

ON THE RELATIVE DISTRIBUTION OF EIGENVALUES OF EXCEPTIONAL HECKE OPERATORS AND AUTOMORPHIC LAPLACIANS

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Dedicated to Ludwig Faddeev on the occasion of his seventieth birthday

ABSTRACT. The relative distribution of the embedded eigenvalues of exceptional Hecke operators and automorphic Laplacians is studied in connection with the Phillips and Sarnak conjectures concerning the violation of the Weyl law.

INTRODUCTION

The problem we consider here is related to the paper [8] of Ludwig Faddeev, where he proved a theorem on expansion in automorphic eigenfunctions of the Laplacian L on the hyperbolic plane H . He studied the most interesting case of automorphic functions defined for a general discrete cofinite subgroup $\Gamma \subset G$ of the isometry group of H . He found an efficient method of analytic continuation of the resolvent of this Laplacian beyond the continuous spectrum.

It was proved by Selberg that for any congruence subgroup of the modular group Γ_Z the automorphic Laplacian, which we denote $A(\Gamma)$, has an infinite sequence of eigenvalues $\{\lambda_i\}$ embedded in the continuous spectrum and satisfying the Weyl law

$$N(\lambda, \Gamma) = \#\{\lambda_i \leq \lambda\} \rightarrow (|F|/4\pi) \cdot \lambda, \quad \lambda \rightarrow \infty.$$

Here $|F|$ is the area of the fundamental domain F of the group Γ in H . The eigenvalues λ_i are counted in accordance with their multiplicity. The same is true for the Laplacian $A(\Gamma, \chi)$, where instead of the usual Γ -automorphic functions we consider Γ -automorphic functions twisted by a one-dimensional unitary representation χ of Γ , and we assume that the kernel of χ is a congruence subgroup again.

It is an important but a very difficult question as to whether the Weyl law for the embedded eigenvalues is a characteristic of the congruence subgroups, or it may be fulfilled also for some more general cofinite groups Γ .

To investigate this problem, Phillips and Sarnak introduced and studied perturbation theory for Laplacians $A(\Gamma, \chi)$, varying the group Γ in the Teichmüller space of a given congruence subgroup Γ_0 , and varying the representation χ in the Jacobi manifold of a given congruence character χ_0 . Applying Kato's perturbation theory of isolated eigenvalues, they found a version of the famous Fermi Golden Rule. The disappearance of an embedded eigenvalue λ_i (disappearance means that λ_i becomes a pole of the scattering matrix) under a perturbation was related to the question as to whether a certain value of a Dirichlet L -series given by the Phillips and Sarnak integral is different from zero.

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The integral was constructed explicitly in terms of modular forms as the Rankin–Selberg convolution.

After the results of Colin de Verdière and Mueller, it must be remembered that on a general Riemannian manifold with cusps the Laplace–Beltrami operator has no eigenvalues embedded in the continuous spectrum. But unlike perturbations in the Teichmüller space, for a general metric we always have infinitely many parameters to manipulate with embedded eigenvalues in order to move all of them away from the spectrum.

To prove that the Phillips–Sarnak integral is not zero for a given embedded eigenvalue λ_i and eigenfunction v_i is a very difficult problem if we consider regular perturbations in the Teichmüller space or in the Jacobi manifold of Γ . Regularity means that the eigenvalue λ_i remains isolated under a small perturbation. This problem can be handled by averaging over i . In the remarkable paper [9], Luo indicated that, under certain multiplicity conjectures for the embedded eigenvalues, the Weyl law is violated for $A(\Gamma_\varepsilon)$, where Γ_ε is a small regular perturbation of a congruence subgroup Γ_0 . In the 1990s S. Wolpert discovered a special class of singular perturbations of a given congruence group Γ_0 , approaching it from a bigger Teichmüller space. In that case he was able to handle the corresponding Phillips and Sarnak integrals for all embedded eigenvalues and for all corresponding eigenfunctions that belong to the Hecke basis. But unfortunately, the singularity of these perturbations made it impossible to apply Kato’s theory, because each point $\lambda \in [1/4, \infty)$ of the spectrum was an accumulation point of poles of scattering matrices for $A(\Gamma_\varepsilon)$ as $\varepsilon \rightarrow 0$ (Selberg’s phenomenon). Still up to now, it is entirely unclear whether there exists a perturbation theory capable of handling this singularity.

In [1] we introduced and studied character perturbations of the automorphic Laplacian $A(\Gamma_0, \chi_0)$ for the Hecke groups $\Gamma_0(N)$ with primitive congruence character χ_0 . We assumed that $N = 4N_2$ or $N = 4N_3$, where N_2 and N_3 are products of distinct primes, and $N_2 \equiv 2 \pmod{4}$, $N_3 \equiv 3 \pmod{4}$. In these cases we found the regular perturbations of $A(\Gamma_0, \chi_0)$ in the Jacobi manifold of χ_0 . This allows for a rigorous analysis of the problem of stability of embedded eigenvalues. At the same time, the Phillips–Sarnak integral for given λ_i and v_i , an eigenfunction of $A(\Gamma_0, \chi)$, which is also an element of the Hecke basis, can be handled by using ideas almost similar to Wolpert’s, when he studied his singular perturbations. The difference is that we face a new problem, which arises from the exceptional Hecke operators $U(q)$, $q|N$. Let A_{odd} denote the projection of $A(\Gamma_0, \chi_0)$ to the subspace of odd (Γ, χ_0) -automorphic functions. The operators $U(q)$ are unitary [1, Theorem 4.1], so the eigenvalues $\rho(q)$ lie on the unit circle. The basic results on the Phillips–Sarnak integral follow from [1, (7.23), (7.24)]. We formulate this as a theorem.

Theorem 1.I. *Let $\varepsilon_q \neq 0$, $q|N$, $q > 2$, be the fixed parameters of the perturbation (see [1, Theorem 6.2]), and let Φ_n be a common eigenfunction of A_{odd} with eigenvalue λ_n , and of $U(q)$ with eigenvalues $\rho_n(q)$, $q|N$. Then the Phillips–Sarnak integral $I(\Phi_n, \lambda_n)$ is different from 0 if and only if*

$$\rho_n(2) \neq 2^{ir_n} \text{ and } \rho_n(q) \neq \frac{q^{ir_n}}{\varepsilon_q} \text{ for } q > 2.$$

In [1, Theorem 4.3] it was stated that $\rho_n(q) = \pm 1$ for all $q|N$. This gives rise only to the exceptional sequences $r_n = \pi n / \log 2$ and $r_{n,q} = \pi n / \log q$, $n \in \mathbb{Z}$, $q|N$, $q > 2$. However, this lemma [1, Theorem 4.3] is not correct. The eigenvalues of $U(q)$ may lie anywhere on the unit circle. Consequently, [1, Theorem 7.1] should be replaced by Theorem 1.I. This leaves us with the problem of analyzing the conditions of Theorem 1.I. For $q > 2$ we can obtain $\rho_n(q) \neq q^{ir_n} / \varepsilon_q$ by choosing $\varepsilon_q \neq \pm 1$. For $q = 2$ there is no such freedom. *A priori* we might have $\rho_n(2) = 2^{ir_n}$ for all eigenvalues $\lambda_n = 1/4 + r_n^2$ or for no such λ_n .

It is a subtle problem to establish that $\rho_n(2) \neq 2^{ir_n}$ for at least a certain proportion of the eigenvalues λ_n . This is the subject of the present paper.

Here we prove the Weyl law for a certain operator T whose eigenvalues measure the distance $|\rho_n(2) - 2^{ir_n}|$ in the average. From this we deduce that $\rho_n(2) \neq 2^{ir_n}$ asymptotically for at least 1/4 of all eigenvalues λ_n , counted with their multiplicity. Together with the Weyl law for A_{odd} , this implies the following result, which replaces [1, Theorem 8.5].

Theorem 2.I. *We have*

$$\liminf_{\lambda \rightarrow \infty} \# \frac{\{\lambda_n \leq \lambda \mid I(\Phi_n, \lambda_n) \neq 0\}}{\lambda} \geq \frac{|F|}{32\pi},$$

where the eigenvalues λ_n are counted with their multiplicity.

Assuming further that the dimensions of all odd eigenspaces are bounded, we obtain the following result, which replaces [1, Corollary 8.7(c)].

Corollary 1.I. *Suppose that $\dim N(A_{\text{odd}} - \lambda_n) \leq m$ for all n . Let $\tilde{\lambda}_n$ be any eigenvalue of A_{odd} such that, for some $\tilde{\Phi}_n \in N(A_{\text{odd}} - \tilde{\lambda}_n)$, $\tilde{\Phi}_n(\varepsilon)$ is a resonance function for small $\varepsilon \neq 0$. Then*

$$\liminf_{\lambda \rightarrow \infty} \frac{\#\{\tilde{\lambda}_n \leq \lambda\}}{\lambda} \geq \frac{|F|}{32\pi m},$$

where $\tilde{\lambda}_n$ is not counted with its multiplicity. Thus, asymptotically, at least $\frac{1}{4}m$ of the eigenfunctions become resonance functions for $\varepsilon \neq 0$.

Our results remain qualitatively the same as in [1], but the number of eigenvalues that are shown to be unstable is reduced. Similar results can be obtained for $\varepsilon_q = \pm 1$, but with reduction by additional factors.

We want to thank Frederik Stroemberg for pointing out the mistake in [1, Theorem 4.3].

§1. THE HECKE EXCEPTIONAL OPERATORS $U(2), U(2)^*$ AND THEIR SQUARES

We have

$$(*) \quad \begin{aligned} U(2)f(z) &= \frac{1}{\sqrt{2}}[f(\frac{z}{2}) + f(\frac{z+1}{2})], \\ U(2)^*f(z) &= \frac{1}{\sqrt{2}}[f(2z) + f(\frac{2z}{-Nz+1})]. \end{aligned}$$

Recall our discrete group $\Gamma = \Gamma_0(N)$, where $N = 4N_2$ or $N = 4N_3$ [1]. It can be seen that $U(2)^*U(2)f = U(2)U(2)^*f = f$. We identify linear-fractional maps with the corresponding elements of $\text{PSL}(2, \mathbb{R})$. We denote $U = U(2)$, $U^* = U(2)^*$, and we have the following correspondence, where all matrices are taken mod ± 1 .

For U :

$$(1.1) \quad \begin{cases} z \rightarrow \frac{z}{2} & \leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \\ z \rightarrow \frac{z+1}{2} & \leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix}. \end{cases}$$

For U^* :

$$(1.2) \quad \begin{cases} z \rightarrow 2z & \leftrightarrow \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \\ z \rightarrow \frac{2z}{-Nz+1} & \leftrightarrow \begin{pmatrix} \sqrt{2} & 0 \\ \frac{-N}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \end{cases}$$

For U^2 :

$$(1.3) \quad \begin{cases} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}, \\ \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 2 \end{pmatrix}, \\ \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 2 \end{pmatrix}, \\ \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 2 \end{pmatrix}. \end{cases}$$

For U^{*2} :

$$(1.4) \quad \begin{cases} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \\ \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ \frac{-N}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ \frac{-N}{2} & \frac{1}{2} \end{pmatrix}, \\ \begin{pmatrix} \sqrt{2} & 0 \\ \frac{-N}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ \frac{-N}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ \frac{-3N}{2} & \frac{1}{2} \end{pmatrix}, \\ \begin{pmatrix} \sqrt{2} & 0 \\ \frac{-N}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -N & \frac{1}{2} \end{pmatrix}. \end{cases}$$

The four sets of elements of $\mathrm{PSL}(2, \mathbb{R})$ occurring on the right-hand sides of (1.1)–(1.4) will be denoted by P_1, P_2, P_3, P_4 .

Lemma 1. *For $j = 1, 2, 3, 4$, we have $\Gamma P_j = \Gamma P_j \Gamma$.*

Proof. This follows from the definition of the Hecke operators. But since we start with the definition of U, U^* by (*), we can recall the argument. If φ is a continuous function with compact support in H , then for

$$f_\varphi(z) = \sum_{\gamma \in \Gamma} \chi(\gamma) \varphi(\gamma z)$$

we have $f_\varphi(\gamma_0 z) = \chi(\gamma_0) f_\varphi(z)$, where $\gamma_0 \in \Gamma$ and χ is a real primitive character on $\Gamma_0(N)$, $\chi(\gamma_0) = \chi(\gamma_0^{-1})$ (see [1]). If we denote by T_j one of the operators U, U^*, U^2, U^{*2} , then

$$T_j f_\varphi(\gamma_0 z) = \chi(\gamma_0) T_j f_\varphi(z) \quad \text{and} \quad T_j f_\varphi(z) = \sum_{\gamma \in \Gamma} \chi(\gamma) \sum_{p \in P_j} \varphi(\gamma p z).$$

It follows that

$$(1.5) \quad \begin{aligned} \sum_{\gamma \in \Gamma} \chi(\gamma) \sum_{p \in P_j} \varphi(\gamma p \gamma_0 z) &= \sum_{\gamma \in \Gamma} \chi(\gamma_0^{-1} \gamma) \sum_{p \in P_j} \varphi(\gamma p z) \\ &= \sum_{\gamma \in \Gamma} \chi(\gamma) \sum_{p \in P_j} \varphi(\gamma_0 \gamma p z). \end{aligned}$$

Since (1.5) is valid for all functions φ of this type, the lemma follows. \square

Now we study the sets ΓP_j and their conjugacy classes by conjugation from Γ , i.e., $\{\gamma p\}_\Gamma = \{\gamma_1 \gamma p \gamma_1^{-1} \mid \gamma_1 \in \Gamma\}$, $\gamma \in \Gamma$, $p \in P_j$, $j = 1, 2, 3, 4$.

Lemma 2. *There are no parabolic classes $\{\gamma p\}_\Gamma$ in any ΓP_j , $j = 1, 2, 3, 4$.*

Proof. We must check that $\text{tr}(\gamma p) \not\equiv 2 \pmod{\pm 1}$. Take

$$\gamma = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma,$$

$ad - Nbc = 1$. Since N is equal to $4N_2$ or $4N_3$, a and d are odd integers, and the lemma follows. \square

The next result is well known.

Lemma 3.

- 1) *There are at most finitely many elliptic classes $\{\gamma p\}_\Gamma$ in ΓP_j , $j = 1, 2, 3, 4$.*
- 2) *There are infinitely many hyperbolic classes in ΓP_j , $j = 1, 2, 3, 4$.*

Obviously, the unity $e \in \Gamma$ is not in $\Gamma P = \bigcup_{j=1}^4 \Gamma P_j$. We shall study the centralizers $\Gamma_{\gamma p}$ of γp in Γ . By definition,

$$\Gamma_{\gamma p} = \{\gamma_1 \in \Gamma \mid \gamma_1 \gamma p = \gamma p \gamma_1\}$$

for $\gamma \in \Gamma$, $p \in P_j$, $j = 1, 2, 3, 4$.

Clearly, $\Gamma_{\gamma p}$ is a subgroup of Γ , and it is known that each $\Gamma_{\gamma p}$ is a cyclic group, possibly trivial, which means that $\Gamma_{\gamma p} = \{e\}$ for some γp .

Lemma 4. *A hyperbolic element of $\text{PSL}(2, \mathbb{R})$ only commutes with the identity and with hyperbolic elements.*

Proof. To check this, we use the language of linear fractional transformations. Any hyperbolic transformation is conjugated in $\text{PSL}(2, \mathbb{R})$ to a transformation of the type $z \mapsto \lambda^2 z$, $\lambda > 1$, $z \in H$. Assume that

$$z \mapsto \frac{az + b}{cz + d},$$

corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{R})$, commutes with the above transformation, so that

$$\frac{a\lambda^2 z + b}{c\lambda^2 z + d} = \lambda^2 \frac{az + b}{cz + d}, \quad z \in H.$$

This implies that $a \neq 0$, $b = c = 0$, $d = a^{-1}$. Thus, the transformation defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is either hyperbolic or coincides with the identity. \square

Similarly, we have the following statement.

Lemma 5. *A parabolic element of $\text{PSL}(2, \mathbb{R})$ only commutes with the identity and with parabolic elements.*

Also, a simple check shows that the following is true.

Lemma 6. *There are no elliptic classes in Γ .*

Theorem 1 below is a consequence of Lemmas 3–5.

Theorem 1.

- 1) Any elliptic class $\{\gamma p\}_\Gamma$, $\gamma \in \Gamma$, $p \in P_j$, $j = 1, 2, 3, 4$, has only trivial centralizer $\Gamma_{\gamma p} = \{e\}$.
- 2) For a hyperbolic class $\{\gamma p\}$, $\gamma p \in \Gamma P_j$, we have the following alternative:
 - (a) $\Gamma_{\gamma p} = \{e\}$,
 - (b) $\Gamma_{\gamma p}$ is generated by a hyperbolic element in Γ .

We shall study the hyperbolic classes in ΓP_j in more detail and characterize their centralizers.

The proof of Lemma 4 implies the following statement.

