm 2e_j, \pm 2e_k)$	C_ℓ
19	$(\pi/4, 3\pi/4)$	$(e_1 + e_j, -e_1 - e_j)$	$\mathrm{B}_\ell, \mathrm{F}_4$
		$(2e_1, -2e_2)$ or $(2e_2, -2e_1)$	C_ℓ
20	$(\pi/2, 3\pi/4)$	$(\pm e_j \pm e_k, -e_1 \pm e_i)$	F_4
		$(\pm 2e_j, -2e_2)$ or $(\pm 2e_j, -2e_1)$	C_ℓ
21	$(3\pi/4, 3\pi/4)$	$(-e_1 + e_j, -e_1 - e_j)$ up to ordering	$\mathrm{B}_\ell, \mathrm{F}_4$
		$(-2e_2, -2e_1)$ up to ordering	C_ℓ

For the case where $R_J \neq \varnothing$, we get 27 possible configurations. To describe these configurations, we must use both (θ_1, θ_2) and (ψ_1, ψ_2). In Table 4 the angles ψ_1, ψ_2 are usually formed by one and the same root ρ_i with α, β. However, in variants 24, 28, 31, 35, 45, and 48 the angle ψ_1 is formed by ρ_j and α, where $j \in J$, $j \neq i$. To stress this we add 'j' to the corresponding numbers.

TABLE 4

	(θ_1, θ_2), (ψ_1, ψ_2)	(α, β)	Φ
22	$(\pi/4, \pi/4), (\pi/3, \pi/3)$	$(e_1 + e_j, e_1 - e_j)$ up to ordering	$\mathrm{B}_\ell, \mathrm{F}_4$
23	$(\pi/4, \pi/4), (0, \pi/2)$	$(e_1 + e_i, e_1 - e_i)$	$\mathrm{B}_\ell, \mathrm{F}_4$
		$(2e_1, 2e_2)$	C_ℓ
24j	$(\pi/4, \pi/4), (\pi/3, \pi/2)$	$(e_1 + e_i, e_1 - e_i)$	$\mathrm{B}_\ell, \mathrm{F}_4$
25	$(\pi/4, \pi/2), (\pi/3, \pi/3)$	$(e_1 \pm e_j, e_i \pm e_k)$	F_4
26	$(\pi/4, \pi/2), (0, \pi/2)$	$(e_1 + e_i, \pm e_j \pm e_k)$	F_4
		$(2e_1, \pm 2e_j)$	C_ℓ
27	$(\pi/4, \pi/2), (\pi/3, \pi/3)$	$(e_1 \pm e_j, e_i \pm e_k)$	F_4
28j	$(\pi/4, \pi/2), (\pi/3, \pi/2)$	$(e_1 + e_i, e_j \pm e_k)$	F_4
29	$(\pi/4, \pi/2), (\pi/2, \pi/2)$	$(e_1 - e_i, \pm e_j \pm e_k)$	F_4
		$(2e_2, \pm 2e_j)$	C_ℓ
30	$(\pi/4, \pi/2), (\pi/3, 2\pi/3)$	$(e_1 \pm e_j, -e_i \pm e_k)$	F_4

31j	$(\pi/4,\pi/2),(\pi/2,2\pi/3)$	$(e_1-e_j,-e_i\pm e_k)$	F_4
32	$(\pi/2,\pi/2),(\pi/3,\pi/3)$	(e_i+e_k,e_i-e_k) up to ordering	$\mathrm{B}_\ell,\mathrm{F}_4$
33	$(\pi/2,\pi/2),(\pi/2,\pi/2)$	$(\pm(e_j+e_k),\pm(e_j-e_k))$ up to ordering	$\mathrm{B}_\ell,\mathrm{F}_4$
		$(\pm 2e_j,\pm 2e_k)$	C_ℓ
34	$(\pi/2,\pi/2),(\pi/3,2\pi/3)$	$(e_i+e_k,-e_i+e_k)$ or $(e_i-e_k,-e_i-e_k)$	$\mathrm{B}_\ell,\mathrm{F}_4$
35j	$(\pi/2,\pi/2),(\pi/2,2\pi/3)$	$(e_i+e_k,-e_i+e_k)$ or $(e_i-e_k,-e_i-e_k)$	$\mathrm{B}_\ell,\mathrm{F}_4$
36	$(\pi/2,\pi/2),(2\pi/3,2\pi/3)$	$(-e_i+e_j,-e_i-e_j)$ up to ordering	$\mathrm{B}_\ell,\mathrm{F}_4$
37	$(\pi/4,3\pi/4),(0,\pi/2)$	$(e_1+e_i,-e_1+e_i)$	$\mathrm{B}_\ell,\mathrm{F}_4$
		$(2e_1,-2e_2)$	C_ℓ
38	$(\pi/4,3\pi/4),(\pi/3,2\pi/3)$	$(e_1+e_j,-e_1+e_j)$ or $(e_1-e_j,-e_1-e_j)$	$\mathrm{B}_\ell,\mathrm{F}_4$
39	$(\pi/4,3\pi/4),(\pi/2,\pi)$	$(e_1-e_i,-e_1-e_i)$	$\mathrm{B}_\ell,\mathrm{F}_4$
		$(2e_2,-2e_1)$	C_ℓ
40	$(\pi/2,3\pi/4),(\pi/2,\pi/2)$	$(\pm e_j\pm e_k,-e_1+e_i)$	F_4
		$(\pm 2e_j,-2e_2)$	C_ℓ
41	$(\pi/2,3\pi/4),(\pi/3,2\pi/3)$	$(e_i\pm e_j,-e_1\pm e_k)$	F_4
42j	$(\pi/2,3\pi/4),(\pi/2,2\pi/3)$	$(e_i\pm e_k,-e_1+e_j)$	F_4
43	$(\pi/2,3\pi/4),(2\pi/3,2\pi/3)$	$(-e_i\pm e_j,-e_1\pm e_k)$	F_4
44	$(\pi/2,3\pi/4),(\pi/2,\pi)$	$(\pm e_j\pm e_k,-e_1-e_i)$	F_4
		$(\pm 2e_j,-2e_1)$	C_ℓ
45j	$(\pi/2,3\pi/4),(2\pi/3,\pi)$	$(-e_j\pm e_k,-e_1-e_i)$	F_4
46	$(3\pi/4,3\pi/4),(2\pi/3,2\pi/3)$	$(-e_1+e_j,-e_1-e_j)$ up to ordering	$\mathrm{B}_\ell,\mathrm{F}_4$
47	$(3\pi/4,3\pi/4),(\pi/2,\pi)$	$(-e_1+e_i,-e_1-e_i)$	$\mathrm{B}_\ell,\mathrm{F}_4$
		$(-2e_2,-2e_1)$	C_ℓ
48j	$(3\pi/4,3\pi/4),(2\pi/3,\pi)$	$(-e_1+e_i,-e_1-e_i)$	$\mathrm{B}_\ell,\mathrm{F}_4$

§4. Representatives of orbits

In §§4–6 we finish the proof of Theorem 1. In particular, we always assume that the characteristic of the ground field is $\neq 2$. When $\operatorname{char} K = 2$, only very few of the configurations listed in Tables 1–4 may occur; see §7.

From what was said in §3, it follows that each pair (X_1, X_2) in question can be reduced to one of the forms $(X_{\rho,J}, X_\gamma)$ or $(X_{\rho,J}, X_{\alpha,\beta})$ and thus corresponds to one of the variants listed in Tables 1–4. From now on, by '*Variant N*' we mean a pair of subgroups that has the same configuration of roots as that in Variant N of Tables 1–4.

Lemma 3. *Let X_1, X_2 be short root subgroups of $G(\Phi, K)$, where $\Phi = \mathrm{B}_\ell$, C_ℓ, or F_4. Then there exists $f \in G$ such that $fX_2f^{-1} = X_\gamma$ or $X_{\alpha,\beta}$, and one of the following*

statements is true:

1) $fX_1f^{-1} = X_\rho$;
2) $fX_1f^{-1} = X_{\rho,c} = \{x_\rho(t)x_{\rho_i}(c_it),\ t \in K\}$;
3) $fX_1f^{-1} = \{x_\rho(t)x_{\rho_i}(c_it)x_{\rho_j}(c_jt),\ t \in K\}$

for some $\rho_i, \rho_j \in R_J$, $c_i, c_j \in K^*$, $i \neq j$. *Case* 3) *occurs only in Variants* 31 *and* 45.

Proof. By Lemma 2, there exists $g \in G$ such that $gX_1g^{-1} = X_{\rho,J}$ and $gX_2g^{-1} = X_\gamma$ or $X_{\alpha,\beta}$. If $J = \varnothing$, we take $f = g$. This happens in Variants 1–6, 16–21 described in Tables 1 and 3. Suppose $J \neq \varnothing$. We try to find f of the form $\prod x_\tau(k_\tau)g$, where $\tau \in \Phi$ and $k_\tau \in K^*$, such that $\tau + \gamma$ or $\tau + \alpha$ and $\tau + \beta$ are not in Φ. So,

$$x_\tau(k_\tau)X_\gamma x_\tau(k_\tau)^{-1} = X_\gamma \quad \text{and} \quad x_\tau(k_\tau)X_{\alpha,\beta}x_\tau(k_\tau)^{-1} = X_{\alpha,\beta}.$$

Thus, we can eliminate some of the $x_{\rho_i}(c_it)$ from $X_{\rho,J}$ by varying k_τ.

In Variants 8, 10, 13, 22, 25–28, 32, 33, 37, 38, 40, 41, 42, 46 and in Variant 11 (for C_ℓ only), we set $\tau = \rho_i - \rho \in \Phi_s$, where ρ_i runs over R_J and $k_\tau = \mp c_i/2$. Since $\tau + \rho = \rho_i$, $\tau + \rho_i \notin \Phi$, we have

$$x_{\rho_i-\rho}(\mp c_i/2)x_\rho(t)x_{\rho_i}(c_it)x_{\rho_i-\rho}(\pm c_i/2) = x_\rho(t).$$

Since for $\Phi = \mathrm{C}_\ell$ the set Σ consists of one root, this finishes the proof in this case.

