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A RECURSION FORMULA FOR THE CORRELATION FUNCTIONS OF AN INHOMOGENEOUS XXX MODEL

H. BOOS, M. JIMBO, T. MIWA, F. SMIRNOV, AND Y. TAKEYAMA

Dedicated to Ludwig Faddeev on the occasion of his seventieth birthday

Abstract. A new recursion formula is presented for the correlation functions of the integrable spin 1/2 XXX chain with inhomogeneity. It links the correlators involving n consecutive lattice sites to those with $n-1$ and $n-2$ sites. In a series of papers by V. Korepin and two of the present authors, it was discovered that the correlators have a certain specific structure as functions of the inhomogeneity parameters. The formula mentioned above makes it possible to prove this structure directly, as well as to obtain an exact description of the rational functions that were left undetermined in the earlier work.

§1. Introduction

Consider the XXX antiferromagnet given by the Hamiltonian

$$H_{XXX} = \frac{1}{2}\sum_j \left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z\right).$$

This model was solved in the famous paper by Bethe [3] already in 1931, by using what is now called the coordinate Bethe Ansatz. Nevertheless, it took some time before the physical content of this model in the thermodynamic limit was clarified completely. For the first time, the spectrum of excitations was described correctly in the paper [10] by Faddeev and Takhtajan; it was shown that the spectrum contains magnons of spin 1/2. These authors used the algebraic Bethe Ansatz formulated by Faddeev, Sklyanin, and Takhtajan (see [9]) on the basis of R-matrices and the Yang–Baxter equation. The origin of these new techniques goes back to the work of Baxter [1].

Restricting ourselves to our example, we recall the role of R-matrices in solvable models. The XXX model is related to the rational R-matrix that acts on $\mathbb{C}^2\otimes\mathbb{C}^2$ (see (2.7) for the explicit formula). We employ the usual notation $R_{1,2}(\lambda)$, where $1,2$ label the corresponding spaces and λ is the spectral parameter. The relationship between the R-matrix and the XXX Hamiltonian is as follows. Consider the transfer matrix

$$t_N(\lambda) = \mathrm{tr}\left(R_{\alpha,-N}(\lambda)R_{\alpha,-N+1}(\lambda)\cdots R_{\alpha,N}(\lambda)\right),$$

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where the trace is taken with respect to the auxiliary space labeled by α. The transfer matrices commute for different values of spectral parameters, giving rise to a commuting family of operators. If we expand $t_N(\lambda)$ in powers of λ,

$$\log t_N(0)^{-1} t_N(\lambda) = \sum_{k=1}^{\infty} I_k \lambda^k,$$

then we see that I_1 coincides with the XXX Hamiltonian for the periodic chain of length $2N+1$. All other I_k are integrals of motion commuting with I_1.

We have repeated these well-known facts in order to make clear the following remark due to Baxter. Consider the inhomogeneous chain whose integrals of motion are generated by the transfer matrix

$$\operatorname{tr}\left(R_{a,-N}(\lambda)\cdots R_{a,0}(\lambda)R_{a,1}(\lambda-\lambda_1)\cdots R_{a,n}(\lambda-\lambda_n)R_{a,n+1}(\lambda)\cdots R_{a,N}(\lambda)\right),$$

where $\lambda_1,\dots,\lambda_n$ are arbitrary parameters. The model is still exactly solvable, although the interaction is not local anymore. This generalization will be very important for us.

After the calculation of the spectrum, the next important issue is that of the correlation functions. As in [12], we consider general correlators of the form

$$\langle \mathrm{vac}\,|(E_{\epsilon_1,\bar\epsilon_1})_1\cdots(E_{\epsilon_n,\bar\epsilon_n})_n|\,\mathrm{vac}\rangle.$$

These are the averages over the ground state $|\,\mathrm{vac}\rangle$ of products of elementary operators $(E_{\epsilon_j,\bar\epsilon_j})_j$ ($\epsilon_j,\bar\epsilon_j=\pm1$) at site j. For $n=2$, they can be calculated easily from the vacuum energy. The first nontrivial result is due to Takahashi [22], who evaluated the correlators for $n=3$ in terms of $\zeta(3)$, where ζ is the Riemann ζ-function.

Results for general n were brought forth fifteen years later by Jimbo, Miki, Miwa, and Nakayashiki [13], in the framework of the representation theory of quantum affine algebras. Their results and further developments were presented in the book [12]. It was shown that, in the context of a more general XXZ model, the correlators in both homogeneous and inhomogeneous cases are given in terms of multiple integrals in which the number of integrations is equal to the distance n on the lattice. Actually, the inhomogeneous case plays a very important role. In this case, the correlators depend on the parameters λ_j since the vacuum does. As functions of λ_j, the correlators satisfy the quantum Knizhnik–Zamolodchikov equation (qKZ) [11] with level -4. Here an unexpected similarity became apparent between the correlators in lattice models and form factors in integrable relativistic models calculated by Smirnov [20]. The latter also satisfy the qKZ equation but with level 0. The symmetry algebra of the XXX model is the Yangian, and the qKZ equation in this situation was discussed in [21].

One remark is in order here. The algebraic methods of [12] work nicely in the presence of a gap in the spectrum. In the gapless case (such as the XXX model under consideration), formulas for the correlators were obtained by "analytic continuation". However, later the same formulas were derived rigorously by Kitanine, Maillet, and Terras [16] on the basis of the algebraic Bethe Ansatz.

Actually, it is not very simple to obtain the result of Takahashi from the formulas in [13]. For $n=3$, we have three-fold integrals, and the result must be surprisingly simple. This observation was the starting point of the paper [4] by Boos and Korepin. They were able to calculate further the cases of $n=4$ and $n=5$ (for some correlators). In all cases the multiple integrals disappeared mysteriously, and the results were expressed in terms of products of ζ-functions at odd positive integers with rational coefficients. This mystery had to be explained.

The explanation was presented in the series of papers [5, 6, 7] by Boos, Korepin, and Smirnov. Again, the idea was to use the inhomogeneous model and the qKZ equation.

In the paper [5], a conjecture was put forward which states that all the correlators are expressed schematically as follows:

$$\langle \mathrm{vac}\,|(E_{\epsilon_1,\bar\epsilon_1})_1\cdots(E_{\epsilon_n,\bar\epsilon_n})_n|\,\mathrm{vac}\rangle=\sum\prod\omega(\lambda_i-\lambda_j)f(\lambda_1,\dots,\lambda_n). \tag{1.1}$$

Here $\omega(\lambda)$ is a certain transcendental function (a linear combination of logarithmic derivatives of the Γ-function), the $f(\lambda_1,\dots,\lambda_n)$ are rational functions, and the sum is taken over partitions (for the precise formula, see (2.4) and Theorems 3.2 and 3.3). In the homogeneous limit, the odd integer values of the ζ-function arise as the coefficients in the Taylor series of $\omega(\lambda)$.

It was explained that the real reason for the formula (1.1) to be true is the existence of a formula for solutions of the qKZ equation with level -4 different than those given in [12]. This follows from a duality between the level 0 and level -4 cases. In the case of level 0, all solutions in the $\mathfrak{sl}_2$-invariant subspace are known. To get solutions for the level -4 case, the matrix of solutions for the level 0 case must be inverted. This can be done efficiently due to the relationship with the symplectic group. Detailed explanations of this point will bring us too far from the subject of the present paper.

Let us consider (1.1) as an *Ansatz*. The question is to calculate the rational functions $f(\lambda_1,\dots,\lambda_n)$. It turns out that this problem is much more complicated than it appears at first glance. Despite the efforts made in the papers [5, 6, 7], only partial answers were gained for small n. All the correlation functions for the homogeneous case of the XXX model until $n=4$ were calculated by Sakai, Shiroishi, Nishiyama, and Takahashi [19], while the result for these functions in the XXZ case was obtained in the papers [14, 23, 15]. The entire set of correlation functions for $n=5$ both in homogeneous and inhomogeneous cases has been found recently; see [8].

The main results of the present paper are

(i) the calculation of the functions $f(\lambda_1,\dots,\lambda_n)$, and
(ii) the proof of the *Ansatz* (1.1).

Surprisingly enough, the result is expressed in terms of transfer matrices over an auxiliary space of “fractional dimension”. This brings us very close to the theory of the Baxter Q-operators developed by Bazhanov, Lukyanov, and Zamolodchikov [2]. We hope to discuss this issue in future publications.

The text is organized as follows.

We begin in Subsection 2.1 with formulation of the problem of computing the correlation functions. We review the *Ansatz* of [5, 6, 7]. In Subsection 2.2 we prepare basic materials from the algebraic Bethe Ansatz. In Subsection 2.3 we summarize the properties of the solution of the qKZ equation relevant to the correlators.

In Subsection 3.1 we introduce the trace over a space of “fractional dimension”. Using this notion, in Subsection 3.2 we define the rational functions $X^{[i,j]}$ that enter the recursion formula as coefficients. Recursion is stated in Subsection 3.3. Simple examples are given in Subsection 3.4 for the correlation functions obtained from recursion.

In Subsection 4.1, we discuss several properties of $X^{[i,j]}$. Using these properties and recursion, we prove the *Ansatz* in Subsection 4.2. The proof of recursion is started in Subsection 4.3 with calculating the residues of the correlation functions. The proof is completed in Subsection 4.4 by giving an asymptotic estimate.

In Appendix A we discuss the relationship among various gauges of the Hamiltonian used here and in the literature. In Appendix B we give a proof of the analytic and asymptotic properties of the correlators used in the text.

§2. Correlation functions and reduced qKZ equation

2.1. Formulation of the problem. In this subsection, we formulate the problem we are going to address.

Consider the XXX model with the Hamiltonian

$$H_{XXX} = \frac{1}{2}\sum_j \left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z\right). \tag{2.1}$$

Its integrability is due to the Yang–Baxter relation. The problem is to calculate the correlation functions

$$\langle \mathrm{vac}\,|(E_{\epsilon_1,\bar\epsilon_1})_1\cdots(E_{\epsilon_n,\bar\epsilon_n})_n|\,\mathrm{vac}\rangle.$$

These are the averages over the ground state $|\,\mathrm{vac}\rangle$ of products of elementary operators

$$E_{\epsilon_j,\bar\epsilon_j} = \left(\delta_{\epsilon_j\alpha}\delta_{\bar\epsilon_j\beta}\right)_{\alpha,\beta=\pm}, \quad \epsilon_j,\bar\epsilon_j = \pm,$$

acting on the site j. More precisely, we consider the correlation functions of an inhomogeneous model, in which each site j carries an independent spectral parameter λ_j. An exact integral formula for these quantities was found in [12, 16].

It is convenient to pass to the quantity

$$h_n(\lambda_1,\dots,\lambda_n)^{-\epsilon_1,\dots,-\epsilon_n,\bar\epsilon_n,\dots,\bar\epsilon_1} = \prod_{j=1}^n(-\bar\epsilon_j)\langle \mathrm{vac}\,|(E_{\epsilon_1,\bar\epsilon_1})_1\cdots(E_{\epsilon_n,\bar\epsilon_n})_n|\,\mathrm{vac}\rangle. \tag{2.2}$$

In Appendix A, we relate this formula to a similar formula for the XXZ model given in [12].

Denoting by v_+, v_- the standard basis of

$$V = \mathbb{C}^2 = \mathbb{C}v_+ \oplus \mathbb{C}v_-,$$

we regard

$$h_n(\lambda_1,\dots,\lambda_n) = \sum_{\epsilon_1,\dots,\epsilon_n,\bar\epsilon_1,\dots,\bar\epsilon_n} h_n(\lambda_1,\dots,\lambda_n)^{\epsilon_1,\dots,\epsilon_n,\bar\epsilon_n,\dots,\bar\epsilon_1} v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_n}\otimes v_{\bar\epsilon_n}\otimes\cdots\otimes v_{\bar\epsilon_1}$$

as an element of $V^{\otimes 2n}$. This function is obtained from a solution

$$g_{2n}(\lambda_1,\dots,\lambda_{2n}) \in V^{\otimes 2n}$$

of the qKZ equation with level -4 by specializing the arguments as follows:

$$h_n(\lambda_1,\dots,\lambda_n) = (-1)^{[n/2]}g_{2n}(\lambda_1,\dots,\lambda_n,\lambda_n+1,\dots,\lambda_1+1) \tag{2.3}$$

(see, e.g., [12]). In [5, 6], it was found that the functions h_n have the following structure:

$$h_n(\lambda_1,\dots,\lambda_n) = \sum_{m=0}^{[n/2]}\sum_{I,J}\prod_{p=1}^m \omega(\lambda_{i_p}-\lambda_{j_p})f_{n,I,J}(\lambda_1,\dots,\lambda_n). \tag{2.4}$$

Here $I=(i_1,\dots,i_m)$, $J=(j_1,\dots,j_m)$, and the sum is taken over all sequences I, J such that $I\cap J=\varnothing$, $1\le i_p<j_p\le n$ $(1\le p\le m)$, and $i_1<\cdots<i_m$.

A characteristic feature of formula (2.4) is that the transcendental functions enter only through a single function[1]

$$\omega(\lambda) = (\lambda^2-1)\frac{d}{d\lambda}\log\rho(\lambda)+\frac{1}{2} = \sum_{k=1}^\infty(-1)^k\frac{2k(\lambda^2-1)}{\lambda^2-k^2}+\frac{1}{2}, \tag{2.5}$$

[1] The function $\omega(\lambda)$ is related to $G(\lambda)$ in [5] by $\omega(\lambda)=G(i\lambda)+1/2$.

where

$$\rho(\lambda) = -\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(-\frac{\lambda}{2}+\frac{1}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)}. \tag{2.6}$$

The remaining factors $f_{n,I,J}(\lambda_1,\dots,\lambda_n)$ are rational functions of $\lambda_1,\dots,\lambda_n$ with only simple poles along the diagonal $\lambda_i=\lambda_j$. Subsequently, in [6, 7] it was explained that this structure of h_n originates from a duality between solutions of the qKZ equations with level 0 and level -4. On the basis of this relation, (2.4) was derived under certain assumptions. In these papers, the rational functions $f_{n,I,J}$ were left undetermined. Our goal in this paper is to obtain a recursion formula for h_n, which enables us to describe $f_{n,I,J}$ and to give a direct proof of (2.4).

2.2. L-operators and fusion of R-matrices. In this subsection, we introduce our notation concerning R-matrices and L-operators, and we give several formulas for the fusion of R-matrices.

The basic R-matrix relevant to the XXX model is

$$R(\lambda) = \rho(\lambda)\frac{r(\lambda)}{\lambda+1}, \tag{2.7}$$

where $\rho(\lambda)$ is given by (2.6), $r(\lambda)=\lambda+P$, and $P\in\mathrm{End}\big((\mathbb{C}^2)^{\otimes 2}\big)$ is the permutation operator, $P(u\otimes v)=v\otimes u$.

We consider also the R-matrices associated with higher-dimensional representations of $\mathfrak{sl}_2$. Let us fix our convention as follows. We denote by

$$\pi^{(k)}:\ U(\mathfrak{sl}_2)\longrightarrow \mathrm{End}(V^{(k)}),\quad V^{(k)}\simeq\mathbb{C}^{k+1},$$

the $(k+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$. We choose a basis $\{v_j^{(k)}\}_{j=0}^k$ of $V^{(k)}$ on which the standard generators E,F,H act as

$$Ev_j^{(k)}=(k-j+1)v_{j-1}^{(k)},\quad Fv_j^{(k)}=(j+1)v_{j+1}^{(k)},\quad Hv_j^{(k)}=(k-2j)v_j^{(k)}$$

with $v_{-1}^{(k)}=v_{k+1}^{(k)}=0$. For $k=1$, we also write $V=V^{(1)}$ and $v_+=v_0^{(1)}$, $v_-=v_1^{(1)}$. We shall use the singlet vectors in $V^{(k)}\otimes V^{(k)}$ $(k=1,2)$ normalized as

$$s^{(1)}=v_+\otimes v_- - v_-\otimes v_+, \tag{2.8}$$

$$s^{(2)}=v_0^{(2)}\otimes v_2^{(2)}-\frac{1}{2}v_1^{(2)}\otimes v_1^{(2)}+v_2^{(2)}\otimes v_0^{(2)}. \tag{2.9}$$

Let $\{S_a\}_{a=1}^3$, $\{S^a\}_{a=1}^3$ be a dual basis of $\mathfrak{sl}_2$ with respect to the invariant bilinear form $(x|y)$ normalized by the requirement that $(H|H)=2$. It is well known that the element

$$L^{(1)}(\lambda)=\lambda+\frac{1}{2}+\sum_{a=1}^3 S_a\otimes\pi^{(1)}(S^a)\in U(\mathfrak{sl}_2)\otimes\mathrm{End}(V^{(1)}) \tag{2.10}$$

is a solution of the Yang–Baxter relation

$$R_{1,2}(\lambda_1-\lambda_2)L_1^{(1)}(\lambda_1)L_2^{(1)}(\lambda_2)=L_2^{(1)}(\lambda_2)L_1^{(1)}(\lambda_1)R_{1,2}(\lambda_1-\lambda_2).$$

The suffix stands for the tensor components on which the operators act nontrivially. We also use a suffix of the form $(\alpha_1,\dots,\alpha_k)$ to denote the symmetric part $V^{(k)}\subset V_{\alpha_1}\otimes\cdots\otimes V_{\alpha_k}$, where the V_{α_i} are copies of V. Denote by $\mathcal{P}^+_{\alpha_1,\dots,\alpha_k}$ the projection

$$\mathcal{P}^+_{\alpha_1,\dots,\alpha_k}:\ V_{\alpha_1}\otimes\cdots\otimes V_{\alpha_k}\to V^{(k)}_{(\alpha_1,\dots,\alpha_k)},\ \mathcal{P}^+_{\alpha_1,\dots,\alpha_k}(v_1\otimes\cdots\otimes v_k)=\frac{1}{k!}\sum_{\sigma\in\mathcal{S}_k}v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)}.$$

Identifying $V^{(2)}$ with the subspace $V^{(2)}_{(1,2)} = \mathcal{P}^+_{1,2}\,(V_1 \otimes V_2)$, where $\mathcal{P}^\pm = (1/2)(1 \pm P)$, we introduce the fused L-operator

$$L^{(2)}_{(1,2)}(\lambda) = L^{(1)}_1\Bigl(\lambda - \frac{1}{2}\Bigr) L^{(1)}_2\Bigl(\lambda + \frac{1}{2}\Bigr)\mathcal{P}^+_{1,2} \in U(\mathfrak{sl}_2) \otimes \mathrm{End}(V^{(2)}). \tag{2.11}$$

Explicitly, it is given by

$$\begin{aligned} L^{(2)}(\lambda) = \lambda(\lambda+1) - \frac{1}{2} C \otimes \mathrm{id}_{V^{(2)}} \\ + (\lambda+1)\sum_{a=1}^{3} S_a \otimes \pi^{(2)}(S^a) + \frac{1}{2}\sum_{a,b=1}^{3} S_a S_b \otimes \pi^{(2)}\bigl(S^a S^b\bigr). \end{aligned} \tag{2.12}$$

On the right-hand side,

$$C = \sum_{a=1}^{3} S_a S^a \tag{2.13}$$

denotes the Casimir operator.

