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FUNCTIONAL INTEGRATION AND THE TWO-POINT CORRELATION FUNCTION OF THE ONE-DIMENSIONAL BOSE GAS IN A HARMONIC POTENTIAL

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Dedicated to Ludwig Dmitrievich Faddeev

Abstract. A quantum field-theoretical model which describes a spatially nonhomogeneous one-dimensional nonrelativistic repulsive Bose gas in an external harmonic potential is considered. The two-point thermal correlation function of the Bose gas is calculated in the framework of the functional integration approach. The calculations are done in the coordinate representation. The method of successive integration first over the "high-energy" functional variables and then over the "low-energy" ones is used. The effective action functional for the low-energy variables is calculated in one-loop approximation. The functional integral representation for the correlation function is obtained in terms of the low-energy variables and is estimated with the help of stationary phase approximation. The asymptotics of the correlation function is studied in the limit when the temperature tends to zero while the volume occupied by the nonhomogeneous Bose gas increases infinitely. It is demonstrated that the behavior of the thermal correlation function in this limit is power-like and is governed by the critical exponent that depends on the spatial and thermal arguments.

§1. Introduction

A recent burst of interest in the theory of the Bose gas is caused by experimental realization of the Bose condensation in the ultracold vapors of alkali metals confined in magneto-optical traps [1, 2]. In particular, it became possible to study the Bose condensation in systems that are effectively two-dimensional or quasi-one-dimensional [3, 4]. Here partial localization along one or two directions in a three-dimensional system is achieved by making the level spacing of the trapping potential in the corresponding directions larger than the energies of individual atoms. The field models that describe the Bose particles with delta-like interparticle coupling confined by an external harmonic potential provide a good approximation for a theoretical approach to the experimental situation [5]. The theory of nonideal Bose gas attracts traditionally not only physicists but also mathematicians [6]–[8]. For a translation-invariant homogeneous Bose gas, the field model in question corresponds to a quantum nonlinear Schrödinger equation, which admits an exact solution in the one-dimensional case [9, 10]. This fact makes it possible to obtain closed expressions for the correlation functions [11, 12].

As to real physical systems, the interest in transition from three-dimensional to one-dimensional behavior is caused by the fact that the effective density of atoms in the

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one-dimensional Bose gas can be either high or low depending on the parameters of the system [13]. Here a low density implies a strong coupling between the particles [14, 15], while a high density corresponds to weak interaction.

The present paper continues the series of papers [16]–[20] devoted to investigation of the correlation functions of the Bose gas with weak repulsive interparticle coupling in the presence of an external harmonic potential. Since there are no exact solutions in the case of an external potential, in the present paper (as well as in [16]–[20]) the method of functional integration [21]–[24] is adopted for investigation of the correlation functions. We develop the results of the paper [20] where, in contrast to [16]–[19], an explicit dependence of the correlation functions on the imaginary time for nonzero temperature was taken into account. It will be demonstrated that the presence of an external potential (of the *trap*) results in a change of the asymptotic behavior of the two-point correlation function in comparison to a translation-invariant case. The change observed happens in the range of temperatures that are comparable with the inverse of the characteristic length of the trap provided this length tends to infinity.

The paper is organized as follows. §1 is of an introductory nature. A description of the one-dimensional model of the nonrelativistic Bose field in question, as well as a summary of the method of functional integration, are given in §2. An approximation approach to the investigation of functional integrals is also presented in §2. This approach is based on successive integration first over the high-energy overcondensate excitations and then over the variables corresponding to the low-energy quasiparticles. Furthermore, in this section we give a derivation of one-loop effective action for the low-excited quasicondensate fields and obtain the corresponding energy spectrum of the low-lying excitations. The stationary phase method is used in §3 for estimation of the functional integral that expresses the two-point thermal correlation function of the nonhomogeneous Bose gas. The method of asymptotical estimation of correlators, which is used in the present paper, was proposed in [25], where the asymptotical behavior of two-point Green functions of the homogeneous Bose gas was investigated for the spatial dimensions one, two, and three. In the present paper it is demonstrated that the stationary phase method [25] admits a generalization also for the spatially nonhomogeneous Bose gas in the external potential. The asymptotics of two-point correlation functions of nonhomogeneous Bose gas are obtained in §4 both at nonzero and zero temperatures. A short discussion of the results of the present paper is given in §5.

§2. The effective action and the Thomas–Fermi approximation

1. The partition function. In this paper we consider the one-dimensional Bose gas described by the Hamiltonian $\widehat{H}$ defined on the real axis $\mathbb{R} \ni x$:

$$\begin{aligned} \widehat{H} &= \int \left\{ \widehat{\psi}^{\dagger}(x)\, \mathcal{H} \widehat{\psi}(x) + \frac{g}{2} \widehat{\psi}^{\dagger}(x)\, \widehat{\psi}^{\dagger}(x)\, \widehat{\psi}(x)\, \widehat{\psi}(x) \right\} dx, \\ \mathcal{H} &\equiv \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \mu + V(x), \end{aligned} \tag{1}$$

where $\widehat{\psi}^{\dagger}(x)$ and $\widehat{\psi}(x)$ are operator-valued fields which describe the creation and annihilation of the quasiparticles over the Fock vacuum $|0\rangle$. The fields $\widehat{\psi}^{\dagger}(x)$ and $\widehat{\psi}(x)$ obey the commutation relation

$$\widehat{\psi}(x)\widehat{\psi}^{\dagger}(x') - \widehat{\psi}^{\dagger}(x')\widehat{\psi}(x) = \delta(x - x')$$

($\widehat{\psi}^{\dagger}(x)$ and $\widehat{\psi}(x)$ commute), and $\mathcal{H}$ is the "single-particle" Hamiltonian. The following notation is used in equation (1): m is the mass of the Bose particles, μ is the chemical potential, g is the coupling constant corresponding to the weak repulsion (i.e., $g > 0$),

and the external confining potential is chosen in the form of the harmonic potential $V(x) \equiv \frac{m}{2}\Omega^2 x^2$.

We begin with an investigation of the partition function Z. It can be represented in the form of a functional integral [21]–[24]:

$$Z = \int e^{S[\psi,\bar{\psi}]} \mathcal{D}\psi \mathcal{D}\bar{\psi}, \tag{2}$$

where $S[\psi,\bar{\psi}]$ is the action functional of the system in question:

$$S[\psi,\bar{\psi}] = \int_0^\beta d\tau \int dx \left(\bar{\psi}(x,\tau)\Big(\frac{\partial}{\partial\tau} - \mathcal{H}\Big)\psi(x,\tau) - \frac{g}{2}\bar{\psi}(x,\tau)\bar{\psi}(x,\tau)\psi(x,\tau)\psi(x,\tau)\right). \tag{3}$$

The domain of the functional integration in (2) is given by the space of complex-valued functions $\bar{\psi}(x,\tau)$, $\psi(x,\tau)$ depending on two arguments $x \in \mathbb{R}$ and $\tau \in [0,\beta]$. With regard to the first argument x, the functions $\bar{\psi}(x,\tau)$, $\psi(x,\tau)$ belong to the space of square integrable functions $L_2(\mathbb{R})$, and with regard to the imaginary time τ they are finite and periodic with the period $\beta = (k_B T)^{-1}$ (k_B is the Boltzmann constant, and T is an absolute temperature). The variables $\bar{\psi}$, ψ are the independent variables of the functional integration [21], and $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ is the functional integration measure.

At sufficiently low temperatures we may assume that each of the variables $\bar{\psi}(x,\tau)$, $\psi(x,\tau)$ is given by two constituents. One of them, $\bar{\psi}_o(x,\tau)$, $\psi_o(x,\tau)$, corresponds to the *quasicondensate*, while the other one to the high-energy thermal (i.e., over condensate) excitations $\bar{\psi}_e(x,\tau)$, $\psi_e(x,\tau)$:

$$\psi(x,\tau) = \psi_o(x,\tau) + \psi_e(x,\tau), \quad \bar{\psi}(x,\tau) = \bar{\psi}_o(x,\tau) + \bar{\psi}_e(x,\tau). \tag{4}$$

It should be emphasized that here and below precisely the quasicondensate is meant, since a true Bose condensate does not exist in a one-dimensional system [6]. In the exactly solvable case, the existence of the quasicondensate implies that a nontrivial vacuum state (i.e., a ground state) exists. The quasicondensate variables $\bar{\psi}_o(x,\tau)$, $\psi_o(x,\tau)$ can also be represented as sums:

$$\psi_o(x,\tau) = \psi_o(x) + \xi(x,\tau), \quad \bar{\psi}_o(x,\tau) = \bar{\psi}_o(x) + \bar{\xi}(x,\tau), \tag{5}$$

where the field $\psi_o(x)$ describes the ground state of the model at zero temperature, and the field $\xi(x,\tau)$ describes the low-lying excited particles. We require that the variables in (4) be orthogonal in the following sense:

$$\int \psi_o(x,\tau)\bar{\psi}_e(x,\tau)\,dx = \int \bar{\psi}_o(x,\tau)\psi_e(x,\tau)\,dx = 0.$$

As a result, the integration measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ will be replaced by $\mathcal{D}\psi_o\mathcal{D}\bar{\psi}_o\mathcal{D}\psi_e\mathcal{D}\bar{\psi}_e$.

