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CAYLEY–HAMILTON THEOREM FOR QUANTUM MATRIX ALGEBRAS OF $GL(m|n)$ TYPE

D. I. GUREVICH, P. N. PYATOV, AND P. A. SAPONOV

To L. D. Faddeev with appreciation and respect

Abstract. The classical Cayley–Hamilton identities are generalized to quantum matrix algebras of $GL(m|n)$ type.

§0. Introduction

The well-known Cayley–Hamilton theorem of classical matrix analysis claims that any square matrix M with entries in a field $\mathbb{K}$ satisfies the identity

$$\Delta(M) \equiv 0, \tag{0.1}$$

where $\Delta(x) = \det(M - xI)$ is the characteristic polynomial of M. Provided $\mathbb{K}$ is algebraically closed, the coefficients of this polynomial are the elementary symmetric functions in the eigenvalues of M. Below, for simplicity we assume $\mathbb{K}$ to be the field $\mathbb{C}$ of complex numbers.

In [NT, PS, GPS, IOPS, IOP1] the Cayley–Hamilton theorem was generalized consecutively to the case of the quantum matrix algebras of $GL(m)$ type. Any such algebra is constructed via R-matrix representations of $GL(m)$ type Hecke algebras.

In [KT1, KT2], the Cayley–Hamilton identity was established for matrix superalgebras. In the present paper we aim at a further generalization of this result to the case of quantum matrix superalgebras.

We formulate our main result: the Cayley–Hamilton theorem for the quantum matrix algebras of $GL(m|n)$ type. All definitions and notation are explained in the main text below.

Theorem. *Let $M = \|M_i^j\|_{i,j=1}^N$ be the matrix of the generators of the quantum matrix algebra $\mathcal{M}(R,F)$ defined by a pair of compatible, strictly skew-invertible R-matrices R and F, where R is a $GL(m|n)$ type R-matrix* (*see Subsections* 2.1 *and* 2.2). *Then in $\mathcal{M}(R,F)$ the following matrix identity is fulfilled*:

$$\sum_{i=0}^{n+m} M^{\overline{m+n-i}} C_i \equiv 0, \tag{0.2}$$

where $M^{\bar{i}}$ is the ith power of the quantum matrix (*see the definition* (2.27)) *and the coefficients C_i are linear combinations of Schur functions* (*see the definition* (2.21)).

The Schur functions involved in the expression for C_i are homogeneous polynomials of order $mn+i$ in the generators M_i^j. Therefore, any matrix element on the left-hand side of (0.2) *is a homogeneous polynomial in M_i^j of order $mn+m+n$.*

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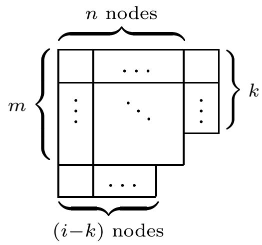
In terms of Schur functions, the expression for the coefficient C_i *can be presented graphically as follows*:

$$C_i = \sum_{k=\max\{0,(i-n)\}}^{\min\{i,m\}} (-1)^k q^{(2k-i)} \tag{0.3}$$



where each Young diagram stands for the corresponding Schur function.

If the matrices F and R are transpositions in an $(m|n)$-dimensional superspace, then the corresponding algebra $\mathcal{M}(R, F)$ is a matrix superalgebra. In this case identity (0.2) coincides with the invariant Cayley–Hamilton relations obtained in [KT1, KT2]. Note that in the supersymmetric case the linear size N of the matrix M is determined by the parameters m and n: $N = m + n$. As was shown in [G], in general this relation does not take place. Our proof of identity (0.2) does not involve such relationships.

The structure of the paper is as follows. §1 contains definitions and necessary facts about Hecke algebras. In §2 we define the quantum matrix algebra of $GL(m|n)$ type, and the notions of the characteristic subalgebra and of the matrix power space are introduced. The last section is devoted to the proof of the Cayley–Hamilton identity (0.2).

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§1. Some facts about Hecke algebras

In this section we give a summary of facts about the structure of the Hecke algebras of the A_{n-1} series. For their proofs the reader is referred to [DJ1, DJ2, M1, M2]. Detailed consideration of the material and the list of references on the theme can also be found in the surveys [OP, R]. When considering quantum matrix algebras, we shall use this section as a basis for our technical tools.

1.1. Young diagrams and tableaux. First of all, we recall some definitions from partition theory. Mainly, we use the terminology and notation of the book [Mac].

A *partition* λ of a positive integer n (notation: $\lambda \vdash n$) is a monotone nonincreasing sequence of nonnegative integers λ_i such that their sum is equal to n:

$$\lambda = (\lambda_1, \lambda_2, \dots), \quad \lambda_i \ge \lambda_{i+1} : \quad |\lambda| \equiv \sum \lambda_i = n.$$

The number $|\lambda| = n$ is called the *weight* of λ. A convenient graphical image for a partition is a *Young diagram* (see [Mac]), and below we shall identify any partition with the corresponding Young diagram.

With each node of a Young diagram located at the intersection of the sth column and the rth row, we associate the number $c = s - r$, which will be called the *content* of the node.

On the set of all the Young diagrams we define an inclusion relation $\subset$ by the agreement that a diagram μ includes a diagram λ (symbolically, $\lambda \subset \mu$) if after superposition the diagram λ is entirely contained inside μ. In other words, $\lambda \subset \mu \iff \lambda_i \le \mu_i$ for $i = 1, 2, \dots$.

Given a Young diagram $\lambda \vdash n$, we can construct $n!$ *Young tableaux* by writing all integers from 1 to n (in an arbitrary order) in the nodes of the diagram. The tableau will be denoted by the symbol

$$\begin{Bmatrix} \lambda \\ \alpha \end{Bmatrix}, \quad \alpha = 1, 2, \ldots, n!, \tag{1.1}$$

where the index α marks different tableaux. The diagram λ will be called the *form* of a tableau $\left\{ {\lambda \atop \alpha} \right\}$.

The set of all tableaux associated with a diagram $\lambda \vdash n$ can be supplied naturally with an action of the nth-order symmetric group S_n. By definition, an element $\pi \in S_n$ acts on a tableau $\left\{ {\lambda \atop \alpha} \right\}$ by replacing any number i with $\pi(i)$ in all the nodes of the tableau. The resulting tableau will be denoted by $\left\{ {\lambda \atop \pi(\alpha)} \right\}$.

In the set of all Young tableaux of the form λ, we distinguish the subset of *standard* tableaux that satisfy an additional requirement: the numbers written in a standard tableau increase in any column from top to bottom and in any row from left to right. From now on we shall use only standard Young tableaux and retain the notation (1.1) for them. However, the index α will vary from 1 to the integer d_λ equal to the total number of all standard Young tableaux of the same form λ. Explicit expressions for d_λ are given, e.g., in [Mac].

Though the subset of standard tableaux is not closed with respect to the action of the symmetric group, it is nevertheless a cyclic set under this action, because an arbitrary standard Young tableau of the form $\lambda \vdash n$ can be obtained from any other standard tableau of the same form by the action of some element π of the group S_n, that is,

$$\forall \alpha, \beta\, \exists \pi \in S_n : \quad \begin{Bmatrix} \lambda \\ \alpha \end{Bmatrix} \mapsto \begin{Bmatrix} \lambda \\ \beta \end{Bmatrix} \equiv \begin{Bmatrix} \lambda \\ \pi(\alpha) \end{Bmatrix}.$$

The relation $\subset$ defined above for the Young diagrams can be extended to the set of standard tableaux. Namely, we say that a tableau includes another tableau,

$$\begin{Bmatrix} \lambda \\ \alpha \end{Bmatrix} \subset \begin{Bmatrix} \mu \\ \beta \end{Bmatrix},$$

if $\lambda \subset \mu$ and the numbers from 1 to $|\lambda|$ occupy the same nodes in the tableau $\left\{ {\mu \atop \beta} \right\}$ as they do in $\left\{ {\lambda \atop \alpha} \right\}$.