We shall make use of the crossing symmetry

$$L^{(k)}_\alpha(\lambda) s^{(k)}_{\alpha,\beta} = L^{(k)}_\beta(-\lambda-1) s^{(k)}_{\beta,\alpha} \tag{2.14}$$

and the quantum determinant relation

$$\mathcal{P}^-_{1,2} L^{(1)}_1(\lambda-1) L^{(1)}_2(\lambda) = \Bigl(\lambda^2 - \frac{1}{4} - \frac{1}{2}C\Bigr)\mathcal{P}^-_{1,2}. \tag{2.15}$$

Taking the images of (2.10), (2.11) in $V^{(k)}$, we obtain the (numerical) R-matrices[2]

$$r^{(k,l)}(\lambda) = (\pi^{(k)} \otimes \mathrm{id}) L^{(l)}(\lambda) \in \mathrm{End}\bigl(V^{(k)} \otimes V^{(l)}\bigr). \tag{2.16}$$

In what follows, we abbreviate $L^{(1)}(\lambda)$ to $L(\lambda)$.

We prepare several formulas about the fusion of R-matrices (see [18]). We have

$$r_{1,\alpha}\Bigl(\lambda - \frac{k-1}{2}\Bigr) r_{2,\alpha}\Bigl(\lambda - \frac{k-3}{2}\Bigr) \cdots r_{k,\alpha}\Bigl(\lambda + \frac{k-1}{2}\Bigr)\mathcal{P}^+_{1,\dots,k} = c_k(\lambda)\, r^{(k,1)}_{(1,\dots,k),\alpha}(\lambda), \tag{2.17}$$

where

$$c_k(\lambda) = \prod_{j=1}^{k-1}\Bigl(\lambda - \frac{k-1}{2} + j\Bigr). \tag{2.18}$$

Also, for all $k \geq 1$ we have

$$r^{(k,1)}_{(\alpha_1,\dots,\alpha_k),\alpha}\Bigl(\lambda - \frac{1}{2}\Bigr) r^{(k,1)}_{(\alpha_1,\dots,\alpha_k),\beta}\Bigl(\lambda + \frac{1}{2}\Bigr)\mathcal{P}^+_{\alpha\beta} = r^{(k,2)}_{(\alpha_1,\dots,\alpha_k),(\alpha\beta)}(\lambda) \tag{2.19}$$

and

$$\begin{aligned} \Bigl(\lambda + \frac{k_1-k_2}{2}\Bigr)\Bigl(\lambda + 1 + \frac{k_1-k_2}{2}\Bigr) r^{(k_1+k_2,2)}_{(1,\dots,k_1+k_2),(\alpha\beta)}(\lambda) \\ = r^{(k_1,2)}_{(1,\dots,k_1),(\alpha\beta)}\Bigl(\lambda - \frac{k_2}{2}\Bigr) r^{(k_2,2)}_{(k_1+1,\dots,k_1+k_2),(\alpha\beta)}\Bigl(\lambda + \frac{k_1}{2}\Bigr)\mathcal{P}^+_{(1,\dots,k_1),(k_1+1,\dots,k_1+k_2)}. \end{aligned} \tag{2.20}$$

[2]We have $r^{(1,2)}_{3,(12)}(\lambda) = (\lambda + \frac{1}{2}) r^{(2,1)}_{(12),3}(\lambda)$.

2.3. Properties of $h_n(\lambda_1,\dots,\lambda_n)$. The function h_n is given by the specialization (2.3) of the solution g_{2n} of the qKZ equation with level -4. In this subsection, we summarize the properties of h_n implied by those of g_{2n}.

In what follows, we deal with various vectors in the tensor product $V^{\otimes 2n}$, along with those obtained by permuting the tensor components. In order to simplify the presentation, we adopt the following convention. Consider the tensor product

$$W = V_1 \otimes \cdots \otimes V_n \otimes V_{\bar n} \otimes \cdots \otimes V_{\bar 1} \tag{2.21}$$

of $2n$ copies of V labeled by $1,\dots,n,\bar n,\dots,\bar 1$. For a vector

$$f = \sum f^{\epsilon_1,\dots,\epsilon_n,\bar\epsilon_n,\dots,\bar\epsilon_1} v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_n}\otimes v_{\bar\epsilon_n}\otimes\cdots\otimes v_{\bar\epsilon_1} \tag{2.22}$$

in $V^{\otimes 2n}$, we denote by $f_{1,\dots,n,\bar n,\dots,\bar 1}$ the same vector (2.22) in (2.21). The vectors obtained by permuting tensor components will be indicated by permuting suffixes from the "standard position". For example, if

$$f = \sum f^{\epsilon_1,\epsilon_2,\bar\epsilon_2,\bar\epsilon_1} v_{\epsilon_1}\otimes v_{\epsilon_2}\otimes v_{\bar\epsilon_2}\otimes v_{\bar\epsilon_1} \in V^{\otimes 4},$$

then $f_{2,\bar 2,1,\bar 1} \in V_1\otimes V_2\otimes V_{\bar 2}\otimes V_{\bar 1}$ is given by

$$f_{2,\bar 2,1,\bar 1} = P_{1,\bar 2}P_{1,2} f_{1,2,\bar 2,\bar 1} = \sum f^{\epsilon_2,\bar\epsilon_2,\epsilon_1,\bar\epsilon_1} v_{\epsilon_1}\otimes v_{\epsilon_2}\otimes v_{\bar\epsilon_2}\otimes v_{\bar\epsilon_1}.$$

By $f_{2,\bar 2,1,\bar 1}$ we do not mean a vector in $V_2\otimes V_{\bar 2}\otimes V_1\otimes V_{\bar 1}$. Note that, in this notation, $s^{(1)}_{\bar 1,1} = -s^{(1)}_{1,\bar 1} \in V_1\otimes V_{\bar 1}$. Sometimes, we need to construct a vector in $V_1\otimes V_2\otimes V_{\bar 2}\otimes V_{\bar 1}$ starting with one in $V_1\otimes V_{\bar 1}$ and one in $V_2\otimes V_{\bar 2}$. Suppose $f,g\in V\otimes V$. Then $f_{1,\bar 1}\in V_1\otimes V_{\bar 1}$, and $g_{2,\bar 2}\in V_2\otimes V_{\bar 2}$. We write $f_{1,\bar 1}g_{2,\bar 2}$ for the vector

$$\sum f^{\epsilon_1,\bar\epsilon_1} g^{\epsilon_2,\bar\epsilon_2} v_{\epsilon_1}\otimes v_{\epsilon_2}\otimes v_{\bar\epsilon_2}\otimes v_{\bar\epsilon_1} \in V_1\otimes V_2\otimes V_{\bar 2}\otimes V_{\bar 1}.$$

Under this convention, the ordering of the tensor product is irrelevant. There is no preference in writing $f_{1,\bar 1}g_{2,\bar 2}$ or $g_{2,\bar 2}f_{1,\bar 1}$ to represent the above vector.

We have three more remarks. First, we use "auxiliary" spaces in addition to the "quantum" spaces $V_1, V_{\bar 1},\dots,V_n,V_{\bar n}$. We use $\alpha,\beta,\alpha_1,\alpha_2$, etc., to label these spaces. Second, we use the index with parenthesis like $(\alpha_1,\dots,\alpha_k)$ to label the completely symmetric subspace $V^{(k)}\subset V^{\otimes k}$. This was already mentioned in Subsection 2.2. If f is a vector in this subspace, we denote by $f_{(\alpha_1,\dots,\alpha_k)}$ the corresponding vector in $V_{\alpha_1}\otimes\cdots\otimes V_{\alpha_k}$. Lastly, we use a similar convention for matrices as well.

The function g_{2n} has the following properties [12]:

$$g_{2n}(\lambda_1,\dots,\lambda_{2n}) \text{ is invariant under the action of } \mathfrak{sl}_2, \tag{2.23}$$

$$\begin{aligned} &g_{2n}(\dots,\lambda_{j+1},\lambda_j,\dots)_{\dots,j+1,j,\dots}\\ &\qquad = R_{j,j+1}(\lambda_{j,j+1}) g_{2n}(\dots,\lambda_j,\lambda_{j+1},\dots)_{\dots,j,j+1,\dots},\end{aligned} \tag{2.24}$$

$$\begin{aligned} &g_{2n}(\lambda_1,\dots,\lambda_{2n-1},\lambda_{2n}+2)_{1,\dots,2n}\\ &\qquad = (-1)^n g_{2n}(\lambda_{2n},\lambda_1,\dots,\lambda_{2n-1})_{2n,1,\dots,2n-1},\end{aligned} \tag{2.25}$$

$$\begin{aligned} &g_{2n}(\lambda_1,\dots,\lambda_{2n-2},\lambda,\lambda-1)_{1,\dots,2n}\\ &\qquad = g_{2n-2}(\lambda_1,\dots,\lambda_{2n-2})_{1,\dots,2n-2} s^{(1)}_{2n-1,2n}.\end{aligned} \tag{2.26}$$

Note that $R(-1) = -1+P = -2\mathcal{P}^-$ and $s^{(1)}_{21} = -s^{(1)}_{12}$. From this, (2.24), and (2.26), we obtain

$$\mathcal{P}^-_{2n-1,2n} g_{2n}(\lambda_1,\dots,\lambda_{2n-2},\lambda,\lambda+1)_{1,\dots,2n} = -\frac{1}{2} g_{2n-2}(\lambda_1,\dots,\lambda_{2n-2})_{1,\dots,2n-2} s^{(1)}_{2n-1,2n}.$$

Here and in the sequel we set $\lambda_{i,j} = \lambda_i - \lambda_j$. For $\alpha = 1$ or $\bar{1}$, set

$$A_\alpha(\lambda_1, \dots, \lambda_n) = (-1)^n R_{\alpha,\bar{2}}(\lambda_{1,2} - 1) \cdots R_{\alpha,\bar{n}}(\lambda_{1,n} - 1) R_{\alpha,n}(\lambda_{1,n}) \cdots R_{\alpha,2}(\lambda_{1,2}). \tag{2.27}$$

The following proposition is deduced easily from the above formulas.

Proposition 2.1. *The function $h_n(\lambda_1, \dots, \lambda_n)$ possesses the following properties*:

(2.28) $h_n(\lambda_1, \dots, \lambda_n)$ *is invariant under the action of* $\mathfrak{sl}_2$,

$$\begin{aligned} &h_n(\dots, \lambda_{j+1}, \lambda_j, \dots)_{\dots,j+1,j,\dots,\bar{j},\overline{j+1},\dots} \\ &\qquad = R_{j,j+1}(\lambda_{j,j+1}) R_{\overline{j+1},\bar{j}}(\lambda_{j+1,j}) h_n(\dots, \lambda_j, \lambda_{j+1}, \dots)_{\dots,j,j+1,\dots,\overline{j+1},\bar{j},\dots}, \end{aligned} \tag{2.29}$$

$$\begin{aligned} &h_n(\lambda_1 - 1, \lambda_2, \dots, \lambda_n)_{1,\dots,n,\bar{n},\dots,\bar{1}} \\ &\qquad = A_{\bar{1}}(\lambda_1, \dots, \lambda_n) h_n(\lambda_1, \lambda_2, \dots, \lambda_n)_{\bar{1},2,\dots,n,\bar{n},\dots,\bar{2},1}, \end{aligned} \tag{2.30}$$

$$\mathcal{P}^-_{1,\bar{1}} \cdot h_n(\lambda_1, \dots, \lambda_n)_{1,\dots,n,\bar{n},\dots,\bar{1}} = \frac{1}{2} s^{(1)}_{1,\bar{1}} h_{n-1}(\lambda_2, \dots, \lambda_n)_{2,\dots,n,\bar{n},\dots,\bar{2}}. \tag{2.31}$$

In particular, (2.30) is a reduced form of the qKZ equation. Note that, in contrast to equation (2.24) for g_{2n}, the coefficients appearing in (2.29) and (2.30) are rational functions. This is a consequence of the relations $\rho(\lambda)\rho(-\lambda) = 1$, $\rho(\lambda-1)\rho(\lambda) = -\lambda/(\lambda-1)$.

An integral formula for the function h_n was constructed in [12, 17]. Using that formula, in Appendix B we derive the following analytic properties of h_n.

Proposition 2.2. *The function $h_n(\lambda_1, \dots, \lambda_n)$ satisfies the following*:

(2.32) $h_n(\lambda_1, \dots, \lambda_n)$ *is meromorphic in $\lambda_1, \dots, \lambda_n$ with at most simple poles at* $\lambda_i - \lambda_j \in \mathbb{Z} \backslash \{0, \pm 1\}$;

for any $0 < \delta < \pi$ *we have*

$$\lim_{\substack{\lambda_1 \to \infty \\ \lambda_1 \in S_\delta}} h_n(\lambda_1, \dots, \lambda_n) = \frac{1}{2} s^{(1)}_{1,\bar{1}} h_{n-1}(\lambda_2, \dots, \lambda_n), \tag{2.33}$$

where $S_\delta = \{\lambda \in \mathbb{C} \mid \delta < |\arg \lambda| < \pi - \delta\}$.

Remark 2.3. Relations (2.30), (2.31) and the analyticity of h_n at $\lambda_1 = \lambda_2$ imply the identity

$$\mathcal{P}^-_{1,2} \cdot h_n(\lambda - 1, \lambda, \dots, \lambda_n)_{1,\dots,n,\bar{n},\dots,\bar{1}} = \frac{1}{2} s^{(1)}_{1,2} s^{(1)}_{\bar{1},\bar{2}} h_{n-2}(\lambda_3, \dots, \lambda_n)_{3,\dots,n,\bar{n},\dots,\bar{3}}. \tag{2.34}$$

As we shall show in the following sections, properties (2.28)–(2.33) determine the functions h_n uniquely.

§3. Recursion formula

In this section we state the main result of this paper: a recursion formula for the correlation functions. The main ingredient of this recursion is a transfer matrix with an auxiliary space of fractional dimension.

3.1. Trace function. We define the "trace over a space of fractional dimension". By this we mean a unique $\mathbb{C}[x]$-linear map

$$\mathrm{Tr}_x \ : \ U(\mathfrak{sl}_2) \otimes \mathbb{C}[x] \longrightarrow \mathbb{C}[x]$$

such that for any nonnegative integer k we have

$$\mathrm{Tr}_{k+1}(A) = \mathrm{tr}_{V^{(k)}}\, \pi^{(k)}(A) \quad (A \in U(\mathfrak{sl}_2)). \tag{3.1}$$

Here tr on the right-hand side stands for the usual trace.

We list some properties of the trace function Tr_x:

$$\mathrm{Tr}_x(AB) = \mathrm{Tr}_x(BA), \quad \mathrm{Tr}_x(1) = x; \tag{3.2}$$

$$\mathrm{Tr}_x(A) = 0 \text{ if } A \text{ has nonzero weight}; \tag{3.3}$$

$$\mathrm{Tr}_x(e^{zH}) = \frac{\sinh(xz)}{\sinh z}; \tag{3.4}$$

$$\mathrm{Tr}_x\Big(\Big(\frac{H^2}{2}+H+2FE\Big)A\Big)=\frac{x^2-1}{2}\,\mathrm{Tr}_x(A) \quad (A \in U(\mathfrak{sl}_2)\otimes\mathbb{C}[x]). \tag{3.5}$$

By the generating series (3.4), the traces $\mathrm{Tr}_x(H^a)$ are known, and we can calculate $\mathrm{Tr}_x(H^aE^bF^c)$ inductively for all $a, b, c \geq 0$ by using (3.5). We emphasize that $\mathrm{Tr}_x(A)$ is determined by the "dimension" $\mathrm{Tr}_x(1) = x$ and the value of the Casimir operator; we have

$$\mathrm{Tr}_x(A) = \mathrm{Tr}_x(A') \quad \text{if } \varpi_x(A) = \varpi_x(A'), \tag{3.6}$$

where ϖ_x is the projection

$$\varpi_x : U(\mathfrak{sl}_2)\otimes\mathbb{C}[x] \to U(\mathfrak{sl}_2)\otimes\mathbb{C}[x]/I_x, \tag{3.7}$$

and I_x denotes the two-sided ideal of $U(\mathfrak{sl}_2)\otimes\mathbb{C}[x]$ generated by $C - (x^2-1)/2$.

The following statements are simple consequences of these rules:

(3.8) $\mathrm{Tr}_{-x}(A) = -\mathrm{Tr}_x(A)$;

(3.9) $\mathrm{Tr}_x(A) - x\varepsilon(A) \in x(x^2-1)\mathbb{C}[x]$, where $\varepsilon : U(\mathfrak{sl}_2)\otimes\mathbb{C}[x] \to \mathbb{C}[x]$ stands for the counit;

(3.10) the degree of $\mathrm{Tr}_x(H^aE^bF^c)$ is at most $m+1$ (m even) or m (m odd) where $m = a+b+c$.

3.2. The functions $X^{[i,j]}$. Now we are going to introduce our main object $X^{[i,j]}$.

For $1 \leq j \leq n$ (respectively, $1 \leq i < j \leq n$), we set

$$W^{[j]} = V_1 \otimes \cdots \overset{j}{\wedge} \cdots \otimes V_n \otimes V_{\bar n} \otimes \cdots \overset{\bar j}{\wedge} \cdots \otimes V_{\bar 1}, \tag{3.11}$$

$$W^{[i,j]} = V_1 \otimes \cdots \overset{i}{\wedge} \cdots \overset{j}{\wedge} \cdots \otimes V_n \otimes V_{\bar n} \otimes \cdots \overset{\bar j}{\wedge} \cdots \overset{\bar i}{\wedge} \cdots \otimes V_{\bar 1}. \tag{3.12}$$

Define the monodromy matrices

$$\begin{aligned} T^{[j]}(\lambda) &= L_{\bar 1}(\lambda-\lambda_1-1)\cdots \overset{\bar j}{\wedge} \cdots L_{\bar n}(\lambda-\lambda_n-1) \\ &\quad \times L_n(\lambda-\lambda_n)\cdots \overset{j}{\wedge} \cdots L_1(\lambda-\lambda_1), \end{aligned} \tag{3.13}$$

$$\begin{aligned} T^{[i,j]}(\lambda) &= L_{\bar 1}(\lambda-\lambda_1-1)\cdots \overset{\bar i}{\wedge} \cdots \overset{\bar j}{\wedge} \cdots L_{\bar n}(\lambda-\lambda_n-1) \\ &\quad \times L_n(\lambda-\lambda_n)\cdots \overset{j}{\wedge} \cdots \overset{i}{\wedge} \cdots L_1(\lambda-\lambda_1). \end{aligned} \tag{3.14}$$

These are elements of $U(\mathfrak{sl}_2)\otimes\mathrm{End}(W^{[j]})$ and $U(\mathfrak{sl}_2)\otimes\mathrm{End}(W^{[i,j]})$, respectively.