To investigate the functional integral (2), we shall perform successive integration over the fields $\bar{\psi}, \psi$. First, we shall integrate over the high-energy constituents, and then over the low-energy ones (see (4)) [21, 24]. At the second step, it is preferable to pass to new functional variables that describe an observable "low-energy" physics [21, 24] in a more adequate way. After substituting the expansion (4) in the action (3), only the terms up to quadratic in $\bar{\psi}_e$, ψ_e will be taken into account in S. This means that we are making an approximation in which the overcondensate quasiparticles do not couple with each other. In this case, it is possible to integrate over the thermal fluctuations $\bar{\psi}_e(x,\tau)$, $\psi_e(x,\tau)$ in a closed form, thus arriving at an effective action functional $S_{\text{eff}}[\psi_o,\bar{\psi}_o]$ that depends only on the quasicondensate variables $\psi_o, \bar{\psi}_o$:

$$S_{\text{eff}}[\psi_o,\bar{\psi}_o] = \ln \int e^{\widetilde{S}[\psi_o+\psi_e,\bar{\psi}_o+\bar{\psi}_e]} \mathcal{D}\psi_e \mathcal{D}\bar{\psi}_e, \tag{6}$$

where the tilde in $\widetilde{S}$ implies that "self-coupling" of the fields $\bar{\psi}_e(x,\tau)$, $\psi_e(x,\tau)$ is excluded. With respect to equation (6), the partition function Z of the model takes an approximate form:

$$Z \approx \int e^{S_{\rm eff}[\psi_o,\bar{\psi}_o]}\mathcal{D}\psi_o\mathcal{D}\bar{\psi}_o. \tag{7}$$

Let us consider the derivation of the effective action $S_{\rm eff}[\psi_o,\bar{\psi}_o]$ (see (6)) in more detail. The splitting (4) allows us to derive $S_{\rm eff}[\psi_o,\bar{\psi}_o]$ in the framework of the field-theoretical approach of the loop expansion (see [26, 27]). We substitute (4) in the initial action $S[\psi,\bar{\psi}]$ (see (3)) and then pass from S to the action $\widetilde{S}$ given by three terms:

$$\widetilde{S} = S_{\rm cond} + S_{\rm free} + S_{\rm int}. \tag{8}$$

In (8), $S_{\rm cond}$ is the action functional of the condensate quasiparticles, which corresponds to tree approximation [26, 27]:

$$\begin{aligned} &S_{\rm cond}[\psi_o,\bar{\psi}_o] \\ &\quad\equiv \int_0^\beta d\tau \int dx \Big\{\bar{\psi}_o(x,\tau)\widehat{K}_+\psi_o(x,\tau) - \frac{g}{2}\bar{\psi}_o(x,\tau)\bar{\psi}_o(x,\tau)\psi_o(x,\tau)\psi_o(x,\tau)\Big\}. \end{aligned} \tag{9}$$

In the approximation chosen, the action for the overcondensate excitations $S_{\rm free}$ takes the form

$$S_{\rm free}[\psi_e,\bar{\psi}_e] \equiv \frac{1}{2}\int_0^\beta d\tau \int dx\, (\bar{\psi}_e,\psi_e)\widehat{G}^{-1}\begin{pmatrix}\psi_e\\ \bar{\psi}_e\end{pmatrix}. \tag{10}$$

Finally, $S_{\rm int}$ is given by the part of the total action functional that describes the coupling of the quasicondensate to the overcondensate excitations:

$$\begin{aligned} &S_{\rm int}[\psi_o,\bar{\psi}_o,\psi_e,\bar{\psi}_e] \\ &\quad\equiv \int_0^\beta d\tau \int dx \Big\{\bar{\psi}_e(x,\tau)[\widehat{K}_+ - g\bar{\psi}_o\psi_o]\psi_o(x,\tau) + \psi_e(x,\tau)[\widehat{K}_- - g\bar{\psi}_o\psi_o]\bar{\psi}_o(x,\tau)\Big\}. \end{aligned} \tag{11}$$

In formulas (9)–(11), we have used the differential operators $\widehat{K}_\pm \equiv \pm\partial/\partial\tau - \mathcal{H}$ (here the Hamiltonian $\mathcal{H}$ is defined in (1)) and the matrix-differential operator $\widehat{G}^{-1}$:

$$\widehat{G}^{-1} \equiv \widehat{G}_0^{-1} - \widehat{\Sigma}, \tag{12}$$

where

$$\widehat{G}_0^{-1} \equiv \begin{pmatrix}\widehat{K}_+ & 0\\ 0 & \widehat{K}_-\end{pmatrix}, \quad \widehat{\Sigma} \equiv \widehat{\Sigma}[\psi_o,\bar{\psi}_o] = g\begin{pmatrix}2\bar{\psi}_o\psi_o & \psi_o^2\\ (\bar{\psi}_o)^2 & 2\bar{\psi}_o\psi_o\end{pmatrix}.$$

Under this approach, it is appropriate to apply the stationary phase method to the functional integral (6). For this, we choose $\bar{\psi}_o,\psi_o$ as the stationarity points of the functional $S_{\rm cond}$ (see (9)), which are defined by the extremum condition $\delta(S_{\rm cond}[\psi_o,\bar{\psi}_o]) = 0$. The corresponding equations take the form of the Gross–Pitaevskii-type equations [28]:

$$\begin{aligned} \Big(\frac{\partial}{\partial\tau} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \mu - V(x)\Big)\psi_o - g(\bar{\psi}_o\psi_o)\psi_o = 0,\\ \Big(-\frac{\partial}{\partial\tau} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \mu - V(x)\Big)\bar{\psi}_o - g(\bar{\psi}_o\psi_o)\bar{\psi}_o = 0. \end{aligned} \tag{13}$$

The contribution of $S_{\rm int}$ (see (11)) drops out from (8) because $\bar{\psi}_o,\psi_o$ are chosen to be solutions of equations (13). Therefore, in the leading approximation, the dynamics of $\psi_e,\bar{\psi}_e$ is described by the action $S_{\rm free}$ (see (10)). The latter depends on $\bar{\psi}_o,\psi_o$ nontrivially through the matrix of the self-energy parts $\widehat{\Sigma}$, which is involved in $\widehat{G}^{-1}$ (see (12)). The *Thomas–Fermi approximation* is essentially used in the present paper in order to

determine the stationarity points $\bar{\psi}_o, \psi_o$. This approximation consists in neglecting the kinetic term $\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ in equations (13) [5, 28]. The Thomas–Fermi approximation is valid for the systems containing a sufficiently large number of particles, and it is widely used in the theoretical approaches to a description of the Bose condensation in the magneto-optical traps [5, 28]. The following condensate solution can be obtained provided only τ-independent solutions of (13) are allowed:

$$\bar{\psi}_o\psi_o = \rho_{TF}(x;\mu) \equiv \frac{1}{g}\left(\mu - V(x)\right)\Theta\left(\mu - V(x)\right), \tag{14}$$

where Θ is the Heaviside function. Now the integration with respect to ψ_e, $\bar{\psi}_e$ in (6) is Gaussian. This leads [21] to the one-loop effective action, which takes the following form in terms of the variables ψ_o, $\bar{\psi}_o$:

$$S_{\text{eff}}[\psi_o, \bar{\psi}_o] \equiv S_{\text{cond}}[\psi_o, \bar{\psi}_o] - \frac{1}{2}\ln\operatorname{Det}(\widehat{G}^{-1}). \tag{15}$$

Here $\widehat{G}^{-1}$ is the matrix operator (12), and ψ_o, $\bar{\psi}_o$ have a meaning of new variables, and their dynamics is governed by the action (15).

In order to assign a meaning to the final expression for the effective action (15), it is necessary to regularize the determinant $\operatorname{Det}(\widehat{G}^{-1})$. In our case, the operator $\widehat{G}^{-1}$ is already written as a (2×2)-matrix Dyson equation (12), where the entries of $\widehat{\Sigma}[\psi_o, \bar{\psi}_o]$ play the role of the normal ($\Sigma_{11} = \Sigma_{22}$) and the anomalous ($\Sigma_{12}$, Σ_{21}) self-energy parts. The Dyson equation (12) determines the matrix $\widehat{G}$, where the entries have a meaning of the Green functions of the fields $\bar{\psi}_e, \psi_e$. The matrix $\widehat{G}$ arises as a formal inverse of the operator $\widehat{G}^{-1}$:

$$\widehat{G} = \left(\widehat{G}_0^{-1} - \widehat{\Sigma}\right)^{-1}. \tag{16}$$

The matrix operator $\widehat{G}^{-1}$ can be diagonalized formally by means of the famous N. N. Bogolyubov's (u, v)-transform [6]. The corresponding equations, which describe the unknown coefficient-functions u and v, result in a compatibility requirement, which, in its turn, determines the quasiclassical spectrum of the elementary excitations [28].

With regard to our purposes, it is appropriate to represent $\widehat{G}^{-1}$ as follows:

$$\widehat{G}^{-1} = \widehat{G}_0^{-1} - \widehat{\Sigma} \equiv \widehat{\mathcal{G}}^{-1} - (\widehat{\Sigma} - 2g\rho_{TF}(x;\mu)\widehat{I}), \tag{17}$$

where $\widehat{I}$ is the unit matrix of size 2×2, and the matrix $\widehat{\mathcal{G}}^{-1}$ is defined as

$$\widehat{\mathcal{G}}^{-1} \equiv \begin{pmatrix} \widehat{K}_+ - 2g\rho_{TF}(x;\mu) & 0 \\ 0 & \widehat{K}_- - 2g\rho_{TF}(x;\mu) \end{pmatrix} \equiv \begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix}. \tag{18}$$

Here $\rho_{TF}(x;\mu)$ is the solution (14), and equation (17) implies that we have simply added and subtracted $2g\rho_{TF}(x;\mu)$ on the principle diagonal of the matrix operator $\widehat{G}^{-1}$. A formal inverse of the operator $\widehat{\mathcal{G}}^{-1}$ can be found from the following equation, which determines the Green functions $\mathcal{G}_\pm$:

$$\begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix}\begin{pmatrix} \mathcal{G}_+ & 0 \\ 0 & \mathcal{G}_- \end{pmatrix} = \delta(x - x')\delta(\tau - \tau')\widehat{I}.$$

Using the relation $\ln\operatorname{Det} = \operatorname{Tr}\ln$, we get

$$-\frac{1}{2}\ln\operatorname{Det}\widehat{G}^{-1} = -\frac{1}{2}\operatorname{Tr}\ln\left(\widehat{I} - \widehat{\mathcal{G}}\big(\widehat{\Sigma} - 2g\rho_{TF}(x;\mu)\widehat{I}\big)\right) - \frac{1}{2}\ln\operatorname{Det}\begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix}. \tag{19}$$

The first term on the right-hand side of (19) is free from divergencies. Consider the determinant of the matrix-differential operator on the right-hand side of (19). The operators $\mathcal{K}_\pm$ can be written as

$$\mathcal{K}_\pm \equiv \pm\frac{\partial}{\partial\tau} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + |V(x)-\mu|. \tag{20}$$

We denote the eigenvalues of the operators $\mathcal{K}_\pm$ as $\pm i\omega_B - \lambda_n$, where ω_B are the bosonic Matsubara frequencies, and the λ_n are the energy levels (labeled by the multi-index n) of the operator $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - |V(x)-\mu|$. The regular part of the logarithm of the determinant in (19) has the meaning of the free energy $\widetilde{F}_{nc}$ of the ideal gas of the overcondensate excitations:

$$\widetilde{F}_{nc}(\mu) \equiv \frac{1}{2\beta}\ln\mathrm{Det}\begin{pmatrix}\mathcal{K}_+ & 0\\ 0 & \mathcal{K}_-\end{pmatrix} = \frac{1}{\beta}\sum_n \ln\Big(2\sinh\frac{\beta\lambda_n}{2}\Big),$$

where the regularized values of the determinants of the operators $\mathcal{K}_\pm$ can be obtained, for instance, with the help of the zeta-regularization approach [27]. Then, in the leading order in g, we obtain

$$\begin{aligned}
&-\frac{1}{2}\ln\mathrm{Det}\,\widehat{G}^{-1}\\
&\quad\approx -\beta\widetilde{F}_{nc}(\mu) + g\int_0^\beta d\tau\int dx\,\big(\mathcal{G}_+(x,\tau;x,\tau)+\mathcal{G}_-(x,\tau;x,\tau)\big)\big(\bar\psi_o\psi_o-\rho_{TF}(x;\mu)\big)\\
&\quad\equiv -\beta F_{nc}(\mu) + 2g\int_0^\beta d\tau\int dx\,\rho_{nc}(x)\bar\psi_o\psi_o.
\end{aligned} \tag{21}$$