1.2. Hecke algebras: Definition and the basis of matrix units. The *Hecke algebra* $\mathcal{H}_n(q)$ of the A_{n-1} series is a unital associative $\mathbb{C}$-algebra generated by the set of elements $\{\sigma_i\}_{i=1}^{n-1}$ subject to the relations

$$\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}, \qquad k = 1, \ldots, n-2, \tag{1.2}$$

$$\sigma_k \sigma_j = \sigma_j \sigma_k, \qquad |k-j| \geq 2, \tag{1.3}$$

$$\sigma_k{}^2 = \mathbf{1} + (q - q^{-1}) \sigma_k, \qquad k = 1, \ldots, n-1, \tag{1.4}$$

where $\mathbf{1}$ is the unit element. The last of these defining relations depends on a parameter $q \in \mathbb{C} \setminus \{0\}$ and is referred to as the *Hecke condition.*

$\mathcal{H}_n(1)$ coincides with the group algebra $\mathbb{C}[S_n]$ of the symmetric group S_n. If

$$k_q := \frac{q^k - q^{-k}}{q - q^{-1}} \neq 0 \quad \text{for all } k = 2, 3, \ldots, n, \tag{1.5}$$

then the algebra $\mathcal{H}_n(q)$ is semisimple and is isomorphic to $\mathbb{C}[S_n]$. In what follows, we assume that condition (1.5) is fulfilled for all $k \in \mathbb{N} \setminus \{0\}$. Under this assumption, the algebra $\mathcal{H}_n(q)$ is isomorphic to the direct sum of matrix algebras

$$\mathcal{H}_n(q) \cong \bigoplus_{\lambda \vdash n} \mathrm{Mat}_{d_\lambda}(\mathbb{C}). \tag{1.6}$$

Here $\mathrm{Mat}_m(\mathbb{C})$ stands for the associative $\mathbb{C}$-algebra generated by m^2 matrix units E_{ab}, $1 \le a, b \le m$, that satisfy the following multiplication law:

$$E_{ab}E_{cd} = \delta_{bc}E_{ad}.$$

The isomorphism (1.6) allows one to construct the set of elements $E^\lambda_{\alpha\beta} \in \mathcal{H}_n(q)$, where $\lambda \vdash n$ and $1 \le \alpha, \beta \le d_\lambda$, that are the images of the matrix units generating the term Mat_{d_λ} in the direct sum (1.6),

$$E^\lambda_{\alpha\beta}E^\mu_{\gamma\tau} = \delta^{\lambda\mu}\delta_{\beta\gamma}E^\lambda_{\alpha\tau}. \tag{1.7}$$

The diagonal matrix units $E^\lambda_{\alpha\alpha} \equiv E^\lambda_\alpha$ are the primitive idempotents of the algebra $\mathcal{H}_n(q)$ and the set of all $E^\lambda_{\alpha\beta}$ forms a linear basis in $\mathcal{H}_n(q)$.

Obviously, the matrix units $E^\lambda_{\alpha\beta}$ can be realized in many different ways. The construction presented in [M1, M2, R] (see also [OP]) satisfies a number of useful additional relations. Precisely this basis of matrix units will be used below. Now we briefly outline its main properties.

First, for this basis, remarkably simple expressions are known for the elements which transform different diagonal matrix units into one another. In order to describe these elements, we need some definitions.

With each generator σ_i of the algebra $H_n(q)$, we associate the element

$$\sigma_i(x) := \sigma_i + \frac{q^{-x}}{x_q}\mathbf{1}, \quad 1 \le i \le n-1. \tag{1.8}$$

Here x is an arbitrary nonnegative integer, and x_q is as in (1.5). The elements $\sigma_i(x)$ obey the relations

$$\sigma_i(x)\sigma_{i+1}(x+y)\sigma_i(y) = \sigma_{i+1}(y)\sigma_i(x+y)\sigma_{i+1}(x), \quad 1 \le i \le n-2, \tag{1.9}$$

$$\sigma_i(x)\sigma_i(-x) = \frac{(x+1)_q(x-1)_q}{x_q^2}\mathbf{1}, \quad 1 \le i \le n-1. \tag{1.10}$$

Consider an arbitrary standard Young tableau $\{{}^\lambda_\alpha\}$ corresponding to a partition $\lambda \vdash n$. For each integer $1 \le k \le n-1$ we define the number $\ell_k\{{}^\lambda_\alpha\}$ to be equal to the difference of the contents of nodes containing the numbers k and $(k+1)$,

$$\ell_k\{{}^\lambda_\alpha\} := c(k) - c(k+1). \tag{1.11}$$

Suppose a tableau $\{{}^\lambda_\beta\}$ (not necessarily standard) differs from the initial tableau $\{{}^\lambda_\alpha\}$ only in the transposition of the nodes with numbers k and $k+1$:

$$\beta = \pi_k(\alpha), \quad 1 \le k \le n-1, \tag{1.12}$$

where $\pi_k \in S_n$ is the transposition of k and $k+1$. If the tableau $\left\{{}^{\lambda}_{\pi_k(\alpha)}\right\}$ is standard, then

$$\sigma_k(\ell_k\{{}^\lambda_\alpha\})E^\lambda_\alpha = E^\lambda_{\pi_k(\alpha)}\sigma_k(-\ell_k\{{}^\lambda_\alpha\}), \tag{1.13}$$

$$E^\lambda_\alpha\sigma_k(\ell_k\{{}^\lambda_\alpha\}) = \sigma_k(-\ell_k\{{}^\lambda_\alpha\})E^\lambda_{\pi_k(\alpha)}. \tag{1.14}$$

Now we recall the cyclic property of the set of standard Young tableaux under the action of the symmetric group and the fact that an arbitrary permutation can be (nonuniquely) expanded in a product of transpositions. This makes it possible to connect any pair E^λ_α and E^λ_β by a chain of transformations (1.13), (1.14). Moreover, relations (1.9), (1.10) ensure the compatibility of all possible formulas expressing dependencies between E^λ_α and E^λ_β.

If the tableau $\left\{ {\lambda \atop \pi_k(\alpha)} \right\}$ is nonstandard, then

$$\sigma_k(\ell_k\{{\lambda \atop \alpha}\})E^\lambda_\alpha = E^\lambda_\alpha \sigma_k(\ell_k\{{\lambda \atop \alpha}\}) = 0. \tag{1.15}$$

The expressions for the nondiagonal matrix units follow from (1.13), (1.14). Assuming that the two tableaux $\{{\lambda \atop \alpha}\}$ and $\left\{ {\lambda \atop \pi_k(\alpha)} \right\}$ are standard, we get

$$E^\lambda_{\alpha\pi_k(\alpha)} := \omega(\ell_k\{{\lambda \atop \alpha}\})E^\lambda_\alpha \sigma_k(\ell_k\{{\lambda \atop \alpha}\}), \tag{1.16}$$

$$E^\lambda_{\pi_k(\alpha)\alpha} := \omega(\ell_k\{{\lambda \atop \alpha}\})\sigma_k(\ell_k\{{\lambda \atop \alpha}\})E^\lambda_\alpha, \tag{1.17}$$

where the normalizing coefficient $\omega(\ell)$ satisfies the relation

$$\omega(\ell)\omega(-\ell) = \frac{\ell_q^2}{(\ell+1)_q(\ell-1)_q}. \tag{1.18}$$

In particular, we can set $\omega(\ell) := \ell_q/(\ell_q+1)$. Note that if the tableau $\left\{ {\lambda \atop \pi_k(\alpha)} \right\}$ is standard, we have $\ell_k\{{\lambda \atop \alpha}\} \neq \pm 1$. Therefore, the above expression for the normalizing coefficient is always well defined.

