

NONLINEAR HYPERBOLIC EQUATIONS IN SURFACE THEORY: INTEGRABLE DISCRETIZATIONS AND APPROXIMATION RESULTS

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Dedicated to L. D. Faddeev on the occasion of his 70th birthday

ABSTRACT. A discretization of the Goursat problem for a class of nonlinear hyperbolic systems is proposed. Local C^∞ -convergence of the discrete solutions is proved, and the approximation error is estimated. The results hold in arbitrary dimensions, and for an arbitrary number of dependent variables. The sine-Gordon equation serves as a guiding example for applications of the approximation theory. As the main application, a geometric Goursat problem for surfaces of constant negative Gaussian curvature (K-surfaces) is formulated, and approximation by discrete K-surfaces is proved. The result extends to the simultaneous approximation of Bäcklund transformations. This rigorously justifies the generally accepted belief that the theory of integrable surfaces and their transformations may be obtained as the continuum limit of a unifying multidimensional discrete theory.

§1. INTRODUCTION

The development of classical differential geometry has led to the introduction and investigation of various classes of surfaces that are of interest both for internal differential-geometric reasons and for applications in other sciences. Well-known examples are minimal surfaces, constant curvature surfaces, isothermic surfaces, etc. To a large extent, the rich theory of such surface classes is a classical heritage. On the other hand, the theory of *discrete differential geometry* is more recent and is nowadays a flourishing area which parallels substantially its classical (continuous) counterpart (see the early book [S] and the recent review [BP2]). Many important classes of surfaces have been discretized up to now. The properties of discrete surfaces are well understood, and they are employed widely for visualization needs and for numerical approximation. Certain convergence theorems are available for problems described by elliptic partial differential equations, such as the Plateau problem in the theory of minimal surfaces (see, e.g., [PP, Hin, DH]).

The characteristic property of various special classes of surfaces studied in classical differential geometry is their *integrability*. One of its manifestations is the existence of a rich theory of transformations, unified under the name of “Bäcklund–Darboux” transformations. Classically, the theory of surfaces and that of their transformations were dealt with separately in great part. In recent times, it became clear that both theories can

2000 *Mathematics Subject Classification*. Primary 65N22, 35L45.

Key words and phrases. Hyperbolic system, Goursat problem, integrability, discretization, K-surface, Bäcklund transformations.

The first and the third authors were partially supported by the DFG Research Center *Matheon* “Mathematics for key technologies”.

The second author was partially supported by the DFG Sonderforschungsbereich 288 “Differential Geometry and Quantum Physics”.

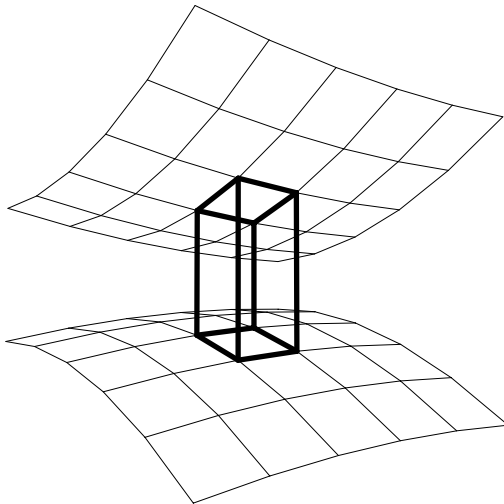


FIGURE 1. Surfaces and their transformations as a limit of multidimensional lattices.

be unified in the framework of discrete differential geometry. In this framework, multidimensional lattices with certain geometric properties become the basic mathematical structures. Passage to the continuum limit in some of the coordinate directions (mesh size $\epsilon \rightarrow 0$) yields the respective smooth surface. The directions where the mesh size remains constant correspond to the transformations of smooth surfaces (see Figure 1).

However, this general picture has not been supported up to now by rigorous convergence results. In this paper, we provide such results for surfaces of constant negative Gaussian curvature (K-surfaces). They are described analytically by the sine-Gordon equation, which is a hyperbolic PDE. Analogously, discrete K-surfaces are described by the hyperbolic difference Hirota equation [BP1]. We develop an approximation theory for equations of this type, which applies to much more general situations than the sine-Gordon equation. As the main geometric application, a rigorous proof of the convergence of discrete K-surfaces to smooth K-surfaces is provided, along with their Bäcklund transformations. Other examples where the general theory has been applied, include conjugate nets and orthogonal coordinate systems, along with their transformations [BMS].

The structure of the paper is as follows. In §2 we recall the well-known geometry of smooth and discrete K-surfaces, along with its analytic description via the sine-Gordon and Hirota equations. In §3 it is pointed out that the notions of discrete K-surfaces, their Bäcklund transformations, and superposition principles for the latter, are all put on an equal footing if they are viewed as multidimensional lattices. Thus, the notion of three-dimensional consistency of discrete two-dimensional equations starts to play a key role; we recall the recent finding that this notion can be put at the very base of the whole theory of integrability. Then, in §4, we formulate our basic Theorem 4.1 on the convergence of discrete K-surfaces, which, in this case, provides a rigorous justification of the general scheme encoded in Figure 1. This theorem is based upon general approximation results for Goursat problems for multidimensional hyperbolic systems of difference equations, which are formulated in §§5 and 6 for two- and m -dimensional systems, respectively. §7 contains technical proofs of the approximation theorems for general discrete Goursat problems. The last §8 is devoted to the proof of Theorem 4.1.

§2. CONTINUOUS AND DISCRETE SINE-GORDON EQUATIONS

2.1. Sine-Gordon equation and differential geometry. The celebrated sine-Gordon equation reads:

$$(1) \quad \partial_x \partial_y \phi = \sin \phi.$$

We consider local solutions $\phi : \mathcal{B}(r) = [0, r] \times [0, r] \rightarrow \mathbb{R}$, for some $r > 0$. The integrability of the sine-Gordon equation has many manifestations, two of which will be of special importance for us: the zero-curvature representation and the existence of Bäcklund transformations.

To formulate the *zero-curvature representation* of the sine-Gordon equation [FT], consider the matrices $U, V : \mathcal{B}(r) \rightarrow g[\lambda]$ defined by the formulas

$$(2) \quad U(x, y; \lambda) = \frac{i}{2} \begin{pmatrix} \partial_x \phi & -\lambda \\ -\lambda & -\partial_x \phi \end{pmatrix},$$

$$(3) \quad V(x, y; \lambda) = \frac{i}{2} \begin{pmatrix} 0 & \lambda^{-1} \exp(i\phi) \\ \lambda^{-1} \exp(-i\phi) & 0 \end{pmatrix},$$

which take values in the twisted loop algebra

$$g[\lambda] = \{ \xi : \mathbb{R}_* \rightarrow \mathfrak{su}(2) : \xi(-\lambda) = \sigma_3 \xi(\lambda) \sigma_3 \}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the zero curvature condition

$$(4) \quad \partial_y U - \partial_x V + [U, V] = 0$$

is satisfied identically in λ if and only if ϕ is a solution of (1). This condition assures the solvability of the following system of linear differential equations:

$$(5) \quad \partial_x \Psi = U \Psi, \quad \partial_y \Psi = V \Psi,$$

for a function $\Psi : \mathcal{B}(r) \rightarrow G[\lambda]$ with values in the twisted loop group

$$G[\lambda] = \{ \Xi : \mathbb{R}_* \rightarrow \mathrm{SU}(2) : \Xi(-\lambda) = \sigma_3 \Xi(\lambda) \sigma_3 \}.$$

A *geometric interpretation* of the sine-Gordon equation is as follows. Let $F : \mathcal{B}(r) \rightarrow \mathbb{R}^3$ be a surface parametrized along its asymptotic lines. This means that the vectors $\partial_x F, \partial_y F, \partial_x^2 F, \partial_y^2 F$ are orthogonal to the normal vector $N : \mathcal{B}(r) \rightarrow S^2$. Surfaces of constant negative Gaussian curvature $K = -1$ (*K-surfaces*, for short) in the asymptotic lines parametrization are characterized by the additional requirement that $|\partial_x F|$ should not depend on y , and $|\partial_y F|$ should not depend on x . Reparametrizing the asymptotic lines of a K-surface, if necessary, we can (and shall) assume that $|\partial_x F| = |\partial_y F| = 1$. Then the angle $\phi = \phi(x, y)$ between the vectors $\partial_x F$ and $\partial_y F$ satisfies the sine-Gordon equation (1).

The zero-curvature representation allows us to reconstruct a K-surface corresponding to a solution ϕ of the sine-Gordon equation. Given a solution $\phi : \mathcal{B}(r) \rightarrow \mathbb{R}$, we introduce the matrices (2), (3) satisfying (4) and define a function $\Psi : \mathcal{B}(r) \rightarrow G[\lambda]$ as the solution of equations (5) with initial condition $\Psi(0, 0; \lambda) = \mathbf{1}$. Then the immersion $F : \mathcal{B}(r) \rightarrow \mathbb{R}^3$ obtained by the *Sym formula*

$$(6) \quad F(x, y) = (2\lambda \Psi^{-1}(x, y; \lambda) \partial_\lambda \Psi(x, y; \lambda)) |_{\lambda=1},$$

under the canonical identification of $\mathfrak{su}(2)$ with \mathbb{R}^3 , is a K-surface parametrized along asymptotic lines, with the angle ϕ between the asymptotic directions. The function Ψ is known as the *extended frame* of F .

Moreover, for $\lambda \neq 1$ the right-hand side of (6) delivers an entire family of immersions $F_\lambda : \mathcal{B}(r) \rightarrow \mathbb{R}^3$, all of which turn out to be K-surfaces parametrized along asymptotic lines. These surfaces F_λ constitute the so-called *associated family* of F .

The classical *Bäcklund transformation* is given by the following construction. For a fixed solution ϕ of equation (1), a new solution $\phi^{(1)}$ can be constructed by solving the following system of differential equations:

$$(7) \quad \partial_x \phi^{(1)} + \partial_x \phi = \frac{2}{\alpha} \sin \frac{\phi^{(1)} - \phi}{2}, \quad \partial_y \phi^{(1)} - \partial_y \phi = 2\alpha \sin \frac{\phi^{(1)} + \phi}{2}.$$

This system is compatible, $\partial_y(\partial_x \phi^{(1)}) = \partial_x(\partial_y \phi^{(1)})$, provided ϕ is a solution of the sine-Gordon equation, and then $\phi^{(1)}$ is also a solution. It is determined by the parameter α and the value $\phi_0^{(1)} = \phi^{(1)}(0, 0)$ at one point.

Geometrically, a Bäcklund transformation $F^{(1)} : \mathcal{B}(r) \rightarrow \mathbb{R}^3$ of a given K-surface $F : \mathcal{B}(r) \rightarrow \mathbb{R}^3$ is characterized as follows: the straight line segments $[F(x, y), F^{(1)}(x, y)]$ are tangent to both surfaces F and $F^{(1)}$, and their length is independent of (x, y) .

It is not difficult to see that equations (7) are equivalent to the following matrix differential equations:

$$(8) \quad \partial_x \mathcal{W} = U^{(1)} \mathcal{W} - \mathcal{W} U, \quad \partial_y \mathcal{W} = V^{(1)} \mathcal{W} - \mathcal{W} V,$$

where the matrix \mathcal{W} is given by the formula

$$(9) \quad \mathcal{W}(x, y; \lambda) = \begin{pmatrix} \exp(i(\phi^{(1)} - \phi)/2) & -i\alpha\lambda \\ -i\alpha\lambda & \exp(-i(\phi^{(1)} - \phi)/2) \end{pmatrix}.$$

On the other hand, equations (8) constitute a solvability condition for the system consisting of (5) and similar equations for the matrix function

$$(10) \quad \Psi^{(1)} = \mathcal{W} \Psi.$$

It can be shown that $\Psi^{(1)}$ serves as the extended frame of the Bäcklund transformed surface $F^{(1)}$.