Using the trace function Tr_x, we define the functions

$$X^{[i,j]}(\lambda_1,\ldots,\lambda_n) \in V_i \otimes V_{\bar i} \otimes V_j \otimes V_{\bar j} \otimes \mathrm{End}(W^{[i,j]}) \quad (1 \leq i < j \leq n)$$

by the formula

$$
\begin{aligned}
&X^{[i,j]}(\lambda_1,\dots,\lambda_n)\\
&\quad=\frac{1}{\lambda_{i,j}\prod_{p\neq i,j}\lambda_{i,p}\lambda_{j,p}}\\
&\qquad\times R_{i,i-1}(\lambda_{i,i-1})\cdots R_{i,1}(\lambda_{i,1})R_{\overline{i-1},\bar i}(\lambda_{i-1,i})\cdots R_{\bar 1\bar i}(\lambda_{1,i})\,\mathrm{Tr}_{\lambda_{i,j}}\\
&\qquad\times\Bigl(T^{[i]}\Bigl(\frac{\lambda_i+\lambda_j}{2}\Bigr)\Bigr)R_{j,j-1}(\lambda_{j,j-1})\cdots\widehat{R_{j,i}(\lambda_{j,i})}\cdots R_{j,1}(\lambda_{j,1})\\
&\qquad\times R_{\overline{j-1},\bar j}(\lambda_{j-1,j})\cdots\widehat{R_{\bar i,\bar j}(\lambda_{i,j})}\cdots R_{\bar 1,\bar j}(\lambda_{1,j})s^{(2)}_{(i,\bar i),(j,\bar j)}.
\end{aligned}
\tag{3.15}
$$

One can think of $\mathrm{Tr}_{\lambda_{i,j}}(T^{[i]}(\frac{\lambda_i+\lambda_j}{2}))$ as a transfer matrix with "$\lambda_{i,j}$-dimensional auxiliary space".

The functions $X^{[i,j]}$ are not independent. By (3.15) and (3.18), the $X^{[i,j]}$ with general i,j can be expressed in terms of one of them, e.g.,

$$
X^{[1,2]}(\lambda_1,\dots,\lambda_n)=\frac{1}{\lambda_{1,2}\prod_{p=3}^n\lambda_{1,p}\lambda_{2,p}}\,\mathrm{Tr}_{\lambda_{1,2}}\Bigl(T^{[1]}\Bigl(\frac{\lambda_1+\lambda_2}{2}\Bigr)\Bigr)s^{(2)}_{(1,\bar 1),(2,\bar 2)},
\tag{3.16}
$$

as follows:

$$
\begin{aligned}
&X^{[i,j]}(\lambda_1,\dots,\lambda_n)\\
&\quad=R_{i,i-1}(\lambda_{i,i-1})\cdots R_{i,1}(\lambda_{i,1})R_{j,j-1}(\lambda_{j,j-1})\cdots\widehat{R_{j,i}(\lambda_{j,i})}\cdots R_{j,1}(\lambda_{j,1})\\
&\qquad\times R_{\overline{i-1},\bar i}(\lambda_{i-1,i})\cdots R_{\bar 1\bar i}(\lambda_{1,i})R_{\overline{j-1},\bar j}(\lambda_{j-1,j})\cdots\widehat{R_{\bar i,\bar j}(\lambda_{i,j})}\cdots R_{\bar 1,\bar j}(\lambda_{1,j})\\
&\qquad\times X^{[1,2]}(\lambda_i,\lambda_j,\lambda_1,\dots,\widehat{\lambda_i},\dots,\widehat{\lambda_j},\dots,\lambda_n)_{i,j,1,\dots,\hat i,\dots,\hat j,\dots,n,\bar n,\dots,\hat{\bar j},\dots,\hat{\bar i},\dots,\bar 1,\bar j,\bar i}.
\end{aligned}
\tag{3.17}
$$

Nonetheless, for the description of the results, it is convenient to use all of $X^{[i,j]}$.

We list the main properties of $X^{[i,j]}$.

Transformation law. For an element σ of the symmetric group $\mathfrak{S}_n$, we use the abbreviation

$$
(X^{[i,j]})^\sigma=X^{[i,j]}(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(n)})_{\sigma(1),\dots,\sigma(n),\overline{\sigma(n)},\dots,\overline{\sigma(1)}}.
$$

Then the functions $X^{[i,j]}=(X^{[i,j]})^{\mathrm{id}}$ obey the transformation law

$$
\begin{aligned}
&R_{k,k+1}(\lambda_{k,k+1})R_{\overline{k+1},\bar k}(\lambda_{k+1,k})X^{[i,j]}\\
&=\begin{cases}
(X^{[i,j]})^{(k,k+1)}R_{k,k+1}(\lambda_{k,k+1})R_{\overline{k+1},\bar k}(\lambda_{k+1,k}) & (k\neq i,i-1,j,j-1),\\
(X^{[i,j+1]})^{(j,j+1)} & (k=j),\\
(X^{[i,j-1]})^{(j-1,j)} & (i<k=j-1),\\
(X^{[i-1,j]})^{(i-1,i)} & (k=i-1),\\
(X^{[i+1,j]})^{(i,i+1)} & (k=i<j-1),\\
(X^{[i,i+1]})^{(i,i+1)} & (i=k,j=i+1).
\end{cases}
\end{aligned}
\tag{3.18}
$$

Zero and pole structure. The function

$$
\frac{\prod_{p(\neq i,j)}\lambda_{i,p}\lambda_{j,p}}{\lambda_{i,j}^2-1}\cdot X^{[i,j]}(\lambda_1,\dots,\lambda_n)
$$

is a polynomial in $\lambda_1,\dots,\lambda_n$.

Regularity at ∞. For each $1 \le k \le n$ and an $\mathfrak{sl}_2$-invariant vector $v \in (W^{[i,j]})^{\mathfrak{sl}_2}$, we have

$$\frac{1}{\lambda_{i,j}^2 - 1} X^{[i,j]}(\lambda_1, \dots, \lambda_n) v = O(1) \quad (\lambda_k \to \infty). \tag{3.19}$$

Difference equation. $X^{[1,2]}(\lambda_1, \dots, \lambda_n)$ satisfies the difference equation

$$X^{[1,2]}(\lambda_1 - 1, \lambda_2, \dots, \lambda_n) = -A_{\bar{1}}(\lambda_1, \dots, \lambda_n) P_{1,\bar{1}} X^{[1,2]}(\lambda_1, \lambda_2, \dots, \lambda_n), \tag{3.20}$$

where $A_{\bar{1}}$ is defined in (2.27).

These properties will be proved in §4 (see Lemmas 4.2–4.5 and 4.9).

3.3. Main result. Now we are in a position to state the recursion formula that determines the functions $h_n(\lambda_1, \dots, \lambda_n)$ starting with the initial condition

$$h_0 = 1, \quad h_1(\lambda_1) = \frac{1}{2} s_{1,\bar{1}}^{(1)}. \tag{3.21}$$

The proofs of the statements in this subsection will be given in §4.

In this subsection we write $X_n^{[i,j]}$ for $X^{[i,j]}$ to indicate the relevant number of sites n.

Theorem 3.1. *We have the following recursion formula:*

$$\begin{aligned} &h_n(\lambda_1, \dots, \lambda_n)_{1,\dots,n,\bar{n},\dots,\bar{1}} \\ &\qquad = \frac{1}{2} s_{1,\bar{1}}^{(1)} \cdot h_{n-1}(\lambda_2, \dots, \lambda_n)_{2,\dots,n,\bar{n},\dots,\bar{2}} \\ &\qquad\quad - \sum_{j=2}^{n} Z_n^{[1,j]}(\lambda_1, \dots, \lambda_n) \cdot h_{n-2}(\lambda_2, \dots, \widehat{\lambda_j}, \dots, \lambda_n)_{2,\dots,\hat{j},\dots,n,\bar{n},\dots,\hat{\bar{j}},\dots,\bar{2}}. \end{aligned} \tag{3.22}$$

Here

$$\begin{aligned} Z_n^{[1,j]}(\lambda_1, \dots, \lambda_n) &= \oint_{\mathcal{C}} \frac{d\sigma}{2\pi i} \frac{\omega(\sigma - \lambda_j)}{\sigma - \lambda_1} \frac{1}{(\sigma - \lambda_j)^2 - 1} X_n^{[1,j]}(\sigma, \lambda_2, \dots, \lambda_n) \\ &= \frac{\omega(\lambda_{1,j})}{\lambda_{1,j}^2 - 1} X_n^{[1,j]}(\lambda_1, \lambda_2, \dots, \lambda_n) \\ &\quad + \sum_{p(\ne 1,j)} \frac{\omega(\lambda_{p,j})}{\lambda_{p,1}(\lambda_{p,j}^2 - 1)} \operatorname{res}_{\sigma=\lambda_p} X_n^{[1,j]}(\sigma, \lambda_2, \dots, \lambda_n), \end{aligned} \tag{3.23}$$

where $\mathcal{C}$ is a simple closed curve encircling $\lambda_1, \dots, \lambda_n$ counterclockwise, $\omega(\lambda)$ is given by (2.5), *and $X_n^{[1,j]}(\lambda_1, \dots, \lambda_n)$ is defined in* (3.15).

In the last line of (3.23), we used the fact that $X_n^{[1,j]}(\sigma, \lambda_2, \dots, \lambda_n)$ has no poles at $\sigma = \lambda_j$.

Theorem 3.2. *The function $h_n(\lambda_1, \dots, \lambda_n)$ has the structure*

$$h_n(\lambda_1, \dots, \lambda_n) = \sum_{m=0}^{[n/2]} \sum_{I,J} \prod_{p=1}^{m} \omega(\lambda_{i_p} - \lambda_{j_p}) f_{n,I,J}(\lambda_1, \dots, \lambda_n), \tag{3.24}$$

where $f_{n,I,J}(\lambda_1, \dots, \lambda_n) \in V^{\otimes 2n}$ are rational functions, and $I = (i_1, \dots, i_m)$, $J = (j_1, \dots, j_m)$ run over the sequences satisfying $I \cap J = \varnothing$, $i_1 < \dots < i_m$, $1 \le i_p < j_p \le n$ $(1 \le p \le m)$. A representation of h_n in the above form is unique.

Theorem 3.3. *In the notation of Theorem* 3.2, *the rational functions* $f_{n,I,J}(\lambda_1,\dots,\lambda_n)$ *are uniquely determined by the recursion relation*

$$f_{n,iI',jJ'}(\lambda_1,\dots,\lambda_n) = \frac{1}{1-\lambda_{i,j}^2} X_n^{[i,j]}(\lambda_1,\dots,\lambda_n) f_{n-2,I',J'}(\lambda_1,\dots,\widehat{\lambda_i},\dots,\widehat{\lambda_j},\dots,\lambda_n), \tag{3.25}$$

$$f_{n,\varnothing,\varnothing}(\lambda_1,\dots,\lambda_n) = \frac{1}{2^n} s^{(1)}_{1,\bar{1}} \cdots s^{(1)}_{n,\bar{n}}. \tag{3.26}$$

In addition, they enjoy the following properties:

(3.27) $f_{n,I,J}(\lambda_1,\dots,\lambda_n)$ *is invariant under the action of* $\mathfrak{sl}_2$;

(3.28)
$$f_{n,I,J}(\lambda_1,\dots,\lambda_{j+1},\lambda_j,\dots,\lambda_n)_{\dots,j+1,j,\dots,\bar{j},\overline{j+1},\dots} = R_{j,j+1}(\lambda_{j,j+1}) R_{\overline{j+1},\bar{j}}(\lambda_{j+1,j}) \times f_{n,\widetilde{I},\widetilde{J}}(\lambda_1,\dots,\lambda_j,\lambda_{j+1},\dots,\lambda_n)_{\dots,j,j+1,\dots,\overline{j+1},\bar{j},\dots},$$
where $\widetilde{I}=\sigma(I)$ *and* $\widetilde{J}=\sigma(J)(\sigma=(j,j+1))$ *except for the following cases*: *if* $i_m=j\in I$, $j_m=j+1\in J$ *for some* m, *we have* $\tilde{I}=I$, $\tilde{J}=J$, *and if* $i_m=j, i_{m+1}=j+1$ *for some* m, *we have* $\tilde{I}=I$ *and* $\tilde{J}$ *is given by* $\tilde{j}_l = \begin{cases} j_l & \text{if } l\neq m,m+1,\\ j_{m+1} & \text{if } l=m,\\ j_m & \text{if } l=m+1;\end{cases}$

(3.29) $f_{n,I,J}(\lambda_1,\dots,\lambda_n)$ *is regular at* ∞ *in each* λ_j;

(3.30) $f_{n,I,J}(\lambda_1,\dots,\lambda_n)$ *has at most simple poles at* $\lambda_i-\lambda_j=0$, *where* $1\le i<j\le n$, $i\in I$ *or* $j\in J$, $(i,j)\neq(i_p,j_p)$ $(1\le p\le m)$.

In general, $f_{n,I,J}$ is expressed as follows. Suppose $I=(i_1,\dots,i_m)$, $J=(j_1,\dots,j_m)$ and $\{1,\dots,n\}\backslash I\cup J=\{k_1,\dots,k_l\}$, where $k_1<\dots<k_l$, $n=2m+l$. We denote the corresponding permutation by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & \cdots & \cdots & \cdots & \cdots & n \\ i_1 & j_1 & \cdots & i_m & j_m & k_1 & \cdots & k_l \end{pmatrix} \in \mathfrak{S}_n,$$

and let $\sigma=\sigma_{a_1}\circ\dots\circ\sigma_{a_N}$ be a reduced decomposition into transpositions $\sigma_a=(a,a+1)$. With σ we associate the R-matrices

$$R^\sigma = R_{b'_1,b_1}(\lambda_{b'_1,b_1})\cdots R_{b'_N,b_N}(\lambda_{b'_N,b_N}),$$
$$\overline{R}^\sigma = R_{\overline{b_1},\overline{b'_1}}(\lambda_{b_1,b'_1})\cdots R_{\overline{b_N},\overline{b'_N}}(\lambda_{b_N,b'_N}),$$

where $b_i=\sigma_{a_1}\cdots\sigma_{a_{i-1}}(a_i)$, $b'_i=\sigma_{a_1}\cdots\sigma_{a_{i-1}}(a_i+1)$. Then

$$f_{n,I,J}(\lambda_1,\dots,\lambda_n) = \frac{1}{D} R^\sigma \overline{R}^\sigma \tau_1\tau_2\cdots\tau_m \mathbf{s},$$

where

$$D = 2^l \prod_{a<b} \lambda_{i_a,j_b}\lambda_{j_a,i_b}\lambda_{i_a,i_b}\lambda_{j_a,j_b} \prod_{a,c} \lambda_{i_a,k_c}\lambda_{j_a,k_c},$$

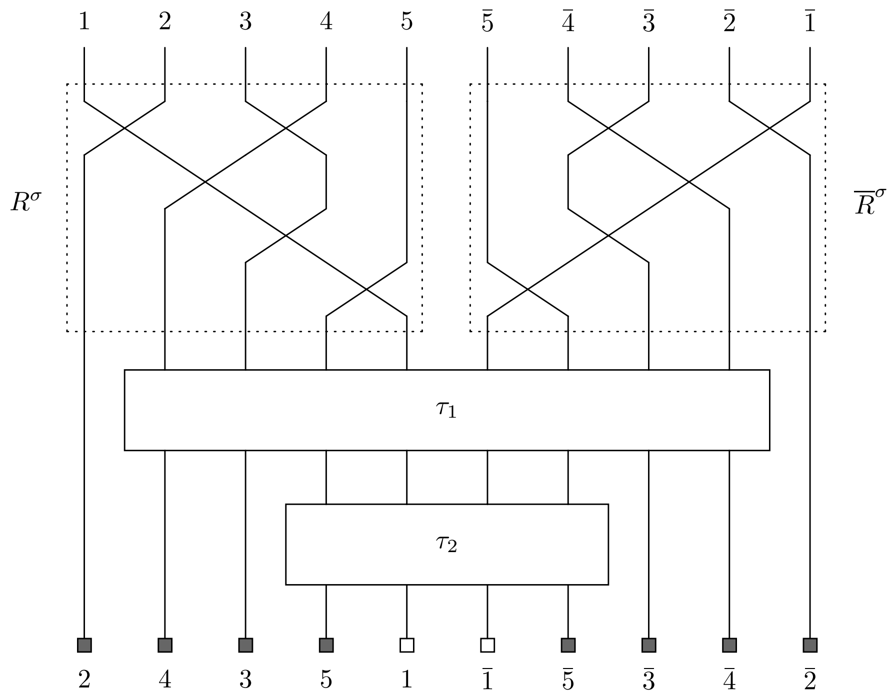

FIGURE 1. The case where $n = 5$, $I = (23)$, $J = (45)$.

$\mathbf{s} = s^{(2)}_{(i_1,\overline{i_1}),(j_1,\overline{j_1})} \cdots s^{(2)}_{(i_m,\overline{i_m}),(j_m,\overline{j_m})} s^{(1)}_{k_1,\overline{k_1}} \cdots s^{(1)}_{k_l,\overline{k_l}}$ is the product of singlet vectors, and the τ_a are the transfer matrices

$$\tau_a = \frac{1}{\lambda_{i_a,j_a}(1-\lambda^2_{i_a,j_a})} \operatorname{Tr}_{\lambda_{i_a,j_a}}\Big(T_{n-2a+2}\Big(\frac{\lambda_{i_a}+\lambda_{j_a}}{2}\Big)\Big),$$
$$T_{n-2a+2}(\lambda) = L_{\overline{\sigma(2a)}}(\lambda-\lambda_{\sigma(2a)}-1)\cdots L_{\overline{\sigma(n)}}(\lambda-\lambda_{\sigma(n)}-1)$$
$$\times L_{\sigma(n)}(\lambda-\lambda_{\sigma(n)})\cdots L_{\sigma(2a)}(\lambda-\lambda_{\sigma(2a)}).$$

3.4. Examples. We write out the recursion relation in simple cases.

First, let $n = 2$. Observing that

$$\operatorname{Tr}_x(AB) = \frac{1}{6}x(x^2-1)(A|B) \quad (A, B \in \mathfrak{sl}_2)$$

and using the initial condition (3.21), we find

$$X^{[1,2]}(\lambda_1,\lambda_2) = \frac{\lambda_{12}^2-1}{3} s^{(2)}_{(1,\bar{1}),(2,\bar{2})}.$$

Along with the initial condition (3.21), the recursion formula gives

$$h_2(\lambda_1,\lambda_2) = \frac{1}{4} s^{(1)}_{1,\bar{1}} s^{(1)}_{2,\bar{2}} - \frac{1}{3}\omega(\lambda_{1,2}) s^{(2)}_{(1,\bar{1}),(2,\bar{2})}.$$

Next, consider the case of $n = 3$. We have

$$\operatorname{Tr}_x(ABC) = \frac{1}{12}x(x^2-1)([A,B]|C) \quad (A, B, C \in \mathfrak{sl}_2).$$

Take a basis of the $\mathfrak{sl}_2$-invariants $(V^{\otimes 6})^{\mathfrak{sl}_2}$ as follows:

$$u_0 = s^{(1)}_{1,\bar{1}} s^{(1)}_{2,\bar{2}} s^{(1)}_{3,\bar{3}},$$
$$u_i = s^{(1)}_{i,\bar{i}} s^{(2)}_{(j,\bar{j}),(k,\bar{k})} \quad (i,j,k = 1,2,3 \text{ are distinct}),$$
$$u_4 = u,$$

where u is a unique $\mathfrak{sl}_2$-invariant vector in $V^{(2)}_{(1,\bar{1})} \otimes V^{(2)}_{(2,\bar{2})} \otimes V^{(2)}_{(3,\bar{3})}$ with coefficient 1 in the component $v^{(2)}_0 \otimes v^{(2)}_1 \otimes v^{(2)}_2$.