To formulate the second property of matrix units to be used below, we consider the chain of Hecke algebra embeddings

$$\mathcal{H}_2(q) \hookrightarrow \cdots \hookrightarrow \mathcal{H}_k(q) \hookrightarrow \mathcal{H}_{k+1}(q) \hookrightarrow \cdots \tag{1.19}$$

that are defined on generators by the formula

$$\mathcal{H}_k(q) \ni \sigma_i \mapsto \sigma_{i+1} \in \mathcal{H}_{k+1}(q), \quad i = 1, \dots, k-1. \tag{1.20}$$

In each subalgebra $\mathcal{H}_k(q)$, a basis of matrix units $E^\lambda_{\alpha\beta}$ can be chosen in such a way that the diagonal matrix units $E^\lambda_\alpha \in \mathcal{H}_k(q)$ are decomposable into sums of diagonal matrix units belonging to any enveloping algebra $\mathcal{H}_m(q) \supset \mathcal{H}_k(q)$,

$$E^{\lambda\vdash k}_\alpha = \sum_{\{{\lambda \atop \alpha}\} \subset \{{\mu \atop \beta}\}} E^{\mu\vdash m}_\beta, \quad m \geq k. \tag{1.21}$$

Here summation is over all the standard Young tableaux $\{{\mu \atop \beta}\}$, $\mu \vdash m$, containing the standard tableau $\{{\lambda \atop \alpha}\}$.

§2. Quantum matrix algebra $\mathcal{M}(R,F)$

2.1. R-matrix representations of the Hecke algebra. Let V be a finite-dimensional vector space over the field of complex numbers, and let $\dim V = N$. With any element $X \in \mathrm{End}(V^{\otimes p})$, $p = 1, 2, \dots$, we associate a sequence of endomorphisms $X_i \in \mathrm{End}(V^{\otimes k})$, $k = p, p+1, \dots$:

$$X_i = \mathrm{Id}_V^{i-1} \otimes X \otimes \mathrm{Id}_V^{k-p-i+1}, \quad 1 \leq i \leq k-p+1, \tag{2.1}$$

where Id_V is the identical automorphism of V.

An operator $R \in \mathrm{Aut}(V^{\otimes 2})$ satisfying the *Yang–Baxter equation*

$$R_1R_2R_1 = R_2R_1R_2 \tag{2.2}$$

will be called an *R-matrix*. It is easily seen that any R-matrix such that

$$(R - q\,\mathrm{Id}_{V^{\otimes 2}})(R + q^{-1}\,\mathrm{Id}_{V^{\otimes 2}}) = 0, \quad q \in \mathbb{C} \setminus \{0\}, \tag{2.3}$$

generates representations ρ_R for the series of the Hecke algebras $\mathcal{H}_k(q)$, $k = 2, 3, \ldots$:

$$\rho_R : \ \mathcal{H}_k(q) \to \mathrm{End}(V^{\otimes k}), \ \sigma_i \mapsto \rho_R(\sigma_i) = R_i, \ 1 \le i \le k-1, \tag{2.4}$$

where the σ_i are the generators of $\mathcal{H}_k(q)$; see (1.2)–(1.4). Such R-matrices will be called the *Hecke type* (or simply *Hecke*) R-matrices, and the corresponding representations ρ_R are the *R-matrix* representations.

Let R be a Hecke R-matrix. Suppose that the corresponding representations ρ_R : $\mathcal{H}_k(q) \to \mathrm{End}(V^{\otimes k})$ are faithful for all $2 \le k < (m+1)(n+1)$ while the representation $\rho_R : \mathcal{H}_{(m+1)(n+1)}(q) \to \mathrm{End}(V^{\otimes (m+1)(n+1)})$ possesses a kernel generated by any matrix unit corresponding to the rectangular Young diagram $((n+1)^{(m+1)})$; that is,

$$\begin{aligned} &\rho_R(E_\alpha^{((n+1)^{(m+1)})}) = 0, \\ &\qquad\qquad \rho_R(E_\alpha^\mu) \neq 0 \quad \text{for any } \mu \vdash (m+1)(n+1), \ \mu \neq ((n+1)^{(m+1)}). \end{aligned} \tag{2.5}$$

Such an R-matrix will be referred to as an R-matrix of $GL(m|n)$ *type*.

2.2. R-trace and pairs of compatible R-matrices. Let $\{v_i\}_{i=1}^N$ be a fixed basis in the space V. In the basis $\{v_i \otimes v_j\}_{i,j=1}^N$, any operator $X \in \mathrm{End}(V^{\otimes 2})$ is determined by its matrix X_{ij}^{kl}: $X(v_i \otimes v_j) := \sum_{k,l=1}^N X_{ij}^{kl} v_k \otimes v_l$.

We say that an operator $X \in \mathrm{End}(V^{\otimes 2})$ is *skew-invertible* if there exists a matrix ${\Psi^X}_{ij}^{kl}$ such that

$$\sum_{a,b=1}^N X_{ia}^{kb} {\Psi^X}_{bj}^{al} = \delta_i^l \delta_j^k. \tag{2.6}$$

It is easily seen that skew-invertibility is invariant with respect to the change of the basis $\{v_i\}$, so that there is an operator $\Psi^X \in \mathrm{End}(V^{\otimes 2})$ corresponding to the matrix ${\Psi^X}_{ij}^{kl}$. We denote

$$D^X := \mathrm{Tr}_{(2)} \Psi^X \in \mathrm{End}(V), \tag{2.7}$$

where the symbol $\mathrm{Tr}_{(2)}$ means taking the trace over the second space in the product $V \otimes V$. A skew-invertible operator X is said to be *strictly* skew-invertible if the operator D^X is invertible.

Consider the set $\mathrm{Mat}_N(W)$ of $(N \times N)$-matrices whose entries are vectors of a linear space W over $\mathbb{C}$. The set $\mathrm{Mat}_N(W)$ is endowed with the structure of a linear space over $\mathbb{C}$ in a standard way.

Let R be a skew-invertible R-matrix. The linear map

$$\mathrm{Tr}_R : \ \mathrm{Mat}_N(W) \to W$$

defined by

$$\mathrm{Tr}_R(M) = \sum_{i,j=1}^N {D^R}_i^j M_j^i, \quad M \in \mathrm{Mat}_N(W), \tag{2.8}$$

will be called the operation of taking the *R-trace*.

Remark 2.1. If R is the transposition operator on the $(m|n)$-dimensional superspace (see (2.32)), then the operation Tr_R evaluates the supertrace of the argument.

An ordered pair $\{R, F\}$ of two R-matrices R and F will be called a *compatible pair* if the following relations are satisfied:

$$R_1 F_2 F_1 = F_2 F_1 R_2, \quad R_2 F_1 F_2 = F_1 F_2 R_1. \tag{2.9}$$

Now we list some properties of compatible pairs of R-matrices.