Let $\Phi = \mathrm{B}_\ell$ or F_4. If R_J consists of one root, we are done. Thus, for the remaining variants we may assume that R_J contains at least two elements. Fix a root ρ_i in R_J that forms the maximal angle with γ or β. Then in Variants 7, 9, 11 (for B_ℓ, F_4), 12, 14, 15, 29, 30, 36, 39, 43, 44 we set $\tau = \rho_j - \rho_i \in \Phi_l$, $j \in J$, $j \neq i$, and $k_\tau = \mp c_j/c_i$, respectively. We have

$$x_{-i+j}(\mp c_j/c_i)x_1(t)x_{1+i}(c_it)x_{1+j}(c_jt)x_{-i+j}(\pm c_j/c_i) = x_1(t)x_{1+i}(c_it).$$

In Variants 24, 34, 35, 47, 48 we set $\tau = e_j$, $i \neq j \in J$, and $k_\tau = \mp c_j/2$, respectively. We have

$$x_j(\mp c_j/2)x_1(t)x_{1+i}(c_it)x_{1+j}(c_jt)x_j(\pm c_j/2) = x_1(t)x_{1+i}(c_it).$$

In Variant 23 the set R_J consists of a single root ρ_i, and we are done. In Variant 38 we fix ρ_j such that the angle between ρ_j and α is maximal and conjugate by $x_k(\mp c_k/2)$, $k \in J$, $k \neq j$.

It remains to consider Variants 31 and 45. We may assume that R_J consists of three roots. Fix a root ρ_i that forms the maximal angle with β, and a root ρ_j that has the same property with respect to α. Then we may set $\tau = \rho_k - \rho_i \in \Phi_l$, $k \neq i, j$, and $k_\tau = \mp c_k/c_i$. This completes the proof of the lemma. □

We call a *coppia*[1] of Variant N, $1 \leq N \leq 48$, a pair of sets of roots occurring in the description of X_2 and X_1 in Lemma 3, where (X_1, X_2) belongs to Variant N. Lemma 3 implies that a *coppia* may have one of the following five possible forms:

$$(\{\rho\}, \{\gamma\}),\ (\{\rho\}, \{\alpha, \beta\}),\ (\{\rho, \rho_i\}, \{\gamma\}),\ (\{\rho, \rho_i\}, \{\alpha, \beta\}),\ (\{\rho, \rho_i, \rho_j\}, \{\alpha, \beta\}).$$

The next lemma shows that, essentially, a variant determines its *coppia* uniquely.

Lemma 4. *All coppie belonging to the same variant of Tables* 1–4 *are conjugate by an element of the Weyl group.*

Proof. The proof in all cases proceeds as follows. First, we observe that for any variant the subsystems generated by the corresponding *coppie* always have the same type. Second, since the Weyl group usually acts transitively on the subsystems of the same type, with the exceptions explicitly known (see [D]), we only need to make sure that we do not deal with an exception in the variant considered.

[1] Plural: *coppie*. In Italian 'coppia' is a 'pair' (suggested by N. A. Vavilov).

Variants 1–6, 8, 10, 11 (for C_ℓ), and 13 have *coppie* of the form $(\{\rho\},\{\gamma\})$, where ρ and γ generate one of the subsystems $\widetilde{\mathrm{A}}_1$, $\widetilde{\mathrm{A}}_2$, $2\widetilde{\mathrm{A}}_1$, or C_2.

The *coppie* associated with Variants 7, 9, 11 (for $\mathrm{B}_\ell, \mathrm{F}_4$), 12, 14, and 15 are of the form $(\{\rho,\rho_i\},\{\gamma\})$, where ρ_i and γ generate either B_2 or $\widetilde{\mathrm{A}}_1+\mathrm{A}_1$. These root subsystems are conjugate in W by a transformation that leaves ρ invariant.

Variants 16–21, 22, 25–28, 32, 33, 37, 38, 40–42, and 46 lead to $(\{\rho\},\{\alpha,\beta\})$. Consider one of the pairs (α,β), $(\alpha\pm\rho,\beta)$, $(\alpha,\beta\pm\rho)$, $(\alpha\pm\rho,\beta\pm\rho)$, where we take $\alpha\pm\rho$ or $\beta\pm\rho$ if and only if it is a root in the variant in question. Such pairs of roots are conjugate in the Weyl group.

Variants 23, 24, 29, 30, 34–36, 39, 43, 44, 47, and 48 lead to $(\{\rho,\rho_i\},\{\alpha,\beta\})$. We consider root subsystems generated by the triples of roots $(\rho_i-\rho,\alpha,\beta)$, $(\rho_i-\rho,\alpha\pm\rho,\beta)$, $(\rho_i-\rho,\alpha,\beta\pm\rho)$, $(\rho_i-\rho,\alpha\pm\rho,\beta\pm\rho)$ as above. These triples generate root systems $\widetilde{\mathrm{A}}_1$, $\widetilde{\mathrm{A}}_1+\mathrm{A}_1$, B_2, B_3.

Finally, the sets $(\rho_i-\rho,\rho_j-\rho,\alpha-\rho,\beta)$, $(\rho_i-\rho,\rho_j-\rho,\alpha,\beta+\rho)$ in Variants 31, 45 generate B_3, and it is easily seen that they are conjugate in W. □

Thus, without loss of generality we may assume that $\rho_i=\delta$, and $j=3$ and $k=4$ whenever these indices occur in Tables 1–4.

Lemmas 3 and 4 show that some variants may be identified because the corresponding *coppie* have the same form. We write $M(N)$ to indicate that Variant N is conjugate to Variant M and does not lead to a new orbit. Then

Variants	Coppie
1, 2(8), 3(10), 4(11,C_ℓ), 5(13), 6	$(\{\rho\},\{\gamma\})$
7, 9, 11(B_ℓ, F_4), 12, 14, 15	$(\{\rho,\rho_i\},\{\gamma\})$
16(22), 17(25–28), 18(32,33), 19(37,38), 20(40–42), 21(46)	$(\{\rho\},\{\alpha,\beta\})$
23(24), 29, 30, 34(35), 36, 39, 43, 44, 47(48)	$(\{\rho,\rho_i\},\{\alpha,\beta\})$
31, 45	$(\{\rho,\rho_i,\rho_j\},\{\alpha,\beta\})$

Lemma 5. *Let X_1, X_2 be as above. Then, apart from Variants* 23, 34, 47 *and Variant* 39 (*for* C_ℓ), *there exists $g\in G$ such that*

$$gX_1g^{-1}=X_\rho \quad \text{or} \quad \{x_\rho(t)x_\delta(t)\} \quad \text{or} \quad \{x_\rho(t)x_{\rho_2}(t)x_{\rho_3}(t)\},$$

and

$$gX_2g^{-1}=X_\gamma \quad \text{or} \quad X_{\alpha,\beta}.$$

Proof. In the variants with *coppie* $(\{\rho,\delta\},\{\gamma\})$, we conjugate the pair of subgroups $(\{x_\rho(t)x_\delta(c_2t)\},X_\gamma)$ by $h_\lambda(c_2^{-1})\in H$, where $\lambda=e_1-e_2$ for B_ℓ, F_4 and $\lambda=2e_2$ for C_ℓ. Since $\langle\rho,\lambda\rangle=0$ and $\langle\delta,\lambda\rangle=1$, we are done.

In the variants that have *coppie* $(\{\rho,\delta\},\{\alpha,\beta\})$, we conjugate the pair of subgroups $(\{x_\rho(t)x_\delta(c_2t)\},X_{\alpha,\beta})$ by $h_\lambda(r)\in H$, where λ is a long root forming equal angles with α and β, and r is determined as follows: $r=c_2^\varepsilon$ with $\varepsilon=-1$ if $\langle\rho,\lambda\rangle=0$, $\langle\delta,\lambda\rangle=1$ and $\varepsilon=1$ if $\langle\rho,\lambda\rangle=1$, $\langle\delta,\lambda\rangle=0$.

Finally, in Variants 31 and 45, we conjugate $(\{x_\rho(t)x_{\rho_i}(c_it)x_{\rho_j}(c_jt)\},X_{\alpha,\beta})$ by the elements $h_{2+4}(c_2^{-1})h_{-3+4}(c_3)$ and $h_{-2+4}(c_2)h_{3-4}(c_3^{-1})$, respectively. □

It is easy to check that it is impossible to find $h\in H$ so as to eliminate the coefficients c_i in Variants 23, 34, 47, and 39 (for C_ℓ). The point is that there exists no root nonorthogonal to exactly one root in a given *coppia*. These cases are considered in §6 in more detail.

Lemma 6. *The subgroups of the following variants are conjugate in G: 7 and 23 with $c = \pm 2$; 9 and 17; 11 and 33 for C_ℓ; 11 and 34 for B_ℓ, F_4; 12 and 37; 14 and 20; 15 and 47 with $c = \pm 2$.*

Proof. Let (X_1, X_2) be a pair of subgroups belonging to one of Variants 17, 20, 23 ($c = \pm 2$), 37, 47 ($c = \pm 2$), or to Variant 11, and let (Y_1, Y_2) be a pair of subgroups belonging to one of Variants 9, 14, 7, 12, 15 or to Variant 33 for C_ℓ and Variant 34 for B_ℓ, F_4, respectively.

In each case we take an element $g \in G$ from the following table (by Lemma 4, we can take $i = 2$, $j = 3$, $k = 4$ in Tables 1–4).

Variants	g		
23, 7	$wx_{\delta-\rho}(\pm 1)$	$w(2\rho - \delta) = \delta$	
47, 15	$wx_{\delta-\rho}(\pm 1)$	$w(\rho) = -\rho$, $w(\rho - \delta) = \delta - \rho$	
37, 12	$wx_{\rho}(1)$	$w(\rho) = \rho - \delta$, $w(\delta - \rho) = \rho$	
20, 14	$wx_{2+3}(1)$	$w(-2+3) = 1+2$, $w(-2) = 1$, $w(1+2) = -1+3$	C_ℓ
	$wx_{1+2+3+4}(1)$	$w(-1+2) = 1{+}2$, $w(-1+2+3+4) = 1$, $w(1) = -1-2+3+4$	F_4
17, 9	$wx_{1-3}(1)$	$w(3) = 1$, $w(1+3) = 1+2$, $w(1+2) = 2+3$	C_ℓ
	$wx_{-1+2+3+4}(1)$	$w(2{+}4) = 1{+}2$, $w(1{+}2{+}3{+}4) = 1$, $w(1) = 1\text{-}2{+}3{+}4$	F_4
11, 33	$x_{3-4}(-1)x_{-3+4}(1)$		C_ℓ
11, 34	$wx_{-2}(-1)$	$w(3) = 1$, $w(1) = 3$, $w(2) = -2$, $w(-2) = 2$	$\mathrm{B}_\ell, \mathrm{F}_4$

Then in all cases except Variants 11 and 33 (34) we have

$$gX_1g^{-1} = Y_2, \quad gX_2g^{-1} = Y_1.$$

In Variants 11 and 33 (34) we have $gX_1g^{-1} = Y_1$, $gX_2g^{-1} = Y_2$. □

Summarizing, we are in a position to explicitly list representatives (possibly non-unique) of every orbit of G on pairs of short root subgroups. For the variants identified by Lemmas 4 and 6, in the subsequent tables we cite the representatives associated with the roots in Tables 1 and 2, apart from one case (Variant 17).