After some calculation, we obtain

$$X^{[1,2]}(\lambda_1,\lambda_2,\lambda_3) s^{(1)}_{3,\bar{3}} = (\lambda_{12}^2 - 1)\Big(\frac{1}{3}u_3 + \frac{1}{3\lambda_{1,3}\lambda_{2,3}}u_2 - \frac{\lambda_{1,2}}{6\lambda_{1,3}\lambda_{2,3}}u_4\Big),$$

which gives

$$\begin{aligned} h_3(\lambda_1,\lambda_2,\lambda_3) = \frac{1}{8}u_0 &- \omega(\lambda_{1,2})\Big(\frac{1}{6}u_3 + \frac{1}{6\lambda_{1,3}\lambda_{2,3}}u_2 - \frac{\lambda_{1,2}}{12\lambda_{1,3}\lambda_{2,3}}u_4\Big) \\ &- \omega(\lambda_{1,3})\Big(\frac{1}{6}\Big(1 - \frac{1}{\lambda_{1,2}\lambda_{2,3}}\Big)u_2 + \frac{\lambda_{1,2} - \lambda_{2,3}}{12\lambda_{1,2}\lambda_{2,3}}u_4\Big) \\ &- \omega(\lambda_{2,3})\Big(\frac{1}{6}u_1 + \frac{1}{6\lambda_{1,2}\lambda_{1,3}}u_2 + \frac{\lambda_{2,3}}{12\lambda_{1,2}\lambda_{1,3}}u_4\Big). \end{aligned}$$

§4. Derivation from the qKZ equation

Our purpose in this section is to prove Theorems 3.1–3.3.

4.1. Properties of $X^{[i,j]}(\lambda_1,\dots,\lambda_n)$. In this subsection, we study the properties of the rational function $X^{[i,j]}(\lambda_1,\dots,\lambda_n)$.

We begin with an identity for L-operators.

Lemma 4.1. *We have*

(4.1)
$$\varpi_\lambda\Big(L_2\Big(\frac{\lambda}{2}\Big)L_{\bar{2}}\Big(\frac{\lambda}{2}-1\Big)\Big)s^{(2)}_{(1\bar{1}),(2\bar{2})} = \frac{1}{\lambda+2}r^{(2,1)}_{(1,\bar{1}),\bar{2}}\Big(\lambda+\frac{1}{2}\Big)\varpi_\lambda\Big(L^{(2)}_{(2,\bar{2})}\Big(\frac{\lambda+1}{2}\Big)\Big)s^{(2)}_{(1,\bar{1}),(2,\bar{2})}.$$

Proof. We prove identity (4.1) by projecting it to the symmetric and skew-symmetric subspaces of $V_2 \otimes V_{\bar{2}}$ separately. To simplify the notation, in what follows we shall write $A \sim B$ to indicate that $\varpi_\lambda(A) = \varpi_\lambda(B)$. Note that

$$C \sim \frac{\lambda^2 - 1}{2}. \tag{4.2}$$

By (2.11), we have

$$\mathcal{P}^+_{2,\bar{2}} L_2\Big(\frac{\lambda}{2}\Big)L_{\bar{2}}\Big(\frac{\lambda}{2}-1\Big)s^{(2)}_{(1\bar{1}),(2\bar{2})} = L^{(2)}_{(2\bar{2})}\Big(\frac{\lambda-1}{2}\Big)s^{(2)}_{(1\bar{1}),(2\bar{2})}.$$

Hence, on the symmetric subspace, (4.1) reduces to

$$\begin{aligned} &L^{(2)}_{(2\bar{2})}\Big(\frac{\lambda-1}{2}\Big)s^{(2)}_{(1\bar{1}),(2\bar{2})} \\ &\quad \sim \frac{1}{\lambda+2}\Big(\lambda+1+\frac{1}{2}\sum_a (S_a)_{(1\bar{1})}(S^a)_{(2\bar{2})}\Big)L^{(2)}_{(2\bar{2})}\Big(\frac{\lambda+1}{2}\Big)s^{(2)}_{(1\bar{1}),(2\bar{2})}. \end{aligned} \tag{4.3}$$

To simplify the right-hand side, observe the relation

$$\frac{1}{2}\sum_a S_a \otimes S^a = P^{(2)} - 3K^{(2)}, \tag{4.4}$$

where $P^{(2)} \in \mathrm{End}_{\mathfrak{sl}_2}(V^{(2)} \otimes V^{(2)})$ is the permutation operator and $K^{(2)} \in \mathrm{End}_{\mathfrak{sl}_2}(V^{(2)} \otimes V^{(2)})$ is the projection onto the singlet subspace. More explicitly, we have

$$K^{(2)} = \frac{1}{3} s^{(2)} \otimes s^{(2)} \in (V^{(2)} \otimes V^{(2)})^{\otimes 2} \simeq \mathrm{End}(V^{(2)} \otimes V^{(2)}),$$

where we have made identification through the invariant bilinear form $(\cdot,\cdot) : V^{(2)} \otimes V^{(2)} \to \mathbb{C}$ normalized as $(v_0^{(2)}, v_2^{(2)}) = 1$.

By using (4.4), the right-hand side of (4.3) can be rewritten as

$$\frac{1}{\lambda+2}\Big((\lambda+1)L^{(2)}_{(2\bar{2})}\Big(\frac{\lambda+1}{2}\Big) + L^{(2)}_{(2\bar{2})}\Big(-\frac{\lambda+3}{2}\Big) - \mathrm{tr}_{(2\bar{2})}\Big(L^{(2)}_{(2\bar{2})}\Big(\frac{\lambda+1}{2}\Big)\Big)\Big) s^{(2)}_{(1\bar{1}),(2\bar{2})}.$$

In the second term, we have used the crossing symmetry (2.14). The trace is evaluated as follows:

$$\mathrm{tr}_{(2\bar{2})}\Big(L^{(2)}_{(2\bar{2})}\Big(\frac{\lambda+1}{2}\Big)\Big) \sim (\lambda+1)(\lambda+2). \tag{4.5}$$

Now, (4.3) can be checked directly by using (2.12).

Next, let us project to the singlet subspace. The right-hand side becomes

$$\mathcal{P}^-_{2,\bar{2}}(\mathrm{RHS}) = \frac{1}{2(\lambda+2)} \sum_a (S_a)_{(1,\bar{1})} \left((S^a)_{\bar{2}} - (S^a)_2\right) L^{(2)}_{(2,\bar{2})}\Big(\frac{\lambda+1}{2}\Big) s^{(2)}_{(1\bar{1}),(2\bar{2})}. \tag{4.6}$$

Using the matrix representation

$$L^{(2)}_{(2\bar{2})}\left(\frac{\lambda+1}{2}\right) \sim \begin{pmatrix} \frac{(H+\lambda+1)(H+\lambda+3)}{4} & (H+\lambda+3)F & F^2 \\ \frac{E(H+\lambda+3)}{2} & \frac{(\lambda+1)^2 - H^2}{2} & -\frac{F(H-\lambda-3)}{2} \\ E^2 & -(H-\lambda-3)E & \frac{(H-\lambda-1)(H-\lambda-3)}{4} \end{pmatrix},$$

we obtain

$$\mathcal{P}^-_{2,\bar{2}}(\mathrm{RHS}) \sim \big(-F(v_0^{(2)})_{(1\bar{1})} + \frac{1}{2}H(v_1^{(2)})_{(1\bar{1})} + E(v_2^{(2)})_{(1\bar{1})}\big) s^{(1)}_{2,\bar{2}}.$$

The left-hand side becomes

$$\mathcal{P}^-_{2,\bar{2}}(\mathrm{LHS}) = \Big(\mathcal{P}^-_{2,\bar{2}} L_{\bar{2}}\Big(\frac{\lambda-2}{2}\Big) L_2\Big(\frac{\lambda}{2}\Big) + \mathcal{P}^-_{2\bar{2}}\Big[L_2\Big(\frac{\lambda}{2}\Big), L_{\bar{2}}\Big(\frac{\lambda-2}{2}\Big)\Big]\Big) s^{(2)}_{(1\bar{1}),(2\bar{2})}.$$

The first term is zero because of (2.15). Therefore, we have

$$\mathcal{P}^-_{2,\bar{2}}(\mathrm{LHS}) \sim \mathcal{P}^-_{2,\bar{2}} \sum_{a,b} [S_a, S_b](S^a)_2 (S^b)_{\bar{2}} s^{(2)}_{(1\bar{1}),(2\bar{2})}. \tag{4.7}$$

Now, the relation $\mathcal{P}^-_{2,\bar{2}}(\mathrm{LHS}) \sim \mathcal{P}^-_{2,\bar{2}}(\mathrm{RHS})$ is easily seen. □

We have the following symmetry of $X^{[1,2]}$.

Lemma 4.2. *The function $X^{[1,2]}(\lambda_1, \dots, \lambda_n)$ possesses the symmetry property*

$$\begin{aligned} &R_{2,1}(\lambda_{2,1}) R_{\bar{1},\bar{2}}(\lambda_{1,2}) X^{[1,2]}(\lambda_2, \lambda_1, \dots, \lambda_n)_{2,1,\dots,n,\bar{n},\dots,\bar{1},\bar{2}} \\ &\qquad = X^{[1,2]}(\lambda_1, \lambda_2, \dots, \lambda_n)_{1,2,\dots,n,\bar{n},\dots,\bar{2},\bar{1}}. \end{aligned} \tag{4.8}$$

Proof. Recall the definition

$$X^{[1,2]}(\lambda_1, \dots, \lambda_n) = \frac{1}{\lambda_{1,2} \prod_{p \neq 1,2} \lambda_{1,p} \lambda_{2,p}} \mathrm{Tr}_{\lambda_{1,2}} \Big(T^{[1]}\Big(\frac{\lambda_1+\lambda_2}{2}\Big)\Big) s^{(2)}_{(1,\bar{1}),(2,\bar{2})}, \tag{4.9}$$

where

$$T^{[1]}(\lambda) = L_{\bar{2}}(\lambda - \lambda_2 - 1) \cdots L_{\bar{n}}(\lambda - \lambda_n - 1) L_n(\lambda - \lambda_n) \cdots L_2(\lambda - \lambda_2). \tag{4.10}$$

This formula shows that (4.8) will be proved if we verify the relation

$$\begin{aligned}\varpi_\lambda\Big(R_{2,1}(-\lambda)R_{\bar 1,\bar 2}(\lambda)L_1\Big(-\frac{\lambda}{2}\Big)L_{\bar 1}\Big(-\frac{\lambda}{2}-1\Big)s^{(2)}_{(1\bar 1),(2\bar 2)}\Big)\\ =\varpi_\lambda\Big(L_2\Big(\frac{\lambda}{2}\Big)L_{\bar 2}\Big(\frac{\lambda}{2}-1\Big)s^{(2)}_{(1\bar 1),(2\bar 2)}\Big),\end{aligned}\tag{4.11}$$

where $\lambda_{1,2}$ is denoted by λ. As before, we write $A\sim B$ to mean $\varpi_\lambda(A)=\varpi_\lambda(B)$. By Lemma 4.1, (4.11) can be rewritten as

$$\begin{aligned}&R_{2,1}(-\lambda)R_{\bar 1,\bar 2}(\lambda)\frac{1}{\lambda-2}r^{(2,1)}_{(2,\bar 2),\bar 1}\Big(-\lambda+\frac12\Big)L^{(2)}_{(1,\bar 1)}\Big(\frac{-\lambda+1}{2}\Big)s^{(2)}_{(1\bar 1),(2\bar 2)}\\ &\sim-\frac{1}{\lambda+2}r^{(2,1)}_{(1,\bar 1),\bar 2}\Big(\lambda+\frac12\Big)L^{(2)}_{(2,\bar 2)}\Big(\frac{\lambda+1}{2}\Big)s^{(2)}_{(1\bar 1),(2\bar 2)},\end{aligned}$$

which is further equivalent to

$$r^{(2,2)}_{(1\bar 1),(2\bar 2)}(\lambda)L^{(2)}_{(2,\bar 2)}\Big(\frac{\lambda+1}{2}\Big)s^{(2)}_{(1\bar 1),(2\bar 2)}\sim(\lambda+2)(\lambda+1)L^{(2)}_{(1,\bar 1)}\Big(\frac{-\lambda+1}{2}\Big)s^{(2)}_{(1\bar 1),(2\bar 2)},\tag{4.12}$$

by (2.17) and (2.19). The proof of (4.12) is similar to that of (4.3). We use the spectral expansion

$$r^{(2,2)}(\lambda)=\lambda(\lambda+1)I+2(\lambda+1)P^{(2)}-6\lambda K^{(2)},\tag{4.13}$$

which follows from (4.4) and the formula $(1/2)\sum_{a,b}(S_aS_b)_{(1,\bar 1)}\otimes(S^aS^b)_{(2,\bar 2)}=2-6K^{(2)}$. Here $P^{(2)},K^{(2)}$ have the same meaning as in (4.4).

Substituting (4.13) in the left-hand side of (4.12), we obtain

$$\begin{aligned}&r^{(2,2)}_{(1\bar 1),(2\bar 2)}(\lambda)L^{(2)}_{(2,\bar 2)}\Big(\frac{\lambda+1}{2}\Big)s^{(2)}_{(1\bar 1),(2\bar 2)}\\ &=\Big(\lambda(\lambda+1)L^{(2)}_{(1,\bar 1)}\Big(-1-\frac{\lambda+1}{2}\Big)\\ &\quad+2(\lambda+1)L^{(2)}_{(1,\bar 1)}\Big(\frac{\lambda+1}{2}\Big)-2\lambda(\lambda+1)(\lambda+2)\Big)s^{(2)}_{(1\bar 1),(2\bar 2)}.\end{aligned}$$

Now, relation (4.12) can be verified by a straightforward calculation. □

Lemma 4.3. *The $X^{[i,j]}$ obey the transformation rules* (3.18).

Proof. Except for the case where $k=i=j-1$, the claim is an immediate consequence of the definition and the Yang–Baxter relation. The nontrivial case of $k=i=j-1$ follows from Lemma 4.2 and the Yang–Baxter relation. □

The next lemma concerns the pole structure of $X^{[i,j]}$.

Lemma 4.4. *The function*

$$\frac{\prod_{p(\neq i,j)}\lambda_{i,p}\lambda_{j,p}}{\lambda_{i,j}^2-1}\cdot X^{[i,j]}(\lambda_1,\dots,\lambda_n)$$

is a polynomial.

Proof. First, we consider $X^{[1,2]}$ given by (4.9). We prove that the pole at $\lambda_1=\lambda_2$ is spurious and that $X^{[1,2]}$ has zeros at $\lambda_1=\lambda_2\pm1$. Because of property (3.9) of Tr_x, it suffices to show that $\varepsilon\big(T^{[1]}\big((\lambda_1+\lambda_2)/2\big)\big)$ is divisible by $\lambda_{1,2}^2-1$. This is indeed the case, because $\varepsilon\big(L_{\bar 2}(\lambda_{1,2}/2-1)\big)=(\lambda_{1,2}-1)/2$ and $\varepsilon\big(L_2(\lambda_{1,2}/2)\big)=(\lambda_{1,2}+1)/2$.

Next, consider the case where $i = 1$ and j is general. We use the relationship between $X^{[1,j]}$ and $X^{[1,2]}$,

$$\begin{aligned} &X^{[1,j]}(\lambda_1,\ldots,\lambda_n)\\ &\quad = R_{j,j-1}(\lambda_{j,j-1})\cdots R_{j,2}(\lambda_{j,2})R_{\overline{j-1},\bar{j}}(\lambda_{j-1,j})\cdots R_{\bar{2},\bar{j}}(\lambda_{2,j})\\ &\qquad \times X^{[1,2]}(\lambda_1,\lambda_j,\lambda_2,\ldots,\widehat{\lambda_j},\ldots,\lambda_n)_{1,j,2,\ldots,\hat{j},\ldots,n,\bar{n},\ldots,\hat{\bar{j}},\ldots,\bar{2},\bar{j},\bar{1}}. \end{aligned} \tag{4.14}$$

We need to show that the poles $\lambda_i = \lambda_j \pm 1$ $(2 \le i < j)$ contained in the R-matrices are in fact spurious. Taking the residue of $X^{[1,j]}$ at $\lambda_i = \lambda_j + 1$ $(j > i)$, we find the following fragment:

$$\mathrm{Tr}_{\lambda_{1,j}}\Big(T^{[1]}\Big(\frac{\lambda_1+\lambda_j}{2}\Big)\Big)R_{j,j-1}(\lambda_{j,j-1})\cdots R_{j,i+1}(\lambda_{j,i+1})\mathcal{P}^-_{i,j}.$$

We move the R-matrices to the left by using the Yang–Baxter relation, which permutes the L-operators in $T^{[1]}$. This will bring the L_j next to L_i. Thus,

$$L_j\Big(\frac{\lambda_{1,j}}{2}\Big)L_i\Big(\frac{\lambda_{1,j}}{2}-1\Big)\mathcal{P}^-_{i,j} = 0,$$

where we have used the equation for the quantum determinant (2.15) and the fact that the "dimension" of the auxiliary space is equal to $\lambda_{1,j}$. The pole at $\lambda_i = \lambda_j - 1$ $(j > i)$ can be treated similarly.

For general $X^{[i,j]}$, we use the symmetry (4.8) to obtain two representations

$$\begin{aligned} &X^{[i,j]}(\lambda_1,\ldots,\lambda_n)\\ &\quad = R_{i,i-1}\cdots R_{i,1}R_{\overline{i-1},\bar{i}}\cdots R_{\bar{1},\bar{i}}X^{[1,j]}(\lambda_i,\lambda_1,\ldots,\widehat{\lambda_i},\ldots,\lambda_n)_{i,1,\ldots,\hat{i},\ldots,n,\bar{n},\ldots,\hat{\bar{i}},\ldots,\bar{1},\bar{i}}\\ &\quad = R_{j,j-1}\cdots R_{j,1}R_{i+1,i}\cdots R_{j-1,i}R_{\overline{j-1},\bar{j}}\cdots R_{\bar{1},\bar{j}}R_{\bar{i},\overline{i+1}}\cdots R_{\bar{i},\overline{j-1}}\\ &\qquad \times X^{[1,j]}(\lambda_j,\lambda_1,\ldots,\widehat{\lambda_i},\ldots,\lambda_{j-1},\\ &\qquad\qquad \lambda_i,\lambda_{j+1},\ldots,\lambda_n)_{j,1,\ldots,\hat{i},\ldots,j-1,i,j+1,\ldots,n,\bar{n},\ldots,\overline{j+1},\bar{i},\overline{j-1},\ldots,\hat{\bar{i}},\ldots,\bar{1},\bar{j}}, \end{aligned}$$

where $R_{i,j} = R_{i,j}(\lambda_{i,j})$ and $R_{\bar{j},\bar{i}} = R_{\bar{j},\bar{i}}(\lambda_{j,i})$. From these expressions, it is clear that the poles arising from the R-matrices are spurious. □

The following lemma shows that the function $X^{[i,j]}(\lambda_1,\ldots,\lambda_n)$ is regular at $\lambda_k = \infty$ when it is applied to a singlet vector.