1) If R is a skew-invertible R-matrix and $\{R, F\}$ is a compatible pair, then for all $M \in \mathrm{Mat}_N(W)$ we have

$$\mathrm{Tr}_{R(2)}(F_1^{\pm 1} M_1 F_1^{\mp 1}) = \mathrm{Id}_V \, \mathrm{Tr}_R \, M, \quad \text{where } M_1 = M \otimes \mathrm{Id}_V \, . \tag{2.10}$$

2) If R is skew-invertible, then

$$[R_1, D_1^R D_2^R] = 0. \tag{2.11}$$

A direct consequence of this formula is the *cyclic property* of the R-trace:

$$\mathrm{Tr}_{R(12)}(R_1 U) = \mathrm{Tr}_{R(12)}(U R_1), \quad U \in \mathrm{Mat}_N(W)^{\otimes 2}. \tag{2.12}$$

3) If $\{R, F\}$ is a compatible pair, then $R_f := F^{-1} R^{-1} F$ is an R-matrix and the pair $\{R_f, F\}$ is compatible.
4) Suppose $\{R, F\}$ is a compatible pair, the R-matrix R_f is skew-invertible, and the matrices R and F are strictly skew-invertible. Then the map $\phi : \mathrm{Mat}_N(W) \to \mathrm{Mat}_N(W)$ defined by

$$\phi(M) := \mathrm{Tr}_{R(12)}(F_1 M_1 F_1^{-1} R_1), \quad M \in \mathrm{Mat}_N(W), \tag{2.13}$$

is invertible. An explicit form for its inverse ϕ^{-1} is as follows (see [OP2]):

$$\phi^{-1}(M) = \mathrm{Tr}_{\,R_f(2)}(F_1^{-1} M_1 R_1^{-1} F_1). \tag{2.14}$$

2.3. Quantum matrix algebras: Definition. We consider the linear space $\mathrm{Mat}_N(W)$ and define a series of linear mappings $\mathrm{Mat}_N(W)^{\otimes k} \to \mathrm{Mat}_N(W)^{\otimes (k+1)}$, $k \geq 1$, by the following recurrence relations:

$$M_{\overline{1}} := M, \quad M_{\overline{k}} \mapsto M_{\overline{k+1}} := F_k M_{\overline{k}} F_k^{-1}, \quad M_{\overline{k}} \in \mathrm{Mat}_N(W)^{\otimes k}, \tag{2.15}$$

where M is an arbitrary matrix in $\mathrm{Mat}_N(W)$ and F is a fixed element of $\mathrm{Aut}(V \otimes V)$.

In the role of linear space W we choose an associative algebra $\mathcal{A}$ over $\mathbb{C}$ freely generated by the unit element and the N^2 generators M_i^j:

$$\mathcal{A} = \mathbb{C}\langle M_i^j \rangle, \quad 1 \leq i, j \leq N.$$

Definition 2.2. Let $\{R, F\}$ be a compatible pair of strictly skew-invertible R-matrices, and let the R-matrix $R_f = F^{-1} R^{-1} F$ be skew-invertible. By definition, the quantum matrix algebra $\mathcal{M}(R, F)$ is the quotient of the algebra $\mathcal{A}$ over the two-sided ideal generated by the entries of the matrix relation

$$R_1 M_{\overline{1}} M_{\overline{2}} - M_{\overline{1}} M_{\overline{2}} R_1 = 0, \tag{2.16}$$

where $M = \|M_i^j\| \in \mathrm{Mat}_N(\mathcal{A})$ and the matrices $M_{\overline{k}}$ are constructed as in (2.15) with the R-matrix F.

Remark 2.3. The definition of the quantum matrix algebra presented here differs from that given in [IOP1]. Nevertheless, under the additional assumptions on R and F made above, both definitions are equivalent. As was shown in [IOP2], the matrix $\widetilde{M} = (D^F)^{-1} M D^R$ obeys the relations imposed on the generators of the quantum matrix algebra in [IOP1].

Lemma 2.4. *The matrix M composed of the generators of the quantum matrix algebra $\mathcal{M}(R, F)$ satisfies the relations*

$$R_k M_{\overline{k}} M_{\overline{k+1}} = M_{\overline{k}} M_{\overline{k+1}} R_k, \tag{2.17}$$

where the matrices $M_{\overline{k}}$, $M_{\overline{k+1}}$ are defined by the rules (2.15) *with the R-matrix F.*

The proof of this lemma is presented in [IOP1].

2.4. The characteristic subalgebra and Schur functions. Further in this section we consider the quantum matrix algebras $\mathcal{M}(R,F)$ determined by a Hecke R-matrix R. We shall refer to them as the matrix algebras of *Hecke type*.

Consider the linear span $\mathrm{Char}(R,F) \subset \mathcal{M}(R,F)$ of the unit and the elements

$$y(x^{(k)}) = \mathrm{Tr}_{R(1\cdots k)}(M_{\overline{1}}\cdots M_{\overline{k}}\rho_R(x^{(k)})), \quad k=1,2,\dots, \tag{2.18}$$

where $x^{(k)}$ runs over all elements of $\mathcal{H}_k(q)$. The symbol $\mathrm{Tr}_{R(1\cdots k)}$ stands for the calculation of the R-trace over the spaces from the first to the kth.

Proposition 2.5. *Let $\mathcal{M}(R,F)$ be a Hecke type quantum matrix algebra. The subspace $\mathrm{Char}(R,F)$ is a commutative subalgebra of $\mathcal{M}(R,F)$. The subalgebra $\mathrm{Char}(R,F)$ will be called the characteristic subalgebra.*

The detailed proof is given in [IOP1]. It is based, in particular, on the following technical result.

Lemma 2.6. *Consider an arbitrary element $x^{(k)}$ of the algebra $\mathcal{H}_k(q)$. Let $x^{(k)\uparrow i}$ denote the image of $x^{(k)}$ in the algebra $\mathcal{H}_{k+i}(q)$ under the embedding $\mathcal{H}_k(q) \hookrightarrow \mathcal{H}_{k+i}(q)$ defined by* (1.20)*. Let $\{R,F\}$ be a compatible pair of R-matrices, the Hecke R-matrix R being skew-invertible. Then the following relation is fulfilled*:

$$\mathrm{Tr}_{R(i+1,\dots,i+k)}(M_{\overline{i+1}}\cdots M_{\overline{i+k}}\rho_R(x^{(k)\uparrow i})) = \mathrm{Id}_{V^{\otimes i}}\, y(x^{(k)}). \tag{2.19}$$

Formula (2.19) will be used repeatedly in the sequel. It is an immediate consequence of property (2.10) of the compatible pairs $\{R,F\}$; the proof of (2.18) is also presented in [IOP1].

We consider two sets $\{p_k(M)\}$ and $\{s_\lambda(M)\}$ of elements of the characteristic subalgebra. The elements of these sets will be referred to as *the power sums* and *the Schur functions*, respectively. For an arbitrary integer $k \ge 1$, the power sums are defined by

$$p_k(M) := \mathrm{Tr}_{R(1\cdots k)}(M_{\overline{1}}\cdots M_{\overline{k}}R_{k-1}\cdots R_1). \tag{2.20}$$

Given a Hecke type R-matrix R and an arbitrary partition $\lambda \vdash k$, $k=1,2,\dots$, we define the Schur functions as follows:

$$s_0(M) := 1, \quad s_\lambda(M) := \mathrm{Tr}_{R(1\cdots k)}(M_{\overline{1}}\cdots M_{\overline{k}}\rho_R(E^\lambda_\alpha)). \tag{2.21}$$

Actually, the right-hand side of (2.21) does not depend on the index α of the matrix unit. Indeed, for all $x^{(k)} \in \mathcal{H}_k(q)$ the matrices $\rho_R(x^{(k)})$ and $(M_{\overline{1}}\cdots M_{\overline{k}})$ commute in (2.18) due to (2.17). Then, taking the cyclic property of the R-trace into account, we get

$$y\left(\sigma_k(\ell)E^\lambda_\alpha\sigma_k^{-1}(\ell)\right) = y(E^\lambda_\alpha), \quad y(E^\lambda_{\pi_k(\alpha)\alpha}) = w(\ell)y\left(E^\lambda_{\pi_k(\alpha)}\sigma_k(\ell)E^\lambda_\alpha\right) = 0, \tag{2.22}$$

where $\ell := \ell_k\{{\lambda \atop \alpha}\}$. Next, using (1.8), (1.10), (1.17), and (1.18), we can rewrite relations (1.13) in the form

$$E^\lambda_{\pi_k(\alpha)} = \sigma_k(\ell)E^\lambda_\alpha\sigma_k^{-1}(\ell) - (q-q^{-1})w(-\ell)E^\lambda_{\pi_k(\alpha)\alpha}. \tag{2.23}$$

This leads to $y(E^\lambda_{\pi_k(\alpha)}) = y(E^\lambda_\alpha)$. Finally, the fact that the set of standard Young tableaux is cyclic with respect to the action of the symmetric group guarantees that the definition (2.21) is consistent.