A remarkable property of Bäcklund transformations is given by *Bianchi's permutability theorem*: if $\phi^{(1)}$ is a Bäcklund transformation of ϕ with parameter α and $\phi^{(2)}$ is a Bäcklund transformation of ϕ with parameter β , then there exists a unique solution $\phi^{(12)}$ of the sine-Gordon equation that is simultaneously a Bäcklund transformation of $\phi^{(1)}$ with parameter β and a Bäcklund transformation of $\phi^{(2)}$ with parameter α ; this solution is given by the formula

$$(11) \quad \sin \frac{1}{4}(\phi^{(12)} + \phi^{(2)} - \phi^{(1)} - \phi) = \frac{\beta}{\alpha} \sin \frac{1}{4}(\phi^{(12)} - \phi^{(2)} + \phi^{(1)} - \phi).$$

2.2. Hirota equation and discrete differential geometry. Many different discretizations of the sine-Gordon equation can be imagined. A naive one is obtained by replacing the partial derivatives by their difference quotients,

$$(12) \quad \delta_x^\epsilon \delta_y^\epsilon \phi = \sin \phi.$$

Here, of course, the difference operators δ_x^ϵ and δ_y^ϵ act on functions defined on $\mathcal{B}^\epsilon(r) = \mathcal{B}(r) \cap (\epsilon\mathbb{Z})^2$ in accordance with

$$\delta_x^\epsilon \phi = \frac{1}{\epsilon}(\phi(x + \epsilon, y) - \phi(x, y)), \quad \delta_y^\epsilon \phi = \frac{1}{\epsilon}(\phi(x, y + \epsilon) - \phi(x, y)).$$

However, the discretization (12) turns out to be nonintegrable (in particular, it does not possess Bäcklund transformations), nor does it yield interesting geometry. An integrable one is due to Hirota [H]:

$$(13) \quad \begin{aligned} & \sin \frac{1}{4}(\phi(x + \epsilon, y + \epsilon) - \phi(x + \epsilon, y) - \phi(x, y + \epsilon) + \phi(x, y)) \\ & = \frac{\epsilon^2}{4} \sin \frac{1}{4}(\phi(x + \epsilon, y + \epsilon) + \phi(x + \epsilon, y) + \phi(x, y + \epsilon) + \phi(x, y)). \end{aligned}$$

The discrete zero-curvature representation of equation (13) is formulated in terms of the matrices $\mathcal{U}, \mathcal{V} : \mathcal{B}^\epsilon(r) \rightarrow G[\lambda]$ defined by the formulas

$$(14) \quad \mathcal{U}(x, y; \lambda) = (1 + \epsilon^2 \lambda^2 / 4)^{-1/2} \begin{pmatrix} \exp(i\epsilon a / 2) & -i\epsilon \lambda / 2 \\ -i\epsilon \lambda / 2 & \exp(-i\epsilon a / 2) \end{pmatrix},$$

$$(15) \quad \mathcal{V}(x, y; \lambda) = (1 + \epsilon^2 \lambda^{-2} / 4)^{-1/2} \begin{pmatrix} 1 & (i\epsilon \lambda^{-1} / 2) \exp(ib) \\ (i\epsilon \lambda^{-1} / 2) \exp(-ib) & 1 \end{pmatrix},$$

where the following abbreviations are used:

$$(16) \quad a = \frac{1}{\epsilon} (\phi(x + \epsilon, y) - \phi(x, y)), \quad b = \frac{1}{2} (\phi(x, y + \epsilon) + \phi(x, y)).$$

Then the matrix equation

$$(17) \quad \mathcal{U}(x, y + \epsilon; \lambda) \mathcal{V}(x, y; \lambda) = \mathcal{V}(x + \epsilon, y; \lambda) \mathcal{U}(x, y; \lambda)$$

is equivalent to (13). Formula (17) is the compatibility condition of the following system of linear difference equations:

$$(18) \quad \begin{aligned} \Psi(x + \epsilon, y; \lambda) &= \mathcal{U}(x, y; \lambda) \Psi(x, y; \lambda), \\ \Psi(x, y + \epsilon; \lambda) &= \mathcal{V}(x, y; \lambda) \Psi(x, y; \lambda) \end{aligned}$$

for a function $\Psi : \mathcal{B}^\epsilon(r) \rightarrow G[\lambda]$.

A *geometric interpretation* of the Hirota equation (13) was discovered in [BP1]. An immersion $F^\epsilon : \mathcal{B}^\epsilon(r) \rightarrow \mathbb{R}^3$ is called a *discrete surface parametrized along asymptotic lines* if for each $(x, y) \in \mathcal{B}^\epsilon(r)$ the five points $F^\epsilon(x, y)$, $F^\epsilon(x \pm \epsilon, y)$, and $F^\epsilon(x, y \pm \epsilon)$ lie in a single plane $\mathcal{P}(x, y)$ (called the tangent plane to F^ϵ at the point $F^\epsilon(x, y)$). *Discrete K-surfaces* [W] are characterized as discrete surfaces parametrized along asymptotic lines with an additional requirement that all edges have the same length $\epsilon \ell$, that is $|\delta_x^\epsilon F^\epsilon| = |\delta_y^\epsilon F^\epsilon| = \ell$. For discrete K-surfaces, a certain function ϕ related to the angle between $\delta_x^\epsilon F^\epsilon$ and $\delta_y^\epsilon F^\epsilon$ satisfies equation (13); see [BP1] for details.

Conversely, given a solution ϕ of (13), we can introduce the matrices \mathcal{U}, \mathcal{V} by equations (14), (15). Define a function $\Psi = \Psi^\epsilon : \mathcal{B}^\epsilon(r) \rightarrow G[\lambda]$ as the solution of equations (18) with initial condition $\Psi(0, 0; \lambda) = \mathbf{1}$. Then the Sym formula (6) determines an immersion $F = F^\epsilon : \mathcal{B}^\epsilon \rightarrow \mathbb{R}^3$, which is a discrete K-surface with the characteristic angle function ϕ and with the edge length $\ell = (1 + \epsilon^2 / 4)^{-1}$. Again, the right-hand side of (6) at $\lambda \neq 1$ delivers an associated family F_λ^ϵ of discrete K-surfaces.

The Bäcklund transformation for equation (13) is given by the following difference analogs of (7):

$$(19) \quad \begin{aligned} &\sin \frac{1}{4} (\phi^{(1)}(x + \epsilon, y) - \phi^{(1)}(x, y) + \phi(x + \epsilon, y) - \phi(x, y)) \\ &= \frac{\epsilon}{2\alpha} \sin \frac{1}{4} (\phi^{(1)}(x + \epsilon, y) + \phi^{(1)}(x, y) - \phi(x + \epsilon, y) - \phi(x, y)), \end{aligned}$$

$$(20) \quad \begin{aligned} &\sin \frac{1}{4} (\phi^{(1)}(x, y + \epsilon) - \phi^{(1)}(x, y) - \phi(x, y + \epsilon) + \phi(x, y)) \\ &= \frac{\epsilon\alpha}{2} \sin \frac{1}{4} (\phi^{(1)}(x, y + \epsilon) + \phi^{(1)}(x, y) + \phi(x, y + \epsilon) - \phi(x, y)). \end{aligned}$$

There are statements similar to those for the sine-Gordon equation. Difference equations (19), (20) are compatible, provided ϕ is a solution of (13), and then $\phi^{(1)}$ is also a solution (determined by the parameter α and the value $\phi_0^{(1)} = \phi^{(1)}(0, 0)$ at one point).

Also, the geometric meaning of the Bäcklund transformation is similar to the continuous case: the straight line segments connecting the corresponding points of a discrete

K-surface and its Bäcklund transformation lie in the planes tangent to both surfaces, and their length is independent of $(x, y) \in \mathcal{B}^\epsilon(r)$.

Equations (19), (20) are equivalent to the matrix equations

$$(21) \quad \begin{aligned} \mathcal{W}(x + \epsilon, y; \lambda) \mathcal{U}(x, y; \lambda) &= \mathcal{U}^{(1)}(x, y; \lambda) \mathcal{W}(x, y; \lambda), \\ \mathcal{W}(x, y + \epsilon; \lambda) \mathcal{V}(x, y; \lambda) &= \mathcal{V}^{(1)}(x, y; \lambda) \mathcal{W}(x, y; \lambda), \end{aligned}$$

with the same matrix \mathcal{W} as in (9). At the same time, these equations assure the solvability of the system consisting of (18) and similar equations for the matrix (10). It can be shown that $\Psi^{(1)}$ is none other than the extended frame of the transformed surface.

Bianchi's permutability theorem is formulated exactly as in the continuous case, and is expressed by the same formula (11).

§3. THREE-DIMENSIONAL CONSISTENCY

Quite a remarkable feature shows up at the discrete level. It can be observed that the characterization of the discrete K-surfaces does not differ essentially from the characterization of their Bäcklund transformations. This can be seen both at the geometric level and at the level of equations. The following definition reflects the geometric similarities.

Definition 3.1.

1) An *m-dimensional asymptotic lattice* is a function $F : \mathbb{Z}^m \rightarrow \mathbb{R}^3$ with the property that, for any $\mathbf{n} \in \mathbb{Z}^m$, the point $F(\mathbf{n})$ and all its neighbors $F(\mathbf{n} \pm \mathbf{e}_i)$ are coplanar.

2) An *m-dimensional K-lattice* is an asymptotic lattice satisfying the additional requirement that for any $i \in \{1, \dots, m\}$ the length of the edges $[F(\mathbf{n}), F(\mathbf{n} + \mathbf{e}_i)]$ be one and the same for all \mathbf{n} .

Asymptotic lattices are described by an essentially three-dimensional system, while an additional condition that singles out the K-lattices reduces the dimension of the corresponding system to two. The key observation is that the definitions of asymptotic lattices and of K-lattices are *consistent*, i.e., they can be imposed on multidimensional lattices without internal contradictions.

It was first observed by Wunderlich [W] that three-dimensional K-lattices can be interpreted as Bäcklund transformations for K-surfaces. Actually, K-lattices allow us to put all the notions related to discrete K-surfaces on the same footing: the discrete K-surfaces are K-lattices with $m = 2$, the iterated Bäcklund transformations of discrete K-surfaces form K-lattices with $m = 3$, Bianchi's permutability theorem for discrete K-surfaces refers to K-lattices with $m = 4$.

An important step in understanding multidimensional lattices with similar geometric properties was made by Doliwa and Santini, who studied quadrilateral lattices and their reductions (see, e.g., [DS, D]).

To investigate multidimensional consistency at the level of equations, observe that the following equations are essentially of the same structure: equation (13) which describes discrete K-surfaces, equations (19), (20) for Bäcklund transformations of discrete K-surfaces, and equation (11) for the superposition principle for the latter. Their common structure is captured in the following system for a function $\phi : \mathbb{Z}^m \rightarrow \mathbb{R}$ on an m -dimensional lattice:

$$(22) \quad \sin \frac{1}{4}(\phi_{jk} + \phi_k - \phi_j - \phi) = \frac{\alpha_k}{\alpha_j} \sin \frac{1}{4}(\phi_{jk} - \phi_k + \phi_j - \phi).$$

Here the subscript j stands for the shift in the j th lattice direction, and the parameters α_j are assigned to all edges parallel to the j th lattice direction.

In our context, we are dealing with the case where $m = 4$. We assume that the subscripts 1, 2 label the x -, respectively, y -direction, while the subscripts 3, 4 are used as

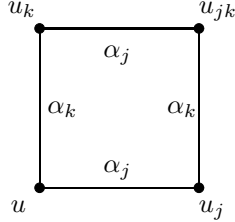


FIGURE 2. 2D equation.