Lemma 4.5. *For any vector* $v \in \big(W^{[i,j]}\big)^{\mathfrak{sl}_2}$ *and* $k = 1,\ldots,n$, *we have*

$$\frac{1}{\lambda_{i,j}^2-1}X^{[i,j]}(\lambda_1,\ldots,\lambda_n)\cdot v = O(1) \quad (\lambda_k \to \infty).$$

Proof. Since $R_{i,j}(\lambda_{i,j})R_{\bar{j},\bar{i}}(\lambda_{j,i})$ is holomorphic and invertible at $\lambda_{i,j} = \infty$, it suffices to consider the case where $(i,j) = (1,2)$. For $k \ge 3$, the statement follows readily from the definition (3.15). We shall treat the case of $k = 1$. The case where $k = 2$ reduces to this case by the symmetry (4.8).

As before, we write $A \sim B$ for $\varpi_x(A) = \varpi_x(B)$. We show the following: for any $\mu_1,\ldots,\mu_{2N} \in \mathbb{C}$ and $u \in \big(V^{\otimes 2N}\big)^{\mathfrak{sl}_2}$, there exist $c_{pqrs} \in \mathbb{C}$ such that

$$L_1(\mu_1+x/2)\cdots L_{2N}(\mu_{2N}+x/2)u \sim \Big(\sum_{p+q+r+s\le N} c_{pqrs}H^pE^qF^rx^s\Big)u. \tag{4.15}$$

Then, the claim follows from property (3.10) of Tr_x by taking $N = n-2$.

We may assume that the μ_j are mutually distinct. It suffices to prove (4.15) for $u = u_\sigma = \sigma u_1$, where $u_1 = s^{(1)}_{1,2}s^{(1)}_{3,4}\cdots s^{(1)}_{2N-1,2N}$ and $\sigma \in \mathfrak{S}_{2N}$. We use induction on the length $\ell(\sigma)$. First, let $\sigma = \mathrm{id}$. Set $\Omega = \sum_{a=1}^{3} S_a \otimes \pi^{(1)}(S^a)$. Since $\Omega_2 s_{1,2}^{(1)} = -\Omega_1 s_{1,2}^{(1)}$ and $\Omega^2 + \Omega = \frac{1}{2} C \otimes \mathrm{id} \sim (x^2-1)/4$, we have

$$
\begin{aligned}
&L_1(\mu_1 + x/2) L_2(\mu_2 + x/2) s_{1,2}^{(1)} \\
&\qquad \sim \Bigl(\mu_1\mu_2 + \frac{x+1}{2}(\mu_1+\mu_2) + (\mu_2-\mu_1+1)\Omega_1 + \frac{(x+1)^2}{4} - \frac{x^2-1}{4}\Bigr) s_{1,2}^{(1)}.
\end{aligned}
$$

We see that the term x^2 cancels. Hence, we have (4.15) for $u = u_1$.

Suppose (4.15) is true for $u = u_\sigma$, and consider $\tau = (i, i+1)\sigma$ with $\ell(\tau) = \ell(\sigma) + 1$. Using the Yang–Baxter relation, we obtain

$$
\begin{aligned}
&P_{i,i+1} r_{i,i+1}(\mu_i - \mu_{i+1}) L_1\Bigl(\mu_1 + \frac{x}{2}\Bigr) \cdots L_{2N}\Bigl(\mu_{2N} + \frac{x}{2}\Bigr) \cdot u_\sigma \\
&\quad = L_1\Bigl(\mu_1 + \frac{x}{2}\Bigr) \cdots L_i\Bigl(\mu_{i+1} + \frac{x}{2}\Bigr) \\
&\qquad \times L_{i+1}\Bigl(\mu_i + \frac{x}{2}\Bigr) \cdots L_{2N}\Bigl(\mu_{2N} + \frac{x}{2}\Bigr) \cdot \bigl((\mu_i - \mu_{i+1}) u_\tau + u_\sigma\bigr).
\end{aligned}
$$

By the induction hypothesis, the left-hand side and the second term on the right-hand side have degree at most N when they are projected by ϖ_x. Therefore, the statement is true also for $u = u_\tau$. □

Using the properties of $X^{[i,j]}$, we check that the pole structure of the recursion (3.22) agrees with that of h_n stated in (2.32). Namely, the following statement is true.

Proposition 4.6. *Assume that h_{n-1} and h_{n-2} satisfy the analyticity property as stated in* (2.32)*. Then the same is true for h_n given by* (3.22)*.*

Proof. In Lemma 4.4 we showed that, on the right-hand side of (3.22), the only possible poles other than $\lambda_{i,j} \in \mathbb{Z}\backslash\{0, \pm 1\}$ are $\lambda_i = \lambda_j$. We rewrite the recursion as follows:

$$
\begin{aligned}
&h_n(\lambda_1, \ldots, \lambda_n)_{1,\ldots,n,\bar{n},\ldots,\bar{1}} \\
&\quad = \frac{1}{2} s_{(1\bar{1})}^{(1)} \cdot h_{n-1}(\lambda_2, \ldots, \lambda_n)_{2,\ldots,n,\bar{n},\ldots,\bar{2}} \\
&\qquad - \sum_{j=2}^{n} \frac{1}{\prod_{p\neq 1,j} \lambda_{jp} \prod_{i=2}^{j-1} (1 - \lambda_{ij}^2)} \\
&\qquad \times \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{d\sigma}{\prod_{p=1}^{n} (\sigma - \lambda_p)} \cdot \frac{\omega(\sigma - \lambda_j)}{(\sigma - \lambda_j)^2 - 1} \mathrm{Tr}_{\sigma-\lambda_j}\Bigl(T^{[1]}\Bigl(\frac{\sigma+\lambda_j}{2}\Bigr)\Bigr) \\
&\qquad\qquad \times r_{j,j-1}(\lambda_{j,j-1}) \cdots r_{j,2}(\lambda_{j,2}) r_{\overline{j-1},\bar{j}}(\lambda_{j-1,j}) \cdots r_{\bar{2}\bar{j}}(\lambda_{2j}) \\
&\qquad\qquad \times s_{(1\bar{1}),(j\bar{j})}^{(2)} h_{n-2}(\lambda_2, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n)_{2,\ldots,\hat{j},\ldots,n,\bar{n},\ldots,\hat{\bar{j}},\ldots,\bar{2}}.
\end{aligned}
$$

Here the contour $\mathcal{C}$ goes around $\lambda_1, \ldots, \lambda_n$. This expression shows that the poles at $\lambda_{1p} = 0$ are spurious.

Consider the pole at $\lambda_k = \lambda_j$ with $k > j > 1$. We have

$$\begin{aligned}
&\mathrm{res}_{\lambda_k=\lambda_j} h_n(\lambda_1,\dots,\lambda_n) \\
&\quad= \frac{1}{\prod_{p\neq 1,j,k}\lambda_{jp}} \\
&\qquad\times \frac{1}{2\pi i}\oint \frac{d\sigma}{\prod_{p=1}^n(\sigma-\lambda_p)}\cdot\frac{\omega(\sigma-\lambda_j)}{(\sigma-\lambda_j)^2-1}\,\mathrm{Tr}_{\sigma-\lambda_j}\Big(T^{[1]}\Big(\frac{\sigma+\lambda_j}{2}\Big)\Big) \\
&\qquad\quad\times\Big\{R_{j,j-1}(\lambda_{j,j-1})\cdots R_{j,2}(\lambda_{j,2})R_{\overline{j-1},\bar{j}}(\lambda_{j-1,j})\cdots R_{\bar{2}\bar{j}}(\lambda_{2j}) \\
&\qquad\qquad\times s^{(2)}_{(1\bar{1}),(j\bar{j})}h_{n-2}(\lambda_2,\dots,\widehat{\lambda_j},\dots,\lambda_n)_{2,\dots,\hat{j},\dots,n,\bar{n},\dots,\hat{\bar{j}},\dots,\bar{2}}\Big|_{\lambda_k=\lambda_j} \\
&\qquad\qquad- R_{k,k-1}(\lambda_{j,k-1})\cdots R_{k,j+1}(\lambda_{j,j+1})R_{\overline{k-1},\bar{k}}(\lambda_{k-1,j})\cdots R_{\overline{j+1},\bar{k}}(\lambda_{j+1,j}) \\
&\qquad\qquad\times P_{j,k}P_{\bar{j},\bar{k}}R_{k,j-1}(\lambda_{j,j-1})\cdots R_{k,2}(\lambda_{j,2})R_{\overline{j-1},\bar{k}}(\lambda_{j-1,j})\cdots R_{\bar{2}\bar{k}}(\lambda_{2j}) \\
&\qquad\qquad\qquad\times s^{(2)}_{(1\bar{1}),(k\bar{k})}h_{n-2}(\lambda_2,\dots,\widehat{\lambda_k},\dots,\lambda_n)_{2,\dots,\hat{k},\dots,n,\bar{n},\dots,\hat{\bar{k}},\dots,\bar{2}}\Big\}.
\end{aligned}$$

In the second term inside the braces, we move $P_{j,k}P_{\bar{j},\bar{k}}$ to the right, and then we apply the symmetry (2.29). After that, the second term cancels the first. □

4.2. Proof of the *Ansatz*.

In this subsection we prove the *Ansatz* (2.4).

Proof of Theorem 3.2. The claim is clear for $n = 0, 1$. By induction, assume that it is true for $h_{n'}$ with $n' < n$. We use the recursion formula in the form

(4.16)
$$\begin{aligned}
&h_n(\lambda_1,\dots,\lambda_n)_{1,\dots,n,\bar{n},\dots,\bar{1}} \\
&\quad= \frac{1}{2}s^{(1)}_{1,\bar{1}}h_{n-1}(\lambda_2,\dots,\lambda_n)_{2,\dots,n,\bar{n},\dots,\bar{2}} \\
&\qquad+\sum_{j=2}^n\frac{\omega(\lambda_{1,j})}{1-\lambda_{1,j}^2}X^{[1,j]}(\lambda_1,\dots,\lambda_n)h_{n-2}(\lambda_2,\dots,\widehat{\lambda_j},\dots,\lambda_n)_{2,\dots,\hat{j},\dots,n,\bar{n},\dots,\hat{\bar{j}},\dots,\bar{2}} \\
&\qquad+\sum_{\substack{2\le j,k\le n\\ j\neq k}}\frac{\omega(\lambda_{k,j})}{\lambda_{k,1}(1-\lambda_{k,j}^2)}\,\mathrm{res}_{\sigma=\lambda_k}X^{[1,j]}(\sigma,\lambda_2,\dots,\lambda_n) \\
&\qquad\qquad\times h_{n-2}(\lambda_2,\dots,\widehat{\lambda_j},\dots,\lambda_n)_{2,\dots,\hat{j},\dots,n,\bar{n},\dots,\hat{\bar{j}},\dots,\bar{2}}.
\end{aligned}$$

By the induction hypothesis, the right-hand side becomes a linear combination of elements of the form

$$\prod_{p=1}^m\omega(\lambda_{i_p}-\lambda_{j_p})f_{n,I,J}(\lambda_1,\dots,\lambda_n),$$

with some rational function $f_{n,I,J}(\lambda_1,\dots,\lambda_n)$ and $1\le i_1\le\cdots\le i_m\le n$, $1\le i_p<j_p\le n$. Since the products of $\omega(\lambda)$'s are linearly independent over the field $\mathbb{C}(\lambda_1,\dots,\lambda_n)$ of rational functions, this representation is unique. If there is a term for which $i_1,\dots,i_m,j_1,\dots,j_m$ are not distinct, then the symmetry (2.29) of h_n implies that there is also a term such that the index 1 appears more than once in $i_1,\dots,i_m,j_1,\dots,j_m$. However, the terms with $i_1 = 1$ only arise from the first term of (4.16), and the corresponding indices are distinct. Therefore, we have (3.24). □

Proof of Theorem 3.3. From the above proof, the recursion relations (3.25) and (3.26) for $f_{n,I,J}$ are clear. Since the rational coefficients in (3.24) are unique, the $\mathfrak{sl}_2$-invariance (3.27) of $f_{n,I,J}$ follows from that of h_n. Similarly, the exchange symmetry (3.28) follows from the symmetry (2.29) of h_n by comparing the coefficients of the ω's. The regularity

(3.29) at ∞ is a consequence of Lemma 4.5 and the $\mathfrak{sl}_2$-invariance (3.27). Finally, the pole structure (3.30) follows from the recursion relations (3.25), (3.26) and Lemma 4.4. $\square$

4.3. Calculation of the residues. In this subsection, we compute the residues of $h_n(\lambda_1,\dots,\lambda_n)$ with respect to λ_1. Our goal is Propositions 4.10 and 4.11. Since the calculation is long, we outline the proof before starting.

We calculate the residues of the meromorphic function $h_n(\lambda_1,\dots,\lambda_n)$ at its poles. The poles are located at the points of the form $\lambda_{1,j}=k\in\mathbb{Z}\backslash\{0,\pm1\}$. Then we show that the right-hand side of (3.22) has the same residues at these poles:

(4.17)

$$\begin{aligned}&\mathrm{res}_{\lambda_{1j}=k}\, h_n(\lambda_1,\dots,\lambda_n)_{1\cdots n\bar{n}\cdots\bar{1}}\\ &\quad=\mathrm{res}_{\lambda_{1j}=k}\left\{\frac{\omega(\lambda_{1,j})}{1-\lambda_{1,j}^2}X^{[1,j]}(\lambda_1,\dots,\lambda_n)\right\}h_{n-2}(\lambda_2,\dots,\widehat{\lambda_j},\dots,\lambda_n)_{2,\dots,\hat{j},\dots,n,\bar{n},\dots,\hat{\bar{j}},\dots,\bar{2}}.\end{aligned}$$

A remarkable fact is that the residues at infinitely many points can be described recursively by using a single analytic function $\omega(\lambda_{1,j})X^{[1,j]}(\lambda_1,\dots,\lambda_n)$. Thus, we show that the difference between the left- and right-hand sides is an entire function of λ_1. Showing that it vanishes at $\lambda_1\to\infty$, we obtain equality.

The calculation of residues goes as follows. Suppose $j=2$. Using (2.30) repeatedly, we can express $h_n(\lambda_1-k-1,\lambda_2,\dots,\lambda_n)$ in terms of several A_1 and $A_{\bar{1}}$ with shifted arguments acting on $h_n(\lambda_1-1,\lambda_2,\dots,\lambda_n)$. This expression allows us to calculate the residue of $h_n(\lambda_1,\lambda_2,\dots,\lambda_n)$ at the pole $\lambda_1=\lambda_2-k-1$. If $k=2$, for instance, then we have

$$\begin{aligned}&\mathrm{res}_{\lambda_1=\lambda_2-3}\, h_n(\lambda_1,\lambda_2,\dots,\lambda_n)\\ &\quad=\mathrm{res}_{\lambda_1=\lambda_2}\, h_n(\lambda_1-3,\lambda_2,\dots,\lambda_n)\\ &\quad=\mathrm{res}_{\lambda_1=\lambda_2}\{A_{\bar{1}}(\lambda_1-2,\dots)A_1(\lambda_1-1,\dots)h_n(\lambda_1-1,\dots)\}.\end{aligned}$$

Because of (2.32), the pole at $\lambda_1=\lambda_2$ in the last expression comes only from one of the R-matrices in $A_1(\lambda_1-1,\dots)$, i.e., from $R_{1,2}(\lambda_{1,2}-1)$. Its residue at $\lambda_{1,2}=0$ is proportional to the projection $\mathcal{P}^-_{1,2}$, so that we can use (2.34). In this way, for any k we get an expression for $\mathrm{res}_{\lambda_1=\lambda_2-k-1}\, h_n(\lambda_1,\dots,\lambda_n)$; a bunch of R-matrices acting on $s^{(1)}_{\gamma 2}s^{(1)}_{\bar{\gamma}\bar{2}}h_{n-2}(\lambda_3,\dots,\lambda_n)$, where $\gamma=1$ (k is even) or $\bar{1}$ (k is odd).

Next, we rewrite the product of R-matrices by using the Yang–Baxter relation. They act on the spaces V_α indexed by $\alpha=1,\dots,n,\bar{n},\dots,\bar{1}$. Three groups of indexes, $\{1,\bar{1}\}$, $\{2,\bar{2}\}$, and $\{3,\bar{3},\dots,n,\bar{n}\}$, play separate roles in the product. An R-matrix $R_{i,j}$ acts on two components V_i and V_j. The first component V_i will be called the *auxiliary space* and the second component V_j, the *quantum space*. For any R-matrix contained in A_α, the auxiliary space is indexed by 1 or $\bar{1}$, and the quantum space, by the other two groups. The second and the third groups are distinct when the residue at $\lambda_{1,2}=0$ is calculated and the vector $s^{(1)}_{\gamma 2}s^{(1)}_{\bar{\gamma}\bar{2}}h_{n-2}(\lambda_3,\dots,\lambda_n)$ is created.

Our goal is to rewrite the product of R-matrices with the help of the transfer matrix $t^{(k)}(\lambda)=\mathrm{tr}_{V^{(k)}}\,\pi^{(k)}\left(T^{[1]}(\lambda)\right)\in\mathrm{End}(W^{[1]})$, where $T^{[1]}(\lambda)$ is given by (3.13), and $W^{[1]}$ by (3.11). We compare the product of transfer matrices $t^{(1)}(\lambda_1-k+1)\cdots t^{(1)}(\lambda_1)$ and the product $A_{\bar{1}}(\lambda_1-k,\dots)A_1(\lambda_1-k+1,\dots)\cdots A_\gamma(\lambda_1-1,\dots)$. In the former, the matrix product is taken only on the quantum spaces indexed by $2,\dots,n,\bar{n},\dots,\bar{2}$; in the latter, not only on the quantum spaces but also on the auxiliary spaces indexed by 1 and $\bar{1}$.

In a graphical representation of the product, we draw vertical lines for quantum spaces and horizontal lines for auxiliary spaces. In the former product we have k horizontal lines, and they form closed circles reflecting the trace; in the latter product we have only two horizontal lines corresponding to 1 and $\bar{1}$. They form spirals. We can rewrite the latter as a trace by introducing k auxiliary spaces indexed by $\alpha_1, \ldots, \alpha_k$. We do the following procedure. We cut the horizontal lines in front of the vertical line corresponding to $\bar{3}$, obtaining k separate horizontal lines. We rename these lines by $\alpha_1, \ldots, \alpha_k$. Specifically, we replace the spaces V_1 and $V_{\bar{1}}$ on these lines by $V_{\alpha_1}, \ldots, V_{\alpha_k}$. We recover the original product by taking the traces on these new auxiliary spaces.

After some manipulation involving the Yang–Baxter relation, the entire expression splits into two parts: the fused monodromy operator

$$\pi^{(k)}_{(\alpha_1,\ldots,\alpha_k)}\Big(T^{[1,2]}(\lambda_2-(k+1)/2)\Big)$$

and the rest. The latter acts on the vector $s^{(1)}_{\gamma 2} s^{(1)}_{\overline{\gamma}\overline{2}}$. This action can be rewritten as

$$\pi^{(k)}_{(\alpha_1,\ldots,\alpha_k)}\Big(L_2\Big(-\frac{k+1}{2}\Big)L_{\bar{2}}\Big(-\frac{k+3}{2}\Big)\Big)s^{(2)}_{(1\bar{1}),(2\bar{2})}.$$

Combining these two expressions inside the trace $\mathrm{tr}_{\alpha_1,\ldots,\alpha_k}$, we obtain (4.17) for $j=2$ and $k \le -3$. The calculation of the residues at the rest of the poles can be done by using symmetries.