Proposition 2.7. *Let $\mathcal{M}(R,F)$ be any quantum matrix algebra of Hecke type.*

i) *Its characteristic subalgebra $\mathrm{Char}(R,F)$ is generated by the unit and the power sums $p_k(M)$, $k=1,2,\dots$.*
ii) *The set of Schur functions $s_\lambda(M)$, $\lambda \vdash k$, $k=0,1,\dots$, forms a linear basis in $\mathrm{Char}(R,F)$.*

Proof. The first part of this proposition was proved in [IOP1]. To prove the second part, we note that the arguments given above to justify relations (2.21) lead to the following identities:

$$y(E^{\lambda}_{\alpha\beta}) = \delta_{\alpha\beta}\, y(E^{\lambda}_{\alpha}), \;\; y(E^{\lambda}_{\alpha}) = y(E^{\lambda}_{\beta}) \quad \text{for all } \alpha, \beta. \tag{2.24}$$

Now, the second part of the proposition follows from the fact that the matrix units $\{E^{\lambda}_{\alpha\beta} \mid \lambda \vdash k\}$ form a linear basis in the algebra $\mathcal{H}_k(q)$. □

2.5. Powers of quantum matrices. Consider the linear span

$$\mathrm{Pow}(R, F) \subset \mathrm{Mat}_N(\mathcal{M}(R, F))$$

of the scalar matrices $\mathrm{Id}_V\, y$, $y \in \mathrm{Char}(R, F)$, and the matrices

$$M^{(x^{(k)})} := \mathrm{Tr}_{R(2\cdots k)}(M_{\overline{1}} \cdots M_{\overline{k}}\, \rho_R(x^{(k)})), \quad k = 1, 2, \ldots, \tag{2.25}$$

where $x^{(k)}$ runs over all elements of $\mathcal{H}_k(q)$. The matrix $M^{(x^{(k)})}$ will be called the $x^{(k)}$*th power* of the matrix M, and the space $\mathrm{Pow}(R, F)$ will be referred to as the *space of matrix powers*.

Proposition 2.8. *The space* $\mathrm{Pow}(R, F)$ *is a right module over the characteristic subalgebra* $\mathrm{Char}(R, F)$.

The proof employs (2.19) and is similar to the proof of the commutativity of $\mathrm{Char}(R, F)$ (see [IOP1]).

In the space $\mathrm{Pow}(R, F)$, we choose a special set of matrices constructed by the following rule. Having fixed a standard Young tableau $\{\begin{smallmatrix}\lambda\\ \alpha\end{smallmatrix}\}$ of the form $\lambda \vdash k$, $k = 1, 2, \ldots$, we introduce the matrices

$$M^{((1);1)} := M, \quad M^{(\lambda;i)} := \mathrm{Tr}_{R(2\cdots k)}(M_{\overline{1}} \cdots M_{\overline{k}}\, \rho_R(E^{\lambda}_{\alpha})), \quad k = 2, 3, \ldots, \tag{2.26}$$

where the index i in $M^{(\lambda;i)}$ is the number of the row of the tableau $\{\begin{smallmatrix}\lambda\\ \alpha\end{smallmatrix}\}$ (counting from the top) that contains k—the largest integer among those filling the tableau. As an example, take the standard tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

Since the largest integer $k = 7$ is contained in the second row, we have $i = 2$ for the diagram involved.

The matrix $M^{(\lambda;i)}$ defined in (2.26) does not depend on the positions of the other integers that are less than k. The proof is similar to the arguments presented in favor of the unambiguity of the Schur function definition (see the text following formula (2.21)), but an additional circumstance should be taken into account. If two matrix units E^{λ}_{α} and E^{λ}_{β}, $\lambda \vdash k$, have the number k standing at the same nodes of the corresponding tableaux, then the chain of transformations linking these matrix units contains only the elements $x^{(k-1)}$ of the subalgebra $\mathcal{H}_{k-1}(q)$. Precisely with respect to the images $\rho_R(x^{(k-1)}) \in \mathrm{Id}_V \otimes \mathrm{Aut}(V^{\otimes(k-1)})$, the operation $\mathrm{Tr}_{R(2\cdots k)}$ possesses the cyclic property.

Besides $M^{(\lambda;i)}$, we also consider a series of matrices $M^{\overline{k}}$:

$$M^{\overline{1}} := M, \quad M^{\overline{k}} := \mathrm{Tr}_{R(2\cdots k)}(M_{\overline{1}} \cdots M_{\overline{k}} R_{k-1} \cdots R_1), \quad k = 2, 3, \ldots. \tag{2.27}$$

The matrices $M^{(\lambda;i)}$ and $M^{\overline{k}}$ will be called (respectively) the $(\lambda; i)$*th* and the k*th powers* of the matrix M.

Proposition 2.9. *The following is true for any quantum matrix algebra $\mathcal{M}(R,F)$ of Hecke type:*

i) *The unit matrix and the family of matrices $M^{\overline{k}}$, $k=1,2,\dots$, is a generating set of the right* $\mathrm{Char}(R,F)$*-module* $\mathrm{Pow}(R,F)$.
ii) *The scalar matrices* $\mathrm{Id}_V\, s\lambda(M)$ *and the matrices* $M^{(\lambda;i)}$, $\lambda\vdash k$, $k=1,2,\dots$, *where i runs over all possible values for each partition $\lambda\vdash k$, form a basis of the linear space* $\mathrm{Pow}(R,F)$.

The proof of items i) and ii) of the above proposition can be given by a quite obvious generalization of the proof of the corresponding items of Proposition 2.7.

To end this subsection, we present another expression for the kth power of the matrix M. By using property (2.10) of compatible pairs, it can be shown that the definition (2.27) is equivalent to the following iterative formulas:

$$M^{\overline{0}}=\mathrm{Id}_V,\quad M^{\overline{k}}=M\cdot\phi(M^{\overline{k-1}}), \tag{2.28}$$

where the map ϕ is given by (2.13) and the dot stands for the usual matrix product. Relations (2.28) can be interpreted as the result of the consecutive application of the operator $\mathsf{M}:\mathrm{Mat}_N(\mathcal{M}(R,F))\to\mathrm{Mat}_N(\mathcal{M}(R,F))$,

$$\mathsf{M}(X):=M\cdot\phi(X),\quad X\in\mathrm{Mat}_N(\mathcal{M}(R,F)),$$

to the unit matrix Id_V. Indeed, since $\phi(\mathrm{Id}_V)=\mathrm{Id}_V$, we have

$$\mathsf{M}(\mathrm{Id}_V)=M,\quad M^{\overline{k}}=\mathsf{M}(M^{\overline{k-1}})=\cdots=\mathsf{M}^k(\mathrm{Id}_V).$$

2.6. Quantum matrix algebras: Examples. Let $P:V^{\otimes 2}\to V^{\otimes 2}$ be the transposition operator,

$$P(v_1\otimes v_2)=v_2\otimes v_1,$$

and let the R-matrix R be strictly skew-invertible. The pair $\{R,P\}$ is compatible and determines a quantum matrix algebra $\mathcal{M}(R,P)$. Let T denote the matrix of its generators. In the case in question we have $T_{\overline{k}}=T_k$ (see (2.1) and (2.15)), and the commutation relations (2.16) take the form

$$R_1T_1T_2-T_1T_2R_1=0. \tag{2.29}$$

Suppose additionally that R is of $GL(m)$ type. The standard example of such an R-matrix is the Drinfeld–Jimbo R-matrix obtained by quantization of classical groups of the A_m series. The corresponding algebra $\mathcal{M}(R,P)=\mathrm{Fun}_q(GL(m))$ is the quantization of the algebra of functions on the general linear group of $(m\times m)$-matrices ($N=m$ in this case) [FRT]. Note that $m\neq N$ in general. Examples of $GL(m)$ type R-matrices acting in spaces V of dimension $N\neq m$ were constructed in [G].