Lemma 7. *Two hyperbolic elements of $\mathrm{PSL}(2, \mathbb{R})$ commute if and only if they have the same fixed points as linear fractional transformations.*

Let

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ Nc_1 & d_1 \end{pmatrix} \in \Gamma$$

be a hyperbolic element. Then $c_1 \neq 0$. The equation $\frac{a_1 z + b_1}{c_1 z + d_1} = z$ has two solutions:

$$(1.6) \quad z_{1,2} = \frac{a_1 - d_1 \pm \sqrt{(a_1 + d_1)^2 - 4}}{2Nc_1}.$$

Since $N = 4N_2$ or $N = 4N_3$, the numbers a_1 and d_1 are odd integers (recall that $a_1 d_1 - N b_1 c_1 = 1$), $a_1 + d_1$ is an even integer, and $(a_1 + d_1)^2$ is divisible by 4. It follows that for any hyperbolic element γ of Γ the integer $(\mathrm{tr} \gamma)^2 - 4$ cannot be the square of an integer.

Now consider a hyperbolic element

$$g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma P.$$

It has two fixed points as a transformation of H :

$$(1.7) \quad t_{1,2} = \frac{a_2 - d_2 \pm \sqrt{(a_2 + d_2)^2 - 4}}{2c_2}.$$

The problem is how for given g_2 to find g_1 with the same fixed points $z_{1,2} = t_{1,2}$. Since $\sqrt{(a_1 + d_1)^2 - 4}$ is always irrational, we have

$$(1.8) \quad \begin{aligned} \frac{a_1 - d_1}{2Nc_1} &= \frac{a_2 - d_2}{2c_2}, \\ \frac{(a_1 + d_1)^2 - 4}{4N^2 c_1^2} &= \frac{(a_2 + d_2)^2 - 4}{4c_2^2}. \end{aligned}$$

This will work, of course, if $(a_2 + d_2)^2 - 4$ is an integer and not a square, which is not necessarily true for all hyperbolic elements of ΓP . We try to solve the system of equations (1.8). From (1.8) we deduce the relations

$$(1.9) \quad \begin{aligned} \frac{a_1 d_1 - 1}{N^2 c_1^2} &= \frac{a_2 d_2 - 1}{c_2^2}, \\ \frac{b_1}{Nc_1} &= \frac{b_2}{c_2}. \end{aligned}$$

From (1.8) and (1.9) it follows that

$$(1.10) \quad \begin{aligned} a_1 &= d_1 + Nc_1 \frac{a_2 - d_2}{c_2}, \\ b_1 &= Nc_1 \frac{b_2}{c_2}. \end{aligned}$$

If

$$\begin{aligned} g &= \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma, \\ p &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in P_i, \end{aligned}$$

then

$$(1.11) \quad \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ Nc\alpha + d\gamma & Nc\beta + d\delta \end{pmatrix}.$$

Since the ratios $\frac{a_2 - d_2}{c_2}$, $\frac{b_2}{c_2}$ occur in (1.10), we can multiply the matrix (1.11) by $\sqrt{2}$ for $i = 1, 2$ or by 2 for $i = 3, 4$, not changing these ratios but getting ratios of integers.

In the first case ($i = 1, 2$) we have

$$(1.12) \quad \begin{aligned} a_1 &= d_1 + Nc_1 \frac{\sqrt{2}(a_2 - d_2)}{\sqrt{2}c_2}, \\ b_1 &= Nc_1 \frac{\sqrt{2}b_2}{\sqrt{2}c_2} \end{aligned}$$

and $\sqrt{2}(a_2 - d_2), \sqrt{2}b_2, \sqrt{2}c_2 \in \mathbb{Z}$. In the second case ($i = 3, 4$),

$$(1.13) \quad \begin{aligned} a_1 &= d_1 + Nc_1 \frac{2(a_2 - d_2)}{2c_2}, \\ b_1 &= Nc_1 \frac{2b_2}{2c_2}, \end{aligned}$$

and $2(a_2 - d_2), 2b_2, 2c_2 \in \mathbb{Z}$. In the first case we assume that

$$(1.14) \quad Nc_1 = 2k_1\sqrt{2}c_2, \quad k_1 \in \mathbb{Z},$$

and we see that

$$(1.15) \quad g_1 = \begin{pmatrix} a_1 & b_1 \\ Nc_1 & d_1 \end{pmatrix} = \begin{pmatrix} d_1 + 2k_1\sqrt{2}(a_2 - d_2) & 2k_1\sqrt{2}b_2 \\ 2k_1\sqrt{2}c_2 & d_1 \end{pmatrix}$$

is an integral matrix with the left lower entry divisible by N . Now we need to prove that there exist $d_1, k_1 \in \mathbb{Z}$ with the property $\det g_1 = 1$. This means we need to prove the existence of integral solutions of the equation

$$(1.16) \quad d_1^2 + 2k_1d_1\sqrt{2}(a_2 - d_2) - 4k_1^2 \cdot 2b_2c_2 = 1,$$

or the equation

$$(1.17) \quad m_1^2 - k_1^2 \cdot 2[(a_2 + d_2)^2 - 4] = 1,$$

where $m_1 = d_1 + k_1\sqrt{2}(a_2 - d_2)$. This is Pell's equation, which has infinitely many integral solutions in $m_1, k_1 \in \mathbb{Z}$ for given a_2, d_2 if $2[(a_2 + d_2)^2 - 4]$ is not the square of an integer (recall that $\sqrt{2}a_2, \sqrt{2}d_2$ are integers). Notice that this is a square if and only if $|\sqrt{2}(a_2 + d_2)| = 3$. If $|\sqrt{2}(a_2 + d_2)| \neq 3$, we can always find a matrix g_1 with $z_{1,2} = t_{1,2}$ that belongs to the centralizer $\Gamma_{\gamma p}$ of a given hyperbolic γp . On the other hand, if $|\sqrt{2}(a_2 + d_2)| = 3$, so that $2[(a_2 + d_2)^2 - 4] = 1$, there is no such matrix g_1 , and the centralizer of γp is $\{e\}$.

For $i = 3, 4$ we obtain a similar result by using (1.13). This leads to the equation

$$m_2^2 - 4k_2^2[(a_2 + d_2)^2 - 4] = 1$$

for the integers m_2, k_2 . Recall that $2a_2, 2d_2$ are given integers in that case. This equation has integral solutions if and only if $4[(a_2 + d_2)^2 - 4]$ is not the square of an integer, i.e., if $2|a_2 + d_2| \neq 5$. In this case the centralizer $\Gamma_{\gamma p}$ of γp is nontrivial and is generated by a hyperbolic element of Γ . If $2|a_2 + d_2| = 5$, then $\Gamma_{\gamma p} = \{e\}$.

We have proved the following.

Theorem 2. *For a hyperbolic class $\{\gamma p\}_\Gamma$ in ΓP from statement 2) of Theorem 1, alternative (a) occurs when $\sqrt{2}|\operatorname{tr} \gamma p| = 3$ for $p \in P_i$, $i = 1, 2$, and when $2|\operatorname{tr} \gamma p| = 5$ for $p \in P_i$, $i = 3, 4$. In those cases, the fixed points of γp in \overline{H} are rational points. The norms of the classes $\{\gamma p\}_\Gamma$ are 2 in case 1 and 4 in case 2. For the other values of $\operatorname{tr}(\gamma p)$, alternative (b) occurs.*

Later we shall see that there are only finitely many classes $\{\gamma p\}_\Gamma$ as in Theorem 2.

§2. THE INVOLUTION $J: z \mapsto -\bar{z}$ AND THE EXCEPTIONAL HECKE OPERATORS

Let $g \in \operatorname{PSL}(2, \mathbb{R})$,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \pmod{\pm 1}.$$

It is easily seen that JgJ acts on H as the matrix

$$\tilde{g} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \pmod{\pm 1}.$$

It is convenient to introduce an isomorphic model of H . This model is well known. We consider the set of positive definite symmetric matrices

$$(2.1) \quad z(x, y) = \begin{pmatrix} y + x^2 y^{-1} & x y^{-1} \\ x y^{-1} & y^{-1} \end{pmatrix},$$

where $y > 0$, $x \in \mathbb{R}$. We define the action of $g \in \operatorname{PSL}(2, \mathbb{R})$ on such matrices by

$$(2.2) \quad gz(x, y) = g[z(x, y)]g^t,$$

where g^t is the transpose of g and the product on the right in (2.2) is the usual product of matrices. It is easily seen that the set $\widehat{H} = \{z(x, y)\}$ with the action (2.2) has the structure of a symmetric space and is isomorphic to H . The isomorphism is given by the map

$$z(x, y) \rightarrow z = x + iy.$$

This model \widehat{H} of the hyperbolic plane has the following useful property, which cannot be seen in the case of H . The reflection J in the model \widehat{H} is given by

$$(2.3) \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \pmod{\pm 1},$$

i.e., it is an element in $\operatorname{GL}(2, \mathbb{R})/(\pm E)$, where E is the identity matrix. Therefore, it makes sense to consider the products Jg, gJ in $\operatorname{PGL}(2, \mathbb{R})$ for $g \in \operatorname{PSL}(2, \mathbb{R})$.

Now we study the relative conjugacy classes $\{gJ\}_G$ under conjugation by elements of $\operatorname{PSL}(2, \mathbb{R}) = G$. Since, obviously, $GJ = GJG$, we can consider the conjugation $g_1 g J g_1^{-1}$, $g_1 \in G$, for fixed $g \in G$. From the trace formula point of view, for the relative conjugacy classes there is an important alternative:

- (1) $\operatorname{tr}(gJ) \neq 0$,
- (2) $\operatorname{tr}(gJ) = 0$ (see [2, §6.5]).

The fixed points of $gJ : H \rightarrow H$ are determined by the equation

$$\begin{aligned} \frac{-a\bar{z} + b}{-c\bar{z} + d} &= z, \\ b - a\bar{z} &= -c|z|^2 + dz. \end{aligned}$$

For $z = x + iy$ we have

$$b - ax = dx - c|z|^2, \quad dy = ay.$$

In case (1), where $\text{tr}(gJ) = d - a \pmod{\pm 1} \neq 0$, we have

$$y = 0, \quad b - ax = dx - cx^2.$$

Thus, for $c = 0$ we obtain $x = \frac{b}{a+d}$.

For $c \neq 0$, the fixed points are

$$(2.4) \quad z_{1,2} = t_{1,2} = \frac{a+d}{2c} \pm \sqrt{\frac{(a+d)^2}{4c^2} - \frac{b}{c}} = \frac{a+d}{2c} \pm \frac{\sqrt{(a-d)^2 + 4}}{2c},$$

where we have used the relation $\det g = 1$.

In case (2), where $\text{tr}(gJ) = d - a = 0$, we have a one-parameter family of fixed points z given by the equation

$$(2.5) \quad c|z|^2 - a(z + \bar{z}) + b = 0,$$

where $z = x + iy$, $\bar{z} = x - iy$, $y > 0$, $x \in \mathbb{R}$.

Now, we consider $gJ, g \in G$, where J is given by (2.3) and $\text{tr}(gJ) \neq 0$. There exists $g_1 \in G$ such that

$$(2.6) \quad g_1(gJ)g_1^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix} \pmod{\pm 1}, \quad \lambda > 1.$$

By definition, the norm $N(gJ)$ is equal to λ^2 .

The next lemma is similar to Lemma 4.

Lemma 8. *An element gJ occurring in (2.6) with $\text{tr}(gJ) \neq 0$ commutes in $\text{PSL}(2, \mathbb{R})$ only with the identity and with hyperbolic elements.*

Later, we shall specify precisely which hyperbolic elements commute with gJ as in Lemma 8. Lemma 1 implies the following statement.

Lemma 9. $\Gamma P_j \Gamma = \Gamma P_j J \Gamma$ for $j = 1, 2, 3, 4$.

Lemma 10. *For any $\gamma \in \Gamma$ and any $p \in P_j$, $j = 1, 2, 3, 4$, we have $\text{tr}(\gamma p J) \neq 0$, where J is given by (2.3).*

Proof. This follows directly from (1.1)–(1.4) and the fact that for any

$$\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma,$$

where a and d are odd integers, we have $N = 4N_2$ or $N = 4N_3$. □

The next lemma is similar to Lemma 7.

Lemma 11. *An element gJ as in Lemma 8 commutes with a hyperbolic element g_1 if and only if they have the same fixed points.*

Now we modify the proof of Theorem 2 for the case where

$$(2.7) \quad g_2 = \begin{pmatrix} a_2 & -b_2 \\ c_2 & -d_2 \end{pmatrix} \in \Gamma P J.$$

The fixed points of g_2 are

$$(2.8) \quad t_{1,2} = \frac{a_2 + d_2}{2c_2} \pm \frac{\sqrt{(a_2 - d_2)^2 + 4}}{2c_2}$$

(see (2.4)). For given g_2 , we must find a hyperbolic $g_1 \in \Gamma$ with the same fixed points $x_{1,2} = t_{1,2}$. Since $\sqrt{(a_1 + d_1)^2 - 4}$ is always irrational, we have (see (1.8))

$$(2.9) \quad \begin{aligned} \frac{a_1 - d_1}{2Nc_1} &= \frac{a_2 + d_2}{2c_2}, \\ \frac{(a_1 + d_1)^2 - 4}{4N^2c_1^2} &= \frac{(a_2 - d_2)^2 + 4}{4c_2^2}, \end{aligned}$$

and, as in (1.9),

$$(2.10) \quad \begin{aligned} \frac{a_1d_1 - 1}{N^2c_1^2} &= \frac{1 - a_2d_2}{c_2^2}, \\ \frac{b_1}{Nc_1} &= -\frac{b_2}{c_2}. \end{aligned}$$

This gives the matrix

$$(2.11) \quad g_1 = \begin{pmatrix} a_1 & b_1 \\ Nc_1 & d_1 \end{pmatrix} = \begin{pmatrix} d_1 + Nc_1 \frac{a_2 + d_2}{c_2} & -\frac{Nc_1 b_2}{c_2} \\ Nc_1 & d_1 \end{pmatrix}.$$

Let $g_2 \in \Gamma P_j J$, where $j = 1, 2$. As in (1.12), we have

$$(2.12) \quad \begin{aligned} a_1 &= d_1 + Nc_1 \frac{\sqrt{2}(a_2 + d_2)}{\sqrt{2}c_2}, \\ b_1 &= Nc_1 \frac{\sqrt{2}b_2}{\sqrt{2}c_2}(-1), \end{aligned}$$

and $\sqrt{2}(a_2 + d_2), \sqrt{2}c_2, \sqrt{2}b_2 \in \mathbb{Z}$. In the same way as in (1.14),

$$(2.13) \quad Nc_1 = 2\kappa_1\sqrt{2}c_2, \quad \kappa_1 \in \mathbb{Z},$$

where

$$(2.14) \quad g_1 = \begin{pmatrix} d_1 + 2\kappa_1\sqrt{2}(a_2 + d_2) & -2\kappa_1\sqrt{2}b_2 \\ 2\kappa_1\sqrt{2}c_2 & d_1 \end{pmatrix}$$

is an integral matrix with $2\kappa_1\sqrt{2}c_2$ divisible by N . In parallel to (1.16) and (1.17), we obtain

$$(2.15) \quad d_1^2 + 2\kappa_1d_1\sqrt{2}(a_2 + d_2) + 4\kappa_1^2 \cdot 2b_2c_2 = 1$$

or

$$(2.16) \quad m_1^2 - 2\kappa_1^2((a_2 - d_2)^2 + 4) = 1,$$

where $m_1 = d_1 + \kappa_1\sqrt{2}(a_2 + d_2)$.

If $g_2 \in \Gamma P_j J$ and $j = 3, 4$, then

$$(2.17) \quad a_1 = d_1 + Nc_1 \frac{2(a_2 + d_2)}{2c_2},$$

$$(2.18) \quad b_1 = -Nc_1 \frac{2b_2}{2c_2},$$

$$Nc_1 = 2\kappa_2 \cdot 2c_2, \quad \kappa_2 \in \mathbb{Z},$$

where

$$(2.19) \quad g_1 = \begin{pmatrix} d_1 + 2\kappa_2 \cdot 2(a_2 + d_2) & -2\kappa_2 \cdot 2b_2 \\ 2\kappa_2 \cdot 2c_2 & d_1 \end{pmatrix}$$

is an integral matrix with $2\kappa_2 \cdot 2c_2$ divisible by N . Since $\det g_1 = 1$, we have

$$(2.20) \quad d_1^2 + 2\kappa_2 d_1 \cdot 2(a_2 + d_2) + 4\kappa_2^2 \cdot 4b_2 c_2 = 1$$

or

$$(2.21) \quad m_2^2 - 4\kappa_2^2((a_2 - d_2)^2 + 4) = 1,$$

where $m_2 = d_1 + \kappa_2 \cdot 2(a_2 + d_2)$.

We can always solve equations (2.16), (2.21) if $2((a_2 - d_2)^2 + 4)$ (or $4((a_2 - d_2)^2 + 4)$ in the second case) is not an integer squared. The latter is true if and only if $\sqrt{2} \operatorname{tr}(\gamma p) \neq \pm 1$ for $p \in P_j$, $j = 1, 2$, and $2 \operatorname{tr}(\gamma p) \neq \pm 3$, $p \in P_j$, $j = 3, 4$.