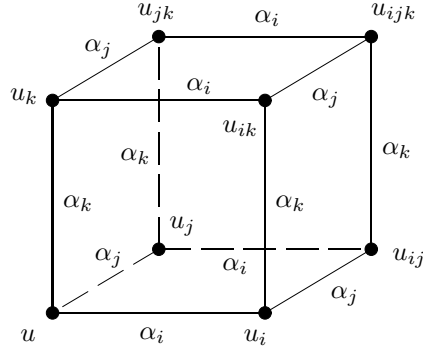


FIGURE 3. 3D consistency.

replacements of the Bäcklund superscripts (1), (2). The relevant values of the parameters are: $\alpha_1 = \epsilon/2$, $\alpha_2 = 2/\epsilon$, $\alpha_3 = \alpha$, and $\alpha_4 = \beta$. Indeed, equations (19), (11) are exactly of the form (22), and equations (13), (20) are brought into this form upon changing the sign of ϕ on every second hyperplane complementary to the y -direction, i.e., upon the change of variables $\phi(\mathbf{n}) \rightarrow (-1)^{n_2} \phi(\mathbf{n})$.

The variable transformation $u = \exp(i\phi/2)$ reshapes (22) to the form

$$(23) \quad \frac{u_{jk}}{u} = \frac{\alpha_j u_j - \alpha_k u_k}{\alpha_j u_k - \alpha_k u_j},$$

which is known as the *Hirota system*. Historically, this system played an important role in understanding quantum integrability in discrete space-time [FV1, FV2].

The Hirota equation (23) is a two-dimensional discrete equation, because it relates the variables u at the vertices of any elementary two-dimensional cell (square) of the m -dimensional lattice. The possibility of imposing this equation everywhere on the m -dimensional lattice hinges on the case of $m = 3$. The corresponding property of *three-dimensional consistency* should be understood as follows: suppose that four values u, u_i, u_j, u_k are given (we refer to the notation of Figure 3). Then equation (23) determines u_{ij}, u_{jk} , and u_{ik} , and further application of this equation gives three *a priori* different values of u_{ijk} . The 3D consistency of the Hirota equation means that these three values coincide automatically for arbitrary initial data. As a consequence, the Hirota equation can be imposed consistently on all elementary squares of a multidimensional lattice.

It was found in [BS1], and independently in [N], that this property is fundamental for integrability, because it yields both the zero-curvature representation and the existence of Bäcklund transformations. This was extended to equations with noncommuting variables (including a noncommutative Hirota equation) in [BS2]. A classification of discrete integrable systems based on the notion of 3D consistency was given in [ABS].

§4. APPROXIMATION THEOREM FOR K-SURFACES

Thus, at the formal level it would be fair to say that the theory of K-surfaces and their Bäcklund transformations is a consequence of the single Hirota equation (22) and its three-dimensional consistency. Moreover, it is easily seen that *equations* describing the continuous surfaces and their transformations are limit versions (as $\epsilon \rightarrow 0$) of the corresponding discrete equations. The remaining part of the paper is devoted to the proof that this is true also for the *solutions* of suitably posed problems for these equations.

Theorem 4.1. *Let $\phi_1, \phi_2 : [0, r] \rightarrow \mathbb{R}$ be smooth functions with $\phi_1(0) = \phi_2(0)$.*

• *There exists a unique, up to Euclidean motions, smooth K-surface $F : \mathcal{B}(r) \rightarrow \mathbb{R}^3$ parametrized along asymptotic lines such that, on the coordinate axes, the angle $\phi : \mathcal{B}(r) \rightarrow \mathbb{R}$ between the asymptotic lines attains the values*

$$(24) \quad \phi(x, 0) = \phi_1(x), \quad \phi(0, y) = \phi_2(y).$$

• *For each $\epsilon > 0$ there exists a unique discrete K-surface $F^\epsilon : \mathcal{B}^\epsilon(r) \rightarrow \mathbb{R}^3$ such that on the coordinate axes its characteristic angle $\phi^\epsilon : \mathcal{B}^\epsilon(r) \rightarrow \mathbb{R}$ attains the values*

$$(25) \quad \phi^\epsilon(x, 0) = \phi_1(x), \quad \phi^\epsilon(0, y) = \phi_2(y).$$

• *The discrete surfaces converge to the smooth one uniformly in C^∞ with rate $\mathcal{O}(\epsilon)$: for each pair (m, n) of nonnegative integers, we have*

$$(26) \quad \sup_{\mathcal{B}^\epsilon(r - (m+n)\epsilon)} |(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n F^\epsilon - \partial_x^m \partial_y^n F| \leq C_{mn} \epsilon,$$

where the numbers $C_{mn} > 0$ are independent of ϵ . For $(m, n) = (0, 0)$ this estimate can be improved to $\mathcal{O}(\epsilon^2)$:

$$(27) \quad \sup_{\mathcal{B}^\epsilon(r)} |F^\epsilon - F| \leq C \epsilon^2.$$

• *Estimates (26) and (27) are valid also for the associated families $F_\lambda, F_\lambda^\epsilon$ in place of F, F^ϵ uniformly in $\lambda \in [\Lambda^{-1}, \Lambda]$ for a suitable $\Lambda > 1$.*

• *Estimates (26) and (27) are valid also for the Bäcklund transformations $F^{(1)}, (F^\epsilon)^{(1)}$ in place of F, F^ϵ if all Bäcklund transformations are determined by the same parameter α and the same initial value $\phi_0^{(1)}$.*

With Theorem 4.1 at hand, Bianchi's permutability theorem for smooth K-surfaces becomes a simple consequence of a similar statement for discrete K-surfaces, which, in its turn, is a consequence of the consistency of the Hirota equation on the four-dimensional lattice.

We illustrate the convergence of discrete K-surfaces by Figure 4 (produced by T. Hoffmann), which presents a continuous Amsler K-surface and an approximating discrete one. Recall that the defining property of the Amsler surfaces (continuous and discrete) is that they contain two straight asymptotic lines [BP1, Hof].

§5. TWO-DIMENSIONAL HYPERBOLIC SYSTEMS

In this section we formulate an approximation theorem for a certain class of hyperbolic differential and difference equations in two dimensions. More general m -dimensional systems are considered in §6.

The following notation will be used. The dependent variables of differential and difference equations under consideration belong to a Banach space \mathcal{X} with norm $|\cdot|$. The independent variables in the two-dimensional situation belong to $\mathcal{B}(r) = [0, r] \times [0, r] \subset \mathbb{R}^2$ in the case of differential equations, and to $\mathcal{B}^\epsilon(r) = [0, r]^\epsilon \times [0, r]^\epsilon \subset (\epsilon\mathbb{Z})^2$ in the case of difference equations; here $[0, r]^\epsilon = [0, r] \cap (\epsilon\mathbb{Z})$. Each $\mathcal{B}^\epsilon(r)$ contains $\mathcal{O}(\epsilon^{-2})$ grid points. It is convenient to assume that ϵ only takes values of the form 2^{-k} with $k \in \mathbb{Z}_+$. Then $\epsilon_1 < \epsilon_2$ implies that ϵ_2 is an integer multiple of ϵ_1 , so that $\mathcal{B}^{\epsilon_2}(r) \subset \mathcal{B}^{\epsilon_1}(r)$. The limiting domains

$$(28) \quad \mathcal{B}^0(r) = \bigcup_{\epsilon=2^{-k}} \mathcal{B}^\epsilon(r)$$

lie densely in $\mathcal{B}(r)$. Each point $x \in \mathcal{B}^0(r)$ belongs to $\mathcal{B}^\epsilon(r)$ with $\epsilon = 2^{-k}$ for all k sufficiently large. Hence, we can talk about convergence of functions $a^\epsilon : \mathcal{B}^\epsilon(r) \rightarrow \mathcal{X}$ as

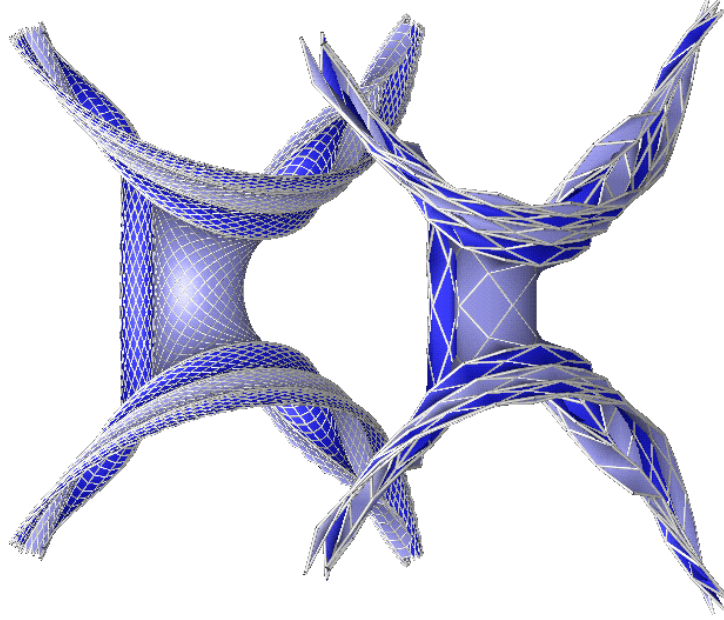


FIGURE 4. A continuous and a discrete Amsler surfaces.

$\epsilon \rightarrow 0$: the limiting function a^0 is defined naturally on $\mathcal{B}^0(r)$. If $a^0 : \mathcal{B}^0(r) \rightarrow \mathcal{X}$ is a Lipschitz function, then it extends to a Lipschitz function $a^0 : \mathcal{B}(r) \rightarrow \mathcal{X}$.

In all our approximation results, we are dealing with systems of *first-order equations* only. This suggests the following definitions.

Definition 5.1. A *continuous 2D hyperbolic system* is a system of partial differential equations for functions $a, b : \mathcal{B}(r) \rightarrow \mathcal{X}$ of the form

$$(29) \quad \partial_x a = f(a, b), \quad \partial_y b = g(a, b),$$

with smooth functions $f, g : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$. A *Goursat problem* consists in prescribing the initial values

$$(30) \quad a(x, 0) = a_0(x), \quad b(0, y) = b_0(y)$$

for $x \in [0, r]$ and $y \in [0, r]$, respectively. The functions $a_0, b_0 : [0, r] \rightarrow \mathcal{X}$ are assumed to belong to some C^k .

Definition 5.2. A *discrete 2D hyperbolic system* is formed by two partial difference equations for $a, b : \mathcal{B}^\epsilon(r) \rightarrow \mathcal{X}$ of the form

$$(31) \quad \delta_x^\epsilon a = f^\epsilon(a, b), \quad \delta_y^\epsilon b = g^\epsilon(a, b),$$

with smooth functions $f^\epsilon, g^\epsilon : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$. A *Goursat problem* for this system consists in prescribing the initial values

$$(32) \quad a(x, 0) = a_0^\epsilon(x), \quad b(0, y) = b_0^\epsilon(y)$$

for $x \in [0, r]^\epsilon$ and $y \in [0, r]^\epsilon$, respectively.

The notation suggests that the variables (a, b) in (31) are attached to the points of the two-dimensional lattice $\mathcal{B}^\epsilon(r)$. But it is more natural to associate them with the *edges* of this lattice: $a^\epsilon(x, y)$ with the horizontal edge $[(x, y), (x + \epsilon, y)]$, and $b^\epsilon(x, y)$ with the vertical edge $[(x, y), (x, y + \epsilon)]$; see Figure 5. Equations (31) give the fields on the right

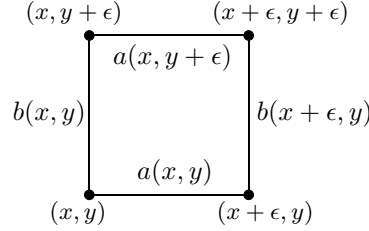


FIGURE 5. An elementary lattice square.

and on the top edges of an elementary square, provided the fields on the left and on the bottom edges are known. Therefore:

A Goursat problem for a discrete 2D hyperbolic system (31) has a unique solution (a^ϵ, b^ϵ) on $\mathcal{B}^\epsilon(r)$.

