

## A KINK IN A FUNNY PLACE

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ABSTRACT. When the 3-dimensional gravitational Chern–Simons term is reduced to two dimensions, a dilation-like gravity theory emerges. Its solutions involve kinks, which therefore describe 3-dimensional, conformally flat spaces.

Two topics—among many—which I have discussed with Ludwig Faddeev over the twenty-five years that I have known him are first, gravity theory, and second, topological entities, both in mathematical settings such as characteristic classes, and in physical realizations such as kink profiles on a line. So, on the occasion of his significant birthday I present to Ludwig an investigation which unites these diverse elements.

Let me begin with lineal kinks. Consider the field equation

$$(1) \quad \square\varphi - C\varphi + \varphi^3 = 0,$$

where  $C$  is a positive constant. When we look for a lineal kink solution, we take  $\varphi$  to depend on a single spatial variable. Then (1) reduces to

$$(2) \quad -\varphi'' - C\varphi + \varphi^3 = 0$$

and has the well-known kink solution,

$$(3) \quad \varphi_k(x) = \sqrt{C} \tanh \sqrt{\frac{C}{2}}x,$$

which interpolates between the “vacuum” solutions  $\varphi_0 = \pm\sqrt{C}$  (see, e.g., [1]). The kink has interesting roles in condensed matter physics, where it triggers fermion fractionization [2]. Other kinks in other models give rise to completely solvable field theories, both in classical and quantal frameworks. These stories do not belong here. But I shall return to the above kink later.

Next let me consider the Chern–Simons characteristic class. In non-Abelian gauge theory it is constructed from a matrix gauge connection  $(A_\alpha)_\nu^\mu$  as [3]

$$(4) \quad W(A) = \frac{1}{4\pi^2} \int d^3x \varepsilon^{\alpha\beta\gamma} \text{tr} \left( \frac{1}{2} A_\alpha \partial_\beta A_\gamma + \frac{1}{3} A_\alpha A_\beta A_\gamma \right).$$

This gauge-theoretic entity finds physical application in the quantum Hall regime, perhaps also in high  $T$  superconductivity. When added with strength  $m$  to the usual Yang–Mills action, the Chern–Simons term gives rise to massive, yet gauge-invariant excitations in  $(2 + 1)$ -dimensional space-time. Also, for consistency in the quantized version of a non-Abelian theory,  $m$  must be an integer multiple of  $2\pi$ . This is a precise field-theoretic analog of Dirac’s celebrated quantization of magnetic monopole strength. Finally, in an important mathematical application, the Chern–Simons term gives a functional integral formula for knot invariants.

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2000 *Mathematics Subject Classification.* Primary 83C80, 83E15.

*Key words and phrases.* Dilation-like gravity theory, kink.

The gauge-theoretic Chern–Simons term (4) can be translated into a 3-dimensional geometric quantity by replacing the matrix gauge connection  $(A_\alpha)_\nu^\mu$  with the Christoffel connection  $\Gamma_{\alpha\nu}^\mu$  [3],

$$(5) \quad W(\Gamma) = \frac{1}{4\pi^2} \int d^3x \varepsilon^{\alpha\beta\gamma} \left( \frac{1}{2} \Gamma_{\alpha\sigma}^\rho \partial_\beta \Gamma_{\gamma\rho}^\sigma + \frac{1}{3} \Gamma_{\alpha\sigma}^\rho \Gamma_{\beta\tau}^\sigma \Gamma_{\gamma\rho}^\tau \right).$$

But it is to be remembered that  $\Gamma_{\alpha\nu}^\mu$  is constructed in the usual way from the metric tensor,  $g_{\mu\nu}$ , which is taken as the fundamental, independent variable. When  $W(\Gamma)$  is varied with respect to  $g_{\mu\nu}$ , there emerges the Cotton tensor, which has an important role in 3-dimensional geometry:

$$(6) \quad \delta W(\Gamma) = -\frac{1}{4\pi^2} \int d^3x \delta g_{\mu\nu} \sqrt{g} C^{\mu\nu},$$

$$(7) \quad C^{\mu\nu} \equiv \frac{1}{2\sqrt{g}} \left( \varepsilon^{\mu\alpha\beta} D_\alpha R_\beta^\nu + \varepsilon^{\nu\alpha\beta} D_\alpha R_\beta^\mu \right).$$