Set $c = c_i$, $x_\alpha = x_\alpha(t)$, and $x_\alpha(c) = x_\alpha(ct)$. In the next three tables, cases (S2) and (S20) arise if $c^2 - 4 \notin K^{*2}$, whereas cases (S3) and (S21) arise if $c^2 - 4 \in K^{*2}$, and finally, cases (S4) and (S19) arise if $c = \pm 2$.

Most cases occur only for a field of characteristic $\neq 2$. The cases for an arbitrary field (independently of characteristic) are labeled by $*$.

We collect our representatives in the following three tables. Table 5 lists the representatives for types C_ℓ. The representatives of orbits for type B_ℓ are collected in Table 6. Type F_4 has the same orbits as B_4 plus the additional orbits listed in Table 7.

The cases listed in Tables 5–7 correspond exactly to the cases of Theorem 1. In §§5–7 we prove that two pairs belonging to different cases are never conjugate. In §6 we prove that in case (S16) two pairs of subgroups are conjugate if and only if the parameters c and c' of their generators differ only by the sign and/or a square factor, $c = \pm d^2 c'$ for some $d \in K^*$. In cases (S2)–(S4), (S8), and (S19)–(S21), two pairs of subgroups are conjugate if and only if the corresponding parameters c and c' differ only by the sign, $c = \pm c'$.

TABLE 5: $\Phi = \mathrm{C}_\ell$

Case	Variants	Representatives	
(S1)*	2(8)	$\{x_{1+2}\}$	$\{x_{1+3}\}$
(S2), (S3)	16	$\{x_{1+2}\}$	$\{x_1x_2(-1)\}$
(S2)–(S4)	23(7)	$\{x_{1+2}x_1(c)\}$	$\{x_1x_2(-1)\}$
(S5)	29	$\{x_{1+2}x_1\}$	$\{x_2x_3(-1)\}$
$(\mathrm{S6})_1^*$	4(11,18,33)	$\{x_{1+2}\}$	$\{x_{3+4}\}$
$(\mathrm{S6})_2$	9(17,26)	$\{x_{1+2}x_1\}$	$\{x_{2+3}\}$
(S7)*	3(10)	$\{x_{1+2}\}$	$\{x_{1-2}\}$
(S10)*	5(13)	$\{x_{1+2}\}$	$\{x_{-1+3}\}$
(S11)	12(19,37)	$\{x_{1+2}x_1\}$	$\{x_{-1+2}\}$
(S12)	14(20,40)	$\{x_{1+2}x_1\}$	$\{x_{-1+3}\}$
(S15)*	6	$\{x_{1+2}\}$	$\{x_{-1-2}\}$
(S16)	39	$\{x_{1+2}x_1(c)\}$	$\{x_2x_{-1}(-1)\}$
(S17)	44	$\{x_{1+2}x_1\}$	$\{x_3x_{-1}(-1)\}$
(S20), (S21)	21	$\{x_{1+2}\}$	$\{x_{-2}x_{-1}(-1)\}$
(S19)–(S21)	47(15)	$\{x_{1+2}x_1(c)\}$	$\{x_{-2}x_{-1}(-1)\}$

TABLE 6: $\Phi = \mathrm{B}_\ell$, F_4

Case	Variants	Representatives	
(S2), (S3)	16(22)	$\{x_1\}$	$\{x_{1+2}x_{1-2}(-1)\}$
(S2)–(S4)	23(7,24)	$\{x_1x_{1+2}(c)\}$	$\{x_{1+2}x_{1-2}(-1)\}$
$(\mathrm{S6})_1$	18(32,33)	$\{x_1\}$	$\{x_{2+3}x_{2-3}(-1)\}$
(S7)*	3(10)	$\{x_1\}$	$\{x_2\}$
(S8)	34(11,35)	$\{x_1x_{1+2}(c)\}$	$\{x_{2+3}x_{-2+3}(-1)\}$
(S11)	12(19,37,38)	$\{x_1x_{1+2}\}$	$\{x_{-2}\}$
(S13)	36	$\{x_1x_{1+2}\}$	$\{x_{-2+3}x_{-2-3}(-1)\}$
(S15)*	6	$\{x_1\}$	$\{x_{-1}\}$
(S16)†	39	$\{x_1x_{1+2}\}$	$\{x_{1-2}x_{-1-2}(-1)\}$
(S20), (S21)	21(46)	$\{x_1\}$	$\{x_{-1+2}x_{-1-2}(-1)\}$
(S19)–(S21)	47(15)	$\{x_1x_{1+2}(c)\}$	$\{x_{-1+2}x_{-1-2}(-1)\}$

† In this case $\ell \geq 3$.

TABLE 7: $\Phi = \mathrm{F}_4$

Case	Variants	Generators/representatives	
(S1)*	2(8)	$\{x_1\}$	$\{x_{1+2+3+4}\}$
(S5)	29	$\{x_1x_{1+2}\}$	$\{x_{1-2}x_{3+4}(-1)\}$
$(\mathrm{S6})_2$	17(9,25–28)	$\{x_1x_{1+2}\}$	$\{x_{1+3}x_{2+4}(-1)\}$
$(\mathrm{S9})_1$	30	$\{x_1x_{1+2}\}$	$\{x_{1+3}x_{-2+4}\}$
$(\mathrm{S9})_2$	31	$\{x_1x_{1+2}x_{1+3}\}$	$\{x_{1-3}x_{-2+4}\}$
(S10)*	5(13)	$\{x_1\}$	$\{x_{-1+2+3+4}\}$
(S12)	14(20,40–42)	$\{x_1x_{1+2}\}$	$\{x_{-1-2+3+4}\}$
(S14)	43	$\{x_1x_{1+2}\}$	$\{x_{-2+3}x_{-1+4}\}$
(S17)	44	$\{x_1x_{1+2}\}$	$\{x_{3+4}x_{-1-2}(-1)\}$
(S18)	45	$\{x_1x_{1+2}x_{1+3}\}$	$\{x_{-3+4}x_{-1-2}\}$

In §5 we study subgroups generated by each pair of short root subgroups. Namely, we determine these subgroups up to isomorphism and, moreover, describe the root subgroups contained in them. It is clear that if either the isomorphism classes or the numbers of long or short root subgroups in $\langle X_1, X_2\rangle$ and $\langle Y_1, Y_2\rangle$ are distinct, then the pairs (X_1, X_2) and (Y_1, Y_2) are nonconjugate. The cases where the subgroups generated by a pair of short root subgroups coincide are considered in §6.

§5. Structure of spans

We start with some general remarks. For a representative (X_1, X_2) of an orbit of G on pairs of short root subgroups listed in Tables 5–7, we denote by X their span $X \sim \langle X_1, X_2\rangle$. Note that sometimes we pass to a conjugate of X by an element of the Weyl group.

As a rule, our calculations do not depend on the signs of the structure constants. When they do, it is always sufficient to calculate inside a Chevalley group of type B_4 or C_2. In these cases we use the following matrix representation (see [C, 11.3]).

Let V be a 9-dimensional vector space, and let Q be a quadratic form defined on V. We choose a base of V such that $Q(x,x) = x_1x_{-1} + x_2x_{-2} + x_3x_{-3} + x_4x_{-4} + x_0^2$. Then the adjoint Chevalley group $G(\mathrm{B}_4, K)$ is isomorphic to the commutant of the orthogonal group $\Omega_9(K, Q)$.

We enumerate the rows and columns of matrices in $\Omega_9(K, Q)$ in the following way: 1, 2, 3, 4, 0, -4, -3, -2, -1. Next, denote by $e_{i,j}$, $-4 \le i \ne j \le 4$, the matrix that has zero everywhere except for the position (i, j). At this position the coefficient is equal to 1. Let I be the identity matrix, and let $0 < i < j$. Then the elements of $G(\mathrm{B}_4, K)$ take the following form:

$$x_{i-j}(t) = I + t(e_{i,j} - e_{-j,-i}),$$

$$x_{j-i}(t) = I + t(e_{j,i} - e_{-i,-j}),$$

$$x_{i+j}(t) = I + t(e_{i,-j} - e_{j,-i}),$$

$$x_{-i-j}(t) = I + t(e_{-j,i} - e_{-i,j}),$$

$$x_i(t) = I + t(2e_{i,0} - e_{0,-i}) - t^2 e_{i,-i},$$

$$x_{-i}(t) = I - t(2e_{-i,0} - e_{0,i}) - t^2 e_{-i,i}.$$

The group $G(\mathrm{C}_2, K)$ is isomorphic to $\mathrm{Sp}(4, K)$. Let V be a 4-dimensional vector space, and let B be a skew-symmetric bilinear form defined on V. We choose a base of V such that

$$B(x, y) = x_1 y_{-1} - x_{-1} y_1 + x_2 y_{-2} - x_{-2} y_2.$$

As above, we enumerate the rows and columns of matrices in $\mathrm{Sp}(4, K)$: 1, 2, -2, -1. Assume that $0 < i < j$; the elements of $G(\mathrm{C}_2, K)$ take the following form:

$$x_{i-j}(t) = I + t(e_{i,j} - e_{-j,-i}),$$

$$x_{j-i}(t) = I + t(e_{j,i} - e_{-i,-j}),$$

$$x_{i+j}(t) = I + t(e_{i,-j} + e_{j,-i}),$$

$$x_{-i-j}(t) = I + t(e_{-i,j} + e_{-j,i}).$$

By definition, an (additive) one-parameter subgroup is isomorphic to the additive group of the field. Every such subgroup has the form $Y = \{x(t),\ t \in K\}$. In particular, the root subgroups are one-parameter subgroups.