Example 1. We illustrate the above definitions for the sine-Gordon equation (1). The canonical way to put (1) into the first-order form (29) is to introduce the dependent variables

$$(33) \quad a = \partial_x \phi, \quad b = \phi,$$

so that equation (1) becomes equivalent to

$$(34) \quad \partial_y a = \sin b, \quad \partial_x b = a.$$

To put the naive discretization (12) of the sine-Gordon equation into the first-order form, we introduce the dependent variables

$$(35) \quad a = \delta_x^\epsilon \phi, \quad b = \phi,$$

which satisfy a discrete 2D hyperbolic system,

$$(36) \quad \delta_y^\epsilon a = \sin b, \quad \delta_x^\epsilon b = a.$$

Similarly, to put the Hirota discretization (13) into the first-order form, introduce the dependent variables a, b as in (16). Then (13) is equivalent to

$$\begin{aligned} b(x + \epsilon, y) - b(x, y) &= \frac{\epsilon}{2}(a(x, y + \epsilon) + a(x, y)), \\ e^{i\epsilon a(x, y + \epsilon)/2} - e^{i\epsilon a(x, y)/2} &= \frac{\epsilon^2}{4}(e^{ib(x + \epsilon, y)} - e^{-ib(x, y)}). \end{aligned}$$

After solving for $a(x, y + \epsilon)$ and $b(x + \epsilon, y)$, we are left with

$$(37) \quad \delta_y^\epsilon a = S_{\epsilon^2/4}(b + \frac{\epsilon}{2}a), \quad \delta_x^\epsilon b = a + \frac{\epsilon}{2}\delta_y^\epsilon a,$$

where the following abbreviation is used:

$$(38) \quad S_\delta(\varphi) = \frac{1}{2i\delta} \log \frac{1 - \delta \exp(-i\varphi)}{1 - \delta \exp(i\varphi)} = \sin \varphi + \mathcal{O}(\delta).$$

The two discrete 2D hyperbolic systems (36), (37) approximate the continuous system (34) in the sense of the next definition.

Definition 5.3. A discrete 2D hyperbolic system (31) approximates the continuous system (29) with rate $\mathcal{O}(\epsilon)$ if the functions f^ϵ, g^ϵ satisfy

$$(39) \quad f^\epsilon(a, b) = f(a, b) + \mathcal{O}(\epsilon), \quad g^\epsilon(a, b) = g(a, b) + \mathcal{O}(\epsilon),$$

uniformly on compact subsets of $\mathcal{X} \times \mathcal{X}$. If (39) is fulfilled for all partial derivatives of the functions f^ϵ, g^ϵ up to order s , we talk of $\mathcal{O}(\epsilon)$ -approximation in C^s .

The basic convergence result is the following statement, which admits various generalizations.

Theorem 5.4. *Let a discrete 2D hyperbolic system (31) approximate the continuous 2D hyperbolic system (29) with rate $\mathcal{O}(\epsilon)$ in C^1 . Also, let the discrete Goursat data (32) approximate the continuous data (30):*

$$(40) \quad a_0^\epsilon(x) = a_0(x) + \mathcal{O}(\epsilon), \quad b_0^\epsilon(y) = b_0(y) + \mathcal{O}(\epsilon),$$

uniformly for $x \in [0, r]^\epsilon$ and $y \in [0, r]^\epsilon$, respectively. Then the solutions (a^ϵ, b^ϵ) of the Goursat problems for equation (31) converge uniformly in $\mathcal{B}(\bar{r})$, with a suitable $\bar{r} \in (0, r]$, to a pair of Lipschitz functions (a, b) ,

$$(41) \quad a^\epsilon(x, y) = a(x, y) + \mathcal{O}(\epsilon), \quad b^\epsilon(x, y) = b(x, y) + \mathcal{O}(\epsilon).$$

The functions a, b solve the Goursat problem for (29) on $\mathcal{B}(\bar{r})$.

One improvement of Theorem 5.4 deals with a higher-order approximation in ϵ .

Definition 5.5. A 2D hyperbolic system (31) approximates the continuous system (29) with rate $\mathcal{O}(\epsilon^2)$ if

$$\begin{aligned} f^\epsilon &= f + \frac{\epsilon}{2} (\partial_a f \cdot f + \partial_b f \cdot g) + \mathcal{O}(\epsilon^2), \\ g^\epsilon &= g + \frac{\epsilon}{2} (\partial_a g \cdot f + \partial_b g \cdot g) + \mathcal{O}(\epsilon^2) \end{aligned}$$

uniformly on compact subsets of $\mathcal{X} \times \mathcal{X}$.

Theorem 5.6. *Suppose that, in addition to the conditions of Theorem 5.4, the discrete hyperbolic system (31) approximate the continuous hyperbolic system (29) with rate $\mathcal{O}(\epsilon^2)$. Let the corresponding Goursat data (32) and (30) satisfy*

$$(42) \quad a_0^\epsilon(x) = a_0(x + \frac{\epsilon}{2}) + \mathcal{O}(\epsilon^2), \quad b_0^\epsilon(y) = b_0(y + \frac{\epsilon}{2}) + \mathcal{O}(\epsilon^2).$$

Then, in addition to the conclusions of Theorem 5.4, we have the estimates

$$(43) \quad a^\epsilon(x, y) = a(x + \frac{\epsilon}{2}, y) + \mathcal{O}(\epsilon^2), \quad b^\epsilon(x, y) = b(x, y + \frac{\epsilon}{2}) + \mathcal{O}(\epsilon^2)$$

uniformly on $\mathcal{B}(\bar{r})$.

Estimate (43) indicates that it is more natural to think of $a^\epsilon(x, y)$ and $b^\epsilon(x, y)$ as associated with (the midpoints of) the edges $[(x, y), (x + \epsilon, y)]$ and $[(x, y), (x, y + \epsilon)]$, respectively.

Another improvement of Theorem 5.4 deals with some additional smoothness assumptions (and conclusions).

Theorem 5.7. *Additionally to the conditions of Theorem 5.4, assume that the functions f^ϵ, g^ϵ approximate f, g with rate $\mathcal{O}(\epsilon)$ in C^{s+1} , $s > 1$. Assume further that the continuous Goursat data a_0, b_0 are $C^{s,1}$ -functions (i.e., their s th derivatives are Lipschitz-continuous), and that the discrete Goursat data $a_0^\epsilon, b_0^\epsilon$ satisfy*

$$(44) \quad (\delta_x^\epsilon)^\ell a_0^\epsilon(x) = \partial_x^\ell a_0(x) + \mathcal{O}(\epsilon), \quad (\delta_y^\epsilon)^\ell b_0^\epsilon(y) = \partial_y^\ell b_0(y) + \mathcal{O}(\epsilon)$$

for all $\ell \leq s$. Then the continuous solutions a, b belong to $C^{s,1}(\mathcal{B}^\epsilon(\bar{r}))$ with the same $\bar{r} > 0$ as in Theorem 5.4. Moreover, estimates similar to (41) are valid for the higher derivatives,

$$(45) \quad (\delta_x^\epsilon)^m (\delta_y^\epsilon)^n a^\epsilon = \partial_x^m \partial_y^n a + \mathcal{O}(\epsilon), \quad (\delta_x^\epsilon)^m (\delta_y^\epsilon)^n b^\epsilon = \partial_x^m \partial_y^n b + \mathcal{O}(\epsilon)$$

for all m, n with $m + n \leq s$.

Note that the assumption (44) is fulfilled, for instance, if the functions $a_0^\epsilon, b_0^\epsilon$ are the restrictions of a_0, b_0 to the lattice $\epsilon\mathbb{Z}$.

All results of this section, contained in Theorems 5.4, 5.6 and 5.7, are applicable to both the naive discretization (36) and the Hirota discretization (37) of the sine-Gordon equation (34), provided the Goursat data are chosen in a proper way. The continuous Goursat data corresponding to (24) are

$$(46) \quad a(x, 0) = \partial_x \phi_1(x), \quad b(0, y) = \phi_2(y).$$

The discrete Goursat data for equation (36) can be chosen as

$$(47) \quad a^\epsilon(x, 0) = \delta_x^\epsilon \phi_1(x), \quad b^\epsilon(0, y) = \phi_2(y),$$

while the proper choice of the discrete Goursat data for equation (37) is

$$(48) \quad a^\epsilon(x, 0) = \delta_x^\epsilon \phi_1(x), \quad b^\epsilon(0, y) = \frac{1}{2}(\phi_2(y + \epsilon) + \phi_2(y)).$$

It will be possible to obtain the approximation results for both discretizations of the sine-Gordon equation. However, these results can be extended to an approximation of the geometric objects (K-surfaces) for the Hirota discretization only.

§6. MULTIDIMENSIONAL HYPERBOLIC SYSTEMS

The approximation theory developed generalizes to higher dimensions without difficulties. We start with notation. Let $\mathbf{r} = (r_1, \dots, r_m)$ consist of positive numbers $r_i > 0$; then the domain of the independent variables for differential equations will be

$$\mathcal{B}(\mathbf{r}) = [0, r_1] \times \dots \times [0, r_m] \subset \mathbb{R}^m.$$

For discrete equations, we use parts of rectangular lattices inside $\mathcal{B}(\mathbf{r})$, with possibly different grid sizes along different coordinate axes, $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_m)$:

$$\mathcal{B}^\epsilon(\mathbf{r}) = [0, r_1]^{\epsilon_1} \times \dots \times [0, r_m]^{\epsilon_m} \subset \prod_{i=1}^m (\epsilon_i \mathbb{Z}).$$

We use the following notation for lower-dimensional subsets of $\mathcal{B}^\epsilon(\mathbf{r})$ determined by indices $\mathcal{S} \subset \{1, \dots, m\}$:

$$\mathcal{B}_{\mathcal{S}}^\epsilon(\mathbf{r}) = \{\mathbf{x} \in \mathcal{B}^\epsilon(\mathbf{r}) : x_i = 0 \text{ if } i \notin \mathcal{S}\}.$$

Definition 6.1. A hyperbolic m -dimensional system of first-order partial difference equations is a system of the form

$$(49) \quad \delta_i^{\epsilon_i} u_k = f_{k,i}(u), \quad i \in \mathcal{E}_k,$$

for functions $u_k : \mathcal{B}^\epsilon(\mathbf{r}) \rightarrow \mathcal{X}_k$, where the \mathcal{X}_k are Banach spaces. For each component u_k of u , equations (49) are given for $i \in \mathcal{E}_k \subset \{1, \dots, m\}$, the nonempty set of *evolution directions* of u_k . Its complement $\mathcal{S}_k = \{1, \dots, m\} \setminus \mathcal{E}_k$ is called the set of *static directions* of u_k .

A *Goursat problem* for the hyperbolic system (49) consists in finding its solution subject to the following initial data:

$$(50) \quad u_k|_{\mathcal{B}_{\mathcal{S}_k}^\epsilon(\mathbf{r})} = u_{k0},$$

where the *Goursat data* $u_{k0} : \mathcal{B}_{\mathcal{S}_k}^\epsilon(\mathbf{r}) \rightarrow \mathcal{X}_k$ are given smooth functions.