Here F_{nc} is the free energy of the nonideal gas of the overcondensate quasiparticles, and the last term in (21) describes the coupling of the overcondensate quasiparticles with the condensate. The density of the overcondensate quasiparticles is $\rho_{nc}(x) \equiv -\mathcal{G}_\pm(x,\tau;x,\tau)$, and it depends only on the spatial coordinate x. At very low temperatures and sufficiently far from the boundary of the domain occupied by the condensate, the quantity $\rho_{nc}(x)$ can be replaced approximately by $\rho_{nc}(0)$, because $\mathcal{G}_\pm(x,\tau;x,\tau)$ is almost constant over a considerable part of the condensate [29]. Finally,

$$\begin{aligned}
S_{\mathrm{eff}}[\psi_o,\bar\psi_o] = -\beta F_{nc}(\mu) + \int_0^\beta d\tau\int dx\,\Big\{&\bar\psi_o(x,\tau)\Big(\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\Lambda-V(x)\Big)\psi_o(x,\tau)\\
&-\frac{g}{2}\bar\psi_o(x,\tau)\bar\psi_o(x,\tau)\psi_o(x,\tau)\psi_o(x,\tau)\Big\},
\end{aligned} \tag{22}$$

where $\Lambda = \mu - 2g\rho_{nc}(0)$ is the renormalized chemical potential. We shall view S_{eff} (see (22)) as one-loop effective action, where the thermal corrections over the "classical" background are taken into account. The "classical" background corresponds to the solution (14). It should be noted that the above derivation of the effective action does not depend on spatial dimension, so that it is also valid for dimensions two and three.

In the effective action (22) it is appropriate to pass to new variables, namely, to the density $\rho(x,\tau)$ and the phase $\varphi(x,\tau)$ of the field $\psi_o(x,\tau)$ [21]:

$$\psi_o(x,\tau) = \sqrt{\rho(x,\tau)}e^{i\varphi(x,\tau)}, \quad \bar\psi_o(x,\tau) = \sqrt{\rho(x,\tau)}e^{-i\varphi(x,\tau)}. \tag{23}$$

We shall regard $\rho(x,\tau)$ and $\varphi(x,\tau)$ as two new independent real-valued variables of the functional integration. Now the integration measure $\mathcal{D}\bar\psi_o\mathcal{D}\psi_o$ is replaced by the measure

$\mathcal{D}\rho\mathcal{D}\varphi$. In these new variables the effective action (22) takes the form

$$S_{\text{eff}}[\rho,\varphi] = -\beta F_{nc}(\mu) + i\int_0^\beta d\tau \int dx \left\{\rho\partial_\tau\varphi + \frac{\hbar^2}{2m}\partial_x(\rho\partial_x\varphi)\right\} + \int_0^\beta d\tau \int dx \left\{\frac{\hbar^2}{2m}(\sqrt{\rho}\partial_x^2\sqrt{\rho} - \rho(\partial_x\varphi)^2) + (\Lambda - V)\rho - \frac{g}{2}\rho^2\right\}. \tag{24}$$

Here and below we denote the partial derivatives of the first order over τ and x by ∂_τ and ∂_x, respectively, whereas the partial derivatives of the second order will be denoted by ∂_τ^2 and ∂_x^2. Notice that equations (22) and (24) remain valid at $V = 0$, so that the renormalized chemical potential is still given by the equation $\Lambda = \mu - 2g\rho_{nc}(0)$, where $\rho_{nc}(0)$ is the "bare" [21] density of the condensate.

2. The excitations spectrum. We turn to the problem of determining the spectrum of the low-energy quasiparticles. We shall apply the stationary phase approximation to the integral (7), where the effective action is given by (22). The corresponding stationarity point is determined from the extremum condition $\delta\big(S_{\text{eff}}[\rho,\varphi]\big) = 0$, which is equivalent to the pair of the Gross–Pitaevskii equations:

$$i\partial_\tau\varphi + \frac{\hbar^2}{2m}\Big(\frac{1}{\sqrt{\rho}}\partial_x^2\sqrt{\rho} - (\partial_x\varphi)^2\Big) + \Lambda - V(x) - g\rho = 0, \qquad -i\partial_\tau\rho + \frac{\hbar^2}{m}\partial_x\left(\rho\partial_x\varphi\right) = 0. \tag{25}$$

Let ρ_0 and φ_0 denote some solutions of (25). Substituting ρ_0, φ_0 in the effective action (24), we obtain

$$S_{\text{eff}}[\rho_0,\varphi_0] = -\beta F_{nc}(\mu) + \frac{g}{2}\int_0^\beta d\tau \int dx\, \rho_0^2. \tag{26}$$

Here $F_{nc}(\mu)$ is the free energy of the nonideal gas of the overcondensate quasiparticles. The total free energy of the system is $F(\mu) = -\frac{1}{\beta}S_{\text{eff}}[\rho_0,\varphi_0]$ [18].

We use the Thomas–Fermi approximation, which is valid at sufficiently low temperatures, and drop out the kinetic term $(\partial_x^2\sqrt{\rho})/\sqrt{\rho}$ in the first equation in (25). A solution with $\partial_\tau\rho = 0 = \partial_\tau\varphi$ appears, provided the velocity field $\mathbf{v} = m^{-1}\partial_x\varphi$ in (25) is taken equal to zero. In this case, equations (25) lead to the density of the condensate described by the solution (14), where the chemical potential μ is replaced by Λ:

$$\rho_{TF}(x) \equiv \frac{\Lambda}{g}\widetilde{\rho}_{TF}(x) = \frac{\Lambda}{g}\Big(1 - \frac{x^2}{R_c^2}\Big)\Theta\Big(1 - \frac{x^2}{R_c^2}\Big). \tag{27}$$

The explicit form of the external potential $V(x) = \frac{m}{2}\Omega^2x^2$ is taken into account in (27). The form of (27) means that the quasicondensate occupies the domain $|x| \le R_c$ at zero temperature. The length R_c determines the boundary of this domain, $R_c^2 \equiv \frac{2\Lambda}{m\Omega^2}$ (in three-dimensional space, this would correspond to a spherical distribution of the condensate). In the homogeneous case given by the limit as $1/R_c \to 0$, the Thomas–Fermi solution $\rho_{TF}(x)$ is transformed into the density $\rho_{TF}(0) = \Lambda/g$, which coincides with the density of the homogeneous Bose gas [21].

Following the initial splitting (5), we suppose that the thermal fluctuations in the vicinity of the stationarity point (27) are small, and therefore a similar splitting can also be written for the condensate density:

$$\rho_0(x,\tau) = \rho_{TF}(x) + \pi_0(x,\tau). \tag{28}$$

Then the Gross–Pitaevskii equations (25) linearized in the vicinity of the equilibrium solution $\rho_{TF} = \rho_{TF}(x), \varphi = \text{const}$ take the form

$$\begin{aligned} i\partial_\tau \varphi_0 - g\pi_0 + \frac{\hbar^2}{4m\rho_{TF}}\partial_x^2 \pi_0 = 0, \\ i\partial_\tau \pi_0 - \frac{\hbar^2}{m}\partial_x(\rho_{TF}\partial_x\varphi_0) = 0. \end{aligned} \tag{29}$$

Eliminating φ_0 and dropping out the terms proportional to $\hbar^4$, we pass from (29) to the *Stringari thermal equation* [30]

$$\frac{1}{\hbar^2 v^2}\partial_\tau^2 \pi_0 + \partial_x\Big(\Big(1 - \frac{x^2}{R_c^2}\Big)\partial_x \pi_0\Big) = 0, \tag{30}$$

where the parameter v has a meaning of the sound velocity in the center of the trap:

$$v^2 \equiv \frac{\rho_{TF}(0)g}{m} = \frac{\Lambda}{m}. \tag{31}$$

The substitution $\pi_0 = e^{i\omega\tau}u(x)$ transforms (30) into the Legendre equation:

$$-\frac{\omega^2}{\hbar^2 v^2}u(x) + \frac{d}{dx}\Big(\Big(1 - \frac{x^2}{R_c^2}\Big)\frac{d}{dx}u(x)\Big) = 0. \tag{32}$$

Since the Thomas–Fermi solution (27) is different from zero only for $|x| \le R_c$, we shall consider equation (32) for $x \in [-R_c, R_c] \subset \mathbb{R}$ as well. After analytic continuation $\omega \to iE$, equation (32) possesses polynomial solutions, namely, those given by the Legendre polynomials $P_n(x/R_c)$, if and only if

$$\Big(\frac{R_c}{\hbar v}\Big)^2 E^2 \equiv \frac{2}{\hbar^2\Omega^2}E^2 = n(n+1), \quad n \ge 0. \tag{33}$$

In other words, equation (32) leads to the spectrum of the low-lying excitations: $E_n = \hbar\Omega\sqrt{\frac{n(n+1)}{2}}$, $n \ge 0$ [31]. Note that the corresponding equation for the homogeneous Bose gas is obtained after a formal limit as $1/R_c \to 0$ in (32) at finite x. Provided the latter equation is still considered for the segment $[-R_c, R_c] \ni x$ with a periodic boundary condition for x, we arrive at the discrete spectrum of the following form: $E_k = \hbar v k$, where k is the wave number, $k = (\pi/R_c)n$, $n \in \mathbb{Z}$.

§3. The two-point thermal correlation function

We pass to the main problem of the present paper—to the calculation of the two-point thermal correlation function of the spatially nonhomogeneous Bose gas described by the Hamiltonian (1):

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \equiv \langle T_\tau \widehat{\psi}^\dagger(x_1, \tau_1)\widehat{\psi}(x_2, \tau_2)\rangle, \tag{34}$$

where T_τ is a "τ-chronological" ordering with respect of the imaginary time τ, and the angular brackets $\langle , \rangle$ correspond to averaging with respect of the Gibbs distribution [6]. We can express the correlator $\Gamma(x_1, \tau_1; x_2, \tau_2)$ as the ratio of two functional integrals [21]–[24]:

$$\Gamma(x_1, \tau_1; x_2, \tau_2) = \frac{\int e^{S[\psi,\bar\psi]}\bar\psi(x_1, \tau_1)\psi(x_2, \tau_2)\mathcal{D}\psi\mathcal{D}\bar\psi}{\int e^{S[\psi,\bar\psi]}\mathcal{D}\psi\mathcal{D}\bar\psi}, \tag{35}$$

where the action $S[\psi, \bar\psi]$ is defined in (3).