When counting the root subgroups contained in $\langle X_1, X_2 \rangle$, in all nontrivial cases we must work inside $U(\Phi, K)$. We proceed as follows: we take an arbitrary one-parameter subgroup of X, say $Y = \{x(t),\ t \in K\}$, and try to solve the following equations in $u \in U$:

$$ux(t)u^{-1} = x_\lambda(s), \quad \lambda \in \Phi_l \cap \Phi^+, \tag{1}$$

$$ux(t)u^{-1} = x_\gamma(s), \tag{2}$$

$$ux(t)u^{-1} = x_\alpha(s) x_\beta(-s), \tag{3}$$

where γ, α, and β have the same meaning as in Lemma 1 and are positive. The first equation is obvious from the Bruhat decomposition of a long root element (see [V1, Theorem 1]), or can be checked directly.

The nonsolvability of (1) or both (2) and (3) means that Y is not a long or a short root subgroup, respectively. As in §4, we tacitly assume that t runs over K and write simply $\{x(t)\}$ instead of $\{x(t),\ t \in K\}$.

Now we pass to the case by case analysis of orbits.

(S1)*. $\Phi = \mathrm{C}_\ell$, F_4. Here $X = \langle x_\rho(t), x_\gamma(s) \rangle$, where $\rho - \gamma \in \Phi$, $\rho + \gamma \notin \Phi$. The subgroups of the form $Y = \{x_\rho(t) x_\gamma(kt)\}$, $k \in K$, and X_γ are short root subgroups.

(S2)–(S4). $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4. Here

$$X = \langle x_\rho(t) x_\delta(ct), x_\delta(s) x_{2\rho-\delta}(-s) \rangle \sim X_1 \times X_2.$$

Let $s = kt$ with $k \in K$ fixed. To count the number of root subgroups contained in X, consider the one-parameter subgroups $Y = x_{\delta-\rho}(r) X x_{\delta-\rho}(-r)$, $r \in K^*$. Then

$$Y = \{x_\rho((1 + k + rk)t) x_\delta((2r + c + 2 + 2rk + kr^2)t) x_{2\rho-\delta}(-kt)\}.$$

Now if $1 + k + rk = 0$ or $2r + c + 2 + 2rk + kr^2 = 0$, but not both, then Y is a short root subgroup, otherwise Y is a long root subgroup. Specifically, set $r = -(1+k)/k$; then the second equation becomes $k^2 - kc + 1 = 0$. Hence, we fall into case (S2) if $c^2 - 4 \notin K^{*2}$, into case (S3) if $c^2 - 4 \in K^{*2}$, and into case (S4) if $c = \pm 2$.

(S5). $\Phi = \mathrm{C}_\ell$, F_4. Here

$$X = \langle x_\rho(t)x_\delta(t), x_{2\rho-\delta}(s)x_\beta(-s)\rangle \sim X_1 \times X_2,$$

where β is orthogonal to ρ and δ. The one-parameter subgroups of X are

$$\{x_{2\rho-\delta}(t)x_\beta(-t)\} \quad \text{and} \quad \{x_\rho(t)x_\delta(t)x_{2\rho-\delta}(kt)x_\beta(-kt)\},$$

where k runs over K. Consider the equations

$$ux_\rho(t)x_\delta(t)x_{2\rho-\delta}(kt)x_\beta(-kt)u^{-1} = x_\lambda(s),$$

where λ is a positive long root. It is readily seen that the factors $x_{2\rho-\delta}(*)$ and $x_\beta(*)$ cannot be eliminated, so that the set $\Omega(X)$ is empty. The same argument as above shows that the relation

$$ux_\rho(t)x_\delta(t)x_{2\rho-\delta}(kt)x_\beta(-kt)u^{-1} = x_\gamma(s),$$

where γ is a positive short root, is impossible. Finally, let

$$ux_\rho(t)x_\delta(t)x_{2\rho-\delta}(kt)x_\beta(-kt)u^{-1} = x_\alpha(s)x_\beta(-s),$$

where α, β are as in Lemma 1 and positive. Since $x_{2\rho-\delta}(*)$ and $x_\beta(*)$ cannot be eliminated, the coefficients of $x_\rho(*)$ and $x_\delta(*)$ must be zero. This is possible only if $u = x_{\delta-\rho}(r)$ and the following is true: $x_\rho((1+kr)t) = 1$ and $x_\delta((1+2r+r^2k)t) = 1$. Consequently, $k = 1$ and $r = -1$, whence $|\Lambda(X)| = 3$.

(S6)$^*_1$. First, let $\Phi = \mathrm{B}_\ell$, F_4. Here

$$X = \langle x_1(t), x_{2+3}(s)x_{2-3}(-s)\rangle \sim X_1 \times X_2.$$

The subgroups of X are $\{x_{2+3}(s)x_{2-3}(-s)\}$, $\{x_1(t)x_{2+3}(kt)x_{2-3}(-kt)\}$, where $k \in K$ is fixed. We must consider the following equations:

$$ux_1(t)x_{2+3}(kt)x_{2-3}(-kt)u^{-1} = x_\lambda(s), \quad \text{or} \quad x_\gamma(s), \quad \text{or} \quad x_\alpha(s)x_\beta(-s).$$

It is easy to see that $x_1(t)$ and $x_{2-3}(-kt)$ cannot be eliminated by conjugating by unipotent elements. Thus, $|\Lambda| = 2$, $|\Omega| = 0$.

If $\Phi = \mathrm{C}_\ell$, then $X = \langle x_{1+2}(t), x_{3+4}(s)\rangle$. The number of long and short root subgroups is as above.

(S6)$_2$. $\Phi = \mathrm{C}_\ell$, F_4. Here $X = \langle x_\rho(t)x_\delta(t), x_\gamma(s)\rangle$, where $\rho - \gamma \in \Phi$, δ and γ are orthogonal. Since $x_\rho(*)$ and $x_\gamma(*)$ are not killed by any $u \in U$, the group X has the required structure.

(S7). $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4. Here $X = \langle x_\rho(t), x_{\delta-\rho}(s)\rangle = X_\rho X_{\delta-\rho} X_\delta$. The one-parameter subgroups of X have the form

$$Y = \{x_\rho(at)x_{\delta-\rho}(bt)x_\delta(kt - abt^2)\}, \quad a, b, k \text{ are fixed.}$$

If $a, b = 0$, $k \neq 0$, we get a long root subgroup Y. If $k = 0$ and exactly one of the elements a and b equals 0, we have short root subgroups X_ρ, $X_{\delta-\rho}$,

if $a = 0$, $k, b \neq 0$, consider $x_\rho(-\frac{1}{2}ka^{-1})Yx_\rho(\frac{1}{2}ka^{-1})$;

if $b = 0$, $k, a \neq 0$, consider $x_{\delta-\rho}(-\frac{1}{2}kb^{-1})Yx_{\delta-\rho}(\frac{1}{2}kb^{-1})$;

if $k = 0$, $a, b \neq 0$, consider $x_{\delta-2\rho}(-b)Yx_{\delta-2\rho}(b)]$;

if $a, b, k \neq 0$, consider $x_{\delta-2\rho}(-b)x_{\delta-\rho}(-\frac{1}{2}ka^{-1})Yx_{\delta-\rho}(\frac{1}{2}ka^{-1})x_{\delta-2\rho}(b)$.

It is easy to check that all these subgroups are short root subgroups.

In particular, for a finite field $\mathbb{F}_q$ we get $1+1+(q-1)+(q-1)+(q-1)+(q-1)^2 = q^2+q$ short root subgroups.

(S8). $\Phi = \mathrm{B}_\ell$, F_4. Here

$$X = \langle x_1(t)x_{1+3}(ct), x_{2+3}(s)x_{2-3}(-s)\rangle \sim X_1X_2Z.$$

The one-parameter subgroups of X are

$$\{x_1(at)x_{1+3}(act)x_{2+3}(bt)x_{2-3}(-bt)x_{1+2}(kt - \tfrac{1}{2}abt^2)\}.$$

If $a, b = 0$, $k \neq 0$, we have a long root subgroup Z. If exactly one of the elements a, b equals zero, we get a short root subgroup. In particular, for a finite field $\mathbb{F}_q$ we have $2q$ short root subgroups. It is easy to check that if $a, b \neq 0$, then the given subgroups are not root subgroups.

(S9). $\Phi = \mathrm{F}_4$. Here

$$X = \langle x_1(t)x_{1+2}(t)x_{1+3}(\varepsilon t), x_{1-3}(s)x_{-2+4}(-s)\rangle \sim X_1X_2Z,$$

where $\varepsilon = 0$ or 1 for $(\mathrm{S9})_1$ or $(\mathrm{S9})_2$, respectively (see Table 7).

The one-parameter subgroups of X are

$$\{x_1(at)x_{1+2}(at)x_{1+3}(a\varepsilon t)x_{1-3}(bt)x_{-2+4}(-bt)x_{1+4}(kt - \tfrac{1}{2}abt^2)\}.$$

The number of root subgroups is calculated exactly as above. Under the conjugation action of U, either $x_1(*)$ and $x_{1+2}(*)$ are nontrivial if $\varepsilon = 0$, or so are $x_{1-3}(*)$ and $x_{1+2}(*)$ if $\varepsilon = 1$.

(S10)*. $\Phi = \mathrm{C}_\ell$, F_4. Here $X = \langle x_\rho(t), x_\gamma(s)\rangle \simeq U(\widetilde{A}_2, K)$ since $\rho + \gamma$ is a short root.

(S11). $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4. Here

$$X = \langle x_\rho(t)x_\delta(t), x_{\rho-\delta}(s)\rangle \simeq U(\mathrm{C}_2, K).$$

This is immediate.