We think of the variable $u_k(\mathbf{x})$ as attached to the elementary cell \mathcal{C}_k of dimension $\#\mathcal{S}_k$ adjacent to the point $\mathbf{x} \in \mathcal{B}^\epsilon(\mathbf{r})$ and parallel to $\mathcal{B}_{\mathcal{S}_k}^\epsilon(\mathbf{r})$:

$$\mathcal{C}_k = \{\mathbf{x} + \mu_i e_i : \mu_i \in [0, \epsilon_i], i \in \mathcal{S}_k\}.$$

To a large extent, the solvability of a Goursat problem is a local phenomenon. Namely, it only makes sense to consider *consistent* hyperbolic systems (49), i.e., those for which the Goursat problem for one elementary m -dimensional cube is uniquely solvable for arbitrary initial data. The following obvious but extremely important statement is true.

Proposition 6.2. *A Goursat problem for a consistent hyperbolic system (49) has a unique solution u^ϵ on $\mathcal{B}^\epsilon(\mathbf{r})$.*

Consistency conditions read: $\delta_j^{\epsilon_j} \delta_i^{\epsilon_i} u_k = \delta_i^{\epsilon_i} \delta_j^{\epsilon_j} u_k$ for all $i \neq j$. Substituting equations (49) herein, we get the equations

$$(51) \quad \delta_j^{\epsilon_j} f_{k,i}(u) = \delta_i^{\epsilon_i} f_{k,j}(u), \quad i \neq j,$$

which are further rewritten as

$$(52) \quad \epsilon_j^{-1} (f_{k,i}(u + \epsilon_j f_j(u)) - f_{k,i}(u)) = \epsilon_i^{-1} (f_{k,j}(u + \epsilon_i f_i(u)) - f_{k,j}(u)).$$

Here $f_i(u)$ is a vector-valued function whose ℓ th component is equal to $f_{\ell,i}(u)$ if $i \in \mathcal{E}_\ell$ and is undefined otherwise.

Lemma 6.3. *For a consistent system of hyperbolic equations (49), the function $f_{k,i}$ only depends on those components u_ℓ for which $\mathcal{S}_\ell \subset \mathcal{S}_k \cup \{i\}$.*

Proof. Equations (51) must be fulfilled identically in u . This implies that the function $f_{k,i}$ can only depend on those components u_ℓ for which $\delta_j^{\epsilon_j} u_\ell$ is defined, i.e., for which $j \in \mathcal{E}_\ell$. Since (51) must be satisfied for all $j \in \mathcal{E}_k$, $j \neq i$, we see that $\mathcal{E}_k \setminus \{i\} \subset \mathcal{E}_\ell$ for these ℓ . \square

Lemma 6.3 implies that for any subset $\mathcal{S} \subset \{1, \dots, m\}$, equations (49) for k with $\mathcal{S}_k \subset \mathcal{S}$ and for $i \in \mathcal{S}$ form a closed subsystem, in the sense that the $f_{k,i}$ only depend on u_ℓ with $\mathcal{S}_\ell \subset \mathcal{S}$.

Definition 6.4. The *essential dimension* d of system (49) is given by

$$(53) \quad d = 1 + \max_k (\#\mathcal{S}_k).$$

If $d < m$, then the d -dimensional subsystems corresponding to \mathcal{S} with $\#\mathcal{S} = d$ are hyperbolic. In this case, the consistency of system (49) is a manifestation of a very special property of its d -dimensional subsystems. We suggest treating this property as the discrete integrability (at least under some further conditions, excluding certain noninteresting situations, like trivial evolution in some of the directions). If $d = m$, system (49) has no lower-dimensional hyperbolic subsystems.

Typically, discrete hyperbolic systems appear as discretizations of continuous hyperbolic systems with n independent variables, possibly with Bäcklund transformations. In this situation, $\epsilon_i = \epsilon$ for $i = 1, \dots, n$, and $\epsilon_i = 1$ for $i = n + 1, \dots, m$, so that the continuum limit is performed in the first $n \leq m$ lattice directions, while the remaining $n' = m - n$ lattice directions are kept discrete and correspond to Bäcklund transformations. For simplicity, it may be assumed that $r_i = r$ for $i = 1, \dots, n$, and $r_i = 1$ for $i = n + 1, \dots, m$. Thus, for fixed m and n , we can still adopt the notation $\mathcal{B}^\epsilon(r)$ for $\mathcal{B}^\epsilon(\mathbf{r})$. The limiting domain is then $\mathcal{B}^0(r) = [0, r]^n \times \{0, 1\}^{m-n}$.

Example 2. To write the Bäcklund transformation (7) in the first-order form, we interpret the Bäcklund superscript (1) as the shift in the z -direction. Along with the variables a, b (see (33)), we introduce the auxiliary function $\theta = (\phi^{(1)} - \phi)/2$, which satisfies the following system of ordinary differential equations:

$$(54) \quad \partial_x \theta = -a + \frac{1}{\alpha} \sin \theta, \quad \partial_y \theta = \alpha \sin(b + \theta).$$

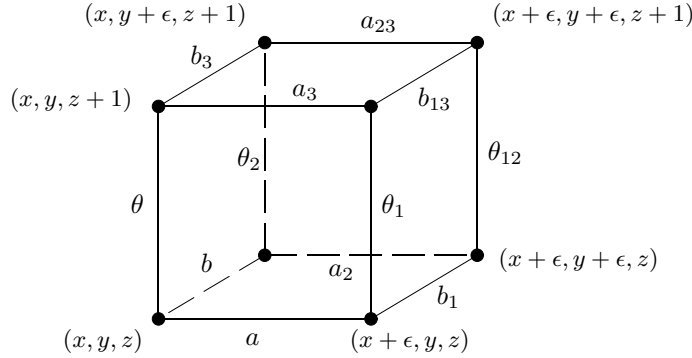


FIGURE 6. Elementary lattice cube.

The consistency condition $\partial_y(\partial_x\theta) = \partial_x(\partial_y\theta)$ is fulfilled provided (a, b) solves system (34), and the initial value $\theta(0, 0) = \theta_0$ determines a unique solution. Then the formulas

$$(55) \quad \delta_z a = 2\partial_x\theta = -2a + \frac{2}{\alpha}\sin\theta, \quad \delta_z b = 2\theta$$

give a new solution $(a^{(1)}, b^{(1)})$ of the 2D hyperbolic system (34) equivalent to the sine-Gordon equation.

Similarly, upon introducing the quantity θ , we can rewrite (19), (20) in the form of the system of first-order equations approximating (54), (55):

$$(56) \quad \delta_x^\epsilon \theta = -a + \frac{1}{\alpha}S_{\epsilon/2\alpha}(\theta - \frac{\epsilon}{2}a), \quad \delta_y^\epsilon \theta = \alpha S_{\epsilon\alpha/2}(b + \theta),$$

and

$$(57) \quad \delta_z a = 2\delta_x^\epsilon \theta, \quad \delta_z b = 2\theta + \epsilon\delta_y^\epsilon \theta.$$

Clearly, equations (56), (57) approximate equations (54), (55) with rate $\mathcal{O}(\epsilon)$.

The variables a , b , and θ live on the *edges* of the three-dimensional lattice $\mathcal{B}^\epsilon(r) = [0, r]^\epsilon \times [0, r]^\epsilon \times \{0, 1\}$ that are parallel to the axes x , y , and z , respectively. The Goursat data for the system (37), (56), (57) consist of the functions $a(x, 0, 0) = a_0^\epsilon(x)$, $b(0, y, 0) = b_0^\epsilon(y)$ for $x \in [0, r]^\epsilon$, $y \in [0, r]^\epsilon$, and of the number $\theta(0, 0, 0) = \theta_0$.

The essential dimension of the system (37), (56), (57) is equal to 2, and its two-dimensional subsystems for the xy -, xz -, and yz -plaquettes are for the variable pairs (a, b) , (a, θ) , and (b, θ) , respectively. Consistency for this system with variables defined on edges is illustrated in Figure 6.

We denote the shifts of the edge variables in the directions of the x -, y -, and z -axes by the subscripts 1, 2, and 3, respectively. Then (a_2, b_1) are determined by (37), (θ_1, θ_2) are determined by (56), and (a_3, b_3) are determined by (57). Now a_{23} is calculated either from a_3, b_3 by (37), or from a_2, θ_2 by the first equations in (57) and (56); consistency guarantees that the two results are identical. The same holds true for b_{13} and θ_{12} .

Example 3. Consider a differential equation with $m = 3$ independent variables:

$$(58) \quad \partial_x \partial_y \partial_z u = F(u, \partial_x u, \partial_y u, \partial_z u, \partial_x \partial_y u, \partial_x \partial_z u, \partial_y \partial_z u).$$

A Goursat problem can be posed by prescribing the values of u on the xy -, yz -, and xz -coordinate planes. Equation (58) can be rewritten as a hyperbolic system of first-order

equations:

$$\begin{cases} \partial_x u = a, & \partial_y u = b, & \partial_z u = c, \\ \partial_y a = h, & \partial_z b = f, & \partial_x c = g, \\ \partial_z a = g, & \partial_x b = h, & \partial_y c = f, \\ \partial_x f = \partial_y g = \partial_z h = F(u, a, b, c, f, g, h). \end{cases}$$

Its naive discretization would be obtained by replacing all partial derivatives ∂_x etc. by the corresponding difference quotients δ_x^ϵ etc. In the resulting difference system it is natural to assume that the variable u lives at the points of the cubic lattice $(\epsilon\mathbb{Z})^3$; the variables a, b, c live on the edges of the lattice adjacent to the point (x, y, z) in the x -, y - and z -direction, respectively; and the variables f, g, h are associated with the two-cells (elementary squares) adjacent to the point (x, y, z) and parallel to the yz -, xz -, and xy -planes, respectively. The essential dimension of this discrete system is $d = 3$. A Goursat problem for this system would be posed by prescribing the values of u at the point $(0, 0, 0)$, the values of a, b , and c on the x -, y -, and z -axes, respectively, and the values of f, g , and h on the yz -, xz -, and xy -planes, respectively.

Theorem 6.5. *Consider a consistent discrete hyperbolic system (49) on $\mathcal{B}^\epsilon(r)$. Suppose that its right-hand sides $f_{k,i} = f_{k,i}^\epsilon$ have smooth C^1 -limits as $\epsilon \rightarrow 0$:*

$$(59) \quad f_{k,i}^\epsilon(u) = f_{k,i}^0(u) + \mathcal{O}(\epsilon)$$

uniformly on any compact subset of $\mathcal{X} = \prod \mathcal{X}_k$. Suppose also that the discrete Goursat data u_{k0}^ϵ have Lipschitz-continuous limits u_{k0}^0 :

$$(60) \quad u_{k0}^\epsilon = u_{k0}^0 + \mathcal{O}(\epsilon)$$

uniformly on $\mathcal{B}_{\mathcal{S}_k}(r)$. Then there exists $\bar{r} \in (0, r]$ such that the solutions $u_k^\epsilon : \mathcal{B}^\epsilon(\bar{r}) \rightarrow \mathcal{X}_k$ of the discrete Goursat problem converge uniformly on $\mathcal{B}^0(\bar{r})$ to Lipschitz-continuous functions $u_k^0 : \mathcal{B}^0(\bar{r}) \rightarrow \mathcal{X}_k$,

$$(61) \quad u_k^\epsilon = u_k^0 + \mathcal{O}(\epsilon).$$

The functions u_k^0 constitute a unique solution of the Goursat problem on $\mathcal{B}^0(\bar{r})$ for the difference-differential hyperbolic system

$$(62) \quad \begin{aligned} \partial_i u_k &= f_{k,i}^0(u), & i \in \mathcal{E}_k, & 1 \leq i \leq n, \\ \delta_i u_k &= f_{k,i}^0(u), & i \in \mathcal{E}_k, & n < i \leq m, \end{aligned}$$

with the Goursat data u_{k0}^0 .