We have proved the following statement.

Theorem 3. *For a relative conjugacy class $\{\gamma p J\}_\Gamma$, $\gamma \in \Gamma$, we have*

- (a) *If $\sqrt{2}|\operatorname{tr}(\gamma p)| = 1$ for $j = 1, 2$ and $2|\operatorname{tr}(\gamma p)| = 3$ for $j = 3, 4$, then the centralizer $\Gamma_{\gamma p J}$ of $\gamma p J$ in Γ is $\{e\}$.*
- (b) *If $\sqrt{2}|\operatorname{tr}(\gamma p)| \neq 1$ for $j = 1, 2$ and $2|\operatorname{tr}(\gamma p)| \neq 3$ for $j = 3, 4$, then $\Gamma_{\gamma p J}$ is a cyclic group generated by a hyperbolic element.*

§3. THE TRACE FORMULA FOR ODD FUNCTIONS AND THE WEYL LAW

We recall the definition of the Eisenstein series (nonholomorphic) for $\Gamma = \Gamma_0(N)$, $N = 4N_2$ or $N = 4N_3$, that correspond to open cusps for the real primitive character χ (see [1, (2.1)]). For the definitions, we have introduced elements g_j of $\operatorname{PSL}(2, \mathbb{R})$. Now we parametrize these elements by the divisors $d \mid N$, $d > 0$. We have

$$g_d = \begin{pmatrix} \sqrt{m_d} & 0 \\ d\sqrt{m_d} & \sqrt{m_d}^{-1} \end{pmatrix}, \quad g_d S_\infty g_d^{-1} = S_d,$$

where S_∞ is the stabilizer of the cusp at ∞ .

For each open cusp $\frac{1}{d}$, we define the Eisenstein series

$$(3.1) \quad E_d(z, s) = E_d(z, s, \Gamma, \chi) = \sum_{\gamma \in \Gamma_d \backslash \Gamma} y^s (g_d^{-1} \gamma z) \chi(\gamma),$$

where $\operatorname{Re} s > 1$, $\chi(\gamma) = \overline{\chi(\overline{\gamma})}$. We calculate $y^s (g_d^{-1} \gamma z)$. Let

$$\gamma = \begin{pmatrix} a & b \\ Nc & h \end{pmatrix} \in \Gamma.$$

We have

$$g_d^{-1} = \begin{pmatrix} \sqrt{m_d}^{-1} & 0 \\ -d\sqrt{m_d} & \sqrt{m_d} \end{pmatrix},$$

$z = x + iy \in H$. With $N = dm_d$ we obtain

$$(3.2) \quad \begin{aligned} g_d^{-1}\gamma &= \begin{pmatrix} -d\sqrt{m_d}a + \sqrt{m_d}Nc & -d\sqrt{m_d}b + \sqrt{m_d}h \\ * & * \end{pmatrix}, \\ y^s(g_d^{-1}\gamma z) &= y^s[\{(-d\sqrt{m_d}a + \sqrt{m_d}Nc)x - d\sqrt{m_d}b + \sqrt{m_d}h\}^2 \\ &\quad + (-d\sqrt{m_d}a + \sqrt{m_d}Nc)^2 y^2]^{-s} \\ &= y^s|m_d|^{-s}[\{(Nc - ad)x - db + h\}^2 + (Nc - ad)^2 y^2]^{-s}. \end{aligned}$$

Observe that (3.2) is unchanged if (d, x, b, c) is replaced by $(-d, -x, -b, -c)$. It follows that

$$(3.3) \quad E_{-d}(-\bar{z}, s) = E_d(z, s).$$

Lemma 12. *The cusps $\frac{1}{d}$ and $\frac{-1}{d}$ are equivalent, $d \mid N$, and*

$$(3.4) \quad E_d(z, s) = E_{-d}(z, s) = E_d(Jz, s).$$

Proof. We want to find a matrix

$$\begin{pmatrix} \alpha & \beta \\ N\gamma & \delta \end{pmatrix} \in \Gamma_0(N),$$

where $N = 4N_2$ or $N = 4N_3$, with $\alpha\delta - N\beta\gamma = 1$ and such that

$$\begin{pmatrix} \alpha & \beta \\ N\gamma & \delta \end{pmatrix} \begin{pmatrix} -1 \\ d \end{pmatrix} = \frac{1}{d}.$$

This yields the equation

$$(3.5) \quad \delta^2 - (\beta d + \gamma m_d)\delta + \beta\gamma dm_d + 1 = 0$$

with the solutions

$$(3.6) \quad \delta = \frac{\beta d + \gamma m_d}{2} \pm \frac{\sqrt{(\beta d - \gamma m_d)^2 - 4}}{2}.$$

Here $(\beta d - \gamma m_d)^2 - 4$ is the square of an integer if and only if

$$(3.7) \quad \beta d - \gamma m_d = \pm 2.$$

It follows that $\alpha = \delta$, because α is a solution of the same equation (3.5). Since $N = dm_d = 4p_1 \cdots p_k$, where the p_j are distinct primes, the following cases are possible:

(1) $d = 2d'$, $m_d = 2m'_d$, $(d', m'_d) = 1$. Then (3.7) yields

$$(3.8) \quad \beta d' - \gamma m'_d = \pm 1.$$

Equation (3.8) has integral solutions β, γ , and $\delta = \beta d' + \gamma m'_d$ by (3.6).

(2) $d = 2d'$, $2 \mid d'$, $(d', m_d) = 1$. Then, by (3.7), γ must be even, $\gamma = 2\gamma'$, and

$$(3.9) \quad \beta d' - \gamma' m_d = \pm 1.$$

Equation (3.9) has integral solutions β, γ' , and $\delta = \beta d' + \gamma' m_d$ by (3.6).

(3) $m_d = 2m'_d$, $2 \mid m'_d$, $(d, m'_d) = 1$. This is similar to (2), upon interchanging d with m_d and β with γ .

Hence, $\frac{1}{d}$ and $\frac{-1}{d}$ are equivalent, and it follows that $E_d(z, s) = E_{-d}(z, s)$. Together with (3.3), this proves the lemma. \square

We recall that the involution $J: z \rightarrow -\bar{z}$ acts on the space of all continuous (Γ, χ) -automorphic functions and splits this space into the sum of even and odd functions given by

$$(3.10) \quad f(Jz) = f(z), \quad f(Jz) = -f(z).$$

The operator J commutes with the automorphic Laplacian $A(\Gamma, \chi)$ and with all Hecke operators. In accordance with (3.10), the Hilbert space $\mathcal{H} = \mathcal{H}(\Gamma, \chi)$ decomposes into the orthogonal sum of two subspaces: $\mathcal{H} = \mathcal{H}_{\text{odd}} \oplus \mathcal{H}_{\text{even}}$. Lemma 12 implies the following statement.

Lemma 13. *Let $\mathcal{D}(A)$ be the domain of $A(\Gamma, \chi)$ in \mathcal{H} , and let $A_{\text{odd}} = A(\Gamma, \chi)|_{\mathcal{D}(A) \cap \mathcal{H}_{\text{odd}}}$. Then the operator A_{odd} has discrete spectrum as a selfadjoint operator in \mathcal{H}_{odd} .*

Let $N_{\text{odd}}(\lambda)$ be the distribution function of the eigenvalues of A_{odd} . Now, we prove the Weyl law

$$(3.11) \quad N_{\text{odd}}(\lambda) \sim \frac{\mu(F)}{8\pi} \lambda, \quad \lambda \rightarrow \infty,$$

where $\mu(F)$ is the area ($d\mu$ -area) of the fundamental domain F of Γ . The proof is an extension of the proof presented in [2] and [7]. We have a preliminary trace formula on the space of odd functions

$$(3.12) \quad \sum_j h_\varepsilon(\lambda_j) = \lim_{Y \rightarrow \infty} \frac{1}{2} \int_{F_Y} \sum_{\gamma \in \Gamma} \chi(\gamma) (k_\varepsilon(u(z, \gamma z)) - k_\varepsilon(u(z, \gamma Jz))) d\mu(z),$$

where $\{\lambda_j\}$ is the set of all eigenvalues of A_{odd} , and F_Y is the cut-off fundamental domain of Γ in H (see (3.32)), $F_Y \rightarrow F$ as $Y \rightarrow \infty$. Here u is the distance function

$$u(z, z') = \frac{|z - z'|^2}{yy'}.$$

The test function $h_\varepsilon(\lambda)$ is given by

$$(3.13) \quad h_\varepsilon(\lambda) = h_\varepsilon\left(\frac{1}{4} + r^2\right) = e^{-\varepsilon r^2}, \quad \varepsilon > 0,$$

and $k_\varepsilon(u)$ is the corresponding Selberg transform of $h_\varepsilon(\lambda)$, given by

$$(3.14) \quad \begin{aligned} g_\varepsilon(u) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iru} h_\varepsilon\left(\frac{1}{4} + r^2\right) dr, \\ h_\varepsilon(\lambda) &= \int_{-\infty}^{\infty} e^{i\lambda u} g_\varepsilon(u) du, \quad \lambda = \frac{1}{4} + r^2, \\ Q_\varepsilon(e^u + e^{-u} - 2) &= g_\varepsilon(u), \\ k_\varepsilon(t) &= -\frac{1}{\pi} \int_t^\infty \frac{dQ_\varepsilon(\omega)}{\sqrt{\omega - t}}, \\ Q_\varepsilon(\omega) &= \int_\omega^\infty \frac{k_\varepsilon(t)}{\sqrt{t - \omega}} dt. \end{aligned}$$

We have

$$(3.15) \quad \int_{F_Y} \sum_{\gamma \in \Gamma} \chi(\gamma) k_\varepsilon(u(z, \gamma z)) d\mu(z) = \sum_{\{\gamma\}_\Gamma} \chi(\gamma) \int_{F_Y^\gamma} k_\varepsilon(u(z, \gamma z)) d\mu(z),$$

where $\{\gamma\}_\Gamma$ is the conjugacy class in Γ with representative γ ,

$$(3.16) \quad F_Y^\gamma = \bigcup_{\gamma' \in \Gamma_\gamma \backslash \Gamma} \gamma' F_Y,$$

Γ_γ is the centralizer of γ in Γ , and $\Gamma_\gamma \backslash \Gamma$ is the left coset. Next, $F_Y^\gamma \xrightarrow{Y \rightarrow \infty} F^\gamma$, where F^γ is a fundamental domain of Γ_γ in H .

By analogy, we have

$$(3.17) \quad \int_{F_Y} \sum_{\gamma \in \Gamma} \chi(\gamma) k_\varepsilon(u(z, \gamma J z)) d\mu(z) = \sum_{\{\gamma J\}_\Gamma} \chi(\gamma) \int_{F_Y^{\gamma J}} k_\varepsilon(u(z, \gamma J z)) d\mu(z),$$

where $\{\gamma J\}_\Gamma$ is the relative conjugacy class in ΓJ by conjugation of Γ ,

$$(3.18) \quad F_Y^{\gamma J} = \bigcup_{\gamma' \in \Gamma_{\gamma J} \setminus \Gamma} \gamma' F_Y \xrightarrow{Y \rightarrow \infty} F^{\gamma J},$$

$\Gamma_{\gamma J}$ is the centralizer of γJ in Γ , and $F^{\gamma J}$ is a fundamental domain of $\Gamma_{\gamma J}$ in H . First, we consider the sum given by (3.15). It is well known that the sum over all conjugacy classes $\{\gamma\}_\Gamma$ in (3.15) splits into $\{e\}_\Gamma$, $\{h\}_\Gamma$, and $\{p\}_\Gamma$, i.e., into the identity, hyperbolic, and parabolic classes (Γ has no elliptic classes). Also, the sum over all parabolic classes splits into two sums according to the character χ ($\chi = 1$, $\chi = -1$) [1]. The contribution from $\{e\}_\Gamma$ is equal to

$$(3.19) \quad \int_{F_Y} k_\varepsilon(u(z, z)) d\mu(z) = \frac{\mu(F_Y)}{4\pi} \int_{-\infty}^{\infty} r \cdot (\tanh \pi r) h_\varepsilon\left(\frac{1}{4} + r^2\right) dr,$$

$$(3.20) \quad \mu(F_Y) \xrightarrow{Y \rightarrow \infty} \mu(F).$$

The contribution from all hyperbolic classes to (3.15) is equal to

$$(3.21) \quad \sum_{\{h\}_\Gamma} \sum_{k=1}^{\infty} \frac{\chi^k(h) \log N(h)}{N(h)^{k/2} - N(h)^{-k/2}} g_\varepsilon(k \log N(h)) + o(1), \quad Y \rightarrow \infty$$

where $\gamma = h^k$ is a positive integral power of a primitive hyperbolic element h , $N(h)$ is the norm of h , and

$$(3.22) \quad g_\varepsilon(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-riu} h_\varepsilon\left(\frac{1}{4} + r^2\right) dr.$$

Let $a(\Gamma, \chi)$ be the number of open cusps for F relative to χ , and $b(\Gamma, \chi)$ the number of closed cusps of F . Then the contribution to (3.15) from all parabolic conjugacy classes is equal to

$$(3.23) \quad a(\Gamma, \chi) \left[g_\varepsilon(0) \log Y - g_\varepsilon(0) \log 2 + \frac{h_\varepsilon\left(\frac{1}{4}\right)}{4} - \frac{1}{2\pi} \int_{-\infty}^{\infty} h_\varepsilon\left(\frac{1}{4} + r^2\right) \frac{\Gamma'}{\Gamma}(1 + ir) dr \right] \\ - b(\Gamma, \chi) g_\varepsilon(0) \log 2 + o(1), \quad Y \rightarrow \infty$$

where $\Gamma(s)$ is the Euler function and $o(1)_{Y \rightarrow \infty}$ means $o(1) \xrightarrow{Y \rightarrow \infty} 0$.

Now we consider the sum given by (3.17). In order to calculate the right-hand side of (3.17), we need to separate two different cases for the conjugacy classes $\{\gamma J\}_\Gamma$, $\gamma \in \Gamma$ (as in §2):

$$(3.24) \quad \sum_{\{\gamma J\}_\Gamma} = \sum_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J) \neq 0}} + \sum_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J) = 0}}.$$

There is a significant difference between the classes $\{\gamma J\}_\Gamma$ and $\{\gamma p J\}_\Gamma$, $\gamma \in \Gamma$, $p \in P_j$, $j = 1, 2, 3, 4$. We shall use the following results about these classes.

Lemma 14.

- 1) For any $\{\gamma J\}_\Gamma$ with the property $\text{tr}(\gamma J) \neq 0$, the centralizer $\Gamma_{\gamma J}$ of γJ in Γ is generated by a hyperbolic element $h = h(\gamma J)$.
- 2) There are classes $\{\gamma J\}_\Gamma$, $\gamma \in \Gamma$, with $\text{tr}(\gamma J) = 0$.

Proof. 1) It suffices to show that $\Gamma_{\gamma J}$ contains a hyperbolic element. Then from the discreteness of Γ it follows that $\Gamma_{\gamma J}$ is a cyclic group generated by a hyperbolic element. We have $\gamma J \gamma J \in \Gamma$ if $\gamma \in \Gamma$. Clearly, $(\gamma J \gamma J) \gamma J = \gamma J (\gamma J \gamma J)$. As in (2.6), we have

$$g_1(\gamma J)g_1^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix} \pmod{\pm 1}, \quad \lambda > 1.$$

Then

$$g_1(\gamma J \gamma J)g_1^{-1} = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \pmod{\pm 1}$$

is a hyperbolic element.

Statement 2) follows from the definition of Γ . \square

Remark. Later we shall see that there are at most finitely many classes $\{\gamma J\}_\Gamma$ falling into case 2) of Lemma 14.

The proof of statement 1) in Lemma 14 implies the following.

Lemma 15. For any $\{\gamma J\}_\Gamma$ with $\text{tr}(\gamma J) \neq 0$ we have $N(h(\gamma J)) \leq N^2(\gamma J)$.

The next lemma is a consequence of the definition of the group $\Gamma = \Gamma_0(N)$, $N = 4N_2$ or $N = 4N_3$, and relation (2.6).

Lemma 16. Let $\gamma \in \Gamma$ and $\{\gamma J\}_\Gamma$ be such that $\text{tr}(\gamma J) \neq 0$. Then γ is a hyperbolic element and $N(\gamma J) = N(\gamma)$.

The following is well known.

Lemma 17. The series

$$\sum_{k=1}^{\infty} \sum_{\{\gamma\}_\Gamma} \frac{1}{N(\gamma)^{ks}} \text{ with } \gamma \text{ hyperbolic primitive}$$

converges absolutely for $\text{Re } s > 1$, where the sum is taken over all hyperbolic conjugacy classes in Γ .

Now we calculate the contribution to (3.24) from the classes with $\text{tr}(\gamma J) = 0$.

Lemma 18. If $\gamma \in \Gamma$ and $\text{tr}(\gamma J) = 0$, then the centralizer $\Gamma_{\gamma J}$ of γJ in Γ either is trivial, $\Gamma_{\gamma J} = \{e\}$, or is a cyclic hyperbolic group.