We are interested in the behavior of the correlators at distances considerably smaller than the size of the entire domain occupied by the condensate. The main contribution to the behavior of the correlation functions is due to the low-lying excitations at sufficiently low temperatures [21]. To calculate $\Gamma(x_1, \tau_1; x_2, \tau_2)$, we use the method of successive

functional integration first over the high-energy excitations $\bar{\psi}_e$, ψ_e, and then over the low-energy excitations $\bar{\psi}_o$, ψ_o [21, 24, 25]. We find that in the leading approximation the correlator we are interested in looks like this [21, 25]:

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \frac{\int e^{S_{\rm eff}[\psi_o,\bar{\psi}_o]}\bar{\psi}_o(x_1,\tau_1)\psi_o(x_2,\tau_2)\mathcal{D}\psi_o\mathcal{D}\bar{\psi}_o}{\int e^{S_{\rm eff}[\psi_o,\bar{\psi}_o]}\mathcal{D}\psi_o\mathcal{D}\bar{\psi}_o}, \tag{36}$$

where $S_{\rm eff}[\psi_o,\bar{\psi}_o]$ is the effective action (22). We can rewrite (36) in terms of the density–phase variables (23), and then represent the integrand in the numerator in the form of a single exponential:

$$\begin{aligned}&\Gamma(x_1,\tau_1;x_2,\tau_2)\\ &\simeq \frac{\int \exp\Big(S_{\rm eff}[\rho,\varphi]-i\varphi(x_1,\tau_1)+i\varphi(x_2,\tau_2)+\frac{1}{2}\ln\rho(x_1,\tau_1)+\frac{1}{2}\ln\rho(x_2,\tau_2)\Big)\mathcal{D}\rho\mathcal{D}\varphi}{\int \exp\big(S_{\rm eff}[\rho,\varphi]\big)\mathcal{D}\rho\mathcal{D}\varphi}.\end{aligned} \tag{37}$$

Here $S_{\rm eff}[\rho,\varphi]$ is the effective action (24).

Since the fluctuations of the density are suppressed at sufficiently low temperatures [28], $\ln\rho(x_1,\tau_1)$ and $\ln\rho(x_2,\tau_2)$ in (37) can be replaced by $\ln\rho_{TF}(x_1)$ and $\ln\rho_{TF}(x_2)$, where the density ρ_{TF} is defined by (27). In accordance with the variational principle suggested in [25], we shall estimate the functional integrals in (37) by the stationary phase method, and we shall consider only a leading approximation. For the correlation function $\Gamma(x_1,\tau_1;x_2,\tau_2)$, we obtain the following:

$$\begin{aligned}\Gamma(x_1,\tau_1;x_2,\tau_2)\simeq&\sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)}\\ &\times\exp\big(-S_{\rm eff}[\rho_0,\varphi_0]+S_{\rm eff}[\rho_1,\varphi_1]-i\varphi_1(x_1,\tau_1)+i\varphi_1(x_2,\tau_2)\big),\end{aligned} \tag{38}$$

where the variables ρ_0,φ_0 are determined by the extremum condition $\delta(S_{\rm eff}[\rho,\varphi])=0$ and satisfy the stationary Gross–Pitaevskii equations (25). For $S_{\rm eff}[\rho_0,\varphi_0]$ we use (26).

The fields ρ_1,φ_1 are determined by the extremum condition:

$$\delta\Big(S_{\rm eff}[\rho,\varphi]-i\varphi(x_1,\tau_1)+i\varphi(x_2,\tau_2)\Big)=0. \tag{39}$$

This variational equation leads to another pair of equations of the Gross–Pitaevskii type. One of these equations turns out to be a nonhomogeneous equation with a δ-like source, and another equation is homogeneous. In fact, the homogeneous equation arises due to the requirement that the coefficient of the variation $\delta\rho(x,\tau)$ should vanish,

$$i\partial_\tau\varphi+\frac{\hbar^2}{2m}\Big(\frac{1}{\sqrt{\rho}}\partial_x^2\sqrt{\rho}-(\partial_x\varphi)^2\Big)+\Lambda-V(x)-g\rho=0. \tag{40}$$

In its turn, the nonhomogeneous equation comes from the vanishing of the coefficient of the variation $\delta\varphi(x,\tau)$:

$$-i\partial_\tau\rho+\frac{\hbar^2}{m}\partial_x\left(\rho\partial_x\varphi\right)=i\delta(x-x_1)\delta(\tau-\tau_1)-i\delta(x-x_2)\delta(\tau-\tau_2). \tag{41}$$

Substituting the solutions ρ_1,φ_1, which satisfy (40) and (41), in the effective action (24), we get

$$S_{\rm eff}[\rho_1,\varphi_1]=-\beta F_{nc}(\mu)-1+\frac{g}{2}\int_0^\beta d\tau\int dx\,\rho_1^2. \tag{42}$$

Next, it can consistently be assumed that the solution $\rho_1(x,\tau)$ can be represented as the sum of $\rho_{TF}(x)$ and a weakly fluctuating part provided the boundary R_c is far from the origin: $\rho_1(x,\tau)=\rho_{TF}(x)+\pi_1(x,\tau)$ (for comparison, see (28)). Therefore, the terms $\sqrt{\pi_1}\partial_x^2\sqrt{\pi_1}$ and $\partial_x\pi_1\partial_x\varphi_1$ are small and can be omitted. Taking into account

linearization near the Thomas–Fermi solution, we can finally rewrite equations (40) and (41) as the following pair of equations:

$$i\partial_\tau\varphi_1 - g\pi_1 - \frac{\hbar^2}{2m}\left(\partial_x\varphi_1\right)^2 = 0, \tag{43.1}$$

$$-i\partial_\tau\pi_1 + \frac{\hbar^2}{m}\partial_x\left(\rho_{TF}\partial_x\varphi_1\right) = i\delta(x-x_1)\delta(\tau-\tau_1) - i\delta(x-x_2)\delta(\tau-\tau_2). \tag{43.2}$$

Differentiating (43.1) in τ, substituting the result in (43.2), and dropping out the terms of higher orders in g and $\hbar^2$, we arrive at the equation

$$\frac{1}{g}\partial_\tau^2\varphi_1 + \frac{\hbar^2}{m}\partial_x\left(\rho_{TF}(x)\partial_x\varphi_1\right) = i\delta(x-x_1)\delta(\tau-\tau_1) - i\delta(x-x_2)\delta(\tau-\tau_2). \tag{44}$$

It is convenient to rewrite it as follows:

$$\frac{1}{\hbar^2 v^2}\partial_\tau^2\varphi_1 + \partial_x\left(\widetilde{\rho}_{TF}(x)\partial_x\varphi_1\right) = i\frac{mg}{\hbar^2\Lambda}\Big\{\delta(x-x_1)\delta(\tau-\tau_1) - \delta(x-x_2)\delta(\tau-\tau_2)\Big\}, \tag{45}$$

where v means the sound velocity at the center of the trap (31), and $\widetilde{\rho}_{TF}$ is defined by (27). More specifically, solutions of (44), (45) depend on the coordinates of the δ-like sources on the right-hand side, i.e., on x_1, τ_1, x_2, τ_2: $\varphi_1(x,\tau) \equiv \varphi_1(x,\tau;x_1,\tau_1,x_2,\tau_2)$. Now, with the help of equations (26), (42), and (43), we can calculate the following contribution occurring in the exponent in (38):

$$\begin{aligned}
&- S_{\text{eff}}[\rho_0,\varphi_0] + S_{\text{eff}}[\rho_1,\varphi_1] \\
&\simeq \frac{g}{2}\int_0^\beta d\tau \int dx\,(\rho_1^2 - \rho_0^2) = \frac{g}{2}\int_0^\beta d\tau \int dx\,(\rho_1-\rho_0)(\rho_1+\rho_0) \\
&\simeq g\int_0^\beta d\tau \int dx\,\pi_1\rho_0 = \int_0^\beta d\tau \int dx\,\Big(i\partial_\tau\varphi_1 - \frac{\hbar^2}{2m}(\partial_x\varphi_1)^2\Big)\rho_0 \\
&= -\frac{\hbar^2}{2m}\int_0^\beta d\tau \int dx\,\rho_0\,(\partial_x\varphi_1)^2 = \frac{\hbar^2}{2m}\int_0^\beta d\tau \int dx\,\varphi_1(x)\partial_x\big(\rho_0(x)\partial_x\varphi_1(x)\big) \\
&= \frac{i}{2}\big(\varphi_1(x_1,\tau_1) - \varphi_1(x_2,\tau_2)\big).
\end{aligned} \tag{46}$$

Substituting (46) in (38), we obtain an approximate formula for the correlator:

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)}\exp\Big(-\frac{i}{2}\big(\varphi_1(x_1,\tau_1) - \varphi_1(x_2,\tau_2)\big)\Big). \tag{47}$$

It is natural to represent the solutions of equations (44), (45) in terms of the solution $G(x,\tau;x',\tau')$ of the equation

$$\frac{1}{\hbar^2 v^2}\partial_\tau^2 G(x,\tau;x',\tau') + \partial_x\Big(\Big(1-\frac{x^2}{R_c^2}\Big)\partial_x G(x,\tau;x',\tau')\Big) = \frac{g}{\hbar^2 v^2}\delta(x-x')\delta(\tau-\tau'). \tag{48}$$

Bearing in mind the homogeneous equation (30), we shall call (48) the *nonhomogeneous Stringari equation*. As a result, the representation (47) can be rewritten as follows:

$$\begin{aligned}
&\Gamma(x_1,\tau_1;x_2,\tau_2) \\
&\quad\simeq \sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)}\exp\Big(-\frac{1}{2}\big(G(x_1,\tau_1;x_2,\tau_2) + G(x_2,\tau_2;x_1,\tau_1)\big) \\
&\qquad\qquad + \frac{1}{2}G(x_1,\tau_1;x_1,\tau_1) + \frac{1}{2}G(x_2,\tau_2;x_2,\tau_2)\Big).
\end{aligned} \tag{49}$$