Assume, moreover, that estimate (59) is valid locally uniformly in C^{s+1} , and that the continuous Goursat data u_{k0}^0 are C^s -smooth and are C^s -approximated by the discrete data u_{k0}^ϵ :

$$(63) \quad \delta_{i_1}^\epsilon \cdots \delta_{i_s}^\epsilon u_{k0}^\epsilon = \partial_{i_1} \cdots \partial_{i_s} u_{k0}^0 + \mathcal{O}(\epsilon),$$

where $i_1, \dots, i_s \in \mathcal{S}_k$ and $i_1, \dots, i_s \leq n$. Then the convergence (61) is in C^s :

$$(64) \quad \delta_{i_1}^\epsilon \cdots \delta_{i_s}^\epsilon u_k^\epsilon = \partial_{i_1} \cdots \partial_{i_s} u_k^0 + \mathcal{O}(\epsilon)$$

for arbitrary $i_1, \dots, i_s \leq n$.

The proof of this theorem is a multidimensional extension of the proofs given in two dimensions. Technical care is needed, but no essentially new ideas enter. A detailed proof can be found in [M].

§7. PROOFS OF APPROXIMATION RESULTS FOR $m = 2$

Proof of Theorem 5.4. In general, we cannot expect that $\bar{r} = r$ because the solutions of the limiting equations may develop blow-ups that are absent in the discretization. Consequently, an essential prerequisite for the proof of Theorem 5.4 is ϵ -independent *a priori* bounds on a^ϵ and b^ϵ .

Lemma 7.1 (Uniform bound). *Suppose the norms of the initial data $a_0^\epsilon, b_0^\epsilon$ are bounded by ϵ -independent constants. Then there exists some $\bar{r} \in (0, r]$ such that the norms of the solutions (a^ϵ, b^ϵ) of the Goursat problem (31), (32) are bounded on $\mathcal{B}^\epsilon(\bar{r})$ independently of ϵ .*

Proof. Let $M_0 > 0$ be such that $|a_0^\epsilon|, |b_0^\epsilon| \leq M_0$, and choose $M_1 > M_0$ arbitrarily. Define

$$\bar{r} = (M_1 - M_0) / \sup_{\epsilon} \sup_{|a|, |b| < M_1} \{|f^\epsilon(a, b)| + |g^\epsilon(a, b)|\}.$$

We show that $|a^\epsilon|, |b^\epsilon| < M_1$ on $\mathcal{B}^\epsilon(\bar{r})$. Rewriting (31) as

$$(65) \quad a^\epsilon(x, y) = a^\epsilon(x, y - \epsilon) + \epsilon f^\epsilon(a^\epsilon(x, y - \epsilon), b^\epsilon(x, y - \epsilon)),$$

$$(66) \quad b^\epsilon(x, y) = b^\epsilon(x - \epsilon, y) + \epsilon g^\epsilon(a^\epsilon(x - \epsilon, y), b^\epsilon(x - \epsilon, y)),$$

we then conclude by induction that

$$|a^\epsilon(x, y)| \leq M_0 + (M_1 - M_0) \frac{y}{\bar{r}} < M_1,$$

$$|b^\epsilon(x, y)| \leq M_0 + (M_1 - M_0) \frac{x}{\bar{r}} < M_1$$

for $(x, y) \in \mathcal{B}^\epsilon(\bar{r})$. □

Remark 1. If f and g possess a global Lipschitz constant, then we can take $\bar{r} = r$ in Lemma 7.1 and also in Theorem 5.4.

Lemma 7.2 (Discrete Gronwall estimate). *Assume that a nonnegative function $\Delta : \mathbb{Z}_+ \rightarrow \mathbb{R}$ satisfies*

$$(67) \quad \Delta(n+1) \leq (1 + \epsilon\mathcal{K})\Delta(n) + \kappa$$

with nonnegative constants \mathcal{K} and κ , for all $n \geq 0$. Then

$$(68) \quad \Delta(n) \leq (\Delta(0) + n\kappa) \exp(\mathcal{K}n\epsilon).$$

Proof. Iterate (67) to confirm (68) by induction on $n > 0$, observing that $\exp(\epsilon\mathcal{K}) \leq (1 + \epsilon\mathcal{K})$ for $\mathcal{K} \geq 0$. □

Lemma 7.3 (Lipschitz bound). *Assume that the continuous Goursat data a_0, b_0 are C^1 functions, and that the discrete Goursat data $a_0^\epsilon, b_0^\epsilon$ satisfy*

$$(69) \quad |a_0^\epsilon(x) - a_0(x)| \leq M\epsilon, \quad |b_0^\epsilon(y) - b_0(y)| \leq M\epsilon$$

with an ϵ -independent constant M . Then the difference quotients

$$\delta_x a^\epsilon, \delta_y a^\epsilon, \delta_x b^\epsilon, \delta_y b^\epsilon$$

are bounded uniformly in ϵ on the respective $\mathcal{B}^\epsilon(\bar{r})$, where $\bar{r} \in (0, r]$ is chosen in accordance with Lemma 7.1.

Proof. By (31) and Lemma 7.1, the difference quotients $\delta_y^\epsilon a^\epsilon$ and $\delta_x^\epsilon b^\epsilon$ are uniformly bounded. Let the solutions (a^ϵ, b^ϵ) of the discrete Goursat problems be bounded by M_1 , and set

$$M_2 = \sup_{\epsilon} \sup_{|a|, |b| \leq M_1} \left\{ |f^\epsilon(a, b)|, |g^\epsilon(a, b)|, |\partial_a f^\epsilon(a, b)|, \dots, |\partial_b g^\epsilon(a, b)| \right\},$$

which is finite because $f^\epsilon \rightarrow f$ and $g^\epsilon \rightarrow g$ locally uniformly in C^1 . Without loss of generality, $M > M_1$ and $M > M_2$. By the mean value theorem,

$$|\delta_x^\epsilon a_0^\epsilon(x)| \leq |\delta_x^\epsilon a_0(x)| + \epsilon^{-1} |a_0^\epsilon(x + \epsilon) - a_0(x + \epsilon)| + \epsilon^{-1} |a_0(x) - a_0^\epsilon(x)| \leq 3M.$$

Proceeding from y to $y + \epsilon$, we have

$$\begin{aligned} |\delta_x^\epsilon a^\epsilon(x, y + \epsilon)| &\leq |\delta_x^\epsilon a^\epsilon(x, y)| + \epsilon |\delta_x^\epsilon f^\epsilon(a^\epsilon(x, y), b^\epsilon(x, y))| \\ &\leq |\delta_x^\epsilon a^\epsilon(x, y)| + \epsilon M (|\delta_x^\epsilon a^\epsilon(x, y)| + |\delta_x^\epsilon b^\epsilon(x, y)|) \\ &\leq (1 + \epsilon M) |\delta_x^\epsilon a^\epsilon(x, y)| + \epsilon M^2. \end{aligned}$$

Now Lemma 7.2 yields the desired estimate:

$$|\delta_x^\epsilon a^\epsilon(x, y)| \leq 4M \exp(M\bar{r}).$$

The same argument applies to $\delta_y^\epsilon b^\epsilon$. \square

Proof of Theorem 5.4, continued. Consider the family $\{(\tilde{a}^\epsilon, \tilde{b}^\epsilon)\}_{\epsilon=2^{-k}}$ of functions $\tilde{a}^\epsilon, \tilde{b}^\epsilon : \mathcal{B}(\bar{r}) \rightarrow \mathcal{X}$ obtained from a^ϵ and b^ϵ by linear interpolation. By Lemma 7.3, there is a Lipschitz constant $L > 0$ such that

$$|\tilde{a}^\epsilon(x', y') - \tilde{a}^\epsilon(x, y)| + |\tilde{b}^\epsilon(x', y') - \tilde{b}^\epsilon(x, y)| \leq L(|x' - x| + |y' - y|).$$

In combination with Lemma 7.1, this shows that the family is *equicontinuous*, i.e., it satisfies the hypotheses of the Arzelà–Ascoli theorem. Consequently, there exist continuous functions $a, b : \mathcal{B}(\bar{r}) \rightarrow \mathcal{X}$ such that $\tilde{a}^{\epsilon'} \rightarrow a$ and $\tilde{b}^{\epsilon'} \rightarrow b$ uniformly for an infinite subsequence $\epsilon' = 2^{-k'}$. Moreover, a and b are Lipschitz functions with Lipschitz constant L .

To show that (a, b) solve the differential equations (29), observe that relation (65) and the Lipschitz-continuity of \tilde{a}^ϵ imply

$$(70) \quad \tilde{a}^{\epsilon'}(x, y) = \tilde{a}_0^{\epsilon'}(x) + \epsilon \sum_{k=0}^{\lfloor y/\epsilon' \rfloor - 1} f^{\epsilon'}[\tilde{a}^{\epsilon'}, \tilde{b}^{\epsilon'}](x, k\epsilon') + \mathcal{O}(\epsilon')$$

for $(x, y) \in \mathcal{B}(\bar{r})$. Since the convergence of $\tilde{a}^{\epsilon'}$ and $\tilde{b}^{\epsilon'}$ is uniform, and $f^\epsilon \rightarrow f$ in C^1 , we can pass to the limit as $\epsilon' \rightarrow 0$ on both sides of (70), obtaining

$$(71) \quad a(x, y) = a_0(x) + \int_0^y f[a, b](x, \eta) d\eta.$$

It follows that a is everywhere differentiable with respect to y , and $\partial_y a = f(a, b)$. The function b is treated in the same way.

Now, the convergence (41) can be proved. For arbitrary $\epsilon = 2^{-k}$ we introduce the approximation error

$$(72) \quad \Delta^\epsilon(n) = \max\{|a^\epsilon(x, y) - a(x, y)| + |b^\epsilon(x, y) - b(x, y)|, (x, y) \in \mathcal{B}^\epsilon(\bar{r}), x + y = n\epsilon\}.$$

Combining formula (65) with the integral representation (71) yields

$$\begin{aligned} \Delta^\epsilon(n+1) &\leq \Delta^\epsilon(n) + \epsilon \max_{x+y=n\epsilon} (|\delta_x^\epsilon(a^\epsilon - a)|(x, y) + |\delta_y^\epsilon(b^\epsilon - b)|(x, y)) \\ &\quad + |a_0^\epsilon - a_0|(n\epsilon + \epsilon) + |b_0^\epsilon - b^0|(n\epsilon + \epsilon) \\ &\leq \Delta^\epsilon(n) + \epsilon \max_{x+y=n\epsilon} (|f^\epsilon[a^\epsilon, b^\epsilon] - f[a, b]|(x, y) + |g^\epsilon[a^\epsilon, b^\epsilon] - g[a, b]|(x, y)) + \mathcal{O}(\epsilon) \\ &\leq (1 + \mathcal{O}(\epsilon))\Delta^\epsilon(n) + \mathcal{O}(\epsilon). \end{aligned}$$

By the Gronwall estimate of Lemma 7.2, $\Delta^\epsilon(n) = \mathcal{O}(\epsilon)$ for $n\epsilon \leq \bar{r}$. This implies (41). \square

Corollary 7.4. *The two-dimensional hyperbolic Goursat problem (29), (30) possesses a unique classical solution.*

Proof. The existence of a classical solution is already part of the conclusions of Theorem 5.4. Uniqueness follows from the proof above: the estimates for Δ^ϵ introduced in (72) are independent of the specific solution (a, b) of (29), (30). In fact, only the integral representation (71) has been used. Hence, every solution of the continuous Goursat problem appears as a uniform limit of discrete solutions (a^ϵ, b^ϵ) as $\epsilon \rightarrow 0$. On the other hand, the discrete solutions are unique, and so is their limit. \square

Proof of Theorem 5.6. Since the initial data a_0, b_0 are Lipschitz-continuous, it is clear that the hypothesis (42) implies (40), and therefore the conclusions of Theorem 5.4 are valid.