Proof. The element γJ is conjugated by $g \in G$ to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \pmod{\pm 1}$. Let $g_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. We consider the commutation condition

$$(3.25) \quad \frac{-a\bar{z} + b}{-c\bar{z} + d} = -\frac{a\bar{z} + b}{c\bar{z} + d},$$

which is assumed to be valid for all $z \in H$. This implies that either g_1 is a hyperbolic element, or $g_1 = e$, or g_1 is elliptic with $\text{tr}(g_1) = 0$. Since Γ is discrete, the result follows from Lemma 6. \square

We calculate the sum in (3.17) in a little more detail since it is less known as compared to the sum (3.15). We start with the sum in (3.24) with $\text{tr}(\gamma J) \neq 0$. Any $\gamma'J$ in ΓJ is an odd positive integral power of a primitive element γJ . Let $h(\gamma J) \in \Gamma$ denote the generator of $\Gamma_{\gamma J}$, and let $F_{\gamma J}$ be a fundamental domain of $\Gamma_{\gamma J}$ in H . If $g(\gamma J) \in G$ brings $h(\gamma J)$ to the diagonal form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ as in (2.6), then as $F_{\gamma J}$ we take the domain

$$(3.26) \quad g(\gamma J)F_{\gamma J} = \{z = re^{i\varphi} \in H \mid 1 \leq r < N(h(\gamma J)), 0 < \varphi < \pi\}.$$

The part of (3.17) with the condition $\text{tr}(\gamma J) \neq 0$ looks like the following:

$$\begin{aligned}
& \sum_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J) \neq 0}} \chi(\gamma) \int_{F_Y^{\gamma J}} k_\varepsilon(u(z, \gamma Jz)) d\mu(z) \\
&= \sum'_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J) \neq 0}} \sum_{k=1}^{\infty} \chi(\gamma)^{2k-1} \int_{F_{\gamma J}} k_\varepsilon(u(z, (\gamma J)^{2k-1}z)) d\mu(z) + o(1)_{Y \rightarrow \infty} \\
&= \sum'_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J) \neq 0}} \sum_{k=1}^{\infty} \chi(\gamma)^{2k-1} \int_1^{N(h)} \frac{dr}{r} \int_0^\pi \frac{d\varphi}{\sin^2 \varphi} k_\varepsilon\left(\frac{|z + N(\gamma J)^{2k-1}\bar{z}|^2}{y^2 N(\gamma J)^{2k-1}}\right) + o(1)_{Y \rightarrow \infty} \\
&= \sum'_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J) \neq 0}} \sum_{k=1}^{\infty} \left[\chi(\gamma)^{2k-1} \log N(h(\gamma J)) \right. \\
&\quad \left. \times 2 \int_0^{\pi/2} \frac{d\varphi}{\sin^2 \varphi} k_\varepsilon\left(\frac{|e^{i\varphi} + N(\gamma J)^{2k-1}e^{-i\varphi}|^2}{\sin^2 \varphi \cdot N(\gamma J)^{2k-1}}\right) \right] + o(1)_{Y \rightarrow \infty} \\
(3.27) \quad &= \sum'_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J) \neq 0}} \sum_{k=1}^{\infty} \left[\chi(\gamma)^{2k-1} \log N(h(\gamma J)) \right. \\
&\quad \left. \times \int_0^\infty \frac{dt}{\sqrt{t}} k_\varepsilon\left(t(N(\gamma J)^{k-\frac{1}{2}} + N(\gamma J)^{\frac{1}{2}-k})^2 \right. \right. \\
&\quad \left. \left. + (N(\gamma J)^{k-\frac{1}{2}} - N(\gamma J)^{\frac{1}{2}-k})^2 \right) \right] + o(1)_{Y \rightarrow \infty} \\
&= \sum'_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J) \neq 0}} \sum_{k=1}^{\infty} \chi(\gamma)^{2k-1} \log N(h(\gamma J)) \frac{Q_\varepsilon(N(\gamma J)^{2k-1} + N(\gamma J)^{1-2k} - 2)}{N(\gamma J)^{k-1/2} + N(\gamma J)^{1/2-k}} + o(1)_{Y \rightarrow \infty} \\
&= \sum'_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J) \neq 0}} \sum_{k=1}^{\infty} \chi(\gamma)^{2k-1} \log N(h(\gamma J)) \frac{g_\varepsilon((2k-1) \log N(\gamma J))}{N(\gamma J)^{k-1/2} + N(\gamma J)^{1/2-k}} + o(1)_{Y \rightarrow \infty},
\end{aligned}$$

where the summation in \sum' is only taken over the primitive relative classes and N is the norm.

Now we consider the situation where $\text{tr}(\gamma J) = 0$ and $\Gamma_{\gamma J}$ is a hyperbolic cyclic group. Let $h(\gamma J)$ be the generator of $\Gamma_{\gamma J}$. Then

$$\begin{aligned}
& \sum_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J)=0 \\ \Gamma_{\gamma J} \text{ nontrivial}}} \chi(\gamma) \int_{F_Y^{\gamma J}} k_\varepsilon(u(z, \gamma J z)) d\mu(z) \\
&= \sum_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J)=0 \\ \Gamma_{\gamma J} \text{ nontrivial}}} \chi(\gamma) \int_{F_{\gamma J}} k_\varepsilon(u(z, \gamma J z)) d\mu(z) + o(1)_{Y \rightarrow \infty} \\
(3.28) \quad &= \sum_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J)=0 \\ \Gamma_{\gamma J} \text{ nontrivial}}} \chi(\gamma) \int_1^{N(h(\gamma J))} \frac{dr}{r} \int_0^\pi \frac{d\varphi}{\sin^2 \varphi} k_\varepsilon(u(z, Jz)) + o(1)_{Y \rightarrow \infty} \\
&= \sum_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J)=0 \\ \Gamma_{\gamma J} \text{ nontrivial}}} \chi(\gamma) \log N(h(\gamma J)) \int_0^\pi \frac{d\varphi}{\sin^2 \varphi} k_\varepsilon\left(\frac{4 \cos^2 \varphi}{\sin^2 \varphi}\right) + o(1)_{Y \rightarrow \infty} \\
&= \sum_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J)=0 \\ \Gamma_{\gamma J} \text{ nontrivial}}} \chi(\gamma) \frac{\log N(h(\gamma J))}{2} g_\varepsilon(0) + o(1)_{Y \rightarrow \infty}.
\end{aligned}$$

The calculation is similar to (3.27). Later we shall see that the sum in (3.28) contains only finitely many terms.

Finally, we calculate the contribution to (3.24) from the classes with $\text{tr}(\gamma J) = 0$ and with trivial centralizer $\Gamma_{\gamma J} = \{e\}$. We need to find the asymptotics of

$$(3.29) \quad \sum_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J)=0 \\ \Gamma_{\gamma J}=\{e\}}} \chi(\gamma) \int_{F_Y^{\gamma J}} k_\varepsilon(u(z, \gamma J z)) d\mu(z), \quad Y \rightarrow \infty,$$

where

$$(3.30) \quad F_Y^{\gamma J} = \bigcup_{\gamma' \in \Gamma_{\gamma J} \backslash \Gamma} \gamma' F_Y.$$

Recall that $\Gamma = \Gamma_0(N)$, $N = 4N_2$ or $N = 4N_3$, and $F = F_0(N)$, a fundamental domain of Γ in H . We introduce $\Gamma(1)$ to be the modular group and $F(1)$ to be a fundamental domain of $\Gamma(1)$ in H . For the purpose of calculation, we take

$$F(1) = \{z \in H, z = x + iy \mid x^2 + y^2 > 1, 0 \leq x \leq \frac{1}{2} \text{ or } x^2 + y^2 \geq 1, \frac{-1}{2} < x < 0\}.$$

Then we take

$$(3.31) \quad F_Y(1) = \{z \in F \mid y \leq Y\}, \quad Y > 1.$$

We have

$$F_0(N) = \bigcup_{\gamma \in \Gamma(1)/\Gamma_0(N)} \gamma F(1),$$

and now we define F_Y by

$$(3.32) \quad F_Y = F_Y^0(N) = \bigcup_{\gamma \in \Gamma(1)/\Gamma_0(N)} \gamma F_Y(1).$$

Thus, (3.30) can be continued:

$$(3.33) \quad \bigcup_{\gamma' \in \Gamma} \gamma' F_Y = \bigcup_{\gamma \in \Gamma(1)} \gamma F_Y(1).$$

In the sum (3.29), first we consider the term with $\gamma = e$,

$$(3.34) \quad \int_{F_Y^J} k_\varepsilon(u(z, Jz)) d\mu(z).$$

We define two sets $\Omega_j \subset H$, $j = 1, 2$. By definition,

$$\begin{aligned} \Omega_1 &= H(Y) = \{z \in H, z = x + iy \mid y > Y\}, \\ \Omega_2 &= \Omega_2(Y) = \left\{z = \frac{-1}{z'} \mid z' \in \Omega_1(Y)\right\} \\ &= \left\{x + iy \mid x^2 + \left(y - \frac{1}{2Y}\right)^2 < \frac{1}{4Y^2}\right\}. \end{aligned}$$

Formulas (3.30)–(3.33) imply

$$(3.35) \quad F_Y^J \subset H \setminus \Omega_1 \cup \Omega_2 \stackrel{\text{def}}{=} \Omega_3.$$

We shall not calculate the integral (3.34) explicitly, but we shall calculate the divergent term and estimate the remainder term in order to prove the Weyl law.

Define

$$(3.36) \quad \Omega_4 = \Omega_4(Y) = \left\{z \in H, z = x + iy \mid \frac{1}{Y} \leq y \leq Y\right\}.$$

We can see that

$$(3.37) \quad \Omega_4(Y) \subset F_Y^J.$$

This follows from (3.30)–(3.33) and the fact that

$$(3.38) \quad \begin{aligned} \max \operatorname{Im}(\gamma z) &\leq \frac{1}{y}, \quad z = x + iy, \\ \gamma &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1), \quad c \neq 0. \end{aligned}$$

We shall prove that the only part of (3.34) divergent as $Y \rightarrow \infty$ is given by

$$(3.39) \quad \int_{\Omega_4(Y)} k_\varepsilon(u(z, Jz)) d\mu(z),$$

which is equal to

$$(3.40) \quad \begin{aligned} 2 \int_0^\infty dx \int_{1/Y}^Y \frac{dy}{y^2} k_\varepsilon\left(\frac{4x^2}{y^2}\right) &= \frac{1}{2} \int_{1/Y}^Y \frac{dy}{y} \int_0^\infty \frac{k_\varepsilon(t)}{\sqrt{t}} dt \\ &= \frac{1}{2} \cdot 2 \log Y \cdot g_\varepsilon(0) = g_\varepsilon(0) \log Y. \end{aligned}$$

Now we estimate the remaining part of the integral (3.34), given by

$$(3.41) \quad \int_{F_Y^J \setminus \Omega_4(Y)} k_\varepsilon(u(z, Jz)) d\mu(z).$$

From (3.14), (3.22), and (3.13) it follows that

$$(3.42) \quad \begin{aligned} g_\varepsilon(u) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iru} e^{-\varepsilon(r^2 + \frac{1}{4})} dr = \frac{1}{2\sqrt{\pi\varepsilon}} e^{-u^2/4\varepsilon} \cdot e^{-\varepsilon/4}, \\ k_\varepsilon(e^v + e^{-v} - 2) &= -\frac{1}{\pi} \int_v^\infty \frac{g'_\varepsilon(u) du}{\sqrt{e^u + e^{-u} - e^v - e^{-v}}}, \\ g'_\varepsilon(u) &= \frac{d}{du} g_\varepsilon(u). \end{aligned}$$

This implies that $k_\varepsilon(t) \geq 0$, $t \geq 0$. Therefore, by (3.35),

$$(3.43) \quad \int_{F_Y^j \setminus \Omega_4(Y)} k_\varepsilon(u(z, Jz)) d\mu(z) \leq \int_{\Omega_3(Y) \setminus \Omega_4(Y)} k_\varepsilon(u(z, Jz)) d\mu(z) = T_1$$

for all $Y > 1$, $\varepsilon > 0$. For the right-hand side of (3.43), we have

$$(3.44) \quad \begin{aligned} T_1 &= 2 \int_0^{1/Y} \frac{dy}{y^2} \int_{y\sqrt{1/yY-1}}^\infty k_\varepsilon\left(\frac{4x^2}{y^2}\right) dx \\ &= \frac{1}{2} \int_0^{1/Y} \frac{dy}{y} \int_{4(1/yY-1)}^\infty \frac{k_\varepsilon(t)}{\sqrt{t}} dt \\ &= \frac{1}{2} \int_1^\infty \frac{d\tau}{\tau} \int_{4(\tau-1)}^\infty \frac{k_\varepsilon(t)}{\sqrt{t}} dt \\ &= -g_\varepsilon(0) \log 2 + \frac{1}{2} \int_0^\infty \log(t+4) \frac{k_\varepsilon(t)}{\sqrt{t}} dt. \end{aligned}$$

For estimation of integrals, (3.44) is sufficiently good, but we can also transform (3.44) to integrals with the function h_ε . This was done in [2, §6.5]. We have

$$(3.45) \quad \begin{aligned} \int_0^\infty \log(t+4) \frac{k_\varepsilon(t)}{\sqrt{t}} dt &= -\frac{1}{\pi} \int_0^\infty dQ_\varepsilon(\omega) \int_0^\omega \frac{\log(t+4)}{\sqrt{t}\sqrt{\omega-t}} dt \\ &= \frac{1}{\pi} g_\varepsilon(0) \log 2 \int_0^1 \frac{1}{\sqrt{t}\sqrt{1-t}} dt + \frac{1}{\pi} \int_0^\infty Q_\varepsilon(\omega) d\omega \int_0^1 \frac{d\tau}{(\omega + 4/\tau)\sqrt{\tau}\sqrt{1-\tau}} \\ &= 2 \log 2 g_\varepsilon(0) + \frac{1}{\pi} \int_0^\infty Q_\varepsilon(\omega) \cdot \pi \cdot \frac{1}{\omega + 4 + 2\sqrt{\omega+4}} d\omega \\ &= 2 \log 2 g_\varepsilon(0) + \int_0^\infty g_\varepsilon(u) \tanh(u/4) du. \end{aligned}$$

In a similar way (cf. Lemma A.3 in the Appendix), we obtain the main term of the asymptotics of (3.29):

$$\int_{\Omega_4(Y)} k_\varepsilon(u(z, Jz)) d\mu(z) = g_\varepsilon(0) \log y$$

and the remaining terms:

$$\int_{c^{-1}(\Omega_3(Y) \setminus \Omega_4(Y))} k_\varepsilon(u(z, Jz)) d\mu(z) = T_1$$

and

$$\int_{c^{-1}\frac{1}{Y}}^{1/Y} k_\varepsilon(u(z, Jz)) d\mu(z);$$

see (4.34), (4.36).

Lemma 19. *The number of classes $\{\gamma J\}_\Gamma$ such that $\text{tr}(\gamma J) = 0$ and $\Gamma_{\gamma J}$ is nontrivial is finite.*

Proof. We can derive a formula similar to (3.12) for a trivial character χ . Then we can repeat the calculation of the contributions from all classes to (3.12). In place of (3.28) we get

$$(3.46) \quad \frac{g_\varepsilon(0)}{2} \sum_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J)=0 \\ \Gamma_{\gamma J} \text{ nontrivial}}} \log N(h(\gamma J)) + o(1)_{Y \rightarrow \infty}.$$

For any fixed $Y > 1$ and any trivial character χ , the integral in (3.12) is finite. It follows that the sum in (3.46) is finite, which can happen only if there are finitely many terms in the sum. \square

Let $a(\Gamma, 1) = a(\Gamma)$ denote the number of pairwise inequivalent open cusps of F relative to χ , and let $m(\Gamma) = \#\{\gamma J\}_\Gamma$, $\text{tr}(\gamma J) = 0$, $\Gamma_{\gamma J} = \{e\}$.

Lemma 20. *We have*

- 1) $a(\Gamma) = m(\Gamma)$,
- 2) $a(\Gamma, \chi) = \sum_{\{\gamma J\}_\Gamma, \text{tr}(\gamma J)=0, \Gamma_{\gamma J}=\{e\}} \chi(\gamma)$.