As is clear after [8], the function $G(x_1,\tau_1;x_2,\tau_2)$ has a meaning of the correlation function of the phases:

$$G(x_1,\tau_1;x_2,\tau_2) = -\big\langle\varphi(x_1,\tau_1)\varphi(x_2,\tau_2)\big\rangle, \tag{50}$$

where the angle brackets on the right should be understood as averaging with respect to the weighted measure $\mathcal{D}\rho\mathcal{D}\varphi \exp\big(S_{\text{eff}}[\rho,\varphi]\big)$. Substituting (50) in (49), we obtain the known approximate formula for the correlator (34) [8, 21]:

$$\begin{aligned}&\big\langle T_\tau \widehat{\psi}^\dagger(x_1,\tau_1)\widehat{\psi}(x_2,\tau_2)\big\rangle\\ &\qquad\simeq \sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)}\exp\Big(-\frac{1}{2}\big\langle(\varphi(x_1,\tau_1)-\varphi(x_2,\tau_2))^2\big\rangle\Big).\end{aligned} \tag{51}$$

Note that the terms in the exponent in (49) have different meanings. The terms $G(x_1,\tau_1;x_2,\tau_2)$ and $G(x_2,\tau_2;x_1,\tau_1)$ depend on the differences of the arguments of the two-point correlation function and, therefore, are responsible for the behavior of the correlator at large distances. Each of the terms $G(x_1,\tau_1;x_1,\tau_1)$, $G(x_2,\tau_2;x_2,\tau_2)$ depends only on one set of the coordinates and, thus, they contribute to the amplitudes only. The Green function $G(x,\tau;x',\tau')$ depends, in fact, on the difference of τ and τ' because of invariance under the shifts of τ (see (48)). Therefore, only dependence on the spatial coordinates remains provided the corresponding thermal arguments coincide. Thus, the correlation function can be represented as follows:

$$\Gamma(x_1,\tau_1;x_2,\tau_2)\simeq\sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}\exp\Big(-\frac{1}{2}\left(G(x_1,\tau_1;x_2,\tau_2)+G(x_2,\tau_2;x_1,\tau_1)\right)\Big), \tag{52}$$

where $\widetilde{\rho}(x_1)$, $\widetilde{\rho}(x_2)$ are the renormalized densities [18]. The solution $G(x_1,\tau_1;x_2,\tau_2)$ of equation (48) is defined up to a purely imaginary additive constant, which has a meaning of a global phase. As is seen from (51), this constant does not influence the fluctuations.

§4. The asymptotics of the correlation function

So, the problem concerning the study of the asymptotic behavior of the two-point thermal correlation function $\Gamma(x_1,\tau_1;x_2,\tau_2)$ (34), written as in (37), reduces to the solution of the nonhomogeneous Stringari equation (48). Subsequently, the corresponding solution (or its asymptotics) should be substituted in (52). In the present section, we shall obtain explicit representations for solutions of (48), and we shall consider the corresponding representations for the asymptotics of $\Gamma(x_1,\tau_1;x_2,\tau_2)$. We begin with the limiting case of equation (48), which corresponds to a homogeneous Bose gas.

1. The homogeneous Bose gas. In this subsection we consider the asymptotic behavior of the correlation function of the homogeneous Bose gas. As was mentioned above, the homogeneous case corresponds to $V(x)\equiv 0$, and the related equation is obtained from (48) as $1/R_c\to 0$:

$$\frac{1}{\hbar^2v^2}\partial_\tau^2 G(x,\tau;x',\tau')+\partial_x^2 G(x,\tau;x',\tau')=\frac{g}{\hbar^2v^2}\delta(x-x')\delta(\tau-\tau'). \tag{53}$$

We consider (53) for the domain $[-R_c,R_c]\times[0,\beta]\ni(x,\tau)$, with periodic boundary conditions for each variable (in other words, we consider (53) on the torus $S^1\times S^1\ni(x,\tau)$). The δ-functions on the right in (53) are treated as the corresponding periodic δ-functions. This allows us to represent the solution of this equation as a formal double Fourier series:

$$\begin{aligned}G(x,\tau;x',\tau')&=\Big(\frac{-g}{\hbar^2v^2}\Big)(2\beta R_c)^{-1}\sum_{\omega,k}\frac{e^{i\omega(\tau-\tau')+ik(x-x')}}{\omega^2/(\hbar^2v^2)+k^2}\\ &=\Big(\frac{-g}{2\beta R_c}\Big)\sum_{\omega,k}\frac{e^{i\omega(\tau-\tau')+ik(x-x')}}{\omega^2+E_k^2},\end{aligned} \tag{54}$$

where $\omega = (2\pi/\beta)l$, $l \in \mathbb{Z}$. The notation for the energy $E_k = \hbar v k$, where $k = (\pi/R_c)n$, $n \in \mathbb{Z}$, is used in the representation (54). Also, the representation (54) requires regularization, which consists in neglecting the term given by $\omega = k = 0$.

By using (54), two important asymptotic representations for the Green function can be deduced. In the limit corresponding to zero temperature and to infinite size of the domain occupied by the Bose gas, we can come to the asymptotics of $\Gamma(x_1, \tau_1; x_2, \tau_2)$. If a strong inequality $\beta^{-1} \equiv k_B T \gg \hbar v/R_c$ is valid, we obtain

$$\begin{aligned} &G(x,\tau;x',\tau') \\ &\quad \simeq \frac{g}{2\pi\hbar v} \ln\Big\{2\Big|\sinh\frac{\pi}{\hbar\beta v}(|x-x'| + i\hbar v(\tau-\tau'))\Big|\Big\} - \frac{g}{4\beta R_c}\frac{|x-x'|^2}{\hbar^2 v^2} + \mathcal{C}, \end{aligned} \tag{55}$$

where $|x-x'| \le 2R_c$, $|\tau-\tau'| \le \beta$, and $\mathcal{C}$ is some constant, which is not written explicitly. If the reverse inequality $\beta^{-1} \equiv k_B T \ll \hbar v/R_c$ is fulfilled, then

$$\begin{aligned} &G(x,\tau;x',\tau') \\ &\quad \simeq \frac{g}{2\pi\hbar v} \ln\Big\{2\Big|\sinh\frac{i\pi}{2R_c}(|x-x'| + i\hbar v(\tau-\tau'))\Big|\Big\} - \frac{g}{4\beta R_c}|\tau-\tau'|^2 + \mathcal{C}', \end{aligned} \tag{56}$$

where $|x-x'| \le 2R_c$, $|\tau-\tau'| \le \beta$, and $\mathcal{C}'$ is another constant.

We substitute estimate (55) in the representation (52) and simultaneously take the limit as $\beta\hbar v/R_c \to 0$ (the size is growing faster than the inverse temperature). This yields the following expression for the correlator in question:

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}\Big|\sinh\frac{\pi}{\hbar\beta v}\left(|x_1-x_2| + i\hbar v(\tau_1-\tau_2)\right)\Big|^{-g/(2\pi\hbar v)}. \tag{57}$$

Next, applying (56) and taking the limit as $R_c/(\beta\hbar v) \to 0$ (the inverse temperature grows faster than the size), for $\Gamma(x_1, \tau_1; x_2, \tau_2)$ we obtain:

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}\Big|\sinh\frac{i\pi}{2R_c}\left(|x_1-x_2| + i\hbar v(\tau_1-\tau_2)\right)\Big|^{-g/(2\pi\hbar v)}. \tag{58}$$

From (57) and (58) it follows that in the limit of zero temperature, $(\hbar\beta v)^{-1} \to 0$, and of infinite size, $1/R_c \to 0$, the two-point correlation function behaves like

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \frac{\sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}}{||x_1-x_2| + i\hbar v(\tau_1-\tau_2)|^{1/\theta}}. \tag{59}$$

The latter formula is valid in the limit $\beta\hbar v/R_c \to 0$, as well as in the limit $R_c/(\beta\hbar v) \to 0$. In (59), θ denotes the critical exponent: $\theta \equiv 2\pi\hbar v/g$, and the arguments x_1, x_2, τ_1, and τ_2 are assumed to be sufficiently close to each other. Using the notation $v = \sqrt{\Lambda/m}$ for the sound velocity and $\rho = \Lambda/g$ for the density of the homogeneous ideal Bose gas, we obtain the following universal expression for the critical exponent [11, 12]:

$$\theta = \frac{2\pi\hbar\rho}{mv}. \tag{60}$$

2. The trapped Bose gas. High temperature case: $k_B T \gg \hbar v/R_c$. Consider the case of a nonhomogeneous Bose gas, which is described by the Hamiltonian (1) with the external potential $V(x) \equiv \frac{m}{2}\Omega^2 x^2$. Mainly, in this subsection we follow the content of the paper [20]. We return to the nonhomogeneous Stringari equation (48) and write it again, for convenience:

$$\frac{1}{\hbar^2 v^2}\partial_\tau^2 G(x,\tau;x',\tau') + \partial_x\Big(\Big(1-\frac{x^2}{R_c^2}\Big)\partial_x G(x,\tau;x',\tau')\Big) = \frac{g}{\hbar^2 v^2}\delta(x-x')\delta(\tau-\tau'). \tag{61}$$

We consider (61) for the arguments $(x,\tau) \in [-R_c, R_c] \times [0,\beta]$, with periodic boundary condition only with respect to τ (contrary to equation (53), $\delta(x-x')$ is the usual Dirac

δ-function with support at $x' \in \mathbb{R}$). The Green function satisfying (61) can be written as a formal Fourier series:

$$G(x,\tau;x',\tau') = \frac{1}{\beta}\sum_{\omega} e^{i\omega(\tau-\tau')} G_\omega(x,x'), \tag{62}$$

where $\omega = (2\pi/\beta)l$, $l \in \mathbb{Z}$. The spectral density $G_\omega(x,x')$ in (62) is then governed by the equation

$$-\frac{\omega^2}{\hbar^2 v^2} G_\omega(x,x') + \frac{d}{dx}\Big(\Big(1-\frac{x^2}{R_c^2}\Big)\frac{d}{dx}G_\omega(x,x')\Big) = \frac{g}{\hbar^2 v^2}\delta(x-x'). \tag{63}$$

The solution of (63) can be obtained in terms of Legendre functions of the first and second kind, $P_\nu(x/R_c)$, $Q_\nu(x/R_c)$, which are linearly independent solutions of the homogeneous Legendre equation (32). As a result, we get

$$G_\omega(x,x') = \frac{gR_c}{2\hbar^2 v^2}\epsilon(x-x')\Big\{Q_\nu\Big(\frac{x}{R_c}\Big)P_\nu\Big(\frac{x'}{R_c}\Big) - Q_\nu\Big(\frac{x'}{R_c}\Big)P_\nu\Big(\frac{x}{R_c}\Big)\Big\}, \tag{64}$$

where

$$\nu = -\frac{1}{2} + \sqrt{\frac{1}{4} - \Big(\frac{R_c}{\hbar v}\Big)^2\omega^2},$$

and $\epsilon(x-x')$ is the sign function $\epsilon(x) \equiv \operatorname{sgn}(x)$. The validity of the solution (64) can be verified by direct substitution of (64) in (63), with the use of the following expression for the Wronskian of two linearly independent solutions P_ν and Q_ν [32]:

$$P_\nu(y)\frac{d}{dy}Q_\nu(y) - Q_\nu(y)\frac{d}{dy}P_\nu(y) = (1-y^2)^{-1}$$

(we recall the rule of differentiation of the sign function: $(d/dx)\epsilon(x) = 2\delta(x)$).