We set

$$t_\alpha(\lambda_2) = r_{\alpha\bar{3}}(\lambda_{23}-1)\cdots r_{\alpha\bar{n}}(\lambda_{2n}-1)r_{\alpha n}(\lambda_{2n})\cdots r_{\alpha 3}(\lambda_{23}), \tag{4.18}$$

and recall that

$$T^{[1,2]}(\lambda) = L_{\bar{3}}(\lambda-\lambda_3-1)\cdots L_{\bar{n}}(\lambda-\lambda_n-1)L_n(\lambda-\lambda_n)\cdots L_3(\lambda-\lambda_3),$$
$$T^{[1]}(\lambda) = L_{\bar{2}}(\lambda-\lambda_2-1)T^{[1,2]}(\lambda)L_2(\lambda-\lambda_2).$$

Using (2.17), we obtain

$$\begin{aligned} &t_{\alpha_k}(\lambda_2-k)\cdots t_{\alpha_1}(\lambda_2-1)\mathcal{P}^+_{\alpha_1,\ldots,\alpha_k} \\ &\qquad = \frac{\prod_{j=3}^n\prod_{l=1}^k\{(\lambda_{2j}-l)^2-1\}}{\prod_{j=3}^n \lambda_{2j}(\lambda_{2j}-k-1)}\pi^{(k)}_{(\alpha_1,\ldots,\alpha_k)}\Big(T^{[1,2]}\Big(\lambda_2-\frac{k+1}{2}\Big)\Big). \end{aligned} \tag{4.19}$$

Here $\mathcal{P}^+_{\alpha_1,\ldots,\alpha_k}$ stands for the projection onto the completely symmetric tensors.

Now we start the calculation. We rewrite A_α given by (2.27):

$$A_\alpha(\lambda_1,\lambda_2,\ldots,\lambda_n) = \frac{-1}{\prod_{j=2}^n(\lambda_{1j}^2-1)}r_{\alpha\bar{2}}(\lambda_{12}-1)t_\alpha(\lambda_1)r_{\alpha 2}(\lambda_{12}). \tag{4.20}$$

In this form it is easy to show that

$$\mathrm{res}_{\lambda_{12}=-1}\,A_\alpha(\lambda_1,\ldots,\lambda_n) = \frac{-1}{\prod_{j=3}^n\lambda_{2j}(\lambda_{2j}-2)}r_{\alpha\bar{2}}(-2)t_\alpha(\lambda_2-1)\mathcal{P}^-_{\alpha 2}. \tag{4.21}$$

Using (2.30), (4.21), and (2.34), we obtain

$$\begin{aligned} &\mathrm{res}_{\lambda_{12}=-2}\,h_n(\lambda_1,\lambda_2,\ldots,\lambda_n)_{12\cdots n\bar{n}\cdots\bar{2}\bar{1}} \\ &\quad = \frac{1}{2\prod_{j=3}^n\lambda_{2,j}(\lambda_{2,j}-2)}r_{\bar{1}\bar{2}}(-2)t_{\bar{1}}(\lambda_2-1)s^{(1)}_{\bar{1}2}s^{(1)}_{1\bar{2}}h_{n-2}(\lambda_3,\ldots,\lambda_n)_{3\cdots n\bar{n}\cdots\bar{3}}. \end{aligned}$$

Now we consider the residue at $\lambda_{12} = -k-1$, where $k \ge 2$. We set

$$\gamma = \begin{cases} 1 & \text{if } k \text{ is even},\\ \bar{1} & \text{if } k \text{ is odd}.\end{cases} \tag{4.22}$$

The only poles in (4.20) are $\lambda_{1j} = \pm 1$. Applying (2.30) repeatedly, we obtain

$$\mathrm{res}_{\lambda_{12}=-k-1}\,h_n(\lambda_1,\lambda_2,\ldots,\lambda_n)_{12\cdots n\bar{n}\cdots\bar{2}\bar{1}} = X_k h_{n-2}(\lambda_3,\ldots,\lambda_n)_{3\cdots n\bar{n}\cdots\bar{3}} \tag{4.23}$$

for $k \geq 1$, where

$$\begin{aligned} X_k = \frac{1}{2\prod_{j=3}^n \lambda_{2,j}(\lambda_{2,j}-2)} & A_{\bar{1}}(\lambda_2-k,\lambda_2,\dots,\lambda_n)A_1(\lambda_2-k+1,\lambda_2,\dots,\lambda_n) \\ & \times\cdots\times A_{\bar{\gamma}}(\lambda_2-2,\lambda_2,\dots,\lambda_n)r_{\gamma\bar{2}}(-2)t_\gamma(\lambda_2-1)s^{(1)}_{\gamma 2}s^{(1)}_{\bar{\gamma}\bar{2}}. \end{aligned} \tag{4.24}$$

If $k \geq 2$, we use (4.20) to rewrite (4.24) as

$$X_k = \frac{(-1)^{k+1}}{(k-1)!(k+1)!\prod_{l=1}^k\prod_{j=3}^n((\lambda_{2,j}-l)^2-1)} r_{\bar{1}\bar{2}}(-k-1)r_{1\bar{2}}(-k)\bullet Y_k, \tag{4.25}$$

where

$$\begin{aligned} Y_k = {} & t_{\bar{1}}(\lambda_2-k)r_{\bar{1}2}(-k)r_{\bar{1}\bar{2}}(-k+1)t_1(\lambda_2-k+1)r_{12}(-k+1)r_{1\bar{2}}(-k+2) \\ & \times\cdots\times t_\gamma(\lambda_2-3)r_{\gamma 2}(-3)r_{\gamma\bar{2}}(-2)t_{\bar{\gamma}}(\lambda_2-2)r_{\bar{\gamma}2}(-2)t_\gamma(\lambda_2-1)s^{(1)}_{\gamma 2}s^{(1)}_{\bar{\gamma}\bar{2}}. \end{aligned} \tag{4.26}$$

We use the following simple identity, which follows from (2.15):

$$r_{12}(-1)r_{13}(-2)r_{23}(-1) = 0. \tag{4.27}$$

Lemma 4.7. *We can insert $\mathcal{P}^+_{1\bar{1}}$ at the position $\bullet$ in* (4.25).

Proof. Since $\mathcal{P}^+_{1\bar{1}} + \mathcal{P}^-_{1\bar{1}} = \mathrm{id}_{1\bar{1}}$ and $-2\mathcal{P}^-_{1\bar{1}} = r_{1\bar{1}}(-1)$, it suffices to show that if we insert $r_{1\bar{1}}(-1)$ at the position $\bullet$, then (4.25) vanishes. Note that $s^{(1)} = -\frac{1}{2}r(-1)s^{(1)}$. If $\gamma = 1$, we can find $r_{1\bar{1}}(-1)r_{\bar{1}2}(-2)r_{12}(-1)$ therein by using the Yang–Baxter relation, and if $\gamma = \bar{1}$, we find $r_{1\bar{1}}(-1)r_{1\bar{2}}(-2)r_{\bar{1}2}(-1)$ instead. □

Then, using (2.19), we obtain

$$X_k = \frac{(-1)^k k}{(k-1)!(k+1)!\prod_{l=1}^k\prod_{j=3}^n((\lambda_{2,j}-l)^2-1)} r^{(2,1)}_{(1\bar{1}),\bar{2}}\Big(-k-\frac{1}{2}\Big)Y_k. \tag{4.28}$$

Now we use the identity

$$t_1(\lambda-j) = \mathrm{tr}_{V_\alpha}\left(t_\alpha(\lambda-j)P_{\alpha,1}\right). \tag{4.29}$$

We can rewrite (4.26) as

$$\begin{aligned} Y_k = \mathrm{tr}_{\alpha_1\cdots\alpha_k}\Big(& t_{\alpha_k}(\lambda_2-k)\cdots t_{\alpha_1}(\lambda_2-1)\bullet \\ & \times r_{\alpha_k 2}(-k)\cdots r_{\alpha_3 2}(-3)r_{\alpha_k\bar{2}}(-k+1)\cdots r_{\alpha_3\bar{2}}(-2) \\ & \times r_{\alpha_2 2}(-2)P_{\alpha_k\bar{1}}P_{\alpha_{k-1}1}\cdots P_{\alpha_2\bar{\gamma}}P_{\alpha_1\gamma}\bullet\Big)s^{(1)}_{\gamma 2}s^{(1)}_{\bar{\gamma}\bar{2}}, \end{aligned} \tag{4.30}$$

where $\mathrm{tr}_{\alpha_1,\dots,\alpha_k}$ stands for the trace over $V_{\alpha_1}\otimes\cdots\otimes V_{\alpha_k}$.

Lemma 4.8. *We can insert $\mathcal{P}^+_{\alpha_k\cdots\alpha_1}$ at the position $\bullet$ in* (4.30) *in both places.*

Proof. We prove the claim for the first $\bullet$. Then, the assertion for the second $\bullet$ follows from the cyclicity of the trace.

We define an element of $\mathrm{End}(W)$ by

$$\widetilde{Y}_k = \mathrm{tr}_{\alpha_k,\dots,\alpha_1}\left((t_k)_{\alpha_k,\dots,\alpha_1}r_{\alpha_k,\dots,\alpha_1;2,\bar{2}}P_{\alpha_k\bar{1}}P_{\alpha_{k-1}1}\cdots P_{\alpha_2\bar{\gamma}}P_{\alpha_1\gamma}\right), \tag{4.31}$$

where

$$(t_k)_{\alpha_k,\dots,\alpha_1} = t_{\alpha_k}(\lambda_2-k)\cdots t_{\alpha_1}(\lambda_2-1), \tag{4.32}$$

$$r_{\alpha_k,\dots,\alpha_1;2,\bar{2}} = r_{\alpha_k 2}(-k)\cdots r_{\alpha_1 2}(-1)r_{\alpha_k\bar{2}}(-k+1)\cdots r_{\alpha_2\bar{2}}(-1). \tag{4.33}$$

Let t_k^{sym} be the symmetrization of t_k,

$$t_k^{\mathrm{sym}}(v_{\tau_k}\otimes\cdots\otimes v_{\tau_1}) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} t(\lambda_2-k)^{\tau'_k}_{\tau_{\sigma(k)}}\cdots t(\lambda_2-1)^{\tau'_1}_{\tau_{\sigma(1)}}v_{\tau'_k}\otimes\cdots\otimes v_{\tau'_1}. \tag{4.34}$$

By definition, the matrix element $(t_k^{\rm sym})^{\tau'_k\cdots\tau'_1}_{\tau_k\cdots\tau_1}$ depends on $\tau_1,\dots,\tau_k$ only through $\tau_k+\cdots+\tau_1=i$. We write it $(t_k^{\rm sym})^{\tau'_k\cdots\tau'_1}_{i}$. We have the following symmetry of $t_k^{\rm sym}$:

$$(t_k^{\rm sym})^{\tau'_k\cdots\tau'_1}_{i}=(t_k^{\rm sym})^{\tau'_{\sigma(k)}\cdots\tau'_{\sigma(1)}}_{i}\,. \tag{4.35}$$

Let $\widetilde{Y}'_k$ denote the matrix $\widetilde{Y}_k$ in which t_k is replaced with $t_k^{\rm sym}$. We must show that $\widetilde{Y}_k=\widetilde{Y}'_k$. We use the following equivalent definition of $\widetilde{Y}_k$ (and a similar one for $\widetilde{Y}'_k$):

$$\begin{aligned}\widetilde{Y}_k(v_{\varepsilon_1}\otimes v_{\varepsilon_{\bar 1}})=\sum_{\varepsilon'_1,\varepsilon'_{\bar 1}}\sum_{\substack{\tau_1,\dots,\tau_k\\ \tau'_1,\dots,\tau'_{k-2}}}&((t_k)_{\alpha_k,\dots,\alpha_1})^{\varepsilon'_{\bar 1}\varepsilon'_1\tau'_{k-2}\cdots\tau'_1}_{\tau_k\tau_{k-1}\tau_{k-2}\cdots\tau_1}\\ &\times\left(r_{\alpha_k,\dots,\alpha_1;2,\bar 2}\right)^{\tau_k\tau_{k-1}\tau_{k-2}\cdots\tau_1}_{\tau'_{k-2}\cdots\tau'_1\varepsilon_{\bar\gamma}\varepsilon_\gamma}(v_{\varepsilon'_1}\otimes v_{\varepsilon'_{\bar 1}}).\end{aligned} \tag{4.36}$$

We shall show the following identity for each i:

$$\begin{aligned}&\sum_{\substack{\tau_1,\dots,\tau_k\\ \tau_1+\cdots+\tau_k=i\\ \tau'_1,\dots,\tau'_{k-2}}}((t_k)_{\alpha_k,\dots,\alpha_1})^{\varepsilon'_{\bar 1}\varepsilon'_1\tau'_{k-2}\cdots\tau'_1}_{\tau_k\tau_{k-1}\tau_{k-2}\cdots\tau_1}\left(r_{\alpha_k,\dots,\alpha_1;2,\bar 2}\right)^{\tau_k\tau_{k-1}\tau_{k-2}\cdots\tau_1}_{\tau'_{k-2}\cdots\tau'_1\varepsilon_{\bar\gamma}\varepsilon_\gamma}\\ &\quad=\sum_{\substack{\tau_1,\dots,\tau_k\\ \tau_1+\cdots+\tau_k=i}}\Bigg\{\sum_{\tau'_1,\dots,\tau'_{k-2}}(t_k^{\rm sym})^{\varepsilon'_{\bar 1}\varepsilon'_1\tau'_{k-2}\cdots\tau'_1}_{i}\left(r_{\alpha_k,\dots,\alpha_1;2,\bar 2}\right)^{\tau_k\tau_{k-1}\tau_{k-2}\cdots\tau_1}_{\tau'_{k-2}\cdots\tau'_1\varepsilon_{\bar\gamma}\varepsilon_\gamma}\Bigg\}.\end{aligned} \tag{4.37}$$

We prove this by induction on k. The cases of $k=2,3$ are immediate: by using the relations

$$r_{\alpha_2,\alpha_1}(-1)r_{\alpha_2,2}(-2)r_{\alpha_1,2}(-1)=0, \tag{4.38}$$

$$r_{\alpha_3\alpha_2}(-1)r_{\alpha_3 2}(-3)r_{\alpha_2 2}(-2)r_{\alpha_3\bar 2}(-2)r_{\alpha_2\bar 2}(-1)=0, \tag{4.39}$$

we see that $\left(r_{\alpha_k,\dots,\alpha_1;2,\bar 2}\right)^{\tau_k\tau_{k-1}\tau_{k-2}\cdots\tau_1}_{\tau'_{k-2}\cdots\tau'_1\varepsilon_{\bar\gamma}\varepsilon_\gamma}$ is totally symmetric in $\tau_1,\dots,\tau_k$. For $k\geq 4$ the last expression is not totally symmetric.

The identity in question is clear if $i=k$ or $i=-k$, because there is only one term for the summation of $\tau_1,\dots,\tau_k$, and t_k and $t_k^{\rm sym}$ are the same in this sector. Now, assume that $i\neq\pm k$.

Using the Yang–Baxter equation and (4.38), (4.39), we see that the summand $\{\cdots\}$ on the right-hand side of (4.37) is independent of $\tau_1,\tau_2,\dots,\tau_k$. Since $i\neq\pm k$, we can choose it as $\tau_k=+$ and $\tau_{k-1}=-$. Then, the right-hand side becomes

$$\begin{aligned}\frac{4k(k-1)}{(k-i)(k+i)}\sum_{\substack{\tau_1,\dots,\tau_{k-2}\\ \tau_1+\cdots+\tau_{k-2}=i}}\Bigg\{\sum_{\tau'_1,\dots,\tau'_{k-2}}&\Big(\frac{k+i}{2k}t(\lambda_2-k)^{\varepsilon'_{\bar 1}}_{+}(t_{k-1}^{\rm sym})^{\varepsilon'_1\tau'_{k-2}\cdots\tau'_1}_{i-1}\\ &+\frac{k-i}{2k}t(\lambda_2-k)^{\varepsilon'_{\bar 1}}_{-}(t_{k-1}^{\rm sym})^{\varepsilon'_1\tau'_{k-2}\cdots\tau'_1}_{i+1}\Big)\left(r_{\alpha_k,\dots,\alpha_1;2,\bar 2}\right)^{+-\tau_{k-2}\cdots\tau_1}_{\tau'_{k-2}\cdots\tau'_1\varepsilon_{\bar\gamma}\varepsilon_\gamma}\Bigg\}.\end{aligned} \tag{4.40}$$

Consider the first half,

$$\frac{2(k-1)}{k-i}\sum_{\substack{\tau_1,\dots,\tau_k\\ \tau_1+\cdots+\tau_{k-2}=i}}\Bigg\{\sum_{\tau'_1,\dots,\tau'_{k-2}}t(\lambda_2-k)^{\varepsilon'_{\bar 1}}_{+}(t_{k-1}^{\rm sym})^{\varepsilon'_1\tau'_{k-2}\cdots\tau'_1}_{i-1}\left(r_{\alpha_k,\dots,\alpha_1;2,\bar 2}\right)^{+-\tau_{k-2}\cdots\tau_1}_{\tau'_{k-2}\cdots\tau'_1\varepsilon_{\bar\gamma}\varepsilon_\gamma}\Bigg\}. \tag{4.41}$$

By the same argument as before, we show that the summand $\{\cdots\}$ is invariant under permutation of the indices $-,\tau_{k-2},\dots,\tau_1$. Therefore, we can rewrite (4.41) as

$$\sum_{\substack{\tau_1,\dots,\tau_{k-1}\\ \tau_1+\cdots+\tau_{k-1}=i-1}}\Bigg\{\sum_{\tau'_1,\dots,\tau'_{k-2}}t(\lambda_2-k)^{\varepsilon'_{\bar 1}}_{+}(t_{k-1}^{\rm sym})^{\varepsilon'_1\tau'_{k-2}\cdots\tau'_1}_{i-1}\left(r_{\alpha_k,\dots,\alpha_1;2,\bar 2}\right)^{+\tau_{k-1}\tau_{k-2}\cdots\tau_1}_{\tau'_{k-2}\cdots\tau'_1\varepsilon_{\bar\gamma}\varepsilon_\gamma}\Bigg\}. \tag{4.42}$$

Now, by the induction hypothesis, this is equal to

$$\sum_{\substack{\tau_1,\dots,\tau_{k-1}\\ \tau_1+\cdots+\tau_{k-1}=i-1}} \Big\{ \sum_{\tau'_1,\dots,\tau'_{k-2}} (t_k)^{\varepsilon'_{\bar 1}\varepsilon'_1\tau'_{k-2}\cdots\tau'_1}_{+\tau_{k-1}\cdots\tau_1} \big(r_{\alpha_k,\dots,\alpha_1;2,\bar 2}\big)^{+\tau_{k-1}\tau_{k-2}\cdots\tau_1}_{\tau'_{k-2}\cdots\tau'_1\varepsilon_{\bar\gamma}\varepsilon_\gamma} \Big\}. \tag{4.43}$$