(S12). $\Phi = \mathrm{C}_\ell$, F_4. Again,

$$X = \langle x_\rho(t)x_\delta(t), x_\gamma(s)\rangle \simeq U(\mathrm{C}_2, K),$$

where ρ, γ, $\rho + \gamma \in \Phi_s$ and $\{\delta, \gamma, \delta + \gamma, \delta + 2\gamma\} = \Phi^+(\mathrm{C}_2)$, and

$$[x_\rho(t)x_\delta(t), x_\gamma(s)] = x_{\rho+\gamma}(\pm ts)x_{\delta+\gamma}(\pm ts)x_{\delta+2\gamma}(\pm ts^2),$$

$$[[x_\rho(t)x_\delta(t), x_\gamma(s)], x_\gamma(r)] = x_{\delta+2\gamma}(\pm 2tsr).$$

Thus, the subgroup X is generated by the elements of the form $x_\gamma(t)$, $x_\rho(t)x_\delta(t)$, $x_{\delta+2\gamma}(t)$, and $x_{\rho+\gamma}(t)x_{\delta+\gamma}(t)$, which satisfy the same defining relations as the standard generators of $U(\mathrm{C}_2, K)$. In other words, the map defined by

$$x_\gamma(t) \mapsto x_\gamma(t),$$

$$x_\rho(t)x_\delta(t) \mapsto x_\delta(t),$$

$$x_{\rho+\gamma}(t)x_{\delta+\gamma}(t) \mapsto x_{\delta+\gamma}(t),$$

$$x_{\delta+2\gamma}(t) \mapsto x_{\delta+2\gamma}(t)$$

is an isomorphism of between X and $U(\mathrm{C}_2, K)$ as abstract groups.

(S13). $\Phi = \mathrm{B}_\ell$, F_4. Here

$$X = \langle x_1(t)x_{1+2}(t), x_{-2+3}(s)x_{-2-3}(-s)\rangle \sim X_1X_2YZ.$$

It suffices to calculate the commutators:

$$[x_1(t)x_{1+2}(t), x_{-2+3}(s)x_{-2-3}(-s)] = x_{1+3}(-ts)x_{1-3}(ts)x_{1-2}(-ts^2),$$

$$[x_{-2+3}(s)x_{-2-3}(-s), x_{1+3}(-r)x_{1-3}(r)x_{1-2}(rs)] = x_{1-2}(-2rs).$$

(S14). $\Phi = \mathrm{F}_4$. Here

$$X = \langle x_1(t)x_{1+2}(t), x_{-1+4}(s)x_{-2+3}(-s) \simeq U(\mathrm{C}_2, K) \times Z.$$

We calculate the commutators:

$$[x_1(t)x_{1+2}(t), x_{-1+4}(s)x_{-2+3}(-s)] = x_{2+4}(ts)x_{1+3}(-ts)x_4(ts)x_{1+4}(-t^2s)x_{3+4}(ts^2),$$

$$[x_{2+4}(rs)x_{1+3}(-rs)x_4(rs)x_{1+4}(-r^2s)x_{3+4}(rs^2), x_1(t)x_{1+2}(t)] = x_{1+4}(2rst),$$

$$[x_{2+4}(rs)x_{1+3}(-rs)x_4(rs)x_{1+4}(-r^2s)x_{3+4}(rs^2), x_{-1+4}(t)x_{-2+3}(-t)] = x_{3+4}(2rst).$$

The quotient group X/X_{3+4} is isomorphic to $U(\mathrm{C}_2, K)$. On the generators, an isomorphism can be defined as follows:

$$x_1(t)x_{1+2}(t)X_{3+4} \mapsto x_\alpha(t),$$

$$x_{-1+4}(t)x_{-2+3}(-t)X_{3+4} \mapsto x_\beta(t),$$

$$x_{2+4}(t)x_{1+3}(-t)x_4(t)X_{3+4} \mapsto x_{\alpha+\beta}(t),$$

$$x_{1+4}(t)X_{3+4} \mapsto x_{2\alpha+\beta}(t),$$

where α and β are the short and the long fundamental roots, respectively.

(S15)*. $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4. Clearly, $X = \langle x_\rho(t), x_{-\rho}(s)\rangle \simeq \mathrm{SL}_2(K)$.

(S16). $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4. Here

$$X = \langle x_\rho(t)x_\delta(ct), x_{2\rho-\delta}(s)x_{-\delta}(-s)\rangle \simeq \mathrm{SL}_2(K) \times Z \quad \text{or} \quad \mathrm{SL}_2(K).$$

Conjugating generators by $x_{\rho-\delta}(-1/c)$, we get

$$\langle x_{2\rho-\delta}(-t/c)x_\delta(ct), x_{2\rho-\delta}(s)x_{-\delta}(-s)\rangle.$$

First, suppose that $K \neq \mathbb{F}_3$. Then the commutator subgroup $[X, X]$ is isomorphic to $\mathrm{SL}_2(K)$, and we are done.

Let $K = \mathbb{F}_3$. Then each element of the group X can be written as yu, where $y \in \langle X_\delta, X_{-\delta}\rangle \simeq \mathrm{SL}_2(K)$, $u \in U(\mathrm{C}_2, K)$.

Consider the map ϕ from X to $\mathrm{SL}_2(K)$ defined by $yu \mapsto y$. It is a surjective homomorphism. We calculate the kernel $U_0 = \ker\phi$ of this homomorphism.

It is well known that the defining relations of $\mathrm{SL}_2(K)$ look like this:

$$x_\alpha(t)x_\alpha(s) = x_\alpha(t+s),$$
$$w_\alpha(t)x_\alpha(s)w_\alpha(-t) = x_{-\alpha}(-t^{-2}s),$$
$$h_\alpha(t)h_\alpha(s) = h_\alpha(ts),$$

where $\alpha \in \Phi(\mathrm{A}_1)$ and t, s run over K.

Let $x(t) = x_\rho(t)x_\delta(\pm t)$, $y(s) = x_{2\rho-\delta}(s)x_{-\delta}(-s)$. We set

$$w(t) = x(t)y(-t^{-1})x(t), \quad h(t) = w(t)w(-1).$$

Then

$$U_0 = \langle w(t)x(s)w(-t)y(t^{-2}s), h(t)h(s)h(t^{-1}s^{-1})\rangle,$$

and straightforward calculations show that $\ker\phi$ is trivial.

(S17). $\Phi = \mathrm{C}_\ell$, F_4. Here

$$X = \langle x_\rho(t)x_\delta(t), x_{-\delta}(s)x_\beta(-s)\rangle,$$

where β is orthogonal to ρ and δ. Conjugating X by $x_{\rho-\delta}(-1)$, we get

$$\langle x_{2\rho-\delta}(t)x_\delta(t), x_{-\delta}(s)x_\beta(-s)\rangle.$$

Again, our statement for $K \neq \mathbb{F}_3$ follows from the fact that $[X, X] \simeq \mathrm{SL}_2(K)$.

Let $K = \mathbb{F}_3$. Proceeding as in (S16), we deduce that $U_0 = \ker\phi = \langle x_{2\rho-\delta}(t)x_\beta(t)\rangle$ is the cyclic group of order 3. Note that for F_4 the subgroup U_0 is a short root subgroup.

(S18). $\Phi = \mathrm{F}_4$. Here

$$X = \langle x_1(t)x_{1-2}(t)x_{1-3}(t), x_{3-4}(s)x_{-1+2}(-s)\rangle.$$

Now X may be regarded as a subgroup of $\Omega_9(K, Q)$, the commutator subgroup of the orthogonal group with respect to the form Q mentioned at the beginning of this section. Let $x(t)$ stand for the first generator and $y(t)$ for the second. It is clear that X is isomorphic to the semidirect product $\mathrm{SL}_2(K) \leftthreetimes U_0$, where U_0 is a subgroup of $U(\mathrm{B}_4, K)$. To determine U_0, we proceed as above (compare (S16)). Calculations with $x(t)$ and $y(t)$ (in the matrix representation) show that U_0 is generated by

$$w(t)x(s)w(-t)y(t^{-2}s) = x_{1+2}(s)x_{2-4}(t^{-2}s)x_{3-4}(-t^{-2}s),$$

$$h(t)h(s)h(t^{-1}s^{-1}) = x_{1+2}(2f(t,s))x_{2-4}(-f(t^{-1},s^{-1}))x_{3-4}(f(t^{-1},s^{-1})),$$

where $f(t,s) = t^{-1}s^{-1} + s + t - 3$. Varying t and s, we conclude that U_0 is generated by the long root subgroups X_{1+2} and $\{x_{2-4}(t)x_{3-4}(t)\}$.

(S19)–(S21). $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4. Here

$$X = \langle x_\rho(t)x_\delta(ct), x_{\delta-2\rho}(-s)x_{-\delta}(s)\rangle.$$

Obviously, X may be viewed as a subgroup of $\mathrm{Sp}(V)$, where $\dim(V) = 4$. Set $x(t,c) = x_\rho(t)x_\delta(ct)$, $y(s) = x_{\delta-2\rho}(-s)x_{-\delta}(s)$. Then

$$x(t,c) = \begin{pmatrix} 1 & 0 & t & ct \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad y(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -s & 1 & 0 \\ s & 0 & 0 & 1 \end{pmatrix}.$$

Let $\overline{V}$ be a 3-dimensional projective space whose points are 1-dimensional subspaces of V. Denote by $\overline{x}(t,c)$, $\overline{y}(s)$ the projective transformations induced by $x(t,c)$, $y(s)$, respectively, and let $\overline{X}$ be the group generated by $\overline{x}(t,c)$, $\overline{y}(s)$.

A point of $\overline{V}$ is determined by its homogeneous coordinates $(p_1, p_2, p_3, p_4)^t$, and by duality the same is true for the planes of $\overline{V}$: a plane is determined by (p^1, p^2, p^3, p^4). The lines of $\overline{V}$ can be described by nonzero skew-symmetric (4×4)-matrices P with $\det(P) = 0$. Note that a line is determined uniquely up to a scalar factor.

Now we find the subspaces of $\overline{V}$ invariant under the action of $\overline{X}$. Let $p = (p_1, p_2, p_3, p_4)^t$ be a point, let $q = (q^1, q^2, q^3, q^4)$ be a plane, and let $P = (p_{ij})$ be a line of $\overline{V}$, where $p_{ij} = -p_{ji}$ and $p_{12}p_{34} + p_{13}p_{24} - p_{14}p_{23} = 0$.