If R is an R-matrix of $GL(m)$ type (including the Drinfeld–Jimbo R-matrix), the matrix T of generators of $\mathcal{M}(R,P)$ obeys a polynomial Cayley–Hamilton identity of order m [IOPS, IOP1]. The coefficients of the polynomial are proportional to the Schur functions $s_{(1^k)}(T)$, $0\le k\le m$. In the classical limit as $q\to 1$, the Drinfeld–Jimbo R-matrix tends to the transposition matrix $\lim_{q\to 1}R=P$, and the algebra (2.29) becomes the commutative matrix algebra $\mathcal{M}(P,P)$. The corresponding Cayley–Hamilton identity coincides with (0.1). In particular, the Schur functions $s_{(1^k)}(T)$ turn into the elementary symmetric functions in the eigenvalues of the matrix T.

Another example of a compatible pair is given by $\{R,R\}$, where R is a strictly skew-invertible R-matrix. The commutation relations (2.16) for the generators of the quantum matrix algebra $\mathcal{M}(R,R)$ look like this:

$$R_1L_1R_1L_1-L_1R_1L_1R_1=0, \tag{2.30}$$

where by $L = \|L_i^j\|$ we denote the matrix composed of the algebra generators. The algebra $\mathcal{M}(R,R)$ is called the *reflection equation algebra* [KS]. Having appeared first in the theory of integrable systems with boundaries, this algebra has since found applications in the differential geometry of quantum groups (see, e.g., [IP, FP, GS, GS2]).

In this case the power of the quantum matrix coincides with the usual matrix power $L^{\overline{k}} = L^k$. The characteristic subalgebra $\mathrm{Char}(R,R)$ belongs to the center of $\mathcal{M}(R,R)$. The Cayley–Hamilton identity for the reflection equation algebra with Drinfeld–Jimbo R-matrix was obtained in [NT, PS]. Its generalization to the case of an arbitrary R-matrix of $GL(m)$ type can be found in [GPS].

Assuming that R is a strictly skew-invertible R-matrix of Hecke type, we make a linear shift of generators of $\mathcal{M}(R,R)$:

$$L \mapsto \mathcal{L} = L + \frac{1}{q - q^{-1}} \,\mathrm{Id}_V\,.$$

After this change, relations (2.30) take the quadratic-linear form

$$R_1\mathcal{L}_1R_1\mathcal{L}_1 - \mathcal{L}_1R_1\mathcal{L}_1R_1 = R_1\mathcal{L}_1 - \mathcal{L}_1R_1. \tag{2.31}$$

The above basis of generators is convenient in constructing quantum analogs of orbits of the coadjoint action of a Lie group on the space dual to its Lie algebra. The Cayley–Hamilton identity for this basis was obtained in [GS]. If R is the Drinfeld–Jimbo R-matrix, then, in the limit as $q \to 1$, relations (2.31) turn into commutation relations for the generators of the universal enveloping algebra $U(gl_N)$, and the Cayley–Hamilton identity transforms into the well-known identity for the matrix composed of the $U(gl_N)$ generators (see, e.g., [Gou]).

Let $P_{m|n}$ be the transposition operator on the $(m|n)$-dimensional superspace

$$P_{m|n}(v_1 \otimes v_2) = (-1)^{|v_1||v_2|} v_2 \otimes v_1, \tag{2.32}$$

where v_i is a homogeneous element of the superspace and $|v_i|$ denotes the corresponding parity. The algebra $\mathcal{M}(R,R)$ with $R = P_{m|n}$ was considered in [KT1, KT2]. Note that the matrix superalgebra $\mathcal{M}(P_{m|n}, P_{m|n})$ is a limit case of the algebra $\mathcal{M}(R_{m|n}, R_{m|n})$ (2.30) as $q \to 1$, where $R_{m|n}$ is the Drinfeld–Jimbo R-matrix obtained by the quantization of the classical supergroup $GL(m|n)$ (an explicit form of this R-matrix can be found in [DKS, Isa]). The so-called invariant Cayley–Hamilton identity established in [KT1, KT2] is a limit case of the Cayley–Hamilton identity for the quantum matrix algebras of $GL(m|n)$ type. The latter is proved in the next section.

§3. Proof of the Cayley–Hamilton identities

In this section we give an outline of the proof and then formulate the Cayley–Hamilton identity for the quantum matrix algebras $\mathcal{M}(R,F)$ determined by an R-matrix R of $GL(m|n)$ type. Below, these algebras will be referred to as the $GL(m|n)$ *type matrix algebras.*

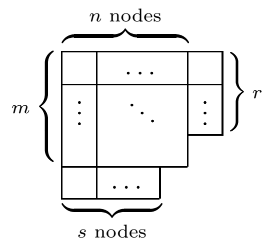
We fix a pair of integers $m \geq 1$, $n \geq 1$ and denote $A := (m+1)(n+1)$. We also introduce special notation for the following partitions:

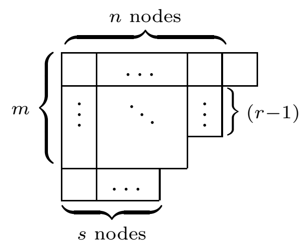
$$\Lambda(r,s) := \big((n+1)^r, n^{(m-r)}, s\big), \qquad r = 0,\dots,m,\ \ s = 0,\dots,n, \tag{3.1}$$

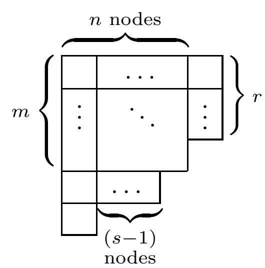
$$\Lambda^+(r,s) := \big(n+2, (n+1)^{(r-1)}, n^{(m-r)}, s\big), \quad r = 1,\dots,m,\ \ s = 0,\dots,n, \tag{3.2}$$

$$\Lambda_+(r,s) := \big((n+1)^r, n^{(m-r)}, s, 1\big), \qquad r = 0,\dots,m,\ \ s = 1,\dots,n. \tag{3.3}$$