If the dependence on τ is neglected, then the equation arising as a result of calculation of the correlation function in accordance with [25] looks much as (61), but the factor β^{-1} arises on its right-hand side instead of $\delta(\tau-\tau')$. In this case, the corresponding solution of the nonhomogeneous equation (i.e., the fundamental solution) $G(x;x')$ takes the form

$$G(x;x') = \frac{1}{\beta}G_0(x,x'), \tag{65.1}$$

where

$$\begin{aligned} G_0(x,x') &= \frac{gR_c}{2\hbar^2 v^2}\epsilon(x-x')\Big\{Q_0\Big(\frac{x}{R_c}\Big) - Q_0\Big(\frac{x'}{R_c}\Big)\Big\} \\ &= \frac{gR_c}{(2\hbar v)^2}\ln\left[\frac{\Big(1+\frac{|x-x'|}{2R_c}\Big)^2 - \frac{(x+x')^2}{4R_c^2}}{\Big(1-\frac{|x-x'|}{2R_c}\Big)^2 - \frac{(x+x')^2}{4R_c^2}}\right]. \end{aligned} \tag{65.2}$$

The explicit form for the simplest Legendre functions $P_0(x) = 1$ and $Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$ is essential for obtaining formulas (65). The fundamental solution (65) becomes equal to zero when its arguments coincide. Substituting (65) in the representation (49), we get the following result for the stationary correlation function [17, 18, 33, 34]:

$$\Gamma(x_1;x_2) \simeq \sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)}\exp\left(-\frac{gR_c}{\beta(2\hbar v)^2}\ln\left[\frac{1+\frac{|x_1-x_2|}{R_c} - \frac{x_1x_2}{R_c^2}}{1-\frac{|x_1-x_2|}{R_c} - \frac{x_1x_2}{R_c^2}}\right]\right). \tag{66}$$

Before studying the behavior of the correlation function, which depends on τ, it should be noted that the solutions of the homogeneous Legendre equation (32) can be added to the Green function (62), because the latter respects the nonhomogeneous equation. In

order to ensure a valid asymptotic behavior of the spectral density in question at large $|\omega|$, we are free to add such a term at $|\omega| \neq 0$; this yields the following expression:

$$\begin{aligned} G_\omega(x,x') = {} & \frac{gR_c}{2\hbar^2 v^2}\epsilon(x-x')\Big\{Q_\nu\Big(\frac{x}{R_c}\Big)P_\nu\Big(\frac{x'}{R_c}\Big) - Q_\nu\Big(\frac{x'}{R_c}\Big)P_\nu\Big(\frac{x}{R_c}\Big)\Big\} \\ & - i\frac{gR_c}{2\hbar^2 v^2}\Big\{\frac{2}{\pi}Q_\nu\Big(\frac{x}{R_c}\Big)Q_\nu\Big(\frac{x'}{R_c}\Big) + \frac{\pi}{2}P_\nu\Big(\frac{x'}{R_c}\Big)P_\nu\Big(\frac{x}{R_c}\Big)\Big\}. \end{aligned} \tag{67.1}$$

The Green function given by (62) and (67.1) can be represented in a form that allows us to study the corresponding asymptotic behavior. In the case of strong inequality $\beta^{-1} = k_B T \gg \hbar v/R_c$, for the nonzero frequencies we obtain approximately $|\omega| \gg \hbar v/(2R_c)$. We use the following asymptotics of the Legendre functions [32, 36]:

$$\begin{aligned} \begin{Bmatrix} P_\nu \\ Q_\nu \end{Bmatrix}(\cos\theta) &= \frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)}\Big(\frac{2}{\pi}\Big)^{\delta/2}\frac{1}{\sin^{1/2}\theta}\begin{Bmatrix} \sin \\ \cos \end{Bmatrix}\Big[\Big(\nu+\frac{1}{2}\Big)\theta + \frac{\pi}{4}\Big] + O(\nu^{-1}) \\ &\approx \Big(\frac{2}{\pi}\Big)^{\delta/2}\frac{1}{\big(\nu\sin\theta\big)^{1/2}}\begin{Bmatrix} \sin \\ \cos \end{Bmatrix}\Big[\Big(\nu+\frac{1}{2}\Big)\theta + \frac{\pi}{4}\Big], \end{aligned} \tag{67.2}$$

where $\varepsilon < \theta < \pi - \varepsilon$, $\varepsilon > 0$, $|\arg\nu| < \pi/2$, and the notation $\cos\theta \equiv x/R_c$ is adopted. In (67.2), only the up or the down expressions must be chosen simultaneously inside the curly brackets $\{\cdots\}$. Here $\delta = 1$ corresponds to P_ν, and $\delta = -1$ corresponds to Q_ν. Substituting (67.2) in (67.1), we determine the behavior of $G_\omega(x,x')$ at large $|\omega|$:

$$G_\omega(x,x') \simeq -\frac{g}{2\hbar v|\omega|}\frac{1}{\sqrt{\sin\theta\sin\theta'}}\exp\Big(-\frac{R_c}{\hbar v}|\omega|\,|\theta-\theta'|\Big). \tag{68}$$

When the coordinates x_1, x_2 are chosen to be far from the boundary of the trap, i.e., x_1, $x_2 \ll R_c$, but at the same time the inequalities $|x_1 - x_2| \ll \frac{x_1+x_2}{2}$ and $|x_1 - x_2| \ll R_c$ are fulfilled, we say that the corresponding limit is *quasihomogeneous*. In this case, up to the second order inclusive, the function $G_0(x,x')$ (65.2) can be approximated as follows:

$$G_0(x,x') \simeq \frac{\Lambda}{2\hbar^2 v^2 \rho_{TF}(S)}|x-x'|. \tag{69}$$

Here S denotes the half-sum of the spatial arguments of the correlator, $S \equiv \frac{x_1+x_2}{2}$, and v is the sound velocity (31). In the quasihomogeneous limit, equation (68) can be rewritten in the form

$$G_\omega(x,x') \simeq -\frac{\Lambda}{2\hbar v\rho_{TF}(S)}\frac{\exp\left(-(\hbar v)^{-1}|\omega||x-x'|\right)}{|\omega|}. \tag{70}$$

Substitution of (69) and (70) in the series (62) leads to the answer for the Green function (50) (i.e., for the correlator of the phases):

$$G(x,\tau;x',\tau') \simeq \frac{\Lambda}{2\pi\hbar v\rho_{TF}(S)}\ln\Big\{2\Big|\sinh\frac{\pi}{\hbar\beta v}(|x-x'| + i\hbar v(\tau-\tau'))\Big|\Big\}. \tag{71}$$

Therefore, at $\beta^{-1} \gg \hbar v/R_c$ the Green function (52) takes the form

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \frac{\sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}}{|\sinh\frac{\pi}{\hbar\beta v}(|x_1-x_2| + i\hbar v(\tau_1-\tau_2))|^{1/\theta(S)}}, \tag{72}$$

where the critical exponent $\theta(S)$ depends now only on the half-sum S of the coordinates:

$$\theta(S) = \frac{2\pi\hbar\rho_{TF}(S)}{mv}. \tag{73}$$

The result (72), which is valid for the spatially nonhomogeneous case, is in agreement with estimate (57) obtained above for the homogeneous Bose gas. Therefore, estimate

(72) is also related to the condition that the size of the domain occupied by the Bose condensate grows faster than the inverse temperature, i.e., $\hbar\beta v/R_c \to 0$.

Formula (72) can be simplified in two important limiting cases. If the condition

$$1 \ll \frac{|x_1 - x_2|}{\hbar\beta v} \ll \frac{R_c}{\hbar\beta v} \tag{74}$$

is fulfilled in the quasihomogeneous case, from (72) we see that the correlation function decays exponentially:

$$\begin{gathered}\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}\exp\Big(-\frac{1}{\xi(S)}\big||x_1-x_2| + i\hbar v(\tau_1-\tau_2)\big|\Big),\\ \xi^{-1}(S) = \frac{\Lambda}{2\beta\hbar^2 v^2 \rho_{TF}(S)}.\end{gathered} \tag{75}$$

Relation (75) involves the correlation length $\xi(S)$, which now depends on the half-sum of the coordinates:

$$\xi(S) \equiv \frac{\hbar\beta v}{\pi}\theta(S) = \frac{2\hbar^2\beta\rho_{TF}(S)}{m}. \tag{76}$$

In the opposite case, where

$$\frac{|x_1-x_2|}{\hbar\beta v}, \frac{|\tau_1-\tau_2|}{\beta} \ll 1 \ll \frac{R_c}{\hbar\beta v}, \tag{77}$$

the asymptotics of $\Gamma(x_1,\tau_1;x_2,\tau_2)$ takes the form

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \frac{\sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}}{||x_1-x_2| + i\hbar v(\tau_1-\tau_2)|^{1/\theta(S)}}. \tag{78}$$

The asymptotics (78) is similar to estimate (59), which characterizes the spatially homogeneous Bose gas. But the critical exponent $\theta(S)$ given by (73) differs from θ in (60), since it depends on spatial coordinates.

3. The trapped Bose gas. Low temperature case: $k_BT \ll \hbar v/R_c$. Let us pass to another case which also admits investigation of the asymptotic behavior of the two-point correlator $\Gamma(x_1,\tau_1;x_2,\tau_2)$. As in the preceding subsection, we begin with the nonhomogeneous Stringari equation (48), which can be written in the form

$$\begin{gathered}\partial_\tau^2 G(x,\tau;x',\tau') + \frac{1}{\alpha^2}\partial_{(x/R_c)}\Big(\Big(1-\frac{x^2}{R_c^2}\Big)\partial_{(x/R_c)}G(x,\tau;x',\tau')\Big)\\ = \frac{g}{R_c}\delta\Big(\frac{x-x'}{R_c}\Big)\delta(\tau-\tau'),\end{gathered} \tag{79}$$

where the notation $\alpha \equiv R_c/(\hbar v)$ is introduced. The asymptotic behavior can be investigated in two cases, and these cases can be characterized in terms of α: $\beta/\alpha \ll 1$ (the preceding subsection) and $\beta/\alpha \gg 1$ (see below). The functions

$$\sqrt{n+\frac{1}{2}}P_n\Big(\frac{x}{R_c}\Big), \quad n \geq 0,$$

where the $P_n(x/R_c)$ are the Legendre polynomials, constitute a complete orthonormal system in the space $L_2[-R_c,R_c]$. This fact allows us to obtain a representation of the Green function $G(x,\tau;x',\tau')$ in the form of a generalized double Fourier series:

$$G(x,\tau;x',\tau') = \Big(\frac{-g}{\beta R_c}\Big)\sum_\omega\sum_{n=0}^\infty \frac{n+1/2}{\omega^2+E_n^2}P_n\Big(\frac{x}{R_c}\Big)P_n\Big(\frac{x'}{R_c}\Big)e^{i\omega(\tau-\tau')}. \tag{80}$$

In (80), as in (54), (62), the notation $\sum_\omega$ denotes the sum over the Bose frequencies $\omega = (2\pi/\beta)l$, $l \in \mathbb{Z}$, and the following notation for the energy levels (33) is adopted:

$$E_n = \hbar\Omega\sqrt{\frac{n(n+1)}{2}} = \frac{\sqrt{n(n+1)}}{\alpha}. \tag{81}$$

We note that passage from (54) to (80) consists in an appropriate substitution of the expressions for the wave functions and the dispersion.