Then the action of $\overline{z} \in \mathrm{PSp}(V)$ is determined as follows:

$$p \mapsto \overline{z}p, \quad q \mapsto q\overline{z}^{-1}, \quad P \mapsto \overline{z}^t P\overline{z}.$$

It is easy to verify that there are no points and planes in $\overline{V}$ invariant under $\overline{X}$. Moreover, a line P in $\overline{V}$ is invariant under $\overline{X}$ if and only if

$$\overline{x}^t(t,c)P = aP\overline{x}(-t,c), \quad a \in K^*,$$

$$\overline{y}^t(s)P = bP\overline{y}(-s), \quad b \in K^*.$$

Thus, P must have the form

$$\begin{pmatrix} 0 & 0 & p_{13} & cp_{13}+p_{23} \\ 0 & 0 & p_{23} & -p_{13} \\ -p_{13} & -p_{23} & 0 & 0 \\ -cp_{13}-p_{23} & p_{13} & 0 & 0 \end{pmatrix},$$

where $p_{13}^2 + cp_{13}p_{23} + p_{23}^2 = 0$. The solvability of this equation is equivalent to that of $t^2 + ct + 1 = 0$. This leads to three separate cases.

(S19). The equation $t^2 + ct + 1 = 0$ has a unique root t_0 in K, i.e., $c = \pm 2$. Then the subgroup X is conjugate to $\langle x_\rho(t)x_\delta(t), x_{-\rho}(s)\rangle$ and isomorphic to $\mathrm{SL}_2(K) \times U_0$, where U_0 is a subgroup of $U(\mathrm{C}_2, K)$.

As above, straightforward calculations yield

$$U_0 = \langle x_\delta(-f(t^{-1}, s^{-1}))x_{\delta-2\rho}(f(t,s)), x_{\delta-\rho}(-t^{-1}s)x_{\delta-2\rho}(-t^{-2}s + t^{-3}s^2)\rangle,$$

where $f(t,s) = t^2s^2 - ts^2 - ts + s^2 - s + 1$. Setting $s = t$, we conclude that $\langle x_{\delta-\rho}(1)\rangle$ is contained in U_0. Set $s = kt$, where k is an element of the prime subfield of K. Then $x_{\delta-2\rho}(kt^{-1}(k-1))$ lies in U_0 for any $t \in K^*$; therefore, U_0 is generated by $X_{\delta-2\rho}$, $X_{\delta-\rho}$, X_δ.

(S20). The equation $t^2 + ct + 1 = 0$ has two distinct roots t_1 and t_2 in K, i.e., $\sqrt{c^2-4}$ belongs to K^*. Then the subgroup X has two 2-dimensional invariant subspaces in V, namely, the planes spanned by $(1, t_i, 0, 0)$ and $(0, 0, -t_i, 1)$, $i = 1, 2$. Thus, the generators $x(t,c)$ and $y(s)$ are conjugate to $x_{12}(t_2t)x_{34}(t_1t)$ and $x_{21}(s)x_{43}(s)$ by the matrix z,

$$z = \begin{pmatrix} 1 & 0 & 1 & 0 \\ t_1 & 0 & t_2 & 0 \\ 0 & -t_1 & 0 & -t_2 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4, K),$$

where $x_{ij}(t)$, $1 \le i \ne j \le 4$, is an elementary transvection, i.e., the matrix with unit elements at the main diagonal, with t at the position (i,j), and with zeros elsewhere.

Since $t_2t_1 = 1$, it suffices to determine the subgroup X_0 generated by

$$x_0(t,k) = x_{12}(kt)x_{34}(t) \quad \text{and} \quad y_0(s) = x_{21}(s)x_{43}(s),$$

where $k = t_2^2$. We set $w(t) = x_0(t,k)y_0(-t^{-1})x_0(t,k)$ and calculate the product

$$u(t,s) = w(t)x_0(s,k)w(-t)y_0(-t^{-2}s).$$

Putting $u_0(s) = u(ks, s)$, we get

$$u_0(s) = \begin{pmatrix} k^{-1} & k(k-1)^2s & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $[u_0(s), u_0(-2s)] = x_{12}((1+k^{-2})(k-1)^2s)$, the group X_0 contains all transvections of the form $x_{12}(t)$, $t \in K$. Also, it is easy to check that

$$x_{21}(\tfrac{1}{2}f(k)s^{-1}) = u_0(-k(k-1)^2s^2(ks - k^2s + s)^{-1})u(-s,s),$$

where $f(k) = (k^2 - k - 1)(1 - k^2)(k^3 - 2k^2 - k + 2)$. Thus, X_0 contains $\mathrm{SL}_2(K) = \langle x_{12}(t), x_{21}(s)\rangle$.

A similar argument for $x_{34}(t)$ and $x_{43}(s)$ completes the analysis of this case.

(S21). The equation $t^2 + ct + 1 = 0$ has no roots in K, i.e., $\sqrt{c^2 - 4}$ does not belong to K. Then we pass to the quadratic extension $L = K(\sqrt{c^2 - 4})$ and use the same argument as in case (S20).

Remark. If we do not pass to a quadratic extension of the ground field, it can be proved that X is a primitive subgroup containing no long root subgroups and leaving a regulus in a projective space invariant (see [M]).

§6. Completion of proof of Theorem 1

To finish the proof of Theorem 1, it only remains to show that subgroups falling into different cases (S1)–(S21) are not conjugate and to calculate the number of nonconjugate subgroups of the same case.

Clearly, we may restrict our attention to the cases where the structure of subgroups $\langle X_1, X_2\rangle$ in the sense of §5 (i.e., the isomorphism class plus the numbers of short and long root subgroups contained in $\langle X_1, X_2\rangle$) is one and the same for distinct representatives of Tables 5–7.

We start with some terminology. Two roots α and β are said to be *strictly orthogonal* if $\alpha \pm \beta \notin \Phi \cup \{0\}$. For a subset Δ of roots in Φ, we denote by $\Delta^\perp$ (respectively, $\Delta_0^\perp$) the set of roots orthogonal (respectively, strictly orthogonal) to Δ. The closure of the set Δ is the minimal closed set of roots that contains Δ.

For $u \in U$ we write $u = \prod x_\sigma(c_\sigma)$, $\sigma \in \Phi^+$, and the product is taken in the increasing order of roots. Then the closure of the set $\{\sigma \in \Phi^+ \mid c_\sigma \neq 0\}$ is called the *support* of u and will be denoted by $\Gamma(u)$.

Recall also that the reduced Bruhat decomposition asserts that any element $g \in G$ can be uniquely written in the form $g = uhn_wv$, where $u \in U$, $h \in H$, $v \in U \cap w^{-1}Vw$, and $n_w \in N$. Since N/H is isomorphic to W, we ignore the diagonal factor and write simply w instead of n_w.

We start with cases (S2)–(S4), (S8), (S19)–(S21), and (S16) (the latter for $\Phi = \mathrm{C}_\ell$). These pairs are parameterized by an element of the field K. Thus, we need to know when two pairs of the form $(X_{\rho,c}, X_{\alpha,\beta})$ and $(X_{\rho,c'}, X_{\alpha,\beta})$ are conjugate. We may assume that $c \neq 0$ and c' is an arbitrary element of K.

First, consider the effect of conjugation by elements of $H(\Phi)$. There is no loss of generality in assuming that $\Phi = \mathrm{F}_4$ or C_2. Any element of $H(\mathrm{F}_4)$ can be decomposed as

$$h_{1-2-3-4}(k_1)h_{2-3}(k_2)h_{3-4}(k_3)h_4(k_4), \quad k_i \in K^*, \ 1 \le i \le 4$$

(see [St, Lecture 38]), and any element of $H(\mathrm{C}_2)$ has the form $h_{1-2}(k_1)h_2(k_2)$, $k_i \in K^*$, $i = 1, 2$. Therefore, the condition

$$hx_\rho(t)x_\delta(ct)h^{-1} = x_\rho(t')x_\delta(c't')$$

amounts to the following system of equations:

$$\begin{cases} k_1 t = t', \\ k_2 ct = c't' \end{cases} \text{(for } \mathrm{F}_4\text{)}, \quad \text{and} \quad \begin{cases} k_2 t = t', \\ k_1^2 ct = c't' \end{cases} \text{(for } \mathrm{C}_2\text{)}. \tag{1}$$

In the same way, the condition

$$hx_\alpha(s)x_\beta(-s)h^{-1} = x_\alpha(s')x_\beta(-s')$$

leads to a certain second system of equations. Now we analyze these two systems case by case.

(S2)–(S4), $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4. We have

$$\begin{cases} k_2 s = s', \\ k_1^2 k_2^{-1} s = s', \end{cases} \text{(for } \mathrm{F}_4\text{)}, \quad \text{and} \quad \begin{cases} k_1^2 s = s' \\ k_1^{-2} k_2^2 s = s', \end{cases} \text{(for } \mathrm{C}_2\text{)}. \tag{2}$$

Comparing (1) and (2), we conclude that $c = \pm c'$.

(S8), $\Phi = \mathrm{B}_\ell$, F_4. Then

$$\begin{cases} k_1^2 k_3 s = s', \\ k_2^{-2} k_3 s = s'. \end{cases} \tag{3}$$

Again, comparison of (1) and (3) shows that $c = \pm c'$.

(S19)–(S21), $\Phi = \mathrm{B}_\ell$, C_ℓ, F_4. Then

$$\begin{cases} k_1^{-2} k_2 s = s', \\ k_2^{-1} s = s' \end{cases} \text{(for } \mathrm{F}_4\text{)} \quad \text{and} \quad \begin{cases} k_1^2 k_2^{-2} s = s', \\ k_1^{-2} s = s' \end{cases} \text{(for } \mathrm{C}_2\text{)}. \tag{4}$$

Again, we have $c = \pm c'$.

(S16), $\Phi = \mathrm{C}_\ell$. Then

$$\begin{cases} k_1^{-2} k_2^2 s = s', \\ k_1^{-2} s = s'. \end{cases} \tag{5}$$

Thus, $c = \pm k^2 c'$ for any $k \in K$.

This finishes the analysis of conjugation by the elements of H, and we pass to conjugation by an arbitrary element.