The corresponding Young diagrams are

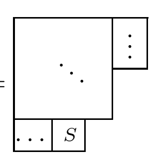
$\Lambda(r,s) =$ 

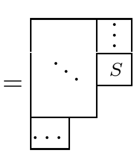
$\Lambda^+(r,s) =$ 

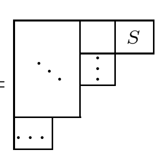
$\Lambda_+(r,s) =$ 

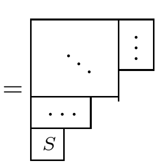
Let the symbol $E_{\rm row}^{\Lambda(r,s)}$, $s \ge 1$ (respectively, $E_{\rm col}^{\Lambda(r,s)}$, $r \ge 1$, $E^{\Lambda^+(r,s)}$, $E^{\Lambda_+(r,s)}$) denote a diagonal matrix unit that corresponds to a standard tableau of the form $\Lambda(r,s)$ (respectively, $\Lambda(r,s)$, $\Lambda^+(r,s)$, $\Lambda_+(r,s)$) containing a standard Young tableau of the form $\Lambda(r,s-1)$ (respectively, $\Lambda(r-1,s)$, $\Lambda(r,s)$, $\Lambda(r,s)$). In other words, in the diagonal matrix units $E_{\rm row}^{\Lambda(r,s)}$, $E_{\rm col}^{\Lambda(r,s)}$, $E^{\Lambda^+(r,s)}$, and $E^{\Lambda_+(r,s)}$ the position of the largest number S (S is equal to $mn+r+s$, $mn+r+s$, $mn+r+s+1$, or $mn+r+s+1$, respectively) is fixed as shown below:

(3.4)

$E_{\rm row}^{\Lambda(r,s)} =$ 

$E_{\rm col}^{\Lambda(r,s)} =$ 

$E^{\Lambda^+(r,s)} =$ 

$E^{\Lambda_+(r,s)} =$ 

We emphasize that the further calculations do not depend on the positions of the other numbers in the nodes of the matrix units (3.4). Finally, we introduce special notation for the following elements of the space $\mathrm{Mat}_N(\mathcal{M}(R,F))$:

$$P_{\mathrm{row}}(r,s) := \mathrm{Tr}_{R(2\cdots A)}\big(M_{\overline{2}}\cdots M_{\overline{A}}\rho_R(E^{\Lambda(r,s)}_{\mathrm{row}})R_tR_{t-1}\cdots R_1\big), \tag{3.5}$$

$$P_{\mathrm{col}}(r,s) := \mathrm{Tr}_{R(2\cdots A)}\big(M_{\overline{2}}\cdots M_{\overline{A}}\rho_R(E^{\Lambda(r,s)}_{\mathrm{col}})R_tR_{t-1}\cdots R_1\big), \tag{3.6}$$

$$P^{+}(r,s) := \mathrm{Tr}_{R(2\cdots A)}\big(M_{\overline{2}}\cdots M_{\overline{A}}\rho_R(E^{\Lambda^{+}(r,s)})R_{t-1}R_{t-2}\cdots R_1\big), \tag{3.7}$$

$$P_{+}(r,s) := \mathrm{Tr}_{R(2\cdots A)}\big(M_{\overline{2}}\cdots M_{\overline{A}}\rho_R(E^{\Lambda_{+}(r,s)})R_{t-1}R_{t-2}\cdots R_1\big), \tag{3.8}$$

where $t := (m-r)+(n-s)+1$.

We take the following linear combinations of the above matrices:

$$\begin{aligned}\Phi_i := &\sum_{k=\max\{0,i-n\}}^{\min\{i-1,m\}}(-1)^k\frac{(i-k)_q(m+n-i+k+2)_q}{(m+n-i+2)_q}P_{\mathrm{row}}(k,i-k)\\ &-\sum_{k=\max\{1,i-n\}}^{\min\{i,m\}}(-1)^k\frac{k_q(m+n-k+2)_q}{(m+n-i+2)_q}P_{\mathrm{col}}(k,i-k),\end{aligned} \tag{3.9}$$

where the index i runs from 1 to $m+n$. Our immediate task is to prove the relations

$$\Phi_{i+1}-\Phi_i = \phi(M^{\overline{m+n-i}})\sum_{k=\max\{0,i-n\}}^{\min\{i,m\}}(-1)^kq^{2k-i}s_{\Lambda(k,i-k)}(M), \quad i=1,2,\dots,m+n-1. \tag{3.10}$$

For this, we reshape the expression for Φ_i to the form

$$\begin{aligned}\Phi_i = &\sum_{k=\max\{0,i-n\}}^{\min\{i-1,m\}}(-1)^k\frac{(i-k)_q(m+n-i+k+2)_q}{(m+n-i+2)_q}\\ &\quad\times\Big\{\frac{q^{(m+n+k-i+2)}}{(m+n+k-i+2)_q}P^{+}(k,i-k)\\ &\qquad-\frac{q^{-(i-k)}}{(i-k)_q}P_{+}(k,i-k)+qP_{\mathrm{row}}(k,i-k+1)\\ &\qquad\quad+\frac{q^{(m+n-i+1)}}{(m+n-i+1)_q}P_{\mathrm{col}}(k+1,i-k)\Big\}\\ &-\sum_{k=\max\{1,i-n\}}^{\min\{i,m\}}(-1)^k\frac{k_q(m+n-k+2)_q}{(m+n-i+2)_q}\\ &\quad\times\Big\{\frac{q^k}{k_q}P^{+}(k,i-k)-\frac{q^{-(m+n-k+2)}}{(m+n-k+2)_q}P_{+}(k,i-k)\\ &\qquad-\frac{q^{-(m+n-i+1)}}{(m+n-i+1)_q}P_{\mathrm{row}}(k,i-k+1)-q^{-1}P_{\mathrm{col}}(k+1,i-k)\Big\}.\end{aligned} \tag{3.11}$$

Here we extend the definition of the elements (3.5)–(3.8) to the boundary values of its indices, that is, to the cases where the matrix units used in (3.5)–(3.8) are not defined:

$$P_{\mathrm{row}}(k,n+1) = P_{\mathrm{col}}(m+1,k) = P^{+}(k,0) = P_{+}(0,k) = 0, \quad k=0,1,\dots.$$

When deriving (3.11), we apply formulas (1.21) to the matrix units (3.4),

$$E^{\Lambda(r,s)}_{\mathrm{row/col}} = E^{\Lambda(r,s+1)}_{\mathrm{row}} + E^{\Lambda(r+1,s)}_{\mathrm{col}} + E^{\Lambda^{+}(r,s)} + E^{\Lambda_{+}(r,s)}, \tag{3.12}$$

and then use (1.14) in the form

$$\rho_R(E^\lambda_\alpha)R_k = \omega^{-1}(\ell_k)\rho_R(E^\lambda_{\alpha,\pi_k(\alpha)}) - \frac{q^{-\ell_k}}{(\ell_k)_q}\rho_R(E^\lambda_\alpha), \quad \lambda \vdash (k+1), \ \ \ell_k \equiv \ell_k\{{}^\lambda_\alpha\}, \tag{3.13}$$

in order to transform the terms $\rho_R(E^{\Lambda(r,s)}_{\mathrm{row/col}})R_t$ in the expressions for $P_{\mathrm{row/col}}(k, i-k)$. By using calculations similar to (2.24), it can be shown that the term with the nondiagonal matrix unit involved in the right-hand side of (3.13) does not contribute to (3.11).

Note that the expressions (3.12) for $E^{\Lambda(r,s)}_{\mathrm{row}}$ and $E^{\Lambda(r,s)}_{\mathrm{col}}$ are *formally* the same, since our notation (3.4) only takes into account the position of the largest integer among those filling the Young tableau. One should keep in mind that the matrix units $E^{\Lambda(r,s)}_{\mathrm{row}}$ and $E^{\Lambda(r,s)}_{\mathrm{col}}$ on the right-hand side of (3.12) differ in the position of the number preceding the largest one. This difference manifests itself in the application of relation (3.13).