After summation over the frequencies, and after regularization that consists in neglecting the term corresponding to the zero values of ω and n, formula (80) for $G(x,\tau;x',\tau')$ takes the form

$$\begin{aligned} &G(x,\tau;x',\tau') \\ &\quad = \Big(\frac{-g}{\beta R_c}\Big)\Big[\Big(\frac{\beta}{2\pi}\Big)^2 \sum_{l=1}^{\infty} \frac{\cos\big(\frac{2\pi\Delta\tau}{\beta}l\big)}{l^2} + \frac{\beta}{2}\sum_{n=1}^{\infty}\frac{n+1/2}{E_n}P_n\Big(\frac{x}{R_c}\Big)P_n\Big(\frac{x'}{R_c}\Big) \\ &\qquad\qquad \times\Big(\coth\Big(\frac{\beta}{2}E_n\Big)\cosh(E_n\Delta\tau) - \sinh(E_n\Delta\tau)\Big)\Big], \end{aligned} \tag{82}$$

where $\Delta\tau \equiv |\tau - \tau'|$. The representation (82) admits investigation in two limiting cases: $\beta/\alpha \ll 1$ (this case agrees with estimate (75) in the preceding subsection) and $\beta/\alpha \gg 1$.

Indeed, putting $\tau = \tau'$ in the case where $\beta/\alpha \ll 1$, we obtain the following relation (see [35]):

$$\begin{aligned} G(x,\tau;x',\tau) &= -\frac{g\beta}{24R_c} - \frac{g}{\beta R_c}\sum_{n=1}^{\infty}\frac{n+1/2}{E_n^2}P_n\Big(\frac{x}{R_c}\Big)P_n\Big(\frac{x'}{R_c}\Big) \\ &= \frac{gR_c}{2\beta\hbar^2v^2}\ln\Big(1 + \frac{|x-x'|}{R_c} - \frac{xx'}{R_c^2}\Big) + \frac{gR_c}{\beta\hbar^2v^2}\Big(\frac{1}{2} - \ln 2 - \frac{\beta^2}{24\alpha^2}\Big). \end{aligned} \tag{83}$$

Observe that the right-hand side of (83) satisfies the nonhomogeneous equation

$$\frac{1}{\alpha^2}\partial_{(x/R_c)}\Big(\Big(1 - \frac{x^2}{R_c^2}\Big)\partial_{(x/R_c)}G(x,\tau;x',\tau)\Big) = \frac{g}{\beta R_c}\Big(\delta\Big(\frac{x-x'}{R_c}\Big) - \frac{1}{2}\Big). \tag{84.1}$$

Provided the spatial arguments in (83) are equated, the following identity arises:

$$\begin{aligned} &\frac{G(x,\tau;x,\tau) + G(x',\tau;x',\tau)}{2} \\ &\quad = \frac{g\alpha^2}{4\beta R_c}\ln\Big[\Big(1 - \frac{x^2}{R_c^2}\Big)\Big(1 - \frac{(x')^2}{R_c^2}\Big)\Big] + \frac{gR_c}{\beta\hbar^2v^2}\Big(\frac{1}{2} - \ln 2 - \frac{\beta^2}{24\alpha^2}\Big). \end{aligned} \tag{84.2}$$

Direct substitution demonstrates that (84.2) satisfies the equation in which the left-hand side is the same as in (84.1), while only the constant term $\frac{-g}{2\beta R_c}$ is present on the right-hand side. Therefore, subtracting the right-hand side of (84.2) from that of (83), we arrive precisely at the fundamental solution $G(x;x')$ (see (65)). The Green function $G(x;x')$ becomes equal to zero for $x = x'$, and it satisfies an equation similar to (84.1) but with $\frac{g}{\beta}\delta(x-x')$ on the right.

With the help of the series (83), the exponent in (49) can be written as

$$\Big(\frac{-g}{2\beta R_c}\Big)\sum_{n=1}^{\infty}\frac{n+1/2}{E_n^2}\Big(P_n\Big(\frac{x}{R_c}\Big) - P_n\Big(\frac{x'}{R_c}\Big)\Big)^2. \tag{85}$$

As is seen from (83) and (84.2), expression (85) is none other than the fundamental solution $G(x;x')$ (see (65)) taken with the opposite sign. Thus, if we substitute the representation (83) in (49), we obtain a formula for the "equal-time" correlator $\Gamma(x_1,\tau;x_2,\tau)$,

which has the same form as the right-hand side of (66). Taking into account the quasi-homogeneity conditions discussed in the preceding subsection, and recalling the corresponding estimate (69), for $G(x;x')$ we get

$$G(x,x') \simeq \frac{\Lambda}{2\beta\hbar^2 v^2 \rho_{TF}(S)}|x-x'|.$$

In its turn, the latter formula leads to the estimate

$$\Gamma(x_1,\tau;x_2,\tau) \simeq \sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)}\exp\Big(-\frac{1}{\xi(S)}|x_1-x_2|\Big), \tag{86}$$

where $\xi(S)$ is the correlation length (76). As a result, equations (75) and (86) demonstrate an agreement between the estimates based on two different representations for the Green function $G(x,\tau;x',\tau')$: the first is in the form of the trigonometric Fourier series (62) (where either (64) or (67) is used to express $G_\omega(x,x')$), and the second is in the form of the series (82) over the principle quantum numbers (this series is obtained from the generalized double Fourier series (80)).

Now, we turn to the case where $\beta/\alpha \gg 1$, which means that $\beta E_n \gg 1$, $n=1,2,\dots$. In other words, we assume that $k_BT \ll E_n$, so that $k_BT \ll \hbar\Omega$. Then, by (82),

$$\begin{aligned} &G(x,\tau;x',\tau') \\ &\quad= \frac{-g\beta}{4R_c}\Big[\Big(\frac{1}{2}-\frac{\Delta\tau}{\beta}\Big)^2-\frac{1}{12}\Big] \\ &\quad- \frac{g}{2\hbar v}\sum_{n=1}^{\infty}\frac{n+1/2}{\sqrt{n(n+1)}}P_n\Big(\frac{x}{R_c}\Big)P_n\Big(\frac{x'}{R_c}\Big)\exp\Big(-\sqrt{n(n+1)}\frac{\Delta\tau}{\alpha}\Big). \end{aligned} \tag{87}$$

Note that the difference between two neighboring energy levels (81) can be estimated. After appropriate series expansions, which are possible for $n>1$, we obtain

$$\begin{aligned} E_{n+1}-E_n &\approx \frac{1}{\alpha}\Big[1+\frac{1}{8}\frac{1}{(n+1)^2}+\frac{7}{128}\frac{1}{(n+1)^4}\cdots\Big] \\ &\approx \frac{1}{\alpha}\Big[1+\frac{1}{8n^2}-\frac{1}{4n^3}+\frac{55}{128n^4}\cdots\Big]. \end{aligned} \tag{88}$$

The right-hand side of (88) demonstrates that, approximately, the levels (81) can be treated as equidistant, provided the inverse powers of n are neglected in (88), say, for the values $n>n_0=10$. In its turn, the following series expansion is valid:

$$\frac{n+1/2}{\sqrt{n(n+1)}} = 1+\frac{1}{8n^2}-\frac{1}{8n^3}+\frac{15}{128n^4}-\cdots. \tag{89}$$

It is remarkable that the terms equivalent to n^{-1} are absent both in (88) and in (89). We recall that the leading asymptotic estimates obtainable with so-called *logarithmic accuracy* are important for physical applications. Below it will be seen that precisely such an estimation with leading logarithmic accuracy is available for the problem at hand. In this case, the inverse powers of n can be omitted with the same accuracy in (88) and in (89) for $n>n_0=10$. For such an approximation, the energy levels (81) turn out to be equidistant, while the corresponding ratio $\frac{n+1/2}{\sqrt{n(n+1)}}$ in (87) becomes equal to unity. The convergence of the series (87) is not affected in this situation, and the term that we dropped out can be estimated.

Consider the exponent in (87):

$$\sqrt{n(n+1)}\frac{\Delta\tau}{\alpha} = \frac{\Delta\tau}{\alpha}\Big(n+\frac{1}{2}\Big)-\frac{\Delta\tau}{\alpha}\Big(\frac{1}{8n}-\frac{1}{16n^2}+\frac{5}{128n^3}-\cdots\Big). \tag{90}$$

The second term in (90) can also be neglected for $n > n_0$, provided $\Delta\tau/\alpha \ll 1$. This means that, approximately, the series occurring in (87) can be written as follows:

$$\begin{aligned}\sum_{n=1}^{n_0} \frac{n+1/2}{\sqrt{n(n+1)}} P_n\Big(\frac{x}{R_c}\Big) P_n\Big(\frac{x'}{R_c}\Big) \exp\Big(-\sqrt{n(n+1)}\frac{\Delta\tau}{\alpha}\Big) \\ + e^{-\Delta\tau/(2\alpha)} \sum_{n=n_0+1}^{\infty} P_n\Big(\frac{x}{R_c}\Big) P_n\Big(\frac{x'}{R_c}\Big) \Big(e^{-\Delta\tau/\alpha}\Big)^n,\end{aligned} \tag{91}$$

where n_0 is a fixed number (its specific value should not go to infinity). The correction, say $\widetilde{c}$, omitted in the representation (91) can be estimated:

$$\begin{aligned}|\widetilde{c}| \le \Big(\Big(1+\frac{1}{n_0^2}\Big)\Big(1+\frac{\Delta\tau}{\alpha}\frac{\mathrm{const}}{n_0}\Big)-1\Big) \\ \times e^{-\Delta\tau/(2\alpha)} \sum_{n=n_0+1}^{\infty} \Big|P_n\Big(\frac{x}{R_c}\Big) P_n\Big(\frac{x'}{R_c}\Big)\Big| \Big(e^{-\Delta\tau/\alpha}\Big)^n.\end{aligned}$$