To obtain (43), we modify the proof of estimate (41). By an obvious analogy with Δ^ϵ defined in (72), let

$$\Delta^\epsilon(n) = \sup\{|a^\epsilon(x, y) - a(x + \frac{\epsilon}{2}, y)| + |b^\epsilon(x, y) - b(x, y + \frac{\epsilon}{2})| : (x, y) \in \mathcal{B}^\epsilon(\bar{r}), x + y = n\epsilon\}.$$

As before,

$$\begin{aligned} \Delta^\epsilon(n+1) &\leq \Delta^\epsilon(n) + \epsilon \max_{x+y=n\epsilon} (|\delta_x(a^\epsilon - a)|(x, y) + |\delta_y(b^\epsilon - b)|(x, y)) \\ &\quad + |a_0^\epsilon((n+1)\epsilon) - a_0((n + \frac{3}{2})\epsilon)| + |b_0^\epsilon((n+1)\epsilon) - b_0((n + \frac{3}{2})\epsilon)|. \end{aligned}$$

The quantity $\delta^\epsilon a(x + \frac{\epsilon}{2}, y)$ is now analyzed up to $\mathcal{O}(\epsilon^3)$. For short, set $\bar{a} = a(x + \frac{\epsilon}{2}, y)$ and $\bar{b} = b(x, y + \frac{\epsilon}{2})$. We have

$$\begin{aligned} &a(x + \frac{\epsilon}{2}, y + \epsilon) - a(x + \frac{\epsilon}{2}, y) \\ &= \int_0^\epsilon f[a, b](x + \frac{\epsilon}{2}, y + \eta) d\eta \\ &= \epsilon f[a, b](x + \frac{\epsilon}{2}, y + \frac{\epsilon}{2}) + \mathcal{O}(\epsilon^3) \\ &= \epsilon(f(\bar{a}, \bar{b}) + \partial_a f(\bar{a}, \bar{b}) \cdot (a(x + \frac{\epsilon}{2}, y + \frac{\epsilon}{2}) - \bar{a}) \\ &\quad + \partial_b f(\bar{a}, \bar{b}) \cdot (b(x + \frac{\epsilon}{2}, y + \frac{\epsilon}{2}) - \bar{b})) + \mathcal{O}(\epsilon^3) \\ &= \epsilon(f(\bar{a}, \bar{b}) + \frac{\epsilon}{2} \partial_a f(\bar{a}, \bar{b}) \cdot f(\bar{a}, \bar{b}) + \frac{\epsilon}{2} \partial_b f(\bar{a}, \bar{b}) \cdot g(\bar{a}, \bar{b})) + \mathcal{O}(\epsilon^3). \end{aligned}$$

Using the condition that discrete equations approximate continuous ones with rate $\mathcal{O}(\epsilon^2)$, we find:

$$|\delta_x a^\epsilon(x, y) - \delta_x a(x + \frac{\epsilon}{2}, y)| \leq C\epsilon \Delta^\epsilon(n) + \mathcal{O}(\epsilon^3),$$

where the constant C depends on the functions f, g and their first-order derivatives only. The same argument applies to b . Therefore,

$$\Delta^\epsilon(n) \leq (1 + \mathcal{O}(\epsilon))\Delta^\epsilon(n) + \mathcal{O}(\epsilon^3).$$

Finally, $\Delta^\epsilon(n) = \mathcal{O}(\epsilon^2)$ by Lemma 7.2. \square

Proof of Theorem 5.7. First, we deduce *a priori* estimates for higher-order difference quotients of a^ϵ, b^ϵ . As a discrete analog of the C^s -norm, for $u : \mathcal{B}^\epsilon(r) \rightarrow \mathcal{X}$ we define

$$(73) \quad \|u\|_s = \max_{k+\ell \leq s} \sup_{\mathcal{B}^\epsilon(r-s\epsilon)} |(\delta_x^\epsilon)^k (\delta_y^\epsilon)^\ell u|.$$

Recall that a^ϵ and b^ϵ are bounded on $\mathcal{B}^\epsilon(\bar{r})$: $|a^\epsilon|, |b^\epsilon| \leq M_1$ independently of $\epsilon > 0$. For a smooth function $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$, we introduce

$$\|h\|_s = \max_{m \leq s} \sup_{|a|, |b| < M_1} |D^m h(a, b)|,$$

which is the C^s -norm of f on the ball of radius M_1 . The following is essential to estimate the norm of compositions with smooth functions. \square

Lemma 7.5. *Let $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ be a smooth function, and let $a, b : \mathcal{B}^\epsilon(r) \rightarrow \mathcal{X}$ be bounded by M_1 . Then for any pair (m, n) of nonnegative integers we have*

$$(74) \quad \begin{aligned} & |(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n h[a, b](x, y)| \\ & \leq \left(|(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n a(x, y)| + |(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n b(x, y)| + Q_{mn} \right) \|h\|_{m+n+1}, \end{aligned}$$

where Q_{mn} is a continuous, monotone nondecreasing function of $\|a\|_{m+n-1}$, $\|b\|_{m+n-1}$.

A technical proof of this lemma, based on a discrete generalization of the chain rule, can be found in [M].

Lemma 7.6. *Under the conditions of Theorem 5.7,*

$$(75) \quad \sup_\epsilon \|a^\epsilon\|_{s+1} < \infty, \quad \sup_\epsilon \|b^\epsilon\|_{s+1} < \infty.$$

Proof. We proceed by induction on s . So, assume that (75) is already proved with s in place of $s+1$. Let $m, n \geq 0$ be such that $m+n = s+1$. If $n > 0$, then

$$|(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n a^\epsilon(x, y)| = |(\delta_x^\epsilon)^m (\delta_y^\epsilon)^{n-1} f^\epsilon[a^\epsilon, b^\epsilon](x, y)| \leq Q_{mn} < \infty$$

by Lemma 7.5, and similarly for $(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n b^\epsilon$ if $m > 0$. To estimate $(\delta_x^\epsilon)^m a^\epsilon$, observe that

$$(\delta_x^\epsilon)^m a^\epsilon(x, y) = (\delta_x^\epsilon)^m a^\epsilon(x, y - \epsilon) + \epsilon (\delta_x^\epsilon)^m f^\epsilon[a^\epsilon, b^\epsilon](x, y - \epsilon),$$

so that, by Lemma 7.5,

$$|(\delta_x^\epsilon)^m a^\epsilon(x, y)| \leq (1 + \epsilon \|f^\epsilon\|_{m+1}) |(\delta_x^\epsilon)^m a^\epsilon(x, y - \epsilon)| + \epsilon \|f^\epsilon\|_{m+1} (|(\delta_x^\epsilon)^m b^\epsilon(x, y)| + Q_{mn}).$$

By (44), we have

$$|(\delta_x^\epsilon)^m a^\epsilon(x, 0)| \leq M < \infty.$$

Since we know already that the norm of $(\delta_y^\epsilon)^m b^\epsilon$ is bounded uniformly in ϵ , the Gronwall Lemma 7.2 yields an ϵ -independent bound for $(\delta_x^\epsilon)^m a^\epsilon$. The same argument applies to $(\delta_y^\epsilon)^n b^\epsilon$. \square

Proof of the smoothness of a and b . Recall that the continuous function a was obtained as the uniform limit of a suitable subsequence of a^ϵ as $\epsilon \rightarrow 0$. Under the assumptions of Theorem 5.7, this subsequence can be chosen so that also $(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n a^\epsilon$ converges for $m+n \leq s$ uniformly on $\mathcal{B}^\epsilon(\bar{r})$ to some Lipschitz functions $a^{(m,n)}$. This follows directly from the Arzelà–Ascoli theorem: by estimate (75), each family $\{(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n a^\epsilon\}$ possesses an ϵ -independent Lipschitz constant as long as $m+n \leq s$. It is then easily seen that $a^{(mn)} = \partial_x a^{(m-1,n)} = \partial_y a^{(m,n-1)}$. We can conclude that a is s times differentiable with $\partial_x^m \partial_y^n a = a^{(m,n)}$, which are Lipschitz functions. The same argument is true with b in place of a . \square

To prove estimates (45), another technical lemma is needed.

Lemma 7.7. *Let $h^\epsilon, h : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ be smooth functions, and let the functions $u^\epsilon, u : \mathcal{B}^\epsilon(r) \rightarrow \mathcal{X}$ be bounded by M_1 . For any pair (m, n) of nonnegative integers and for all $(x, y) \in \mathcal{B}^\epsilon(r)$ we have*

$$\begin{aligned} & |(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n (h^\epsilon(u^\epsilon) - h(u))(x, y)| \\ & \leq C_s Q_s \|h^\epsilon - h\|_s + \left(|(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n (u^\epsilon - u)(x, y)| + C_s \|u^\epsilon - u\|_{s-1} Q_s \right) \|h\|_{s+1} \end{aligned}$$

with $s = m+n$, with some constant C_s , where Q_s is a continuous and monotone nondecreasing function of $\|u^\epsilon\|_s$ and $\|u\|_s$.

As in Lemma 7.5, the proof follows from the chain rule for difference operators on lattices.

Proof of estimate (45). Again, we proceed by induction on s . Convergence in C^0 , i.e., the case where $s = 0$, was already settled by Theorem 5.4. Assume that (45) is fulfilled for $s - 1$ in place of s . To estimate the quantity

$$A_{mn}^\epsilon(x, y) = (\delta_x^\epsilon)^m (\delta_y^\epsilon)^n a^\epsilon(x, y) - \partial_x^m \partial_y^n a(x, y),$$

three cases will be considered.

Case (i): $y = 0$ and $n = 0$. Then $A_{m0}^\epsilon(x, 0) = \mathcal{O}(\epsilon)$ by the assumption (44).

Case (ii): $n \geq 1$. Since a, b are $C^{s,1}$ -smooth and f is C^∞ , we have

$$\partial_x^m \partial_y^{n-1} f[a, b](x, y) = (\delta_x^\epsilon)^m (\delta_y^\epsilon)^{n-1} f[a, b](x, y) + \mathcal{O}(\epsilon)$$

uniformly on $B(\bar{r})$. Consequently:

$$(76) \quad A_{mn}^\epsilon(x, y) = (\delta_x^\epsilon)^m (\delta_y^\epsilon)^{n-1} (f^\epsilon[a^\epsilon, b^\epsilon] - f[a, b])(x, y) + \mathcal{O}(\epsilon).$$

Recall that f^ϵ $\mathcal{O}(\epsilon)$ -approximates f in C^{s+1} , and that, by the induction hypothesis, $\|a^\epsilon - a\|_{s-1} = \mathcal{O}(\epsilon)$, $\|b^\epsilon - b\|_{s-1} = \mathcal{O}(\epsilon)$. Application of Lemma 7.7 to (76) now yields $A_{mn}^\epsilon(x, y) = \mathcal{O}(\epsilon)$.