Proof. As in the proof of Lemma 19, we compare the coefficients of the $\log Y$ terms in (3.12) (1) for trivial χ to prove 1), and for the primitive nontrivial character coming from the classes $\{\gamma\}_\Gamma$ and $\{\gamma J\}_\Gamma$ to prove 2). \square

Theorem 4. *The Weyl law (see (3.11)) is valid.*

Proof. To prove the theorem, we establish the asymptotics of each term in (3.12) as $\varepsilon \rightarrow +0$ and then apply a Tauberian theorem. As in [2], we have

$$(3.47) \quad \begin{aligned} \frac{\mu(F)}{4\pi} \int_{-\infty}^{\infty} r(\tanh \pi r) h_\varepsilon\left(\frac{1}{4} + r^2\right) dr &= \frac{\mu(F)}{4\pi} \cdot \frac{1}{\varepsilon} + O(1)_{\varepsilon \rightarrow +0}, \\ \sum_{\{h\}_\Gamma} \sum_{k=1}^{\infty} \frac{\chi^k(h) \log N(h)}{N(h)^{k/2} - N(h)^{-k/2}} g_\varepsilon(k \log N(h)) &= o(1)_{\varepsilon \rightarrow +0}, \\ g_\varepsilon(0) &= O\left(\frac{1}{\sqrt{\varepsilon}}\right)_{\varepsilon \rightarrow +0}, \\ \int_{-\infty}^{\infty} h_\varepsilon\left(\frac{1}{4} + r^2\right) \frac{\Gamma'}{\Gamma}(1 + ir) dr &= O\left(\frac{|\log \varepsilon|}{\sqrt{\varepsilon}}\right)_{\varepsilon \rightarrow +0}. \end{aligned}$$

Also,

$$(3.48) \quad \int_0^{\infty} g_\varepsilon(u) \tanh\left(\frac{u}{4}\right) du = o(1)_{\varepsilon \rightarrow 0}.$$

Applying Lemmas 15, 16, and 17, we obtain (see (3.27))

$$(3.49) \quad \sum'_{\substack{\{\gamma J\}_\Gamma \\ \text{tr}(\gamma J) \neq 0}} \sum_{k=1}^{\infty} \chi(\gamma)^{2k-1} \log N(h(\gamma J)) \frac{g_\varepsilon((2k-1) \log N(\gamma J))}{N(\gamma J)^{k-1/2} + N(\gamma J)^{1/2-k}} = o(1)_{\varepsilon \rightarrow +0}.$$

Finally, using (3.43) and Lemmas 19, 20, from (3.12) we deduce

$$(3.50) \quad \sum_j h_\varepsilon(\lambda_j) = \int_0^{\infty} e^{-\varepsilon \lambda} dN_{\text{odd}}(\lambda) = \frac{\mu(F)}{8\pi} \cdot \frac{1}{\varepsilon} + O\left(\frac{|\log \varepsilon|}{\sqrt{\varepsilon}}\right)_{\varepsilon \rightarrow +0},$$

which implies (3.11) and Theorem 4. \square

§4. A MORE ADVANCED TRACE FORMULA

We introduce three functions

$$(4.1) \quad \begin{aligned} h_1(\lambda) &= h_1\left(\frac{1}{4} + r^2\right) = e^{-\varepsilon\left(\frac{1}{4} + r^2\right)}, \\ h_2(\lambda) &= h_2\left(\frac{1}{4} + r^2\right) = 4 \cos(r \log 2) e^{-\varepsilon\left(\frac{1}{4} + r^2\right)}, \\ h_3(\lambda) &= h_3\left(\frac{1}{4} + r^2\right) = 2 \cos(2r \log 2) e^{-\varepsilon\left(\frac{1}{4} + r^2\right)}. \end{aligned}$$

All these depend on a parameter $\varepsilon > 0$, as in (3.13). For each $h_j(\lambda)$ we denote by k_j the corresponding Selberg transform (see (3.14)), $j = 1, 2, 3$. Also, we introduce

$$(4.2) \quad K_j(z, z', \Gamma, \chi) = \sum_{\gamma \in \Gamma} \chi(\gamma) k_j(u(z, \gamma z')),$$

where γ runs over the entire group $\Gamma = \Gamma_0(N)$, $N = 4N_2$, $N = 4N_3$, and χ is our real primitive character, $u(z, z') = \frac{|z - z'|^2}{yy'}$.

Let $K_j(\Gamma, \chi) = K_j$ be the corresponding integral operator with kernel given by (4.2) on the Hilbert space $\mathcal{H}(\Gamma) = L_2(F, d\mu)$. We shall study the operator

$$(4.3) \quad T = (4 + U^2(2) + U^{*2}(2))K_1 - (U(2) + U^*(2))K_2 + K_3$$

on the space \mathcal{H}_{odd} of odd functions (see Lemma 13 above). We have $K_j = h_j(A(\Gamma, \chi))$, $j = 1, 2, 3$, and $U(2)K_j = K_jU(2)$, $U^*(2)K_j = K_jU^*(2)$, $K_j\mathcal{H}_{\text{odd}} \subset \mathcal{H}_{\text{odd}}$, $U(2)\mathcal{H}_{\text{odd}} \subset \mathcal{H}_{\text{odd}}$, $U^*(2)\mathcal{H}_{\text{odd}} \subset \mathcal{H}_{\text{odd}}$, whence

$$(4.4) \quad T\mathcal{H}_{\text{odd}} \subset \mathcal{H}_{\text{odd}}.$$

Lemma 12 shows that $A(\Gamma, \chi)$ has only discrete spectrum in $\mathcal{H}_{\text{odd}} \cap \mathcal{D}(A(\Gamma, \chi))$. From Theorem 4.2 [1] it follows that there exists a common basis of eigenfunctions of $A(\Gamma, \chi)$, $U(2)$, and $U^*(2)$ in \mathcal{H}_{odd} . Let $\{v_j(z, \Gamma, \chi)\}_{j=1}^{\infty}$ denote the orthonormal basis of common eigenfunctions in \mathcal{H}_{odd} . If

$$A(\Gamma, \chi)v_j(z) = \lambda_j v_j(z), \quad \lambda_j = \lambda_j(\Gamma, \chi), \quad v_j(z) = v_j(z, \Gamma, \chi)$$

and

$$\begin{aligned} U(2)v_j(z) &= \nu_j v_j(z), \quad \nu_j = \nu_j(\Gamma, \chi), \\ \lambda_j &\in \mathbb{R}, \quad \nu_j \in \mathbb{C}, \quad |\nu_j| = 1, \quad \nu_j = e^{i\eta_j}, \quad \eta_j \in \mathbb{R}, \end{aligned}$$

then $U^*(2)v_j = e^{-i\eta_j}v_j$, and we have

$$(4.5) \quad \begin{aligned} Tv_j &= [(4 + e^{2i\eta_j} + e^{-2i\eta_j})h_1(\lambda_j) - (e^{i\eta_j} + e^{-i\eta_j})h_2(\lambda_j) + h_3(\lambda_j)]v_j \\ &= [(4 + e^{2i\eta_j} + e^{-2i\eta_j}) \\ &\quad - (e^{i\eta_j} + e^{-i\eta_j}) \cdot 4 \cos(r_j \log 2) + 2 \cos(2r_j \log 2)]e^{-\varepsilon\lambda_j}v_j, \end{aligned}$$

where $\lambda_j = \frac{1}{4} + r_j^2$. We can continue (4.5):

$$(4.6) \quad \begin{aligned} Tv_j &= (e^{i\eta_j} + e^{-i\eta_j} - 2^{ir_j} - 2^{-ir_j})^2 e^{-\varepsilon\lambda_j}v_j, \\ &= (2 \cos \eta_j - 2 \cos(r_j \log 2))^2 e^{-\varepsilon\lambda_j}v_j. \end{aligned}$$

Let $\omega_j = \cos \eta_j - \cos(r_j \log 2)$. It is not difficult to see that the operator T is of trace class on \mathcal{H}_{odd} , and its spectral trace is equal to

$$(4.7) \quad \text{tr } T = 4 \sum_{j=1}^{\infty} \omega_j^2 e^{-\varepsilon\lambda_j}.$$

Using the trace formula, we can calculate the matrix trace of T and obtain the asymptotics as $\varepsilon \rightarrow +0$. Then we apply the Tauberian theorem to get information on a bound for ω_j .

From (1.4), (4.2), and (4.3) it follows that the kernel $\widehat{T}(z, z')$ of the operator T viewed as an integral operator on the space of odd functions \mathcal{H}_{odd} is given by

$$\begin{aligned}
(4.8) \quad \widehat{T}(z, z') &= \frac{1}{2}(T(z, z') - T(z, Jz')) \\
&= 2 \sum_{\gamma \in \Gamma} \chi(\gamma)[k_1(u(z, \gamma z')) - k_1(u(z, \gamma Jz'))] \\
&\quad + \frac{1}{2} \sum_{\gamma \in \Gamma} \sum_{p \in P_3} \chi(\gamma)[k_1(u(z, \gamma p z')) - k_1(u(z, \gamma p Jz'))] \\
&\quad + \frac{1}{2} \sum_{\gamma \in \Gamma} \sum_{p \in P_4} \chi(\gamma)[k_1(u(z, \gamma p z')) - k_1(u(z, \gamma p Jz'))] \\
&\quad - \frac{1}{2} \sum_{\gamma \in \Gamma} \sum_{p \in P_1} \chi(\gamma)[k_2(u(z, \gamma p z')) - k_2(u(z, \gamma p Jz'))] \\
&\quad - \frac{1}{2} \sum_{\gamma \in \Gamma} \sum_{p \in P_2} \chi(\gamma)[k_2(u(z, \gamma p z')) - k_2(u(z, \gamma p Jz'))] \\
&\quad + \frac{1}{2} \sum_{\gamma \in \Gamma} \chi(\gamma)[k_3(u(z, \gamma z')) - k_3(u(z, \gamma Jz'))].
\end{aligned}$$

Using (4.7) and (4.8), we can construct a trace formula similar to (3.12):

$$(4.9) \quad 4 \sum_{j=1}^{\infty} \omega_j^2 e^{-\varepsilon \lambda_j} = \lim_{Y \rightarrow \infty} \int_{F_Y} \widehat{T}(z, z) d\mu(z),$$

where the λ_j are as in (4.5). In accordance with the decomposition (4.3), the right-hand side of (4.8) is the sum of six automorphic kernels. We denote them by $\widehat{T}_j(z, z')$, $j = 1, 2, \dots, 6$, starting with

$$\widehat{T}_1(z, z') = 2 \sum_{\gamma \in \Gamma} \chi(\gamma)[k_1(u(z, \gamma z')) - k_1(u(z, \gamma Jz'))]$$

and finishing with

$$\widehat{T}_6(z, z') = \frac{1}{2} \sum_{\gamma \in \Gamma} \chi(\gamma)[k_3(u(z, \gamma z')) - k_3(u(z, \gamma Jz'))].$$

It is easily seen that for each $j = 1, \dots, 6$ there exists a finite limit

$$(4.10) \quad \lim_{Y \rightarrow \infty} \int_{F_Y} \widehat{T}_j(z, z) d\mu(z) = \int_F \widehat{T}_j(z, z) d\mu(z) = I_j(\varepsilon)$$

for any fixed $\varepsilon > 0$. We must find an asymptotics (or bound) for all $I_j(\varepsilon)$, $\varepsilon \rightarrow +0$. We did that in §3 for I_1 (see (3.50)). We have

$$(4.11) \quad I_1(\varepsilon) = \frac{\mu(F)}{2\pi} \cdot \frac{1}{\varepsilon} + O\left(\frac{\|\log \varepsilon\|}{\sqrt{\varepsilon}}\right).$$

The next integral we consider is $I_6(\varepsilon)$, because it is close to the previous case. The contribution from the identity (see (3.12) and (3.13)) is equal to

$$(4.12) \quad \frac{\mu(F)}{8\pi} \int_{-\infty}^{\infty} r(\tanh \pi r) h_3\left(\frac{1}{4} + r^2\right) dr.$$

We estimate this integral as $\varepsilon \rightarrow +0$. We have

$$(4.13) \quad \begin{aligned} & \int_{-\infty}^{\infty} r(\tanh \pi r) h_3\left(\frac{1}{4} + r^2\right) dr \\ &= 4 \int_0^{\infty} r(\tanh \pi r) (e^{2ir \log 2} + e^{-2ir \log 2}) e^{-(\frac{1}{4} + r^2)\varepsilon} dr. \end{aligned}$$

Since

$$\int_0^{\infty} r(1 - \tanh \pi r) h_3\left(\frac{1}{4} + r^2\right) dr = O(1), \quad \varepsilon \rightarrow +0$$

we evaluate

$$(4.14) \quad \begin{aligned} & \int_0^{\infty} r(2 \cos(2r \log 2)) e^{-r^2\varepsilon} dr = \frac{1}{\varepsilon} \int_0^{\infty} t(2 \cos(2\frac{t}{\sqrt{\varepsilon}} \log 2)) e^{-t^2} dt \\ &= \frac{2}{\varepsilon} \int_0^{\infty} t \cdot e^{-t^2} d(\sin(\frac{t}{\sqrt{\varepsilon}} 2 \log 2)) \frac{\sqrt{\varepsilon}}{2 \log 2} \\ &= -\frac{1}{\sqrt{\varepsilon} \log 2} \int_0^{\infty} \frac{d}{dt} (t \cdot e^{-t^2}) \sin(\frac{t}{\sqrt{\varepsilon}} 2 \log 2) dt = O(\frac{1}{\sqrt{\varepsilon}}). \end{aligned} \quad \varepsilon \rightarrow +0$$

From (4.14) it follows that (4.12) is $O(\frac{1}{\sqrt{\varepsilon}})_{\varepsilon \rightarrow +0}$, which is smaller than the leading term in (4.11). To see the contribution from hyperbolic elements similar to (3.21) and (3.47), we need to find

$$(4.15) \quad \begin{aligned} g_3(u) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iru} h_3\left(\frac{1}{4} + r^2\right) dr \\ &= \frac{1}{2\sqrt{\pi\varepsilon}} \left[e^{-\frac{(u-2\log 2)^2}{4\varepsilon}} + e^{-\frac{(u+2\log 2)^2}{4\varepsilon}} \right] e^{-\frac{\varepsilon}{4}}. \end{aligned}$$

The worst that could happen is the existence of $\{h\}_{\Gamma}$ (see the second line in (3.47)) with the property

$$(4.16) \quad \kappa \log N(h) = 2 \log 2,$$

which gives us the estimate $O(\frac{1}{\sqrt{\varepsilon}})_{\varepsilon \rightarrow +0}$ coming from the $g_3(2 \log 2)$ term instead of $o(1)_{\varepsilon \rightarrow +0}$ (we know that at most finitely many $\{h\}_{\Gamma}$ may have the same norm). Then we see that the contribution from hyperbolic classes to the integral I_6 is bounded by $O(\frac{1}{\sqrt{\varepsilon}})_{\varepsilon \rightarrow +0}$, which again is smaller than the leading term in (4.11). The contribution from parabolic classes to I_6 is estimated much as in the case of I_1 and is $O(\frac{1}{\sqrt{\varepsilon}})_{\varepsilon \rightarrow +0}$.

Estimations of the contributions from the $\{\gamma J\}_{\Gamma}$ classes also proceed by analogy with the previous case with obvious modifications. For instance, in (3.43) we estimate the numerical value

$$(4.17) \quad \int_{F_Y^{\vee} \setminus \Omega_4(Y)} |k_3(u(z, Jz))| d\mu(z) \leq \int_{\Omega_3(Y) \setminus \Omega_4(Y)} |k_3(u(z, Jz))| d\mu(z),$$

and, instead of $o(1)_{\varepsilon \rightarrow +0}$, in (3.49) we will get the estimate $O(\frac{1}{\sqrt{\varepsilon}})_{\varepsilon \rightarrow +0}$ by using an argument similar to that in the proof of (4.15). Finally, we obtain

$$(4.18) \quad I_6(\varepsilon) = O\left(\frac{\|\log \varepsilon\|}{\sqrt{\varepsilon}}\right)_{\varepsilon \rightarrow +0}.$$