It can also be shown that the absolute value of the first term in (91) does not exceed $2n_0$. Using (91), we put (87) in the form

$$\begin{aligned}&G(x,\tau;x',\tau') \\ &\quad\approx \frac{-g\beta}{4R_c}\Big[\Big(\frac{1}{2}-\frac{\Delta\tau}{\beta}\Big)^2-\frac{1}{12}\Big] \\ &\qquad - \frac{g}{2\hbar v}\sum_{n=1}^{n_0}\Big[\frac{n+1/2}{\sqrt{n(n+1)}}\exp\Big(-\sqrt{n(n+1)}\frac{\Delta\tau}{\alpha}\Big)-\exp\Big(-\Big(n+\frac{1}{2}\Big)\frac{\Delta\tau}{\alpha}\Big)\Big] \\ &\qquad\qquad \times P_n\Big(\frac{x}{R_c}\Big)P_n\Big(\frac{x'}{R_c}\Big) \\ &\qquad - \frac{g}{2\hbar v}e^{-\Delta\tau/(2\alpha)}\Big[\sum_{n=0}^{\infty} t^n P_n\Big(\frac{x}{R_c}\Big)P_n\Big(\frac{x'}{R_c}\Big)-1\Big],\end{aligned} \tag{92}$$

where $t \equiv \exp(-\Delta\tau/\alpha)$. Relation (80) shows that (92) is valid in the case where τ and τ' are close either to zero or to β, as well as in the case where only either τ or τ' is close to β. Moreover, we assume that $\tau \ne \tau'$ in order to keep the convergence in (92).

Using the known series [35]

$$\sum_{n=0}^{\infty} t^n P_n(\cos\vartheta_1)P_n(\cos\vartheta_2) = \frac{4}{\pi}\frac{1}{u_+u_-}\mathbb{K}(\kappa), \quad 0<t<1, \tag{93}$$

$$u_+ \equiv \sqrt{1-2t\cos(\vartheta_1+\vartheta_2)+t^2},\ u_- \equiv \sqrt{1-2t\cos(\vartheta_1-\vartheta_2)+t^2},\ \kappa = \frac{u_+-u_-}{u_++u_-}, \tag{94}$$

where $\mathbb{K}$ is a complete elliptic integral of the first kind, we can estimate the approximate representation (92) for the Green function. Indeed, put $\cos\vartheta_1 \equiv x/R_c \ll 1$ and $\cos\vartheta_2 \equiv x'/R_c \ll 1$. Then the following estimates for u_+ and u_- can be obtained:

$$u_+ \approx 1+t-\frac{t}{1+t}\frac{(x+x')^2}{2R_c^2} \approx 2-\frac{\Delta\tau}{\alpha}-\frac{(x+x')^2}{4R_c^2}, \tag{95.1}$$

$$u_- \approx \Big((1-t)^2+t\frac{(x-x')^2}{R_c^2}\Big)^{1/2} \approx \frac{||x-x'|+i\hbar v\Delta\tau|}{R_c} \equiv u_*, \tag{95.2}$$

where it is assumed that $\Delta\tau/\alpha \ll 1$ and

$$\frac{t}{1+t} \approx \frac{1}{2}\Big(1-\frac{\Delta\tau}{2\alpha}\Big).$$

Provided the terms of the second order of smallness are neglected, for κ (see (94)) we get the estimate $\kappa \simeq 1 - u_*$, where u_* denotes the corresponding approximate value of u_- given by the right-hand side of (95.2). When $\kappa \sim 1$, several leading terms of the asymptotic expansion of the function $\mathbb{K}(\kappa)$ [35] can be written out:

$$\mathbb{K}(\kappa) \approx \mathbb{K}(1-u_*) \approx \frac{u_*}{4}\Big(\Big(\frac{2}{u_*}+1\Big)\ln\frac{8}{u_*}-1\Big), \quad u_* \ll 1. \tag{96}$$

Estimate (96) contains neither terms equivalent to $(u_*)^2$ nor terms with higher powers of u_*. This is due to the fact that under the condition of quasihomogeneity imposed above, the value of u_* given by (95.2) is treated as a quantity of the first order of smallness. Therefore, the corresponding value of the argument κ, i.e., $\kappa \simeq 1 - u_*$, is written by neglecting the contributions of the second order.

We note that the first two terms in (92) are not small, because the number n_0 can be large. Moreover, the inequality $\beta/\alpha \gg 1$ implies that $g\beta/R_c \gg g/(\hbar v)$. However, if the smallness of the quantities x/R_c, x'/R_c, and $\Delta\tau/\alpha$ is taken into account, it can be seen that the first two terms in (92) are less important than the third term, which can be logarithmically large for sufficiently small u_*. That is why we neglect the first two terms and write the leading contribution to the Green function (92) with logarithmic accuracy:

$$\begin{aligned} &G(x,\tau;x',\tau') \\ &\quad\simeq \Big(\frac{-g}{4\pi\hbar v}\Big)\Big(1+\frac{2R_c}{||x-x'|+i\hbar v(\tau-\tau')|}\Big)\ln\frac{8R_c}{||x-x'|+i\hbar v(\tau-\tau')|} \\ &\quad\approx \Big(\frac{-g}{2\pi\hbar v}\Big)\frac{1}{u_*}\ln\frac{8}{u_*}. \end{aligned} \tag{97}$$

Here it is assumed that u_* is given by (95.2) and that the following conditions of validity of logarithmic estimation are respected:

$$n_0 \lesssim \frac{1}{u_*}, \quad 1 \ll \frac{1}{u_*} \ll \frac{1}{u_*}\ln\frac{1}{u_*}. \tag{98}$$

Substitution of (97) in (52) provides the following estimate for the two-point correlator $\Gamma(x_1,\tau_1;x_2,\tau_2)$:

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \frac{\sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}}{||x_1-x_2|+i\hbar v(\tau_1-\tau_2)|^{1/\bar{\theta}}}. \tag{99}$$

In (99), the notation $\bar{\theta}$ for the critical exponent is introduced,

$$\bar{\theta} \equiv \frac{2\pi\hbar v}{g}u_*. \tag{100}$$

The critical exponent $\bar{\theta}$ depends on u_* (see (95.2)), so that it is a function of differences of coordinates. Apart from inequalities (98), the following estimates are true, which characterize relations (99), (100):

$$\begin{aligned} &\frac{1}{\hbar v} \ll \frac{R_c}{\hbar v||x-x'|+i\hbar v(\tau-\tau')|} \ll \frac{\beta}{||x-x'|+i\hbar v(\tau-\tau')|}, \\ &\frac{1}{\hbar v} \ll \frac{\beta}{R_c} \ll \frac{\beta}{||x-x'|+i\hbar v(\tau-\tau')|}. \end{aligned}$$

Estimate (99) (together with the critical exponent $\bar{\theta}$ as in (100)) constitutes the main result of the present subsection, which is devoted to the case of $k_BT \ll \hbar v/R_c$. Comparison with the case of spatially homogeneous Bose gas shows that the derivation of estimate (99) is similar to passage from relations (56), (58) to the final asymptotics (59). The validity of the corresponding limit $R_c/(\hbar\beta v) \to 0$ means that the result (99) is also

related to the fact that the condensate boundary R_c increases slower than the inverse temperature.

Observe that, by (98), a specific value of n_0 can be related to the size of the trap R_c for a fixed range of deviations between the spatial coordinates x and x', and the increase of R_c leads to that of an upper bound for the admissible values of n_0. However, again by (98), estimation (97) obtained for $G(x,\tau;x',\tau')$ does not depend explicitly on a specific choice of the number n_0. On the other hand, for sufficiently large values of n, the following asymptotics for the Legendre polynomials P_n is valid [35]:

$$P_n(\cos\vartheta) = \sqrt{\frac{2}{\pi n \sin\vartheta}}\cos\Big[\Big(n+\frac{1}{2}\Big)\vartheta - \frac{\pi}{4}\Big] + O(n^{-3/2}), \quad 0<\vartheta<\pi. \tag{101}$$

Assume that the number n_0 is sufficiently large to allow the substitution of (101) in the series

$$\sum_{n=1}^{\infty}\Big(e^{-\Delta\tau/\alpha}\Big)^n P_n\Big(\frac{x}{R_c}\Big)P_n\Big(\frac{x'}{R_c}\Big),$$

which is part of the representation (92) (see also (91)), in order to obtain an estimate of the latter series with logarithmic accuracy. As a result, for $G(x,\tau;x',\tau')$ (see (92)) we get

$$G(x,\tau;x',\tau') \simeq \Big(\frac{-g}{2\pi\hbar v}\Big)\Big(1+\frac{S^2}{2R_c^2}\Big)\ln\frac{R_c}{||x-x'|+i\hbar v(\tau-\tau')|}, \tag{102}$$

where $S=\frac{x_1+x_2}{2}$. In the limit $1/R_c\to 0$, the total coefficient of the logarithm in (102) takes the value $-1/\theta$, where the critical exponent θ is defined as in (59), (60).

The use of the asymptotics (101) means that the eigenfunctions are approximated by a base of plane waves that correspond to an almost homogeneous Bose gas. Therefore, replacement of the asymptotics (97) by the asymptotics (102) can be explained as a transition, as R_c increases, to smaller scales characterized by small ratios x/R_c, x'/R_c, and $|x-x'|/R_c$ (the quasihomogeneity condition). Then, the result (99) together with the critical exponent (100) demonstrate an effect of finiteness of the size of the domain occupied by the spatially nonhomogeneous Bose gas. This follows from the employment of the Legendre polynomials as a base of one-particle states.

§5. Conclusion

The model considered in this paper describes a spatially nonhomogeneous one-dimensional Bose gas with a weak repulsive coupling placed in an external harmonic potential. We deal with an application of the functional integration approach to the calculation of the two-point thermal correlation function of the nonhomogeneous Bose gas. We study the temperatures that are sufficiently low for the quasicondensate to be formatted in the Bose system in question (see §2). The functional integral representation for the two-point correlation function in question is estimated by the stationary phase method in the way proposed in [25]. The main results are obtained for the case where the size of the domain occupied by the quasicondensate increases, while the temperature of the system goes to zero. It is demonstrated that the behavior of the correlation function near zero temperature has a power-like dependence, and it is governed by the critical exponent. In contrast with the case of spatial homogeneity of the Bose gas, the presence of the external potential is manifested in the nonhomogeneity of the critical exponent. The latter turns out to be a function of the same spatial and thermal arguments as the correlator itself. The dependence of the critical exponent on these spatial arguments is in correspondence with the limiting behavior of the ratio of the size of the trap to the inverse temperature provided both of them are increasing.

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