Case (iii): $y > 0$ but $n = 0$. Again, the smoothness of $f(a)$ is used to deduce the relation

$$A_{m0}^\epsilon(x, y) = A_{m0}^\epsilon(x, y - \epsilon) + \epsilon (\delta_x^\epsilon)^m (f^\epsilon[a^\epsilon, b^\epsilon] - f[a, b])(x, y - \epsilon) + \mathcal{O}(\epsilon^2).$$

The second term in the sum can be estimated by Lemma 7.7. After trivial manipulations, we obtain

$$|A_{m0}^\epsilon(x, y)| \leq (1 + \epsilon \|f\|_{s-1}) |A_{m0}^\epsilon(x, y - \epsilon)| + \epsilon \|f\|_{s-1} |(\delta_x^\epsilon)^m (b^\epsilon - b)|(x, y - \epsilon) + \mathcal{O}(\epsilon^2).$$

But $(\delta_x^\epsilon)^m (b^\epsilon - b)$ can be estimated along the same lines as $(\delta_y^\epsilon)^n a^\epsilon - \partial_y^n a = \mathcal{O}(\epsilon)$ in case (ii). Hence,

$$|A_{m0}^\epsilon(x, y)| \leq (1 + \epsilon \|f\|_{s-1}) |A_{m0}^\epsilon(x, y - \epsilon)| + \mathcal{O}(\epsilon^2),$$

and application of the Gronwall Lemma 7.2 gives $A_{m0}^\epsilon(x, y) = \mathcal{O}(\epsilon)$ for all $(x, y) \in \mathcal{B}(\bar{r})$. This proves the estimates for a^ϵ , and the same argument applies to b^ϵ . \square

§8. PROOF OF THEOREM 4.1

Asymptotic expansion as $\epsilon \rightarrow 0$ of the right-hand sides of equation (37) gives

$$(77) \quad \delta_y^\epsilon a = \sin b + \frac{\epsilon}{2} a \cos b + \mathcal{O}(\epsilon^2),$$

$$(78) \quad \delta_x^\epsilon b = a + \frac{\epsilon}{2} \sin b + \mathcal{O}(\epsilon^2),$$

so that the discrete equations approximate the hyperbolic system (34) with rate $\mathcal{O}(\epsilon)$ in C^∞ , and also with rate $\mathcal{O}(\epsilon^2)$ in the sense of Definition 5.5. Translation of the Goursat data (25) into the language of the variables a, b of equation (37) is as follows:

$$a_0^\epsilon(x) = \frac{1}{\epsilon} (\phi_1(x + \epsilon) - \phi_1(x)) = \partial_x \phi_1(x + \frac{\epsilon}{2}) + \mathcal{O}(\epsilon^2),$$

$$b_0^\epsilon(y) = \frac{1}{2} (\phi_2(y + \epsilon) + \phi_2(y)) = \phi_2(y + \frac{\epsilon}{2}) + \mathcal{O}(\epsilon^2).$$

Thus, Theorems 5.4, 5.6, and 5.7 yield the existence and uniqueness of discrete solutions (a^ϵ, b^ϵ) , and their convergence to a unique smooth solution (a, b) of system (34) with the Goursat data $a(x, 0) = \partial_x \phi_1(x)$ and $b(0, y) = \phi_2(y)$ on a suitable domain $\mathcal{B}(\bar{r})$. The convergence is of order $\mathcal{O}(\epsilon)$ in C^∞ , and of order $\mathcal{O}(\epsilon^2)$ in C^0 .

It remains to prove that a similar approximation holds also for the immersions F^ϵ , F . The strategy is to approximate the frame Ψ of the smooth K-surface by the frame Ψ^ϵ of the discrete K-surface, and then use the Sym formula. Recall that the frames are solutions of the Cauchy problems for the systems of linear differential and difference

equations (5) and (18), respectively. Since the zero curvature conditions (4) and (17) are satisfied, the existence of Ψ , Ψ^ϵ is guaranteed by standard ODE theory. Furthermore, at any point (x, y) of their domains, $\Psi(\lambda)$ and $\Psi^\epsilon(\lambda)$ are analytic functions of $\lambda \in D$, where D is an arbitrary closed disk in the complex plane that contains 1 in its interior, but does not contain 0. The matrices U , V and $\mathcal{U} = \mathcal{U}^\epsilon$, $\mathcal{V} = \mathcal{V}^\epsilon$ are bounded uniformly with respect to $\lambda \in D$ and $(x, y) \in \mathcal{B}(\bar{r})$.

A natural norm $|\cdot|^D$ on the space of λ -dependent (2×2) -matrices $A = A(\lambda)$ is given by

$$|A|^D = \sup_{\lambda \in D} |A(\lambda)|,$$

and $|\cdot|$ is the usual matrix norm. The discrete C^s -norms $\|\cdot\|_s^D$ are introduced by obvious analogy with (73). Now define

$$U^\epsilon = (\mathcal{U}^\epsilon - \mathbf{1})/\epsilon, \quad V^\epsilon = (\mathcal{V}^\epsilon - \mathbf{1})/\epsilon.$$

Then $U^\epsilon(a; \lambda) = U(a; \lambda) + \mathcal{O}(\epsilon)$ and $V^\epsilon(b; \lambda) = V(b; \lambda) + \mathcal{O}(\epsilon)$ uniformly in λ and in any C^s -norm with respect to (a, b) . As a consequence of the $\mathcal{O}(\epsilon)$ -approximation of (a, b) by (a^ϵ, b^ϵ) , for any $s \geq 0$ we have:

$$(79) \quad \|U^\epsilon - U\|_s^D = \mathcal{O}(\epsilon), \quad \|V^\epsilon - V\|_s^D = \mathcal{O}(\epsilon).$$

By the definition of Ψ^0 and Ψ^ϵ ,

$$(80) \quad \Psi(x + \epsilon, y) = \Psi(x, y) + \epsilon U(x, y) \Psi(x, y) + \mathcal{O}(\epsilon^2),$$

$$(81) \quad \Psi^\epsilon(x + \epsilon, y) = \Psi^\epsilon(x, y) + \epsilon U^\epsilon(x, y) \Psi^\epsilon(x, y).$$

It follows that

$$(82) \quad |\Psi^\epsilon - \Psi|(x + \epsilon, y) = (1 + \epsilon \|U^\epsilon - U\|_0^D) |\Psi^\epsilon - \Psi|(x, y) + \mathcal{O}(\epsilon^2);$$

a similar formula is valid with V , V^ϵ for the shift in the y -direction. The Gronwall estimate yields

$$(83) \quad \|\Psi - \Psi^\epsilon\|_0^D = \mathcal{O}(\epsilon).$$

Choose $\Lambda > 1$ so that $I_\Lambda = [\Lambda^{-1}, \Lambda]$ lies in the interior of D and has a positive distance $\mu > 0$ from ∂D . Since $\Psi(\lambda)$ and $\Psi^\epsilon(\lambda)$ are analytic functions of $\lambda \in D$, the Cauchy formula implies that

$$\sup_{\lambda \in I_\Lambda} |\partial_\lambda \Psi^\epsilon(x, y; \lambda) - \partial_\lambda \Psi(x, y; \lambda)| \leq \mu^{-1} \sup_{\lambda \in D} |\Psi^\epsilon(x, y; \lambda) - \Psi(x, y; \lambda)|,$$

which is $\mathcal{O}(\epsilon)$ uniformly on $\mathcal{B}^\epsilon(r)$ by (83). The Sym formula (6) yields

$$(84) \quad F_\lambda^\epsilon - F_\lambda = 2\lambda(\Psi^\epsilon(\lambda))^{-1} \partial_\lambda \Psi^\epsilon(\lambda) - 2\lambda(\Psi(\lambda))^{-1} \partial_\lambda \Psi(\lambda) = \mathcal{O}(\epsilon)$$

for all $\lambda \in I_\Lambda$ uniformly on $\mathcal{B}^\epsilon(r)$.

Next, we show $\mathcal{O}(\epsilon)$ -approximation of the higher-order partial derivatives of F . As before, an induction argument is used. Assume that the estimate $\|\Psi^\epsilon - \Psi\|_{s-1}^D = \mathcal{O}(\epsilon)$ has already been obtained. Let (m, n) be a pair of nonnegative integers with $m + n = s$. If $m > 0$ (say), then, by the smoothness of U and Ψ ,

$$(85) \quad \begin{aligned} & |(\delta_x^\epsilon)^m (\delta_y^\epsilon)^n (\Psi^\epsilon - \Psi)|(x, y) \\ &= |(\delta_x^\epsilon)^{m-1} (\delta_y^\epsilon)^n (U^\epsilon \Psi^\epsilon - U \Psi)|(x, y) + \mathcal{O}(\epsilon) \leq \|U^\epsilon \Psi^\epsilon - U \Psi\|_{s-1}^D. \end{aligned}$$

Observe that, by iteration of the discrete product rule, namely,

$$\delta_x^\epsilon(u \cdot v)(x) = \delta_x u(x) \cdot v(x) + u(x + \epsilon) \cdot \delta_x^\epsilon v(x),$$

the following estimate can be obtained:

$$\|u \cdot v\|_s \leq C_s \|u\|_s \cdot \|v\|_s.$$

This is applied to (85) in the following way:

$$\|U^\epsilon \Psi^\epsilon - U \Psi\|_{s-1}^D \leq C_{s-1} (\|U^\epsilon - U\|_{s-1}^D \|\Psi\|_{s-1}^D + \|U^\epsilon\|_{s-1}^D \|\Psi^\epsilon - \Psi\|_{s-1}^D).$$

The right-hand side is of order $\mathcal{O}(\epsilon)$ because of (79) and the induction hypothesis. A similar argument applies if $m = 0$ and $n > 0$, with replacement of U by V . As in the proof of (84), the Sym formulas imply (26), also uniformly in $\lambda \in I_\lambda$.

Finally, we prove the $\mathcal{O}(\epsilon^2)$ -approximation in (27). Observe that

$$(86) \quad \begin{aligned} \delta_x^\epsilon \Psi(x, y) &= U(x + \frac{\epsilon}{2}, y) \Psi(x + \frac{\epsilon}{2}, y) + \mathcal{O}(\epsilon^2) \\ &= (U(x + \frac{\epsilon}{2}, y) + \frac{\epsilon}{2} U^2(x + \frac{\epsilon}{2}, y)) \Psi(x, y) + \mathcal{O}(\epsilon^2), \end{aligned}$$

and similarly for the y -shifts. On the other hand, an ϵ -expansion of U^ϵ and V^ϵ as functions of a, b gives:

$$\begin{aligned} U^\epsilon &= \begin{pmatrix} ia/2 - \epsilon(a^2 + \lambda^2)/8 & -i\lambda/2 \\ -i\lambda/2 & -ia/2 - \epsilon(a^2 + \lambda^2)/8 \end{pmatrix} + \mathcal{O}(\epsilon^2) \\ &= U + \frac{\epsilon}{2} U^2 + \mathcal{O}(\epsilon^2), \\ V^\epsilon &= \begin{pmatrix} -\epsilon\lambda^{-2}/8 & i\lambda^{-1} \exp(ib)/2 \\ i\lambda^{-1} \exp(-ib)/2 & -\epsilon\lambda^{-2}/8 \end{pmatrix} + \mathcal{O}(\epsilon^2) \\ &= V + \frac{\epsilon}{2} V^2 + \mathcal{O}(\epsilon^2). \end{aligned}$$

Recall that $a^\epsilon(x, y) = a(x + \frac{\epsilon}{2}, y) + \mathcal{O}(\epsilon^2)$ and $b^\epsilon(x, y) = b(x, y + \frac{\epsilon}{2}) + \mathcal{O}(\epsilon^2)$ by Theorem 5.6. Therefore,

$$\begin{aligned} U^\epsilon(x, y) &= U(x + \frac{\epsilon}{2}, y) + \frac{\epsilon}{2} U^2(x + \frac{\epsilon}{2}, y) + \mathcal{O}(\epsilon^2), \\ V^\epsilon(x, y) &= V(x, y + \frac{\epsilon}{2}) + \frac{\epsilon}{2} V^2(x, y + \frac{\epsilon}{2}) + \mathcal{O}(\epsilon^2). \end{aligned}$$

Thus,

$$(87) \quad \delta_x^\epsilon \Psi^\epsilon(x, y) = U^\epsilon(x, y) \Psi^\epsilon(x, y) = (U(x + \frac{\epsilon}{2}, y) + \frac{\epsilon}{2} U^2(x + \frac{\epsilon}{2}, y)) \Psi^\epsilon(x, y) + \mathcal{O}(\epsilon^2).$$

Compare this with (86). The Gronwall estimate yields $\|\Psi^\epsilon - \Psi\|_0 = \mathcal{O}(\epsilon^2)$, and, by the same argument as above, we eventually arrive at (27).

The statement concerning the approximation of Bäcklund transformed surfaces follows in a completely similar way with a reference to Theorem 6.5. \square

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Received 1/SEP/2004

Originally published in English