(S2)–(S4). The fact that our representatives are conjugate is equivalent to the solvability of the system

$$hwvx_\rho(t)x_\delta(ct)v^{-1}w^{-1}h^{-1} = u^{-1}x_\rho(t')x_\delta(c't')u, \tag{6}$$

$$hwvx_\alpha(s)x_\beta(-s)v^{-1}w^{-1}h^{-1} = u^{-1}x_\alpha(s')x_\beta(-s')u \tag{7}$$

for some $g = uhwv$, $(\alpha, \beta) = (\delta, 2\rho - \delta)$. We rewrite (6) in the form

$$hwx_\rho(t)x_\delta((c + 2c_2)t)\prod_{i>2} x_{\rho_i}(2c_i t)w^{-1}h^{-1} = x_\rho(t')x_\delta((c' + 2c_2')t)\prod_{j>2} x_{\rho_j}(2c_j' t'), \tag{8}$$

where $v = \prod x_\sigma(c_\sigma)$, $u = \prod x_\sigma(c_\sigma')$.

The last equation implies that $w\rho = \rho$, so that $w \in W(\{\rho\}^\perp)$. If $c + 2c_2 \neq 0$ or $c_i \neq 0$ for some $i > 2$, then $c' + 2c_2' \neq 0$, $c_j \neq 0$, $j > 2$. Thus, $w(\rho_i - \rho)$ is a positive root and $\rho_i - \rho \notin \Gamma(v)$. This means that $c_i = 0$, $i > 2$, and either $c_2 = 0$, or $c + 2c_2 = 0$, $c_2 \neq 0$.

Suppose $c_2 = 0$ and $c' + 2c_2' \neq 0$. Then $w\delta = \delta$ and we have

$$g = u_1 x_{\delta-\rho}(c_2')hw_1v_1, \tag{I}$$

where $w_1 \in W(\{\rho, \delta\}^\perp)$, $\Gamma(v_1) \subset \Gamma_1 = \{\rho, \delta\}_0^\perp$, $\Gamma(u_1) \subset \Gamma_2 = \{\sigma \in \Phi^+ \mid \sigma+\rho, \sigma+\delta \notin \Phi\}$.

If $c_2 = 0$, $c' + 2c_2' = 0$, $c_j' \neq 0$ for some $j > 2$, we conclude that $w\delta = \rho_j$ and

$$g = u_1 x_{\rho_j-\rho}(c_j')x_{\delta-\rho}(-c'/2)hw_{\delta-\rho_j}w_1v_2, \tag{II}$$

where u_1, w_1 are as above, and $\Gamma(v_2) \subset \Gamma_1 \cup \{\delta - \rho_3, \ldots, \delta - \rho_j\}$.

Finally, suppose that $c + 2c_2 = 0$, so $c_j' = 0$ for $j > 2$, and $c' + 2c_2' = 0$. The element g has the form

$$g = u_1 x_{\delta-\rho}(-c'/2)hw_\lambda w_1 x_{\delta-\rho}(-c/2)v_3, \tag{III}$$

where u_1, w_1 are as above, $\lambda = \delta + \rho_j - 2\rho$ for some $j > 2$ or $\lambda = \delta - \rho$, and $\Gamma(v_3) \subset \Gamma_1 \cup \{\rho_j - \rho, \rho_j - \rho_k, k > j, \rho_j + \rho_k - 2\rho, 2 \leq k \neq j\}$.

If g is of type (I), (II), or (III) with $j \neq 2$, it is easily seen that (7) is impossible. If g is of type (III) and $j = 2$, we get

$$hx_\delta(s)x_\rho(-cs)x_{2\rho-\delta}(s - c^2 s)h^{-1} = x_\delta(s' - c'^2 s')x_\rho(-c's')x_{2\rho-\delta}(s').$$

The above analysis of conjugation by h implies that $c^2 = c'^2$.

(S19)–(S21). The arguments are similar to those used for cases (S2)–(S4). Equation (7) becomes

$$\begin{aligned} &hw_{\delta-\rho}x_{-\delta}(s)x_{-\rho}(-cs)x_{\delta-2\rho}(s - c^2 s)w_{\delta-\rho}h^{-1} \\ &\quad = x_{-\delta}(s')x_{-\rho}(-c's')x_{\delta-2\rho}(s' - c'^2 s'). \end{aligned}$$

Again, we deduce that $c^2 = c'^2$.

(S8). Here we must consider the system

$$hwvx_1(t)x_{1+3}(ct)v^{-1}w^{-1}h^{-1} = u^{-1}x_1(t')x_{1+3}(c't')u, \tag{9}$$

$$hwvx_{2+3}(s)x_{2-3}(s)v^{-1}w^{-1}h^{-1} = u^{-1}x_{2+3}(s')x_{2-3}(-s')u \tag{10}$$

for some $g = uhwv$.

Proceeding as in cases (S2)–(S4), from (9) we see that g has one of the following forms:

$$g = u_1x_3(c_3')hw_1v_1, \tag{I}$$

where $w_1 \in W(\{\rho, \rho_3\}^\perp)$, and $\Gamma(v_1) \subset \Gamma_1 = \{\rho, \rho_3\}_0^\perp$, $\Gamma(u_1) \subset \Gamma_2 = \{\sigma \in \Phi^+ \mid \sigma + \rho, \sigma + \rho_3) \notin \Phi\}$; or

$$g = u_1x_3(-c_3')x_2(c_2')x_{2-3}(-c_2'/2)x_j(c_j')hw_{3-j}w_1v_2, \tag{II}$$

where $c_j' \neq 0$, $j > 3$, u_1, w_1 are as above, and $\Gamma(v_2) \subset \Gamma_1 \cup \{\rho_3 - \rho_4, \dots, \rho_3 - \rho_j\}$; or

$$g = u_1x_3(-c'/2)x_2(c_2')x_{2-3}(-c_2'/2)hw_{3-j}w_{2+k}w_1x_3(-c/2)x_2(c_2)x_{2-3}(-c_2/2)v_3, \tag{III}$$

where $k \neq j > 3$, u_1, w_1 are as above, and $v_3 \in U$.

Now (10) reduces to the action of $h \in H$ on $\{x_{2+3}(*)x_2(*)x_{2-3}(*)\}$. It follows from the above that $c^2 = c'^2$.

(S16), $\Phi = \mathrm{C}_\ell$. Here we deal with the system

$$hwvx_{1+2}(t)x_2(ct)v^{-1}w^{-1}h^{-1} = u^{-1}x_{1+2}(t')x_2(c't')u,$$

$$hwvx_1(s)x_{-2}(-s)v^{-1}w^{-1}h^{-1} = u^{-1}x_1(s')x_{-2}(-s')u$$

for some $g = uhwv$. The proof for this case is similar to the above.

Now we prove that cases (S6) and (S9) lead to two orbits.

(S6). First, we turn to the root system of type F_4. The representatives of orbits in cases $(\mathrm{S6})_1$ and $(\mathrm{S6})_2$ are the pairs

$$(X_\rho, \{x_{2+3}x_{2-3}(-1)\}) \quad \text{and} \quad (X_\rho, \{x_{1+3}x_{2+4}(-1)\}),$$

respectively (see Tables 6 and 7).

Let $N(X_\rho)$ be the normalizer of X_ρ, and let $g = uhwv \in N(X_\rho)$. Then $w \in W(\{\rho\}^\perp)$, $\Gamma(v) \subset \{\rho\}_0^\perp$, and $\Gamma(u) \subset \{\sigma \in \Phi^+ \mid \sigma + \rho \notin \Phi\}$. For such an element g, the equation

$$hwvx_{2+3}x_{2-3}(-1)v^{-1}w^{-1}h^{-1} = u^{-1}x_{1+3}x_{2+4}(-1)u$$

is unsolvable.

For $\Phi = \mathrm{C}_\ell$ we have (Table 5)

$$(X_\rho, \{x_{3+4}\}) \quad \text{and} \quad (X_\rho, \{x_{1+3}x_{2+3}\}),$$

and the same argument shows that these pairs are not conjugate.

(S9). Here we consider the pairs

$$(x_1x_{1+2}, \{x_{1+3}x_{-2+4}(-1)\}) \quad \text{and} \quad (x_1x_{1+2}x_{1+3}, \{x_{1-3}x_{-2+4}(-1)\}).$$

If $g = uhwv$ transforms x_1x_{1+2} to $x_1x_{1+2}x_{1+3}$, then $w \in \langle w_{2-3}, w_{3-4}, w_4\rangle$ and $\Gamma(v) \subset \{2 \pm 3, 2 \pm 4, 3 \pm 4, 2, 3\}$. If these pairs are conjugate, then X_{1+4} is transformed to itself. Thus, $w = w_{2-3}$, $\Gamma(v) \subset \{2-3\}$, $\{e_3\} \in \Gamma(u)$. Now, we return to the conditions ensuring that $\{x_{1-3}x_{-2+4}(-1)\}$ is carried to $\{x_{1-3}x_{-2+4}(-1)\}$ to see that this is impossible.

It only remains to consider various cases for which $\langle X_1, X_2\rangle$ has the same description in the sense of §5. These are cases (S1) and (S2), and cases (S8) and $(\mathrm{S9})_{1,2}$.

(S1) and (S2). Here we have

$$(X_\rho, X_\gamma) \quad \text{and} \quad (\{x_\rho x_\delta(c)\}, \{x_\delta x_{2\rho-\delta}(-1)\}),$$

where c is such that $c^2 - 4 \notin K^{*2}$. Should these pairs be conjugate, this would also have been true over an algebraically closed field, which is absurd.

(S8) and (S9), $\Phi = \mathrm{F}_4$. Here

$$(x_1x_{1+2}(c), \{x_{2+3}x_{-2+3}(-1)\}) \quad \text{and} \quad (x_1x_{1+2}x_{1+3}(\varepsilon), \{x_{1-3}x_{-2+4}(-1)\}),$$

where $c \in K^*$, $\varepsilon = 0$ or 1. These pairs are not conjugate for the same reason as in case (S6).

This completes the proof of Theorem 1.

Note that the existence of two orbits in cases (S6) and (S9) is explained by the presence in the corresponding X of nonconjugate one-parameter subgroups that are not root subgroups.

§7. Proof of Theorem 2

Lemma 2 shows that now we need to consider only Variants 1–6. For convenience, we extract from Tables 5–7 the representatives of the orbits that may arise in this case.