Now we are going to evaluate the remaining four integrals I_2, I_3, I_4, I_5 . From the point of view of the trace formula, these cases do not differ much from one another. We

consider the case of I_2 in more detail and then explain the changes needed in the other cases. For a fixed $\varepsilon > 0$, we have

$$\begin{aligned}
(4.19) \quad I_2 &= I_2(\varepsilon) \\
&= \lim_{Y \rightarrow \infty} \int_{F_Y} \frac{1}{2} \sum_{\gamma \in \Gamma} \sum_{p \in P_3} \chi(\gamma) [k_1(u(z, \gamma pz)) - k_1(u(z, \gamma pJz))] d\mu(z) \\
&= \frac{1}{2} \lim_{Y \rightarrow \infty} \left\{ \sum_{\gamma \in \Gamma} \chi(\gamma) \sum_{p \in P_3} \int_{F_Y} k_1(u(z, \gamma pz)) d\mu(z) \right. \\
&\quad \left. - \sum_{\gamma \in \Gamma} \chi(\gamma) \sum_{p \in P_3} \int_{F_Y} k_1(u(z, \gamma pJz)) d\mu(z) \right\} \\
&= \frac{1}{2} \lim_{Y \rightarrow \infty} \left\{ \sum_{\{\gamma p\}_\Gamma} \chi(\gamma) \int_{F_Y^{\gamma p}} k_1(u(z, \gamma pz)) d\mu(z) \right. \\
&\quad \left. - \sum_{\{\gamma pJ\}_\Gamma} \int_{F_Y^{\gamma pJ}} k_1(u(z, \gamma pJz)) d\mu(z) \right\},
\end{aligned}$$

where $\{\gamma p\}_\Gamma$, $\{\gamma pJ\}_\Gamma$ are relative conjugacy classes with Γ conjugation (see §§1, 2). As in (3.16), (3.18), we have

$$(4.20) \quad F_Y^{\gamma p} = \bigcup_{\gamma' \in \Gamma_{\gamma p} \setminus \Gamma} \gamma' F_Y, \quad F_Y^{\gamma pJ} = \bigcup_{\gamma' \in \Gamma_{\gamma pJ} \setminus \Gamma} \gamma' F_Y,$$

where $\Gamma_{\gamma p}$ and $\Gamma_{\gamma pJ}$ are the centralizers of γp and γpJ in Γ . We have $F_Y^{\gamma p} \xrightarrow{Y \rightarrow \infty} F^{\gamma p}$, $F_Y^{\gamma pJ} \xrightarrow{Y \rightarrow \infty} F^{\gamma pJ}$, where $F^{\gamma p}$ and $F^{\gamma pJ}$ are fundamental domains of $\Gamma_{\gamma p}$ and $\Gamma_{\gamma pJ}$ in H . We want to find the contribution to (4.19) from different conjugacy classes. From Lemma 2 we know that each ΓP_j contains no parabolic classes $\{\gamma p\}$, $j = 1, 2, 3, 4$. By Lemma 3, there are at most finitely many elliptic classes $\{\gamma p\}_\Gamma$ in ΓP_j , $j = 1, 2, 3, 4$. If an elliptic class $\{\gamma p\}_\Gamma$ has order d , then it is not difficult to see that the contribution to the trace is

$$(4.21) \quad \frac{\chi(\gamma)}{2 \sin \frac{\pi}{d}} \int_{-\infty}^{\infty} \frac{\exp(-2\pi r/d)}{1 + \exp(-2\pi r)} h\left(\frac{1}{4} + r^2\right) dr,$$

where $h = h_1$ for I_2 and I_3 , and $h = h_2$ for I_4 and I_5 . From (4.1) it follows that in all cases the contribution from all elliptic classes is $O(1)_{\varepsilon \rightarrow +0}$. In the first sum in (4.19) we must only evaluate the contributions from hyperbolic classes, since there is no contribution from the identity in all these cases. In accordance with Theorem 2, we have hyperbolic classes of two different types. First we consider the case where $\Gamma_{\gamma p}$ is the nontrivial cyclic group generated by a hyperbolic element $h(\gamma p) \in \Gamma$. We have infinitely many such classes, and the total contribution can be calculated as in (3.21):

$$(4.22) \quad \sum_{\substack{\{\gamma p\}_\Gamma \\ \text{hyperbolic} \\ \Gamma^{\gamma p} \text{ nontrivial}}} \chi(\gamma) \log N(h(\gamma p)) \frac{1}{N(\gamma p)^{1/2} - N(\gamma p)^{-1/2}} \cdot g(\log N(\gamma p)) + o(1)_{Y \rightarrow \infty},$$

where again $g = g_1$ for I_2, I_3 and $g = g_2$ for I_4, I_5 . It is known that the series (4.22) is absolutely convergent. We can take the limit as $Y \rightarrow \infty$ and then evaluate (4.22) by $o(1)_{\varepsilon \rightarrow +0}$ for I_2, I_3 and by $O(\frac{1}{\sqrt{\varepsilon}})_{\varepsilon \rightarrow +0}$ for I_4, I_5 . To complete the study of the first sum in (4.19), it remains to consider the hyperbolic classes with trivial $\Gamma_{\gamma p}$. We have $F_Y^{\gamma p} = \Gamma_Y^0$ (see the Appendix), and we shall consider a more general situation, assuming

that $p \in P_j$, $j = 1, 2, 3, 4$. Then

$$(4.23) \quad \int_{\Gamma_Y^0} k(u(z, \gamma pz)) d\mu(z) = \int_{g\Gamma_Y^0} k(u(z, \alpha z)) d\mu(z),$$

where $g \in \mathrm{PSL}(2, \mathbb{R})$, $\alpha = g\gamma pg^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda > 1$, and $u(z, \alpha z) = \frac{|z - \lambda^2 z|}{y^2 \lambda^2}$. As in (3.40), the leading term of the asymptotics of (4.23) as $Y \rightarrow \infty$ is given by

$$(4.24) \quad \int_{-\infty}^{\infty} dx \int_{1/Y}^Y k\left(\frac{|z - \lambda^2 z|^2}{y^2 \lambda^2}\right) \frac{dy}{y^2} = A$$

(see Lemma A.4 in the Appendix). We have

$$(4.25) \quad \begin{aligned} A &= 2 \int_0^{\infty} dx \int_{1/Y}^Y k\left(a\left(1 + \frac{x^2}{y^2}\right)\right) \frac{dy}{y^2} \\ &= 2(\log(Y) - \log(1/Y)) \int_0^{\infty} dt k(a(1 + t^2)) \\ &= 4 \log Y \int_0^{\infty} k(a(1 + t^2)) dt = \frac{2 \log Y}{\lambda - 1/\lambda} g(2 \log \lambda), \end{aligned}$$

where $a = (\lambda - \frac{1}{\lambda})^2$ and $g(u)$ is the Selberg transform of $k(t)$ (see (3.14)). For k equal to k_1 or k_2 we have

$$(4.26) \quad g(2 \log \lambda) = O\left(\frac{1}{\sqrt{\varepsilon}}\right)_{\varepsilon \rightarrow 0}$$

If $h_0^{(j)}(\Gamma)$ is the number of classes $\{\gamma p\}_{\Gamma}$, $p \in P_j$, with trivial centralizer $\Gamma_{\gamma p} = \{e\}$, we can argue as in Lemma 20 to prove that each $h_0^{(j)}$ is finite, and the total divergent term in the first sum in (4.19) is equal to

$$(4.27) \quad \left(\sum_{\substack{\{\gamma p\}_{\Gamma} \\ \Gamma_{\gamma p} = \{e\} \\ p \in P_j}} \chi(\gamma) \frac{1}{\lambda - 1/\lambda} \right) 2 \log Y g(2 \log \lambda),$$

where the sum consists of $h_0^{(j)}$ terms, $\lambda = \lambda(\gamma p)$. Theorem 2 shows that

- (a) $\mathrm{tr}(\gamma p) = \lambda + \lambda^{-1} = \frac{3}{\sqrt{2}}$ if $p \in P_1$ or P_2 ;
- (b) $\mathrm{tr}(\gamma p) = \lambda + \lambda^{-1} = \frac{5}{2}$ if $p \in P_3$ or P_4 .

It follows that $\lambda = \sqrt{2}$ in case (a) and $\lambda = 2$ in case (b), so that λ is independent of γ . We can rewrite (4.27) as

$$(4.28) \quad (2 \log Y) g(2 \log \lambda) \frac{1}{\lambda - 1/\lambda} \sum_{\substack{\{\gamma p\}_{\Gamma} \\ \Gamma_{\gamma p} = \{e\} \\ p \in P_j}} \chi(\gamma), \quad \lambda = \sqrt{2} \text{ or } \lambda = 2.$$

Now we find the divergent terms in the second sum in (4.19). By Lemma 10, in this sum there is no term with $\mathrm{tr}(\gamma p J) = 0$. We split this sum as follows:

$$(4.29) \quad \sum_{\substack{\{\gamma p J\}_{\Gamma} \\ \Gamma_{\gamma p J} \text{ nontrivial}}} + \sum_{\substack{\{\gamma p J\}_{\Gamma} \\ \Gamma_{\gamma p J} \text{ trivial}}} .$$

The first sum in (4.29) transforms as in (3.27) with $g = g(\varepsilon)$, where $g = g_1$ or $g = g_2$. It is absolutely convergent, has a finite limit as $Y \rightarrow \infty$, and is estimated by $O\left(\frac{1}{\sqrt{\varepsilon}}\right)_{\varepsilon \rightarrow 0}$.

We need to calculate the asymptotics as $Y \rightarrow \infty$ of the second sum in (4.29). We have $F_Y^{\gamma p J} = \Gamma_Y^0$, and

$$(4.30) \quad \int_{\Gamma_Y^0} k(u(z, \gamma p J z)) d\mu(z) = \int_{g\Gamma_Y^0} k(u(z, \beta z)) d\mu(z),$$

where $g^* \in \text{PSL}(2, \mathbb{R})$,

$$\beta = g^* \gamma p J g^{*-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix},$$

$\lambda > 1$, $p \in P_j$.

The leading term of the asymptotics of (4.30) as $Y \rightarrow \infty$ is given by

$$(4.31) \quad \int_{-\infty}^{\infty} dx \int_{1/Y}^Y k\left(\frac{|z + \lambda^2 \bar{z}|^2}{y^2 \lambda^2}\right) \frac{dy}{y^2} = 2(\log(Y) - \log(1/Y)) \int_0^{\infty} k(b^2 x^2 + d^2) dx$$

with $b^2 = (\lambda + \frac{1}{\lambda})^2$, $d^2 = (\lambda - \frac{1}{\lambda})^2$. From Theorem 3 it follows that $\lambda - \lambda^{-1} = \frac{1}{\sqrt{2}}$ if $j = 1, 2$ and $\lambda - \lambda^{-1} = \frac{3}{2}$ if $j = 3, 4$. That means that $\lambda = \sqrt{2}$ in cases (1) and (2), and $\lambda = 2$ in cases (3) and (4). The integral (4.31) is equal to

$$(4.32) \quad 4 \log Y \int_0^{\infty} k(b^2 x^2 + d^2) dx = \frac{\lambda}{1 + \lambda^2} \cdot 2 \log Y g(2 \log \lambda),$$

where g is the Selberg transform of k (see (3.14)). Again, for $k = k_1$ or k_2 we have (4.26). If $h_1^{(j)}(\Gamma)$ is the number of classes $\{\gamma p J\}_{\Gamma}$, $p \in P_j$, with trivial centralizer $\Gamma_{\gamma p J} = \{e\}$, then, as in Lemma 20, we can prove that each $h_1^{(j)}$ is finite, and in a more general situation where $p \in P_j$ the total divergent term in the second sum in (4.19) is equal to

$$(4.33) \quad (2 \log Y) g(2 \log \lambda) \frac{\lambda}{1 + \lambda^2} \sum_{\substack{\{\gamma p J\}_{\Gamma} \\ \Gamma_{\gamma p J} = \{e\} \\ p \in P_j}} \chi(\gamma), \quad \lambda = \sqrt{2} \text{ or } \lambda = 2,$$

where the sum in (4.33) consists of $h_1^{(j)}$ terms. Since the limit (4.10) exists and is finite, we see that the divergent terms (4.28) and (4.33) coincide (not only for a primitive character χ but also for $\chi = 1$). We have proved the following statement.

Lemma 21. *In the notation of (4.27) and (4.33), we have*

- 1) $h_1^{(j)}(\Gamma) = 3h_0^{(j)}(\Gamma)$,
- 2) $\sum_{p \in P_j} \sum_{\{\gamma p J\}_{\Gamma}} \chi(\gamma) = 3 \sum_{p \in P_j} \sum_{\{\gamma p\}_{\Gamma}} \chi(\gamma)$,
- 3) $3h_1^{(j)}(\Gamma) = 5h_0^{(j)}(\Gamma)$,
- 4) $3 \sum_{p \in P_j} \sum_{\{\gamma p J\}_{\Gamma}} \chi(\gamma) = 5 \sum_{p \in P_j} \sum_{\{\gamma p\}_{\Gamma}} \chi(\gamma)$.

Cases 1) and 2) apply when $j = 1, 2$, and cases 3) and 4) apply when $j = 3, 4$.

Now, to complete the evaluation of the terms in (4.19) (for $p \in P_j$), first we need to evaluate the second terms on the right-hand sides of (4.25) and (4.32) as $\varepsilon \rightarrow 0$, and this is done with the help of (4.26). Second, we must evaluate the differences between (4.23) and (4.24), and between (4.30) and (4.31). In both cases, for the integration domain (see Lemmas A.4 and A.5 in the Appendix) we have

$$(4.34) \quad g\Gamma_Y^0 \subset \left\{ z \in H, z = x + iy \mid \frac{1}{cY} < y < Y \right\} \\ \cup \left\{ 0 < y < \frac{1}{cY}, |x| > y \sqrt{\frac{1}{yYc} - 1} \right\}$$

for some $c \geq 1$. It follows that we need to evaluate the following integrals:

$$(4.35) \quad \int_{1/cY}^{1/Y} \frac{dy}{y^2} \int_{-\infty}^{\infty} dxk\left(a\left(1 + \frac{x^2}{y^2}\right)\right),$$

$$(4.36) \quad \int_0^{1/cY} \frac{dy}{y^2} \int_{y\sqrt{1/yY-1}}^{\infty} dxk\left(a\left(1 + \frac{x^2}{y^2}\right)\right),$$

where $c \geq 1$ is a constant (independent of Y and ε , but in general different for γp and γpJ) and $a = (\lambda - \lambda^{-1})^2$ (see (4.25)), and the integrals

$$(4.37) \quad \int_{1/cY}^{1/Y} \frac{dy}{y^2} \int_{-\infty}^{\infty} dxk\left(d^2 + \frac{x^2}{y^2}b^2\right),$$

$$(4.38) \quad \int_0^{1/cY} \frac{dy}{y^2} \int_{y\sqrt{1/yY-1}}^{\infty} dxk\left(d^2 + \frac{x^2}{y^2}b^2\right)$$

with $b = \lambda + \lambda^{-1}$, $d = \lambda - \lambda^{-1}$ (see (4.31)).

It is easy to check (by a calculation similar to (4.25), (4.32)) that the integrals (4.35), (4.37) are independent of Y , and up to a multiplicative constant they are equal to $g(2 \log \lambda)$, which was estimated in (4.26). Now, consider the integral (4.36). By an obvious change of variables we reduce it (up to a multiplicative constant) to

$$(4.39) \quad \begin{aligned} \int_a^{\infty} \frac{dy}{y} \int_y^{\infty} \frac{k(\tau)}{\sqrt{\tau-a}} d\tau &= \int_a^{\infty} d(\log y) \int_y^{\infty} \frac{k(\tau)}{\sqrt{\tau-a}} \\ &= -\log(a)g(2 \log \lambda) + \int_a^{\infty} \log y \frac{k(y)}{\sqrt{y-a}} dy. \end{aligned}$$

We are left with the estimation of

$$(4.40) \quad \begin{aligned} &\int_a^{\infty} \log y \frac{k(y)}{\sqrt{y-a}} dy \\ &= \int_a^{\infty} \frac{\log y}{\sqrt{y-a}} \left(\frac{-1}{\pi} \int_y^{\infty} \frac{dQ(\omega)}{\sqrt{\omega-y}} \right) \\ &= \frac{-1}{\pi} \int_a^{\infty} d\omega \int_a^{\omega} dy \frac{Q'(\omega) \log y}{\sqrt{y-a}\sqrt{\omega-y}} \\ &= \frac{-1}{\pi} \int_a^{\infty} dQ(\omega) \int_a^{\infty} \frac{\log y}{\sqrt{y-a}\sqrt{\omega-y}} dy \\ &= \frac{-1}{\pi} \int_a^{\infty} dQ(\omega) \int_0^1 \frac{\log((\omega-a)y+a)}{\sqrt{y}\sqrt{1-y}} dy \\ &= (\log a)g(2 \log \lambda) + \frac{1}{\pi} \int_a^{\infty} Q(\omega) \int_0^1 \frac{\sqrt{y} dy}{\sqrt{1-y}((\omega-a)y+a)} d\omega \\ &= (\log a)g(2 \log \lambda) + \frac{1}{\pi} \int_a^{\infty} Q(\omega) \frac{\pi}{\omega + \sqrt{a\omega}} d\omega \\ &= (\log a)g(2 \log \lambda) + \int_{2 \log \lambda}^{\infty} g(u) \frac{e^u - e^{-u}}{e^u + e^{-u} - 2 + (\lambda-1/\lambda)(e^{u/2} - e^{-u/2})} du \\ &= (\log a)g(2 \log \lambda) + \int_{2 \log \lambda}^{\infty} g(u) \frac{e^{u/2} + e^{-u/2}}{e^{u/2} - e^{-u/2} + \lambda - 1/\lambda} du. \end{aligned}$$

We have $g = g_1$ or g_2 , and

$$(4.41) \quad \begin{aligned} g_1(\varepsilon) &= \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{\varepsilon}} e^{-u^2/4\varepsilon} e^{-\varepsilon/4}, \\ g_2(\varepsilon) &= e^{-\varepsilon/4} \frac{1}{\sqrt{\pi\varepsilon}} (e^{-(u-\log 2)^2/4\varepsilon} + e^{-(u+\log 2)^2/4\varepsilon}). \end{aligned}$$

From (4.40) and (4.41) it follows that the integral (4.39) is estimated by $O(\frac{1}{\sqrt{\varepsilon}})$, $\varepsilon \rightarrow +0$.