Next we simplify (3.11):

(3.14)

$$\begin{aligned}
\Phi_i = &- \sum_{k=\max\{0,i-n\}}^{\min\{i,m\}} (-1)^k q^{2k-i}\left\{P^+(k,i-k) + P_+(k,i-k)\right\} \\
&+ \sum_{k=\max\{0,i+1-n\}}^{\min\{i,m\}} \frac{(-1)^k}{(m+n-i+2)_q} \\
&\qquad \times \left\{ q^{-(m+n-i+1)}\frac{k_q(m+n-k+2)_q}{(m+n-i+1)_q} + q(i-k)_q(m+n-i+k+2)_q\right\} \\
&\qquad \times P_{\mathrm{row}}(k, i-k+1) \\
&+ \sum_{k=\max\{0,i-n\}}^{\min\{i,m-1\}} \frac{(-1)^k}{(m+n-i+2)_q} \\
&\qquad \times \left\{ q^{(m+n-i+1)}\frac{(i-k)_q(m+n-i+k+2)_q}{(m+n-i+1)_q} + q^{-1}k_q(m+n-k+2)_q\right\} \\
&\qquad \times P_{\mathrm{col}}(k+1, i-k).
\end{aligned}$$

In the calculation of the coefficients of $P^+(\cdot,\cdot)$ and $P_+(\cdot,\cdot)$, the following relation has been useful:

$$q^x y_q + q^{-y}x_q = (x+y)_q, \quad x, y \in \mathbb{Z}. \tag{3.15}$$

Next, we transform the first summand on the right-hand side of (3.14) with the help of the relation

$$P^+(k,i-k) + P_+(k,i-k) + P_{\mathrm{row}}(k,i-k+1) + P_{\mathrm{col}}(k+1,i-k)$$

$$= \mathrm{Tr}_{R(2\cdots A)}\left(M_{\overline{2}}\cdots M_{\overline{A}}\rho_R(E^{\Lambda(r,s)}_\alpha)R_{m+n-i}\cdots R_1\right) \tag{3.16}$$

$$= \mathrm{Tr}_{R(2\cdots(m+n-i+1))}\left(M_{\overline{2}}\cdots M_{\overline{m+n-i+1}}R_{m+n-i}\cdots R_1\right)s_{\Lambda(k,i-k)}(M)$$

$$= \phi(M^{\overline{m+n-i}})s_{\Lambda(k,i-k)}(M). \tag{3.17}$$

In the above computation, first we used (3.12), which yields (3.16). Expression (3.16) turns out to be independent of a particular choice of the index α of the matrix unit. Then we split (3.16) into two factors with the use of (2.19). Finally, the first factor is transformed in the same way that was used to derive relation (2.28) for the matrix powers from the definition (2.27).

Substituting (3.17) in (3.14), we get

$$
\begin{aligned}
\Phi_i = & -\phi(M^{\overline{m+n-i}}) \sum_{k=\max\{0,i-n\}}^{\min\{i,m\}} (-1)^k q^{2k-i} s_{\Lambda(k,i-k)}(M) \\
& + \sum_{k=\max\{0,i+1-n\}}^{\min\{i,m\}} (-1)^k \frac{(i+1-k)_q (m+n-i+k+1)_q}{(m+n-i+1)_q} P_{\mathrm{row}}(k, i+1-k) \\
& + \sum_{k=\max\{0,i-n\}}^{\min\{i,m-1\}} (-1)^k \frac{(k+1)_q (m+n-k+1)_q}{(m+n-i+1)_q} P_{\mathrm{col}}(k+1, i-k).
\end{aligned}
\tag{3.18}
$$

Here, to simplify the coefficients at $P_{\mathrm{row}}(\cdot,\cdot)$ and $P_{\mathrm{col}}(\cdot,\cdot)$, we have used the formula

$$
q^{-x} \frac{(x+y+1)_q z_q}{x_q} + q y_q (x+z+1)_q + q^{z-y}(x+1)_q = \frac{(x+1)_q (y+1)_q (x+z)_q}{x_q}, \tag{3.19}
$$

which can easily be verified with the help of (3.15).

Finally, shifting the summation index $k \mapsto k+1$ in the last summand in (3.18), we arrive at (3.10).

Now it only remains to construct analogs of (3.10) for the boundary values $i = 0$ and $i = m+n$.

For $i = 0$ we have

$$
\Phi_1 = P_{\mathrm{row}}(1,0) + P_{\mathrm{col}}(0,1) = \phi(M^{\overline{m+n}}) s_{\Lambda(0,0)}(M), \tag{3.20}
$$

which coincides with (3.10) if we set $\Phi_0 = 0$.

For $i = m+n$, the transformation of Φ_{m+n} is performed in the same way as we did when deriving (3.10). A peculiarity of this case consists of two points. First, when we carry out the computations, an additional term associated with the Young diagram $((n+1)^{m+1})$ arises, and second, $\Phi_{m+n+1} = 0$. Thus,

$$
\begin{aligned}
\Phi_{m+n} = & (-1)^{m+1} q^{m-n}\, \mathrm{Id}_V\, s_{\Lambda(m,n)}(M) \\
& + (-1)^m (m+1)_q (n+1)_q \mathrm{Tr}_{R(2\dots A)} \big\{ M_{\overline{2}} \cdots M_{\overline{A}} \rho_R \big(E_{m+1}^{((n+1)^{m+1})} \big) \big\},
\end{aligned}
\tag{3.21}
$$

which also has the same form as (3.10) provided that $\rho_R\big(E_{m+1}^{((n+1)^{m+1})}\big) = 0$, which is valid (by definition) in the $GL(m|n)$ case.

Next, we add together the left- and the right-hand sides of all relations (3.10) and (3.20), and then from the sum we subtract the corresponding parts of (3.21). Finally, applying the map ϕ^{-1} (see (2.13), (2.14)) to the result, we arrive at the following theorem.

Theorem 3.1 (The Cayley–Hamilton identity). *Let M be the matrix of generators of a $GL(m|n)$ type quantum matrix algebra $\mathcal{M}(R,F)$. Then the following matrix identity is true*:

$$
\sum_{i=0}^{n+m} M^{\overline{m+n-i}} \sum_{k=\max\{0,i-n\}}^{\min\{i,m\}} (-1)^k q^{2k-i} s_{\Lambda(k,i-k)}(M) \equiv 0. \tag{3.22}
$$

If we multiply both sides of each of the matrix identities (3.10), (3.20), and (3.21) by the matrix M from the left, these identities will turn into relations among the elements of the space $\mathrm{Pow}(R,F)$. Adding these relations together as in the proof of the Cayley–Hamilton theorem, we establish a relationship among the elements of two basis sets of the space $\mathrm{Pow}(R,F)$ that were described in Proposition 2.9.

Proposition 3.2. *Let M be the matrix of generators of a quantum matrix algebra $\mathcal{M}(R,F)$ of Hecke type. Then the λ-powers of M corresponding to the rectangular Young diagrams $\lambda = \big((r+1)^{s+1}\big)$, $r,s = 0,1,\ldots,$ are expressed in terms of its $\overline{k}$th powers as follows*:

$$\begin{aligned}(3.23)\qquad &(-1)^s(s+1)_q(r+1)_q M^{\left(((r+1)^{s+1});s+1\right)}\\ &= \sum_{i=0}^{s+r} M^{\overline{s+r+1-i}} \sum_{k=\max\{0,i-r\}}^{\min\{i,s\}} (-1)^k q^{2k-i} s_{\Lambda(k,i-k)}(M).\end{aligned}$$

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ISTV, Université de Valenciennes, 59304 Valenciennes, France
E-mail address: `Dimitri.Gourevitch@univ-valenciennes.fr`

Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna 141980, Russia
E-mail address: `pyatov@thsun1.jinr.ru`

Department of Theoretical Physics, IHEP, Protvino 142281, Russia
E-mail address: `Pavel.Saponov@ihep.ru`

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Translated by THE AUTHORS