In order to rewrite the second half in a similar way, first we should rewrite the indices $+-$ in $\big(r_{\alpha_k,\dots,\alpha_1;2,\bar 2}\big)^{+-\tau_{k-2}\cdots\tau_1}_{\tau'_{k-2}\cdots\tau'_1\varepsilon_{\bar\gamma}\varepsilon_\gamma}$ as $-+$. This is possible by the same argument again. After that, the argument is similar, and we obtain an expression similar to (4.43). Adding these two expressions, we obtain the left-hand side of (4.37). □

Using (4.19), we obtain

$$\begin{aligned} X_k = {}& d_k r^{(2,1)}_{(1\bar 1),\bar 2}\Big(-k-\frac12\Big) \\ & \times \mathrm{tr}_{\alpha_1\cdots\alpha_k}\Big\{ \pi^{(k)}_{(\alpha_1\cdots\alpha_k)}\Big(T^{[1,2]}\Big(\lambda_2-\frac{k+1}{2}\Big)\Big) \\ & \qquad \times r_{\alpha_k 2}(-k)\cdots r_{\alpha_3 2}(-3) r_{\alpha_k\bar 2}(-k+1)\cdots r_{\alpha_3\bar 2}(-2)\bullet \\ & \qquad\qquad \times P_{\alpha_2\bar\gamma}P_{\alpha_1\gamma}\mathcal{P}^+_{\alpha_k\cdots\alpha_1}\Big\} \\ & \times r_{\bar\gamma 2}(-2) s^{(1)}_{\gamma 2} s^{(1)}_{\bar\gamma\bar 2}, \end{aligned} \tag{4.44}$$

where

$$d_k = \frac{(-1)^k k}{(k-1)!(k+1)!\prod_{j=3}^n \lambda_{2j}(\lambda_{2j}-k-1)}. \tag{4.45}$$

We insert $\mathcal{P}^+_{\alpha_k\cdots\alpha_3}$ and $\mathcal{P}^+_{2\bar 2}$ at the position $\bullet$ in (4.44) and use (2.17) and (2.19) to obtain

$$\begin{aligned} X_k = {}& \frac{(k-2)!(k-1)!d_k}{2} r^{(2,1)}_{(1\bar 1),\bar 2}\Big(-k-\frac12\Big) \\ & \times \mathrm{tr}_{\alpha_1\cdots\alpha_k}\Big\{ \pi^{(k)}_{(\alpha_1\cdots\alpha_k)}\Big(T^{[1,2]}\Big(\lambda_2-\frac{k+1}{2}\Big)\Big) r^{(k-2,2)}_{(\alpha_3\cdots\alpha_k),(2\bar 2)}\Big(-\frac{k+2}{2}\Big) \\ & \qquad \times P_{\alpha_2\bar\gamma}P_{\alpha_1\gamma}\mathcal{P}^+_{\alpha_1\cdots\alpha_k}\Big\} \\ & \times r_{\bar\gamma 2}(-2) s^{(1)}_{\gamma 2} s^{(1)}_{\bar\gamma\bar 2}. \end{aligned} \tag{4.46}$$

Now we rewrite the last part. First we note that

$$r_{\bar\gamma 2}(-2) s^{(1)}_{\gamma 2} s^{(1)}_{\bar\gamma\bar 2} = -2 s^{(2)}_{(1\bar 1),(2\bar 2)}. \tag{4.47}$$

Then we apply the relation

$$P_{\alpha_2\bar\gamma}P_{\alpha_1\gamma}\mathcal{P}^+_{\alpha_1,\alpha_2} s^{(2)}_{(1\bar 1),(2\bar 2)} = \frac12 r^{(2,2)}_{(\alpha_1\alpha_2),(1\bar 1)}(0) s^{(2)}_{(1\bar 1),(2\bar 2)} = \frac12 r^{(2,2)}_{(\alpha_1\alpha_2),(2\bar 2)}(-1) s^{(2)}_{(1\bar 1),(2\bar 2)}, \tag{4.48}$$

where we have used the crossing symmetry (2.14).

Finally, applying (2.20) to $r^{(k-2,2)}_{(\alpha_3\cdots\alpha_k),(2\bar 2)}(-\frac{k+2}{2})$ and $r^{(2,2)}_{(\alpha_1\alpha_2),(2\bar 2)}(-1)$, we obtain

$$\begin{aligned} X_k = {}& \frac{(-1)^{k+1}}{(k^2-1)\prod_{j=3}^n \lambda_{2j}(\lambda_{2j}-k-1)} \\ & \times \mathrm{tr}_{\alpha_1\cdots\alpha_k}\Big\{ \pi^{(k)}_{(\alpha_1\cdots\alpha_k)}\Big(T^{[1,2]}\Big(\lambda_2-\frac{k+1}{2}\Big) r^{(2,1)}_{(1,\bar 1),\bar 2} \\ & \qquad \times\Big(-k-\frac12\Big) L^{(2)}_{(2,\bar 2)}\Big(-\frac{k}{2}\Big)\Big)\Big\} s^{(2)}_{(1,\bar 1),(2,\bar 2)}. \end{aligned} \tag{4.49}$$

This expression can be further simplified by using Lemma 4.1. The cyclicity of the trace allows us to arrive at the final result:

$$X_k = \frac{(-1)^{k+1}}{(k+1)\prod_{j=3}^{n}\lambda_{2j}(\lambda_{2j}-k-1)} \times \mathrm{tr}_{\alpha_1\cdots\alpha_k}\left\{\pi^{(k)}_{(\alpha_1\cdots\alpha_k)}\left(T^{[1]}\left(\lambda_2-\frac{k+1}{2}\right)\right)\right\}s^{(2)}_{(1,\bar{1}),(2,\bar{2})}. \tag{4.50}$$

We summarize the results, using the function ω and $X^{[1,2]}$.

Lemma 4.9. $X^{[1,2]}$ *satisfies the following difference equation*:

$$X^{[1,2]}(\lambda_1-1,\lambda_2,\ldots,\lambda_n) = -A_{\bar{1}}(\lambda_1,\ldots,\lambda_n)P_{1\bar{1}}X^{[1,2]}(\lambda_1,\lambda_2,\ldots,\lambda_n). \tag{4.51}$$

Proof. Equation (4.50) shows that

$$X_k = (-1)^k X^{[1,2]}(\lambda_2-k-1,\lambda_2,\ldots,\lambda_n) \tag{4.52}$$

for all integers $k \geq 2$. On the other hand, the definition of X_k implies that

$$X_k = A_{\bar{1}}(\lambda_2-k,\lambda_2,\ldots,\lambda_n)P_{1\bar{1}}X_{k-1}. \tag{4.53}$$

Therefore, (4.51) is valid if $\lambda_{1,2} = -k$ for all integers $k \geq 2$. Since both sides of (4.51) are rational, it is valid identically. □

Proposition 4.10. *For any positive integer $k \geq 1$ we have*

$$\begin{aligned}&\mathrm{res}_{\lambda_1=\lambda_2-k-1}\, h_n(\lambda_1,\ldots,\lambda_n)_{1,\ldots,n,\bar{n},\ldots,\bar{1}}\\ &\quad= \mathrm{res}_{\lambda_1=\lambda_2-k-1}\left\{\frac{\omega(\lambda_{1,2})}{1-\lambda_{1,2}^2}X^{[1,2]}(\lambda_1,\ldots,\lambda_n)\right\}h_{n-2}(\lambda_3,\ldots,\lambda_n)_{3,\ldots,n,\bar{n},\ldots,\bar{3}}.\end{aligned} \tag{4.54}$$

Proof. Note that $X^{[1,2]}(\lambda_1,\lambda_2,\ldots,\lambda_n)$ has no pole at $\lambda_1 = \lambda_2-k-1$ with $k \geq 1$. Since (4.53) uniquely determines X_{k-1} from X_k for $k \geq 2$, the difference equation (4.51) implies that (4.52) is valid also for $k = 1$. Then, (4.54) is an immediate consequence of (4.52) and

$$\mathrm{res}_{\lambda=-k-1}\,\omega(\lambda) = (-1)^k k(k+2). \tag{4.55}$$

□

It remains to find residues at the poles $\lambda_1 = \lambda_2+k+1$ for $k \geq 1$. The result is formulated in the following proposition.

Proposition 4.11. *For any positive integer $k \geq 1$ we have*

$$\begin{aligned}&\mathrm{res}_{\lambda_1=\lambda_2+k+1}\, h_n(\lambda_1,\ldots,\lambda_n)_{1,\ldots,n,\bar{n},\ldots,\bar{1}}\\ &\quad= \mathrm{res}_{\lambda_1=\lambda_2+k+1}\left\{\frac{\omega(\lambda_{1,2})}{\lambda_{1,2}^2-1}X^{[1,2]}(\lambda_1,\ldots,\lambda_n)\right\}h_{n-2}(\lambda_3,\ldots,\lambda_n)_{3,\ldots,n,\bar{n},\ldots,\bar{3}}.\end{aligned} \tag{4.56}$$

Proof. Observing that $\omega(\lambda)$ is even, we have

$$\begin{aligned}&\mathrm{res}_{\lambda_1=\lambda_2+k+1}\, h_n(\lambda_1,\lambda_2,\ldots,\lambda_n)_{1,2,\ldots,n,\bar{n},\ldots,\bar{2},\bar{1}}\\ &\quad= \mathrm{res}_{\lambda_1=\lambda_2+k+1}\, R_{2,1}(\lambda_{21})R_{\bar{1},\bar{2}}(\lambda_{12})h_n(\lambda_2,\lambda_1,\ldots,\lambda_n)_{2,1,\ldots,n,\bar{n},\ldots,\bar{1},\bar{2}}\\ &\quad= R_{2,1}(-k-1)R_{\bar{1},\bar{2}}(k+1)\\ &\qquad\times \mathrm{res}_{\lambda_1=\lambda_2+k+1}\left(\frac{\omega(\lambda_{1,2})}{1-\lambda_{1,2}^2}\right)X^{[1,2]}(\lambda_2,\lambda_2+k+1,\ldots,\lambda_n)_{2,1,\ldots,n,\bar{n},\ldots,\bar{1},\bar{2}}\\ &\qquad\qquad\times h_{n-2}(\lambda_3,\ldots,\lambda_n)_{3,\ldots,n,\bar{n},\ldots,\bar{3}}.\end{aligned}$$

Hence, the claim follows from the symmetry (4.8). □

Corollary 4.12. *For any $k \in \mathbb{Z}\backslash\{0, \pm 1\}$ and $2 \le j \le n$, we have*

(4.57)

$$\begin{aligned}&\mathrm{res}_{\lambda_1=\lambda_j-k}\, h_n(\lambda_1, \dots, \lambda_n)_{1,\dots,n,\bar{n},\dots,\bar{1}}\\ &\quad = \mathrm{res}_{\lambda_1=\lambda_j-k}\left\{\frac{\omega(\lambda_{1,j})}{1-\lambda_{i,j}^2} X^{[1,j]}(\lambda_1,\dots,\lambda_n)\right\} h_{n-2}(\lambda_2,\dots,\widehat{\lambda_j},\dots,\lambda_n)_{2,\dots,\hat{j},\dots,n,\bar{n},\dots,\hat{\bar{j}},\dots,\bar{2}}.\end{aligned}$$

Proof. This is an immediate consequence of Propositions 4.10 and 4.11, the symmetry (2.29), and relation (4.14). □

4.4. Asymptotics. In this subsection we finish the proof of Theorem 3.1. Let $\Phi_L(\lambda_1) = h_n(\lambda_1, \dots, \lambda_n)$, let $\Phi_R(\lambda_1)$ be the right-hand side of (3.22), and set $\Phi(\lambda_1) = \Phi_L(\lambda_1) - \Phi_R(\lambda_1)$. We are going to show that $\Phi(\lambda_1) = 0$.

By Corollary 4.12, we have

$$\mathrm{res}_{\lambda_1=\lambda_j-k}\, \Phi_L(\lambda_1) = \mathrm{res}_{\lambda_1=\lambda_j-k}\, \Phi_R(\lambda_1)$$

for all $k \in \mathbb{Z}\backslash\{0, \pm 1\}$ and $j = 2, \dots, n$. By (2.32) and the definition of $Z^{[1,j]}(\lambda_1, \dots, \lambda_n)$, there are no other poles in λ_1. Hence $\Phi(\lambda_1)$ is an entire function.

Consider the asymptotic behavior as $\lambda_1 \to \infty$. By (2.33), we know that

$$\lim_{\substack{\lambda_1\to\infty\\ \lambda_1\in S_\delta}} \Phi_L(\lambda_1) = (-1)^{n-1}\frac{1}{2} s_{1,\bar{1}} h_{n-1}(\lambda_2, \dots, \lambda_n)$$

for any $0 < \delta < \pi$, where $S_\delta = \{\lambda \in \mathbb{C} \mid \delta < |\arg \lambda_1| < \pi - \delta\}$. On the other hand, the function $\omega(\lambda)$ satisfies

$$\lim_{\substack{\lambda_1\to\infty\\ \lambda_1\in S_\delta}} \omega(\lambda) = 0.$$

Since the coefficients of $\omega(\lambda_{1,j})$ are regular at $\lambda_1 = \infty$ (Lemma 4.5), the terms in $\Phi_R(\lambda_1)$ except the last vanish in this limit. The last term of $\Phi_R(\lambda_1)$ is chosen so that

$$\lim_{\substack{\lambda_1\to\infty\\ \lambda_1\in S_\delta}} \Phi(\lambda_1) = 0. \tag{4.58}$$

We show that the condition $\lambda_1 \in S_\delta$ can be lifted.

Lemma 4.13. *There exist constants $M, c > 0$ such that*

$$|\Phi(\lambda_1)| \le M e^{c|\lambda_1|} \quad (\lambda_1 \in \mathbb{C}).$$

Proof. Recall that $\Phi_L(\lambda_1)$ satisfies the difference equation $\Phi_L(\lambda_1 + 1) = B(\lambda_1)\Phi_L(\lambda_1)$, where $B(\lambda_1)$ is a matrix of rational functions, holomorphic and invertible at $\lambda_1 = \infty$. We take a small neighborhood U of the set of poles of $B(\lambda_1)^{\pm 1}$ and choose $K' > 0$ such that $\sup_{\lambda_1 \in \mathbb{P}^1\backslash U} |B(\lambda_1)^{\pm 1}| \le K'$. Choose $\lambda_1^0 \in \mathbb{C}$ so that $\lambda_1^0 + \mathbb{Z} + i\mathbb{R} \subset \mathbb{P}^1 \setminus U$. Then for $n \ge 0$ we have

$$|\Phi_L(\lambda_1^0 + it + n)| \le |B(\lambda_1^0 + it + n - 1)| \cdots |B(\lambda_1^0 + it)||\Phi_L(\lambda_1^0 + it)| \le M' {K'}^n, \tag{4.59}$$

where $M' = \sup_{t\in\mathbb{R}} |\Phi_L(\lambda_1^0 + it)|$. Clearly, a similar estimate is valid for $|\Phi_L(\lambda_1^0 + it - n)|$ also.

On the other hand, the function $\omega(\lambda)$ satisfies the difference equation

$$\begin{pmatrix}\omega(\lambda+1)\\ 1\end{pmatrix} = \begin{pmatrix}-\frac{\lambda(\lambda+2)}{\lambda^2-1} & \frac{3}{2(\lambda^2-1)}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\omega(\lambda)\\ 1\end{pmatrix}.$$

Consequently, by the same argument as above, we obtain an estimate of the form (4.59) for $\omega(\lambda_{1,j})$ and, therefore, for $\Phi_R(\lambda_1)$.

In summary, there exist $M, K > 0$ such that

$$\sup_{t\in\mathbb{R}} |\Phi(\lambda_1^0 + it + n)| \le MK^{|n|} \quad (n \in \mathbb{Z}).$$

The lemma follows from this and the maximum principle. □

Now fix $\delta < \pi/2$. By (4.58), there exists $M' \ge M$ such that

$$|\Phi(\lambda_1)| \le M' \quad (\lambda_1 \in \mathbb{C},\ \delta \le |\arg \lambda_1| \le \pi - \delta). \tag{4.60}$$

By the lemma above and the Phragmen–Lindelöf theorem, (4.60) is also fulfilled for $|\arg\lambda_1| \le \delta$ or $\pi - \delta \le |\arg\lambda_1| \le \pi$. Therefore, $\Phi(\lambda_1)$ is bounded in the full neighborhood of $\lambda_1 = \infty$. Hence, $\lambda_1 = \infty$ is a regular point of $\Phi(\lambda_1)$, and we conclude that $\Phi(\lambda_1) = 0$. This completes the proof of Theorem 3.1.

Appendix A. Relationship with the correlators of the XXZ model

We give a relationship between two different gauges, the one used in the XXZ model in the massive regime [12], and the one used in the present paper, which gives the Hamiltonian (2.1).

In [12], the following Hamiltonian was considered:

$$H_{XXZ} = -\frac{1}{2}\sum_{j=1}^{L}\Big(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y - \frac{x+x^{-1}}{2}\sigma_j^z\sigma_{j+1}^z\Big).$$

Here $q = -x$, and the XXX limit is $x \to 1$. The gauge transformation by $\mathcal{K} = \prod_{j:\text{even}} \sigma_j^z$ brings H_{XXZ} to

$$\widetilde{H}_{XXZ} = \mathcal{K}H_{XXZ}\mathcal{K}^{-1} = \frac{1}{2}\sum_{j=1}^{L}\Big(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \frac{x+x^{-1}}{2}\sigma_j^z\sigma_{j+1}^z\Big).$$

The ground states and the correlators are related as follows:

$$|\widetilde{\Omega}\rangle = \mathcal{K}|\Omega\rangle,$$

$$\langle\widetilde{\Omega}|(E_{\epsilon_1,\bar\epsilon_1})_1\cdots(E_{\epsilon_n,\bar\epsilon_n})_n|\widetilde{\Omega}\rangle = \prod_{\substack{1\le j\le n\\ j:\text{even}}} \epsilon_j\bar\epsilon_j \cdot \langle\Omega|(E_{\epsilon_1,\bar\epsilon_1})_1\cdots(E_{\epsilon_n,\bar\epsilon_n})_n|\Omega\rangle,$$

$$G_{2n}(\zeta_1,\dots,\zeta_n,x\zeta_n,\dots,x\zeta_1)^{-\epsilon_1,\dots,-\epsilon_n,\bar\epsilon_n,\dots,\bar\epsilon_1} = \langle\Omega|(E_{\epsilon_1,\bar\epsilon_1})_1\cdots(E_{\epsilon_n,\bar\epsilon_n})_n|\Omega\rangle,$$

$$g_{2n}(\zeta_1,\dots,\zeta_{2n}) = (-1)^n \prod_{\substack{1\le j\le 2n\\ j:\text{even}}} \sigma_j^z \cdot G_{2n}(\zeta_1,\dots,\zeta_{2n}).$$

Therefore,

$$\begin{aligned} &g_{2n}(\zeta_1,\dots,\zeta_n,x\zeta_n,\dots,x\zeta_1)^{-\epsilon_1,\dots,-\epsilon_n,\bar\epsilon_n,\dots,\bar\epsilon_1} \\ &\quad = (-1)^{[n/2]}\prod_{j=1}^{n}(-\bar\epsilon_j)\cdot\langle\widetilde{\Omega}|(E_{\epsilon_1,\bar\epsilon_1})_1\cdots(E_{\epsilon_n,\bar\epsilon_n})_n|\widetilde{\Omega}\rangle. \end{aligned}$$

The Hamiltonian $\widetilde{H}_{XXZ}$ and the ground state $|\widetilde{\Omega}\rangle$ are related to those in §2 by

$$\lim_{x\to 1}\widetilde{H}_{XXZ} = H_{XXX}, \quad \lim_{x\to1}|\widetilde{\Omega}\rangle = |\,\text{vac}\rangle.$$

Therefore, we have (2.2).