The last integral (4.38) reduces easily to the sum of $g(2 \log \lambda)$ and $\int_{d^2}^{\infty} \log(t+4) \frac{k(t)}{\sqrt{t-d^2}} dt$ (up to multiplication by constant coefficients), which is estimated by $O(\frac{1}{\sqrt{\varepsilon}})$, $\varepsilon \rightarrow +0$, as before.

We arrive at the following.

Lemma 22. *We have $I_j = I_j(\varepsilon) = O(\frac{1}{\sqrt{\varepsilon}})$, $\varepsilon \rightarrow +0$, $j = 2, 3, 4, 5$ (see (4.8), (4.10)).*

The next theorem is a consequence of (4.11), (4.18), and Lemma 22.

Theorem 5. *As $\varepsilon \rightarrow +0$, for the trace of the operator T (see (4.7)) we have the following asymptotics:*

$$\operatorname{tr} T = 4 \sum_{j=1}^{\infty} \omega_j^2 e^{-\varepsilon \lambda_j} = \frac{\mu(F)}{2\pi} \cdot \frac{1}{\varepsilon} + O\left(\frac{|\log \varepsilon|}{\sqrt{\varepsilon}}\right).$$

By (3.50) and Theorem 5 with $\omega_j = \cos(\eta_j) - \cos(r_j \log 2)$, we have

$$(4.42) \quad \sum_{j=1}^{\infty} e^{-\varepsilon \lambda_j} = \frac{\mu(F)}{8\pi} \cdot \frac{1}{\varepsilon} + O\left(\frac{|\log \varepsilon|}{\sqrt{\varepsilon}}\right),$$

$$(4.43) \quad \sum_{j=1}^{\infty} \omega_j^2 e^{-\varepsilon \lambda_j} = \frac{\mu(F)}{8\pi} \cdot \frac{1}{\varepsilon} + O\left(\frac{|\log \varepsilon|}{\sqrt{\varepsilon}}\right).$$

The smallest number of terms with $\omega_j \neq 0$ is obtained if $|\omega_j| = 2$ for all j with $\omega_j \neq 0$. Assume this, and also assume that

$$(4.44) \quad \frac{\#\{j \mid 1 \leq j \leq X, \omega_j^2 = 4\}}{X} \xrightarrow{X \rightarrow \infty} \frac{1}{4}.$$

Then we shall see that (4.43) is valid.

By a Tauberian theorem, (4.42) implies that

$$(4.45) \quad \#\{\lambda_j \leq \lambda\} \approx \frac{\mu(F)}{8\pi} \lambda \text{ as } \lambda \rightarrow \infty$$

(see Theorem 4), where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$, and λ_j is repeated in accordance with its multiplicity. Let $j_{k_1} < j_{k_2} < \dots < j_{k_n} < \dots$ be the values of j such that $\omega_{j_k}^2 = 4$. Relation (4.44) implies

$$\frac{\#\{\lambda_{j_k} \leq \lambda\}}{\#\{\lambda_j \leq \lambda\}} \xrightarrow{\lambda \rightarrow \infty} \frac{1}{4},$$

whence, by (4.45),

$$\#\{\lambda_{j_k} \leq \lambda\} \approx \frac{1}{4} \cdot \frac{\mu(F)}{8\pi} \lambda.$$

Then

$$\sum_{j=1}^{\infty} \omega_j^2 e^{-\varepsilon \lambda_j} = 4 \sum_{k=1}^{\infty} e^{-\varepsilon \lambda_{j_k}} \underset{\varepsilon \downarrow 0}{\sim} 4 \frac{1}{4} \frac{\mu(F)}{8\pi} \frac{1}{\varepsilon} = \frac{\mu(F)}{8\pi} \frac{1}{\varepsilon},$$

in agreement with (4.42). The same proof shows that we cannot have

$$\liminf_{X \rightarrow \infty} \frac{\#\{j \mid 1 \leq j \leq X, \omega_j^2 = 4\}}{X} < \frac{1}{4},$$

and we see that the minimal number of j with $\omega_j \neq 0$ is given by (4.44).

We have proved the following statement.

Theorem 6. *We have*

$$(4.46) \quad \liminf_{\lambda \rightarrow \infty} \frac{\#\{\lambda_{j_k} \leq \lambda \mid \omega_{j_k} \neq 0\}}{\#\{\lambda_j \leq \lambda\}} \geq \frac{1}{4},$$

where each λ_j is repeated in accordance with its multiplicity.

Thus, for at least $\frac{1}{4}$ of the eigenvalues $\lambda_j = \frac{1}{4} + r_j^2$ of the automorphic Laplacian with odd eigenfunction Φ_j , the corresponding eigenvalue e^{in_j} of the exceptional Hecke operator $U(2)$ satisfies $\omega_j \neq 0$ or $e^{in_j} \neq 2^{ir_j}$.

Now we consider the other exceptional Hecke operators $U(q)$ for $q > 2$, $q \mid N$, $N = 4N_2$ or $N = 4N_3$. Here we must establish the condition $\rho_j(q) \neq \frac{q^{ir_j}}{\varepsilon_q}$, where $\rho_j(q)$ is the eigenvalue of $U(q)$ corresponding to the eigenvector Φ_j . The parameters ε_q are arbitrary real numbers. We assume that $\varepsilon_q \neq \pm 1$ for all $q \mid N$, $q > 2$. Then from [1, (7.23), (7.24)] and Theorem 6 we obtain the following.

Theorem 7. *Let the forms $\omega(z)$ be defined as in [1, Theorem 6.2] and assume that $\varepsilon_l \neq \pm 1$ for $l = 2, \dots, k$. Then for at least $\frac{1}{4}$ of the eigenvalues λ_j of $A_{\text{odd}}(\Gamma, \chi)$ (in the sense of Theorem 6) with eigenfunctions Φ_j , the Phillips–Sarnak integral $I_j(\Phi_j)$ is not zero, where $I_j(\Phi_j)$ is given by [1, (7.2)].*

Therefore, for each λ_j in the sequence of Theorem 7 at least one eigenvector Φ in the eigenspace $N(A_{\text{odd}}(\Gamma, \chi) - \lambda_j)$ turns into a resonance function under perturbation by the form $\omega(z)$, and the total dimension of this eigenspace is reduced at least by one (cf. [1, Theorem 5.8]).

For each eigenvalue λ_{j_k} with $\omega_{j_k} \neq 0$ there is at least one eigenfunction $\tilde{\Phi}$ with eigenvalue λ_{j_k} such that $\tilde{\Phi}$ turns into a resonance function under perturbation. Let $\{\tilde{\Phi}_i\}_{i=1}^\infty$ be an orthonormal sequence of all such eigenfunctions with increasing eigenvalues λ_i .

Theorem 8. *Assume that $\dim\{N(A_{\text{odd}} - \lambda_j)\} \leq m$ for all eigenvalues λ_j . Then*

$$\liminf_{\lambda \rightarrow \infty} \frac{\#\{\tilde{\Phi}_i \mid \lambda_i \leq \lambda, \omega_i \neq 0\}}{\#\{\Phi_j \mid \lambda_j \leq \lambda\}} \geq \frac{1}{4m}.$$

Proof. The minimal number of eigenfunctions $\tilde{\Phi}_i$ with eigenvalues $\lambda_i \leq \lambda$ occurs if all eigenfunctions Φ_{j_k} with $\omega_{j_k} \neq 0$ are distributed with m in each m -dimensional eigenspace of $A_{\text{odd}}(\Gamma, \chi)$. Then each of such eigenspaces contributes at least one $\tilde{\Phi}_i$, and the result follows from Theorem 7. \square

APPENDIX A. TRANSFORMATION OF THE INTEGRATION DOMAIN OF $k_\varepsilon(u(z, Jz))$ BY $g \in \text{PSL}(2, \mathbb{Q})$

In (3.29)–(3.34) it was shown that $F_Y^J = F_Y^0 = \bigcup_{\gamma \in \Gamma_1} \gamma F_Y(1)$, where Γ_1 is the modular group with fundamental domain $F(1)$ and $F_Y(1) = \{z \in F(1) \mid y \leq Y\}$. Then

$$(A.1) \quad \int_{F_Y^J} k_\varepsilon(u(z, Jz)) d\mu(z) = \int_{F_Y^0} k_\varepsilon(u(z, Jz)) d\mu(z).$$

Let $H(Y) = \{z \in H \mid y > Y\}$, let $\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_1$, and for $c_1 \neq 0$ let $C(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2})$ be the circle centered at $(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2})$ and of radius $\frac{1}{2Yc_1^2}$, touching \mathbb{R} at $\frac{a_1}{c_1}$, with interior $C^0(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2})$.

If $c_1 = 0$, we have $\gamma(H(Y)) = H(Y)$. We prove that $\gamma(H(Y)) = C^0(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2})$ for $c_1 \neq 0$. We have

$$\gamma(x + iY) = \frac{(a_1x + b_1)(c_1x + d_1) + a_1c_1Y^2 + iY}{(c_1x + d_1)^2 + c_1^2Y^2} = x' + iy'.$$

Since $\gamma(\infty) = \frac{a_1}{c_1}$, $\gamma(H(Y))$ is a circle $C^0(\frac{a_1}{c_1}, R)$ with radius R to be determined by the equation

$$\left(x' - \frac{a_1}{c_1}\right)^2 + y'^2 - 2Ry' = 0,$$

implying

$$\left(-x - \frac{d_1}{c_1}\right)^2 + Y^2 = 2RYc_1^2 \left[\left(x + \frac{d_1}{c_1}\right)^2 + Y^2\right],$$

which gives

$$R = \frac{1}{2Yc_1^2}.$$

Lemma A.1.

$$H \setminus F_Y^0 = H(Y) \cup \bigcup_{\substack{\gamma \in \Gamma \\ c_1 \neq 0}} C^0\left(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2}\right).$$

Proof. No point in $F_Y(1)$ is mapped by $\gamma \in \Gamma_1$ to a point in $F(1) \setminus F_Y(1)$, whence $H(Y) \subset H \setminus \Gamma_Y^0$. Therefore, $\gamma(H_Y) = C^0(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2}) \subset H \setminus \Gamma_Y^0$ for $\gamma \in \Gamma_1$, $c_1 \neq 0$, and the lemma follows. \square

In order to estimate the integrals in (3.29), we also need to calculate

$$(A.2) \quad \int_{F_Y^{\gamma J}} k_\varepsilon(u(z, \gamma Jz)) d\mu(z)$$

for $\gamma \in \Gamma_0(N)$ in the case where $\text{tr}(\gamma J) = 0$ and the centralizer $\Gamma_{\gamma J}$ coincides with $\{e\}$. Again we have $F_Y^{\gamma J} = F_Y^0$, and if $g \in \text{PSL}(2, \mathbb{Q})$ is such that $g(\gamma J)g^{-1} = J$, then

$$(A.3) \quad \begin{aligned} \int_{F_Y^{\gamma J}} k_\varepsilon(u(z, \gamma Jz)) d\mu(z) &= \int_{F_Y^0} k_\varepsilon(u(gz, Jgz)) d\mu(z) \\ &= \int_{g\Gamma_Y^0} k_\varepsilon(u(z, Jz)) d\mu(z). \end{aligned}$$

By Lemma A.1,

$$H \setminus g\Gamma_Y^0 = g(H \setminus \Gamma_Y^0) = g(H(Y)) \cup \bigcup_{\substack{\gamma \in \Gamma_1 \\ c_1 \neq 0}} g\left(C^0\left(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2}\right)\right).$$

Therefore, in order to estimate the integral (A.2), we need to calculate $g(H(Y))$ and $g(C^0(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2}))$ for $c_1 \neq 0$. The proof of Lemma A.1 shows that the set $g(H(Y))$ is $C^0(\frac{a}{c}, \frac{1}{2Yc^2})$ if $c \neq 0$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and it is $H(Ya^2)$ if $c = 0$.

Let $\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_1$, $c_1 \neq 0$; we set $r_1 = \frac{a_1}{c_1}$. The equation of $C\left(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2}\right)$ is

$$(x - r_1)^2 + \left(y - \frac{1}{2Yc_1}\right)^2 = \left(\frac{1}{2Yc_1}\right)^2$$

or

$$(A.4) \quad x^2 + y^2 = 2r_1x + \frac{1}{Yc_1^2}y - r_1^2.$$

Two cases are possible:

- 1) $\frac{a_1}{c_1} = -\frac{d}{c}$. Then $g\left(\frac{a_1}{c_1}\right) = \infty$, so that $g\left(C^0\left(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2}\right)\right) = H(Y_0)$ with Y_0 to be determined.
- 2) $\frac{a_1}{c_1} \neq -\frac{d}{c}$. Then $g\left(\frac{a_1}{c_1}\right) = g(r_1) = \frac{ar_1+b}{cr_1+d}$, and

$$g\left(C^0\left(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2}\right)\right) = C^0\left(\frac{ar_1+b}{cr_1+d}, R\right)$$

with R to be determined.

We have

$$(A.5) \quad g(x + iy) = \frac{(ax + b)(cx + d) + acy^2 + iy}{(cx + d)^2 + c^2y^2} = x' + iy'.$$

- 1) $\frac{a_1}{c_1} = -\frac{d}{c}$. Then, by (A.5),

$$\frac{y}{(cx + d)^2 + c^2y^2} = Y_0$$

or

$$(x - r_1)^2 + y^2 = \left(x + \frac{d}{c}\right)^2 = \frac{1}{Y_0c^2}.$$

Therefore, by (A.4),

$$Y_0 = Y \frac{c_1^2}{c^2}.$$

- 2) $\frac{a_1}{c_1} \neq -\frac{d}{c}$. We determine the radius R of the circle

$$C^0\left(\frac{ar_1+b}{cr_1+d}, R\right) = g\left(C\left(r_1, \frac{1}{2Yc_1^2}\right)\right),$$

where $C\left(r_1, \frac{1}{2Yc_1^2}\right)$ is given by (A.4). In terms of x' and y' given by (A.5), the equation of $C^0\left(\frac{ar_1+b}{cr_1+d}, R\right)$ is

$$\left(x' - \frac{ar_1+b}{cr_1+d}\right)^2 + y'^2 = 2Ry'$$

or

$$\begin{aligned} & \left\{ (ax + b)(cx + d) + acy^2 - \frac{ar_1+b}{cr_1+d} [(cx + d)^2 + c^2y^2] \right\}^2 + y^2 \\ & = 2Ry \{ (cx + d)^2 + c^2y^2 \}. \end{aligned}$$

Application of (A.4) reduces this to

$$\left\{ x + \frac{ac}{Yc_1^2}y + \frac{a}{b} \right\}^2 + y^2 = 2Ry \left\{ 2\frac{c}{a}x + \frac{c^2}{Yc_1^2}y + \frac{ad+bc}{a^2} \right\}.$$

Squaring and using (A.4) once again, we get

$$(A.6) \quad \begin{aligned} & \frac{c^2}{(Yc_1^2)^2(cr_1+d)^2}y^2 + \frac{2c}{Yc_1^2(cr_1+d)}xy + \frac{1}{Yc_1^2} \frac{d-cr_1}{cr_1+d}y \\ & = 2Ry \left\{ \frac{c^2}{Yc_1^2}y + 2c(cr_1+d)x + d^2 - c^2r_1^2 \right\}. \end{aligned}$$

Equating the coefficients of y^2 , xy , and y , we see that (A.6) is fulfilled if and only if

$$R = \frac{1}{2Yc_1^2} \frac{1}{(cr_1+d)^2} = \frac{1}{2Y} \frac{1}{(ca_1+dc_1)^2}.$$

This proves the following fact.

Lemma A.2. *Let*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Q}),$$

$c \neq 0$. Then with

$$\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_1,$$

$r_1 = \frac{a_1}{c_1}$, we have

$$H \setminus g\Gamma_Y^0 = C^0\left(\frac{a}{c}, \frac{1}{2Yc^2}\right) \cup \bigcup_{\substack{\gamma \in \Gamma_1 \\ c_1 \neq 0 \\ r_1 \neq -\frac{d}{c}}} C^0\left(\frac{ar_1+b}{cr_1+d}, \frac{1}{2Y} \frac{1}{(ca_1+dc_1)^2}\right) \cup H\left(Y \frac{c_2^2}{c^2}\right),$$

where

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma_1, \quad \frac{a_2}{c_2} = -\frac{d}{c}.$$

In particular,

$$g\left(C^0\left(-\frac{b}{a}, \frac{1}{2Yc^2}\right)\right) = C^0\left(0, \frac{1}{2Y(ca_1+dc_1)^2}\right)$$

for $a \neq 0$, $\frac{a_1}{c_1} = -\frac{b}{a}$, and

$$g(H(Y)) = C^0\left(0, \frac{1}{2Yc^2}\right)$$

for $a = 0$.