Table 8

Case	Variants	$\Phi = \mathrm{C}_\ell$	$\Phi = \mathrm{B}_\ell$, F_4
(T1)	2	$\{x_{1+2}, x_{1+3}\}$	$\{x_1, x_{1+2+3+4}\}$
$(\mathrm{T2})_1$	3	$\{x_{1+2}, x_{1-2}\}$	$\{x_1, x_2\}$
$(\mathrm{T2})_2$	4	$\{x_{1+2}, x_{3+4}\}$	—
(T3)	5	$\{x_{1+2}, x_{-1+3}\}$	$\{x_1, x_{-1+2+3+4}\}$
(T4)	6	$\{x_{1+2}, x_{-1-2}\}$	$\{x_1, x_{-1}\}$

Cases (T1), $(\mathrm{T2})_2$, (T3), and (T4) have already been discussed as cases $(\mathrm{S1})^*$, $(\mathrm{S6})_1^*$, $(\mathrm{S10})^*$, and $(\mathrm{S15})^*$ (respectively) in the proof of Theorem 1. The characteristic plays no role here. The only new case $(\mathrm{T2})_1$ arises when the subgroup $\langle X_1, X_2\rangle$ is generated by $x_\rho(t)$ and $x_\gamma(s)$, where $\rho + \gamma$ is a long root. Then $\langle X_1, X_2\rangle = X_1 \times X_2$, because the corresponding coefficient in the Chevalley commutator formula vanishes in characteristic 2. This proves Theorem 2.

Acknowledgement. I am very grateful to N. A. Vavilov for suggesting the problem and essential help in the final stage of the preparation of the manuscript. Also, I would like to express my special thanks to the referee of the journal *Geometria Dedicata* for his extremely careful reading of the paper, checking all details and suggesting many corrections and improvements concerning contents, English, and otherwise.

References

[B] N. Bourbaki, *Éléments de mathématique.* Fasc. XXXIV. *Groupes et algèbres de Lie.* Chapitre IV–VI, Actualités Sci. Indust., No. 1337, Hermann, Paris, 1968. MR0240238 (39:1590)

[V1] N. A. Vavilov, *The geometry of long root subgroups in Chevalley groups*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **1988**, vyp. 1, 8–11; English transl., Vestnik Leningrad Univ. Math. **21** (1988), no. 1, 5–10. MR0946454 (90d:20078)

[V2] ———, *The relative arrangement of long and short root subgroups in a Chevalley group*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **1989**, vyp. 1, 3–7; English transl., Vestnik Leningrad Univ. Math. **22** (1989), no. 1, 1–7. MR1021468 (90k:20075)

[V3] ———, *Root semisimple elements and triples of root unipotent subgroups in Chevalley groups*, Voprosy Algebry, No. 4, Gomel′skiĭ Univ., Minsk, 1989, pp. 162–173. (Russian) MR1011925 (90i:20046)

[V4] ———, *Subgroups of Chevalley groups containing a maximal torus*, Trudy Leningrad. Mat. Obshch. **1** (1990), 64–109; English transl., Amer. Math. Soc. Transl. Ser. 2, vol. 155, Amer. Math. Soc., Providence, RI, 1993, pp. 59–100. MR1104207 (92f:20046)

[V5] N. A. Vavilov, *Intermediate subgroups in Chevalley groups*, Groups of Lie Type and their Geometries (Como, 1993), London Math. Soc. Lecture Note Ser., vol. 207, Cambridge Univ. Press, Cambridge, 1995, pp. 233–280. MR1320525 (96c:20085)

[D] E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sb. **30 (72)** (1952), no. 2, 349–362. (Russian) MR0047629 (13:904c)

[Z] A. E. Zalesskiĭ, *Linear groups*, Uspekhi Mat. Nauk **36** (1981), no. 5, 57–107; English transl. in Russian Math. Surveys **36** (1981), no. 5. MR0637434 (83d:20032)

[ZS] A. E. Zalesskiĭ and V. N. Serezhkin, *Linear groups generated by transvections*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 1, 26–49; English transl. in Math. USSR-Izv. **10** (1976), no. 1. MR0412295 (54:421)

[K] A. S. Kondrat′ev, *Subgroups of finite Chevalley groups*, Uspekhi Mat. Nauk **41** (1986), no. 1, 57–96; English transl., Russian Math. Surveys **41** (1986), no. 1, 65–118. MR0832410 (87g:20078)

[N1] V. V. Nesterov, *Arrangement of long and short root subgroups in a Chevalley group of type G_2*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **272** (2000), 273–285; English transl., J. Math. Sci. (New York) **116** (2003), no. 1, 3035–3041. MR1811807 (2002m:20080)

[N2] ———, *Pairs of short root subgroups in a Chevalley group of type G_2*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **281** (2001), 253–273; English transl., J. Math. Sci. (New York) **120** (2004), no. 4, 1630–1641. MR1875729 (2002i:20024)

[St] R. Steinberg, *Lectures on Chevalley groups*, Yale Univ., New Haven, Conn., 1968. MR0466335 (57:6215)

[AS] M. Aschbacher and G. M. Seitz, *Involutions in Chevalley groups over fields of even order*, Nagoya Math. J. **63** (1976), 1–91; Correction, Nagoya Math. J. **72** (1978), 135–136. MR0514895 (80b:20058)

[BH] R. Brown and S. P. Humphries, *Orbits under symplectic transvections.* I, II, Proc. London Math. Soc. (3) **52** (1986), 517–531, 532–556. MR0833648 (87i:20095a); MR0833649 (87i:20095b)

[C] R. W. Carter, *Simple groups of Lie type*, Pure Appl. Math., vol. 28, Wiley, London etc., 1972, 331 pp. MR0407163 (53:10946)

[C1] B. N. Cooperstein, *Subgroups of the group $E_6(q)$ which are generated by root subgroups*, J. Algebra **46** (1977), 355–388. MR0460482 (57:475)

[C2] ———, *The geometry of root subgroups in exceptional groups.* I, II, Geom. Dedicata **8** (1979), no. 3, 317–381; **15** (1983), no. 1, 1–45. MR0550374 (81e:20022); MR0732540 (85h:20019)

[C3] ———, *Geometry of long root subgroups in groups of Lie type*, The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, CA, 1979), Proc. Sympos. Pure Math., vol. 37, Amer. Math. Soc., Providence, RI, 1980, pp. 243–248. MR0604589 (82e:20056)

[C4] ———, *Subgroups of exceptional groups of Lie type generated by long root elements.* I, II, J. Algebra **70** (1981), 270–282, 283–298. MR0618394 (83i:20038a); MR0618395 (83i:20038b)

[Cu] H. Cuypers, *The geometry of k-transvection groups*, Preprint, Eindhoven Univ. Technology, 1994, 1–24.

[DV] L. Di Martino and N. A. Vavilov, *$(2,3)$-generation of $SL(n,q)$.* II, Comm. Algebra **24** (1996), 487–515. MR1373489 (97e:20067)

[K1] W. M. Kantor, *Subgroups of classical groups generated by long root elements*, Trans. Amer. Math. Soc. **248** (1979), no. 2, 347–379. MR0522265 (80g:20057)

[K2] ———, *Generation of linear groups*, The Geometric Vein: The Coxeter Festschift, Springer-Verlag, Berlin–New York, 1981, pp. 497–509. MR0661799 (83g:51025)

[L1] Shang Zhi Li, *Maximal subgroups containing root subgroups in finite classical groups*, Kexue Tongbao (English Ed.) **29** (1984), no. 1, 14–18. MR0763638 (86a:20055)

[L2] ———, *Maximal subgroups in $P\Omega(n, F, Q)$ with root subgroups*, Sci. Sinica Ser. A **28** (1985), no. 8, 826–838. MR0815477 (87e:20079)

[L3] ———, *Maximal subgroups containing short-root subgroups in $PSp(2n, \mathbb{F})$*, Acta Math. Sinica (N.S.) **3** (1987), no. 1, 82–91. MR0915853 (88k:20063)

[LS] M. W. Liebeck and G. M. Seitz, *Subgroups generated by root elements in groups of Lie type*, Ann. of Math. (2) **139** (1994), 293–361. MR1274094 (95d:20078)

[M1] J. McLaughlin, *Some groups generated by transvections*, Arch. Math. (Basel) **18** (1967), no. 4, 364–368. MR0222184 (36:5236)

[M2] ———, *Some subgroups of $SL_n(\mathbb{F}_2)$*, Illinois J. Math. **13** (1969), no. 1, 108–115. MR0237660 (38:5941)

[M] H. H. Mitchell, *The subgroups of the quaternary abelian linear group*, Trans. Amer. Math. Soc. **15** (1914), no. 4, 379–396. MR1500986

[Po] H. Pollatsek, *Irreducible groups generated by transvections over finite fields of characteristic two*, J. Algebra **39** (1976), 328–333. MR0401942 (53:5768)

[S1] B. S. Stark, *Some subgroups of $\Omega(V)$ generated by groups of root type* 1, Illinois J. Math. **17** (1973), 584–607. MR0322070 (48:434)

[S2] ———, *Some subgroups of $\Omega(V)$ generated by groups of root type*, J. Algebra **29** (1974), 33–41. MR0342623 (49:7369)

[S3] ———, *Irreducible subgroups of orthogonal groups generated by groups of root type* 1, Pacific J. Math. **53** (1974), 611–625. MR0357642 (50:10110)

[T1] F. G. Timmesfeld, *Groups generated by root-involutions*. I, II, J. Algebra **33** (1975), 75–134; ibid. **35** (1975), 367–441. MR0372019 (51:8236)

[T2] ———, *Groups generated by k-transvections*, Invent. Math. **100** (1990), 167–206. MR1037146 (91b:20059)

[T3] ———, *Groups generated by k-root subgroups*, Invent. Math. **106** (1991), 575–666. MR1134484 (92j:20026)

[T4] ———, *Groups generated by k-root subgroups—a survey*, Groups, Combinatorics and Geometry (Durham, 1990), London Math. Soc. Lecture Note Ser., vol. 165, Cambridge Univ. Press, Cambridge, 1992, pp. 183–204. MR1200260 (94f:20060)

[T5] ———, *Abstract root-subgroups and quadratic action*. I–III (submitted for publication).

[W] A. Wagner, *Groups generated by elations*, Abh. Math. Sem. Univ. Hamburg **41** (1974), 190–205. MR0346072 (49:10798)

Baltic State Technical University of St. Petersburg, 1st Krasnoarmeiskaya 1, St. Petersburg 190005, Russia

E-mail address: vl.nesterov@mail.ru

Received 17/MAR/2004

Translated by THE AUTHOR