Appendix B. Analyticity and the asymptotic properties of h_n

Here we prove Proposition 2.2. We start with the integral formula for the function $h_n(\lambda_1,\dots,\lambda_n)^{\epsilon_1,\dots,\epsilon_n,\bar{\epsilon_n},\dots,\bar{\epsilon_1}}$ given in [12].

Set

$$A=\{j|\epsilon_j=+\},\quad B=\{j|\bar{\epsilon_j}=+\}.$$

We write $A=\{a_1,\dots,a_r\}, B=\{b_1,\dots,b_s\}$ with $a_1<\cdots<a_r, b_1<\cdots<b_s$. Note that $r+s=n$. For all $a\in A$ and $b\in B$, we prepare integration variables t_a and t'_b, respectively. Arrange the variables as follows:

$$(u_1,\dots,u_n)=(t_{a_r},\dots,t_{a_1},t'_{b_1},\dots,t'_{b_s}).$$

Then the formula mentioned above is given by

$$\begin{aligned}(\text{B.1})\qquad & h_n(\lambda_1,\dots,\lambda_n)^{\epsilon_1,\dots,\epsilon_n,\bar{\epsilon}_n,\dots,\bar{\epsilon}_1}\\ &=c^{(n)}_{A,B}\prod_{a\in A}\int_{C^+}\frac{dt_a}{t_a-\lambda_a}\prod_{b\in B}\int_{C^-}\frac{dt'_b}{t'_b-\lambda_b}\\ &\times\prod_{\substack{a\in A\\ j<a}}\frac{t_a-\lambda_j-1}{t_a-\lambda_j}\prod_{\substack{b\in B\\ j<b}}\frac{t'_b-\lambda_j+1}{t'_b-\lambda_j}\\ &\times\prod_{j<k}\Bigl(\frac{\sinh\pi i(u_j-u_k)}{u_j-u_k-1}\frac{\sinh\pi i(\lambda_j-\lambda_k)}{\lambda_j-\lambda_k}\Bigr)\prod_{j,k}\frac{u_j-\lambda_k}{\sinh\pi i(u_j-\lambda_k)}.\end{aligned}$$

Here $c^{(n)}_{A,B}$ is a constant depending on A,B, and n. The integration contour C^+ is parallel to the imaginary axis for $|\operatorname{Im}t_a|\gg 0$ and separates the sequences of the poles of the integrand into the two sets $\lambda_j+\mathbb{Z}_{\le 0}$ and $\lambda_j+\mathbb{Z}_{>0}$. Similarly, C^- is the contour separating the poles into $\lambda_j+\mathbb{Z}_{<0}$ and $\lambda_j+\mathbb{Z}_{\ge 0}$.

Now we check the analyticity property (2.32). The singularity of the integral (B.1) comes from the pinch of the integration contour by some poles of the integrand. Hence, the function h_n is meromorphic with at most poles at $\lambda_i-\lambda_j\in\mathbb{Z}\setminus\{0\}$. We prove that h_n is analytic at $\lambda_i=\lambda_j\pm 1$. By (2.29), it suffices to show that h_n is analytic at $\lambda_1=\lambda_2\pm 1$. The function h_n satisfies (2.30), and $A_{\bar{1}}(\lambda_1,\lambda_2,\dots)$ and $h_n(\lambda_1,\lambda_2,\dots)$ are regular at $\lambda_1=\lambda_2$. Thus, h_n is analytic at $\lambda_1=\lambda_2-1$. Similarly, the regularity at $\lambda_1=\lambda_2+1$ can be checked by using the relation

$$\begin{aligned}& h_n(\lambda_1+1,\lambda_2,\dots,\lambda_n)_{1,\dots,n,\bar{n},\dots,\bar{1}}\\ &=\frac{-1}{\prod_{j=2}^n(\lambda_{1j}^2-1)}r_{12}(-\lambda_{12}-1)\cdots r_{1n}(-\lambda_{1n}-1)r_{1\bar{n}}(-\lambda_{1n})\cdots r_{1\bar{2}}(-\lambda_{12})\\ &\times h_n(\lambda_1,\dots,\lambda_n)_{\bar{1},2,\dots,n,\bar{n},\dots,\bar{2},1},\end{aligned}$$

which is derived from (2.30). By using (2.30) repeatedly, it is easy to check that the poles at $\lambda_i-\lambda_j\in\mathbb{Z}\setminus\{0,\pm1\}$ are simple.

Next, we show the asymptotic property (2.33). We consider the asymptotics in the angular domain

$$(\text{B.2})\qquad \lambda_1\to\infty,\quad \delta<|\arg\lambda_1|<\pi-\delta,$$

where $0<\delta<\pi$. Setting $\lambda_1=\kappa+i\mu$ with $\kappa,\mu\in\mathbb{R}$, in this domain we have

$$|\mu|\le|\lambda_1|\le K|\mu|$$

for a positive constant K. We shall only consider the case where $\mu\to+\infty$; the other case is similar.

First, we deform the contours $C^{\pm}$ suitably and assume that the contours $C^{\pm}, C^{\pm}+1$, and $C^{\pm}-1$ do not cross one another. Taking $\mu \gg 0$, we also assume that $|\operatorname{Im}\lambda_j| \le \mu/6$ for $j = 2, \dots, n$ and that $C^{\pm}$ contains the segment

$$\Gamma^{\pm} = \left\{ \kappa \pm \epsilon + iy \;\middle|\; \frac{2}{3}\mu \le y \le \frac{4}{3}\mu \right\}$$

for small $\epsilon > 0$.

Let $P_{A,B}$ denote the rational part of the integrand, that is,

$$\begin{aligned} &P_{A,B}(u_1, \dots, u_n; \lambda_1, \dots, \lambda_n) \\ &\quad = \prod_{a\in A} \Big(\frac{1}{t_a - \lambda_a} \prod_{j<a} \frac{t_a - \lambda_j - 1}{t_a - \lambda_j} \Big) \prod_{b\in B} \Big(\frac{1}{t'_b - \lambda_b} \prod_{j<b} \frac{t'_b - \lambda_j + 1}{t'_b - \lambda_j} \Big) \\ &\qquad \times \frac{\prod_{j,k}(u_j - \lambda_k)}{\prod_{j<k}(u_j - u_k - 1)(\lambda_j - \lambda_k)}. \end{aligned}$$

We use the relation

$$\begin{aligned} &\prod_{j<k} \sinh \pi i(u_j - u_k) \\ &\quad = 2^{-\binom{n-2}{2}} e^{\pi i(n-1)\lambda_1} \operatorname{Skew} \left(\prod_{j=1}^{n-1} e^{\pi i(n-2j)u_j} \sinh \pi i(u_j - \lambda_1) \cdot e^{\pi i(-n+1)u_n} \right), \end{aligned}$$

where Skew denotes skew-symmetrization with respect to $u_1, \dots, u_n$. Then we find

$$\begin{aligned} &h_n(\lambda_1, \dots, \lambda_n)^{\epsilon_1, \dots, \epsilon_n, \bar\epsilon_n, \dots, \bar\epsilon_1} \\ &\quad = 2^{-\binom{n-2}{2}} c^{(n)}_{A,B} e^{\pi i(n-1)\lambda_1} \prod_{j<k} \sinh \pi i(\lambda_j - \lambda_k) \\ &\qquad \times \sum_{\sigma\in\mathfrak{S}_n} (\operatorname{sgn}\sigma) \prod_{j=1}^{n} \int_{C_{\sigma,j}} du_j \, P_{A,B}(u_{\sigma(1)}, \dots, u_{\sigma(n)}; \lambda_1, \dots, \lambda_n) \\ &\qquad \times \prod_{j=1}^{n-1} \left(\frac{e^{\pi i(n-2j)u_j}}{\prod_{k=2}^{n} \sinh \pi i(u_j - \lambda_k)} \right) \frac{e^{\pi i(-n+1)u_n}}{\prod_{k=1}^{n} \sinh \pi i(u_n - \lambda_k)}. \end{aligned} \tag{B.3}$$

Here the contour $C_{\sigma,j}$ looks like this:

$$C_{\sigma,j} = \begin{cases} C^{+} & \text{if } 1 \le \sigma^{-1}(j) \le r, \\ C^{-} & \text{if } r+1 \le \sigma^{-1}(j) \le n. \end{cases}$$

Note that the factor $e^{\pi i(n-1)\lambda_1} \prod_{j<k} \sinh \pi i(\lambda_j - \lambda_k)$ converges as $\mu \to +\infty$. Below, we prove that each integral in the sum on the right-hand side of (B.3) converges as $\mu \to +\infty$.

We decompose the contour for u_n into two parts: $C_{\sigma,n} = \Gamma + C'$, where Γ is the segment Γ^{+} or Γ^{-} contained in $C_{\sigma,n}$, and $C' = C_{\sigma,n} \setminus \Gamma$. Simultaneously, we decompose the integral:

$$\prod_{j=1}^{n} \int_{C_{\sigma,j}} du_j = \prod_{j=1}^{n-1} \int_{C_{\sigma,j}} du_j \int_{\Gamma} du_n + \prod_{j=1}^{n-1} \int_{C_{\sigma,j}} du_j \int_{C'} du_n.$$

Next, we decompose the integration in the first term over $C_{\sigma,1} \times \cdots \times C_{\sigma,n-1}$ into the following parts:

$$\begin{aligned} D_1 &= \{(u_1, \dots, u_{n-1}) \in C_{\sigma,1} \times \cdots \times C_{\sigma,n-1} \mid -\mu/3 \le \operatorname{Im} u_j \le \mu/3\}, \\ D_2 &= (C_{\sigma,1} \times \cdots \times C_{\sigma,n-1}) \setminus D_1. \end{aligned}$$

Thus, the integral in (B.3) takes the form

$$\int_{D_1}\prod_{j=1}^{n-1}du_j\int_{\Gamma}du_n+\int_{D_2}\prod_{j=1}^{n-1}du_j\int_{\Gamma}du_n+\prod_{j=1}^{n-1}\int_{C_{\sigma,j}}du_j\int_{C'}du_n. \tag{B.4}$$

We treat the limit of these three parts separately.

Consider the first integral in (B.4). After the change $u_n\mapsto u_n+\lambda_1$, the integral becomes

$$\begin{aligned}&\int_{D_1}\prod_{j=1}^{n-1}du_j\int_{\pm\epsilon-\frac{\mu}{3}i}^{\pm\epsilon+\frac{\mu}{3}i}du_nP_{A,B}(\dots,u_n+\lambda_1,\dots;\lambda_1,\dots,\lambda_n)\\&\quad\times\prod_{j=1}^{n-1}\left(\frac{e^{\pi i(n-2j)u_j}}{\prod_{k=2}^{n}\sinh\pi i(u_j-\lambda_k)}\right)\frac{1}{\sinh\pi iu_n}\\&\quad\times\prod_{k=2}^{n}\frac{e^{-\pi i(u_n+\lambda_1)}}{\sinh\pi i(u_n+\lambda_1-\lambda_k)},\end{aligned} \tag{B.5}$$

where the sign $\pm\epsilon$ is $+$ or $-$ in accordance with whether $\Gamma=\Gamma^+$ or Γ^-, respectively.

Note that

$$P_{A,B}(\dots,u_n+\lambda_1,\dots;\lambda_1,\dots,\lambda_n)=O(1)\quad(\lambda_1\to\infty).$$

Consequently, the integrand converges as $\mu\to+\infty$ for fixed $u_1,\dots,u_n$. To apply Lebesgue's convergence theorem, we need to check that the integrand is bounded from above by an integrable function. First, we consider the rational part $P_{A,B}$. Recall that the contours $C^{\pm},C^{\pm}-1$, and $C^{\pm}+1$ do not cross one another. Hence, there exists a positive constant d such that

$$|u_j-u_k\pm1|\geq d \tag{B.6}$$

for $j,k=1,\dots,n-1$. For $(u_1,\dots,u_{n-1})\in D_1$ and $u_n=\pm\epsilon+iy$ $(-\mu/3\leq y\leq\mu/3)$, we have

$$|u_j-u_n-\lambda_1\pm1|\geq|\operatorname{Im}(u_j-u_n-\lambda_1)|\geq\mu/3\quad(j=1,\dots,n-1).$$

Thus, the rational part $P_{A,B}$ is upper bounded:

$$|P_{A,B}(\dots,u_n+\lambda_1,\dots;\lambda_1,\dots,\lambda_n)|\leq\mu^{-2(n-1)}Q_1(|u_1|,\dots,|u_n|;|\lambda_2|,\dots,|\lambda_n|;K;\mu),$$

where Q_1 is a polynomial such that $\deg_\mu Q_1=2(n-1)$. Next, we consider the trigonometric part. Note that the function $e^{\pi ix}/\sinh\pi ix$ is bounded in $\mathbb{C}\setminus U$, where U is the union of small open disks with a fixed radius around integers. Consequently,

$$\begin{aligned}&\left|\prod_{j=1}^{n-1}\left(\frac{e^{\pi i(n-2j)u_j}}{\prod_{k=2}^{n}\sinh\pi i(u_j-\lambda_k)}\right)\frac{1}{\sinh\pi iu_n}\prod_{k=2}^{n}\frac{e^{-\pi i(u_n+\lambda_1)}}{\sinh\pi i(u_n+\lambda_1-\lambda_k)}\right|\\&\qquad\leq M\left|\prod_{j=1}^{n-1}\frac{1}{\sinh\pi i(u_j-\lambda_2)}\frac{1}{\sinh\pi iu_n}\right|\end{aligned} \tag{B.7}$$

for some positive constant M. The right-hand side decays exponentially as $\operatorname{Im}u_j\to\pm\infty$. Therefore we can apply Lebesgue's convergence theorem to (B.5).

Now we consider the second integral in (B.4). After the change $u_n\mapsto u_n+\lambda_1$, the integral becomes equal to (B.5) with D_1 replaced by D_2. We prove that the integral vanishes in the limit $\mu \to +\infty$. For this, we decompose D_2 as follows. Set

$$U_{\pm}^{(k)} = \{\pm \operatorname{Im} u_j > \mu/3\} \cap D_2$$

for $k = 1, \dots, n-1$. Then $D_2 = \bigcup_k (U_+^{(k)} \cup U_-^{(k)})$. We prove that the integral over each set $U_{\pm}^{(k)} \times \{\pm\epsilon + iy | -\mu/3 \le y \le \mu/3\}$ vanishes in the limit. Here we consider the case of $k = 1$; for the other cases the argument is similar. To prove the claim, we bound the integrand by a certain integrable function of the form

$$e^{-c\mu} \sum_{k=-(n-1)}^{n-1} \mu^k R_k(u_1, \dots, u_n; \lambda_2, \dots, \lambda_n), \tag{B.8}$$

where c is a positive constant. First, consider the rational part $P_{A,B}$. We have inequality (B.6), and

$$|u_j - u_n - \lambda_1 \pm 1| \ge d$$

for $j = 1, \dots, n-1$. Hence, we have an upper estimate of the form

$$|P_{A,B}(\dots, u_n + \lambda_1, \dots; \lambda_1, \dots, \lambda_n)| \le \mu^{-(n-1)} Q_2(|u_1|, \dots, |u_n|; |\lambda_2|, \dots, |\lambda_n|; K; \mu),$$

where Q_2 is a polynomial such that $\deg_\mu Q_2 = 2(n-1)$. Next, consider the trigonometric part. We have inequality (B.7). Apply the following inequality to the factor $1/\sinh \pi i(u_1 - \lambda_2)$:

$$\left|\frac{1}{\sinh \pi i u}\right| \le \frac{2e^{-\pi|\operatorname{Im} u|}}{1 - e^{-\pi\mu/3}} \le \frac{2e^{-\pi\mu/12} \cdot e^{-\pi|\operatorname{Im} u|/3}}{1 - e^{-\pi\mu/3}} \quad (\pm \operatorname{Im} u > \mu/6 > 0). \tag{B.9}$$

Thus, we get an upper bound of the form (B.8).

Finally, we consider the third integral in (B.4). We show that it also vanishes in the limit $\mu \to +\infty$. The integrand is given in (B.3). By the same argument as above, for the rational part $P_{A,B}$ we have an upper bound of the form

$$\mu^{-(n-1)} Q_3(|u_1|, \dots, |u_n|; |\lambda_2|, \dots, |\lambda_n|; K; \mu),$$

where Q_3 is a polynomial such that $\deg_\mu Q_3 = n-1$. The trigonometric part is estimated as follows:

$$\begin{aligned} &\left| \prod_{j=1}^{n-1} \left(\frac{e^{\pi i(n-2j)u_j}}{\prod_{k=2}^n \sinh \pi i(u_j - \lambda_k)} \right) \frac{e^{\pi i(-n+1)u_n}}{\prod_{k=1}^n \sinh \pi i(u_n - \lambda_k)} \right| \\ &\qquad \le M \left| \prod_{j=1}^{n-1} \frac{1}{\sinh \pi i(u_j - \lambda_2)} \frac{1}{\sinh \pi i(u_n - \lambda_1)} \right|. \end{aligned}$$

Applying (B.9) to the factor $1/\sinh \pi i(u_n - \lambda_1)$, we can see the required vanishing in the limit $\mu \to +\infty$.

The consideration above shows that h_n converges in the limit (B.2). It remains to check that the limit is equal to the right-hand side of (2.33). We denote this limit by

$\hat{h}_n(\lambda_2, \ldots, \lambda_n)$. From equation (2.30) and the fact that

$$\lim_{\lambda_1 \to \infty} A_{\bar{1}}(\lambda_1, \ldots, \lambda_n) = -1$$

we obtain

$$\hat{h}_n(\lambda_2, \ldots, \lambda_n)_{1,2,\ldots,n,\bar{n},\ldots,\bar{2},\bar{1}} = -\hat{h}_n(\lambda_2, \ldots, \lambda_n)_{\bar{1},2,\ldots,n,\bar{n},\ldots,\bar{2},1}.$$

Namely, the limit $\hat{h}_n$ is a singlet in the space $V_1 \otimes V_{\bar{1}}$. From this and (2.31), we get (2.33).

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Physics Department, University of Wuppertal, D-42097, Wuppertal, Germany, and Institute for High Energy Physics, Protvino 142284, Russia

E-mail address: boos@physik.uni-wuppertal.de

Graduate School of Mathematical Sciences, University of Tokyo, Tokyo 153-8914, Japan

E-mail address: jimbomic@ms.u-tokyo.ac.jp

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: tetsuji@math.kyoto-u.ac.jp

(Membre du CNRS): Laboratoire de Physique Théorique et Hautes Energies, Université Pierre et Marie Curie, Tour 16 1er étage, 4 Place Jussieu, 75252 Paris Cedex 05, France

E-mail address: smirnov@lpthe.jussieu.fr

Graduate School of Pure and Applied Sciences, University of Tsukuba, Tsukuba 305-8571, Japan

E-mail address: takeyama@math.tsukuba.ac.jp

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