In order to estimate (A.3), we apply Lemma A.2 to

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Q}),$$

where $g(\gamma_0 J)g^{-1} = J$,

$$\gamma_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \Gamma_0(N), \quad \mathrm{tr}(\gamma_0 J) = 0, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We solve the equation $g(\gamma_0 J)g^{-1} = J$, which can be written as

$$\begin{pmatrix} a_0a + Nc_0b & -b_0a - a_0b \\ a_0c + Nc_0d & -b_0c - a_0d \end{pmatrix} = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix},$$

or as two pairs of dependent equations

$$(A.7) \quad \begin{aligned} (a_0 - 1)a + Nc_0b &= 0, & -b_0a - (a_0 + 1)b &= 0, \\ (a_0 + 1)c + Nc_0d &= 0, & -b_0c - (a_0 - 1)d &= 0. \end{aligned}$$

Case 1: $b_0 \neq 0, c_0 \neq 0$, so that $a_0 \neq \pm 1$. We find

$$g = \begin{pmatrix} \frac{1-a_0^2}{2b_0}c^{-1} & \frac{a_0-1}{2}c^{-1} \\ c & -\frac{b_0}{a_0-1}c \end{pmatrix}, \quad c \in \mathbb{Q} \setminus \{0\}.$$

Let $e_0 = (1 - a_0, b_0)$ and choose $c = c^* = \frac{1-a_0}{e_0}$. Then $d = d^* = \frac{b_0}{c_0}$, whence we see that c^* and d^* are integers with $(c^*, d^*) = 1$.

Case 2: $b_0 = 0, c_0 \neq 0, a_0 = -1$. We find $c^* = 1, d^* = 0$.

Case 3: $b_0 \neq 0, c_0 = 0, a_0 = -1$. Then:

$$\begin{aligned} c^* &= 2, \quad d^* = b_0 & \text{if } b_0 \text{ is odd, } a^* &= 0; \\ c^* &= 1, \quad d^* = \frac{b_0}{2} & \text{if } b_0 \text{ is even, } a^* &= 0. \end{aligned}$$

Case 4: $b_0 = 0, c_0 \neq 0, a_0 = 1$. We find $c^* = -\frac{Nc_0}{2}, d^* = 1, b = 0$.

Case 5: $b_0 \neq 0, c_0 = 0, a_0 = 1$. We find $c = 0$, and choose $d^* = 1$,

$$g^* = \begin{pmatrix} 1 & -\frac{b_0}{2} \\ 0 & 1 \end{pmatrix}.$$

Applying Lemma A.2 to $g^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$, we obtain the following statement.

Lemma A.3. *Let*

$$\gamma_0 = \begin{pmatrix} a_0 & b_0 \\ Nc_0 & d_0 \end{pmatrix} \in \Gamma_0(N),$$

and let $g^*(\gamma_0 J)g^{*-1} = J$, where

$$g^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \in \text{PSL}(2, \mathbb{Q}),$$

c^* and d^* are integers, and $(c^*, d^*) = 1$. Then

$$\int_{F\gamma_0 J} k_\varepsilon(u(z, \gamma_0 Jz)) d\mu(z) = \int_{g^* \Gamma_Y^0} k_\varepsilon(u(z, Jz)) d\mu(z),$$

where

$$\{y \mid \frac{1}{Y} < y < Y\} \subset g^* \Gamma_Y^0 \subset \{y \mid 0 < y < Y\} \setminus C^0\left(0, \frac{1}{2Yc}\right), \quad c \geq 1,$$

and

$$\begin{aligned} c &= (c^* a_1 + d^* c_1)^2, \quad \frac{a_1}{c_1} = -\frac{b^*}{a^*}, \quad (a_1, c_1) = 1 & \text{for } a^* \neq 0, \text{ and} \\ c &= 1 \text{ for } b_0 \text{ even and } c = 4 \text{ for } b_0 \text{ odd} & \text{if } a^* = 0. \end{aligned}$$

Proof. Suppose that $\gamma \in \Gamma_1, c_1 \neq 0, r_1 = \frac{a_1}{c_1} \neq -\frac{d^*}{c^*}$. Since $(c^*, d^*) = 1$, we have $\min(c^* a_1 + d^* c_1)^2 = 1$ and $\max \frac{1}{2Y} \frac{1}{(c^* a_1 + d^* c_1)^2} = \frac{1}{2Y}$. Also, $\frac{a_2}{c_2} = -\frac{d^*}{c^*}$ in cases 1, 3, and 4, whence $c_2 = \pm c^*$ and $H(Y \frac{c_2^2}{c^{*2}}) = H(Y)$. Clearly, $c = (c^* a_1 + d^* c_1)^2$ with $\frac{a_1}{c_1} = -\frac{b^*}{a^*}$ for $a^* \neq 0$. In case 3 we have $a^* = 0$ and $c = c^{*2} = 1, 4$.

In case 2 we obtain $c^* = 1, d^* = 0$. If $\frac{a_1}{d_1} \neq -\frac{d^*}{c^*} = 0$, then $a_1 \neq 0$. For $a_1 = 0$ we have $c = c^* a_1 + d^* c_1 = 1$. Now c_2 is determined by $\frac{a_2}{c_2} = -\frac{d^*}{c^*} = 0$, so that $a_2 = 0$. Then $c_2 = 1, b_2 = -1$, whence $c_2 = c^* = 1$ and $H(Y \frac{c_2^2}{c^{*2}}) = H(Y)$. In case 5 g^* is translation by $-\frac{b_0}{2}$, and the lemma is proved. \square

In order to estimate the similar integrals from §4, we consider $\gamma_0 p$ with $\gamma_0 \in \Gamma_0(N)$, $p = p_1, p_2, p_3, p_4$, $\Gamma_{\gamma_0 p} = \{e\}$. For

$$\gamma_0 = \begin{pmatrix} a_0 & b_0 \\ Nc_0 & d_0 \end{pmatrix} \in \Gamma_0(N),$$

we shall find

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Q})$$

such that

$$g(\gamma_0 p)g^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

(I) $\gamma_0 p_1$ with

$$p_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & q\frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix}, \quad q = 0, 1.$$

By Theorem 2, $\Gamma_{\gamma_0 p_1} = \{e\}$ if and only if $\sqrt{2} \text{tr}(\gamma_0 p_1) = 3$. Then $\lambda + \lambda^{-1} = \frac{3}{\sqrt{2}}$, $\lambda = \sqrt{2}$, $\lambda = \frac{1}{\sqrt{2}}$. We have

$$(A.8) \quad \gamma_0 p_1 = \begin{pmatrix} \frac{a_0}{\sqrt{2}} & a_0 \frac{q}{\sqrt{2}} + b_0 \sqrt{2} \\ \frac{Nc_0}{\sqrt{2}} & Nc_0 \frac{q}{\sqrt{2}} + d_0 \sqrt{2} \end{pmatrix}.$$

With $\lambda = \sqrt{2}$, we get the following dependent equations for c and d :

$$(A.9) \quad \begin{aligned} (a_0 - 1)c + Nc_0 d &= 0, \\ (a_0 q + 2b_0)c + (Nc_0 q + 2d_0 - 1)d &= 0, \\ a_0 + Nc_0 q + 2d_0 &= 3. \end{aligned}$$

(II)

$$\gamma_0 p_2 = \begin{pmatrix} \sqrt{2}a_0 + \frac{1}{\sqrt{2}}qb_0 & \frac{1}{\sqrt{2}}b_0 \\ \sqrt{2}Nc_0 + \frac{1}{\sqrt{2}}qd_0 & \frac{1}{\sqrt{2}}d_0 \end{pmatrix}, \quad q = 0, -N.$$

By Theorem 2, $\Gamma_{\gamma_0 p_2} = \{e\}$ if and only if $\text{tr}(\gamma_0 p_2) = \frac{3}{\sqrt{2}}$. For the solution $\lambda = \sqrt{2}$ of $\lambda + \lambda^{-1} = \frac{3}{\sqrt{2}}$ we get the following dependent equations for c and d :

$$(A.10) \quad \begin{aligned} b_0 c + (d_0 - 1)d &= 0, \\ (2 - d_0)c + (2Nc_0 + qd_0)d &= 0, \quad q = 0, -N, \\ 2a_0 + qb_0 + d_0 &= 3. \end{aligned}$$

(III)

$$\gamma_0 p_3 = \begin{pmatrix} \frac{1}{2}a_0 & \frac{1}{2}qa_0 + 2b_0 \\ \frac{1}{2}Nc_0 & \frac{1}{2}qNc_0 + 2d_0 \end{pmatrix}, \quad q = 0, 1, 2, 3.$$

By Theorem 2, $\text{tr}(\gamma_0 p_3) = \frac{5}{2}$ if and only if $\Gamma_{\gamma_0 p_3} = \{e\}$. For the solution $\lambda = 2$ of $\lambda + \lambda^{-1} = \frac{5}{2}$ we obtain the dependent equations:

$$(A.11) \quad \begin{aligned} (a_0 - 1)c + Nc_0 d &= 0, \\ (qa_0 + 4b_0)c + (4 - a_0)d &= 0, \quad q = 0, 1, 2, 3, \\ a_0 + qNc_0 + 4d_0 &= 5. \end{aligned}$$

(IV)

$$\gamma_0 p_4 = \begin{pmatrix} 2a_0 + qb_0 & \frac{1}{2}b_0 \\ 2Nc_0 + qd_0 & \frac{1}{2}d_0 \end{pmatrix}, \quad q = 0, -\frac{N}{2}, -N, -\frac{3N}{2}.$$

By Theorem 2, $\text{tr}(\gamma_0 p_4) = \frac{5}{2}$ if and only if $\Gamma_{\gamma_0 p_4} = \{e\}$. For the solution $\lambda = 2$ of $\lambda + \lambda^{-1} = \frac{5}{2}$ we obtain the dependent equations

$$(A.12) \quad \begin{aligned} b_0 c + (d_0 - 1)d &= 0, \\ (2a_0 + qb_0 - \frac{1}{2})c + (2Nc_0 + qd_0)d &= 0, \\ 4a_0 + 2qb_0 + d_0 &= 5. \end{aligned}$$

In all cases I–IV we solve equations (A.9)–(A.12) in the same way as we solved the second set of equations (A.7). Arguing as in Lemma A.3, from Lemma A.2 we deduce the following statement.

Lemma A.4. *Suppose $\gamma_0 \in \Gamma_0(N)$, $\sqrt{2} \text{tr}(\gamma_0 p_i) = 3$ for $i = 1, 2$, and $2 \text{tr}(\gamma_0 p_i) = 5$ for $i = 3, 4$. Then $\Gamma_{\gamma p_i} = \{e\}$ and $F_Y^{\gamma_0 p_i} = F_Y^0$. Let g^* be defined as in I–IV in such a way that*

$$g^*(\gamma_0 p_i)g^{*-1} = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix},$$

where $\lambda_i = \sqrt{2}$ for $i = 1, 2$ and $\lambda_i = 2$ for $i = 3, 4$. Then

$$\int_{F_Y^{\gamma_0 p_i}} k_\varepsilon(u(z, \gamma_0 p_i z)) = \int_{g^* F_Y^0} k_\varepsilon(u(z, \lambda_i^2 z)) d\mu(z),$$

where

$$\left\{ z \in H \mid \frac{1}{Y} < \text{Im } z < Y \right\} \subset g^* F_Y^0 \subset \{z \in H \mid 0 < y < Y\} \setminus C^0\left(0, \frac{1}{2Yc}\right), \quad c \geq 1,$$

and

$$\begin{aligned} c &= (c^* a_1 + d^* c_1)^2, \quad \frac{a_1}{c_1} = -\frac{b^*}{a^*} \quad \text{if } a^* \neq 0, \\ c &\geq 1 \quad \text{if } a^* = 0. \end{aligned}$$

Finally, we consider $\gamma_0 p_i J$, where $\gamma_0 \in \Gamma_0(N)$, $p = p_1, p_2, p_3, p_4$, and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with $\Gamma_{\gamma_0 p_i J} = \{e\}$. By Theorem 3 this is fulfilled if and only if $\sqrt{2} \text{tr}(\gamma_0 p_i J) = 1$ for $i = 1, 2$ and $2 \text{tr}(\gamma_0 p_i J) = 3$ for $i = 3, 4$. We find

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that

$$g(\gamma_0 p_i J)g^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

(I)

$$\gamma_0 p_1 J = \begin{pmatrix} \frac{a_0}{\sqrt{2}} & -(a_0 \frac{q}{\sqrt{2}} + b_0 q) \\ \frac{Nc_0}{\sqrt{2}} & -(-Nc_0 \frac{q}{\sqrt{2}} + d_0 \sqrt{2}) \end{pmatrix}, \quad q = 0, 1.$$

With the solution $\lambda = \sqrt{2}$ of $\lambda - \lambda^{-1} = \frac{1}{\sqrt{2}}$ we get the following dependent equations for c and d :

$$(A.13) \quad \begin{aligned} (a_0 + 1)c + Nc_0 d &= 0, \\ -(a_0 q + 2b_0 + 1)c - a_0 d &= 0, \\ a_0 - (Nc_0 q + 2d_0) &= 1. \end{aligned}$$

(II)

$$\gamma_0 p_2 J = \begin{pmatrix} \sqrt{2}a_0 + \frac{1}{\sqrt{2}}\gamma_0 b_0 & -\frac{1}{\sqrt{2}}b_0 \\ \sqrt{2}Nc_0 + \frac{1}{\sqrt{2}}qd_0 & -\frac{1}{\sqrt{2}}d_0 \end{pmatrix}, \quad q = 0, -N, \quad \text{tr}(\gamma_0 p_2 J) = \frac{1}{\sqrt{2}}.$$

With the solution $\lambda = \sqrt{2}$ of $\lambda - \lambda^{-1} = \frac{1}{\sqrt{2}}$ we get equations for c and d :

$$(A.14) \quad \begin{aligned} -b_0 c - (d_0 + 1)d &= 0, \\ (d_0 + 2)c + (2Nc_0 + qd_0)d &= 0, \\ 2a_0 + \gamma_0 b - d_0 &= 1. \end{aligned}$$

(III)

$$\gamma_0 p_3 J = \begin{pmatrix} \frac{1}{2}a_0 & -(\frac{1}{2}qa_0 + 2b_0) \\ \frac{1}{2}Nc_0 & -(\frac{1}{2}qNc_0 + 2d_0) \end{pmatrix}, \quad q = 0, 1, 2, 3, \quad \text{tr}(\gamma_0 p_3 J) = \frac{3}{2}.$$

With the solution $\lambda = 2$ of $\lambda - \lambda^{-1} = \frac{3}{2}$ we get equations for c and d :

$$(A.15) \quad \begin{aligned} (a_0 + 1)c + Nc_0 d &= 0, \\ -(qa_0 + 4b_0)c - (a_0 - 2)d &= 0, \\ a_0 - (qNc_0 + 4d_0) &= 3. \end{aligned}$$

(IV)

$$\gamma_0 p_4 J = \begin{pmatrix} 2a_0 + qb_0 & -\frac{1}{2}b_0 \\ 2Nc_0 + qd_0 & -\frac{1}{2}d_0 \end{pmatrix}, \quad q = 0, -\frac{N}{2}, -N, -\frac{3N}{2}, \quad \text{tr}(\gamma_0 p_4 J) = \frac{3}{2}.$$

With the solution $\lambda = 2$ of $\lambda - \lambda^{-1} = \frac{3}{2}$ we get the following equations for c and d :

$$(A.16) \quad \begin{aligned} b_0 c + (-d_0 + 1)d &= 0, \\ (d_0 + 4)c + (4Nc_0 + 2qd_0)d &= 0, \\ 4a_0 + 2qb_0 - d_0 &= 3. \end{aligned}$$

In all cases I–IV we solve equations (A.13)–(A.16) in the same way as equations (A.7) and (A.9)–(A.12) were solved. Thus, Lemma A.2 implies the following.

Lemma A.5. *Suppose $\gamma_0 \in \Gamma_0(N)$, $\sqrt{2} \text{tr}(\gamma_0 p_i J) = 1$ for $i = 1, 2$, and $2 \text{tr}(\gamma_0 p_i J) = 3$ for $i = 3, 4$. Then $\Gamma_{\gamma_0 p_i J} = \{e\}$ and $F_Y^{\gamma_0 p_i J} = F_Y^0$. Let $g^* \in \text{PSL}(2, \mathbb{Q})$ be defined as in I–IV in such a way that*

$$g^*(\gamma_0 p_i J)g^{*-1} = \begin{pmatrix} \lambda_i & 0 \\ 0 & -\lambda_i^{-1} \end{pmatrix},$$

where $\lambda_i = \sqrt{2}$ for $i = 1, 2$ and $\lambda_i = 2$ for $i = 3, 4$. Then

$$\int_{F_Y^{\gamma_0 p_i J}} k_\varepsilon(u(z, \gamma_0 p_i J z)) d\mu(z) = \int_{g^* F_Y^0} k_\varepsilon(u(z, \lambda_i^2 z)) d\mu(z),$$

where

$$\left\{ z \mid \frac{1}{Y} < y < Y \right\} \subset g^* F_Y^0 \subset \{z \mid 0 < y < Y\} \setminus C^0\left(0, \frac{1}{2Yc}\right), \quad c \geq 1,$$

and

$$\begin{aligned} c &= (c^* a_1 + d^* c_1)^2, \quad \frac{a_1}{c_1} = -\frac{b^*}{a^*} \quad \text{for } a^* \neq 0, \\ c &\geq 1 \quad \text{for } a^* = 0. \end{aligned}$$

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