(In (7) one may freely replace the Ricci tensor  $R_\nu^\mu$  by the Einstein tensor  $G_\nu^\mu \equiv R_\nu^\mu - \frac{1}{2} \delta_\nu^\mu R^\alpha_\alpha$ .) The Cotton tensor is like the covariant curl of the Ricci or Einstein tensor. It is manifestly symmetric; it is covariantly conserved and traceless because it is the variation of the diffeomorphism and conformally invariant  $W(\Gamma)$ . Furthermore, the Cotton tensor replaces the Weyl tensor, which is absent in three dimensions, as a template for conformal flatness:  $C^{\mu\nu}$  vanishes if and only if the space is conformally flat, i.e.,

$$(8) \quad C^{\mu\nu} = 0 \iff \text{conformally flat space.}$$

The absence of the 3-dimensional Weyl tensor has the consequence that 3-dimensional geometries satisfying Einstein's equation carry nonvanishing curvature only in regions where there are sources. Therefore, there are no propagating excitations. However, upon extending Einstein's gravity equation by adding  $\frac{1}{m} C^{\mu\nu}$  to the Einstein tensor (equivalently, adding  $\frac{4\pi^2}{m} W(\Gamma)$  to the Einstein–Hilbert action) the theory acquires a propagating mode with mass  $m$ , all the while preserving diffeomorphism invariance! Here we have another perspective on the absence of propagating modes in 3-dimensional Einstein theory: to regain the Einstein equations from the modified equations, we must pass  $m$  to infinity, whereupon the supermassive propagating mode decouples.

This is a well-known story, with which I do not concern myself now. Rather, I consider the opposite limit of the extended theory, where only the Cotton tensor survives, and the equation that I shall examine demands its vanishing, i.e., equation (8). But as indicated previously, that equation is not sufficiently restrictive to be interesting: any conformally flat space-time (coordinates  $(t, x, y)$ ) is a solution. So I shall place a further restriction: the solution that I seek should be independent of the  $y$ -coordinate in a Kaluza–Klein dimensional reduction from (2+1) to (1+1) dimensions of the gravitational Chern–Simons term  $W(\Gamma)$  in (5) and of the Cotton tensor  $C^{\mu\nu}$  in (7) (see [4]).

To effect the dimensional reduction, we begin by making a Kaluza–Klein *Ansatz* for the 3-dimensional metric tensor. It is taken in the form

$$(9) \quad \text{3-d metric tensor} = \varphi \begin{pmatrix} g_{\alpha\beta} - a_\alpha a_\beta, & -a_\alpha \\ -a_\beta, & -1 \end{pmatrix},$$

where the 2-dimensional metric tensor  $g_{\alpha\beta}$ , vector  $a_\alpha$ , and scalar  $\varphi$  depend only on  $t$  and  $x$ . (Henceforth, Greek letters from the beginning of the alphabet ( $\alpha, \beta, \gamma, \dots$ ) index 2-dimensional  $(t, x)$ -dependent geometric entities, which are written with lower case letters; in three space-time dimensions geometric entities are capitalized (save the metric tensor) and are indexed by middle Greek alphabet letters ( $\mu, \nu, \rho, \dots$ .)

It is easy to show that under infinitesimal diffeomorphisms that leave the  $y$ -coordinate unchanged,  $g_{\alpha\beta}$ ,  $a_\alpha$  and  $\varphi$  transform as a 2-dimensional coordinate tensor, a vector and a scalar, respectively, and, moreover,  $a_\alpha$  undergoes a gauge transformation.

With the above *Ansatz* for the 3-dimensional metric, the Chern–Simons action becomes

$$(10) \quad CS = -\frac{1}{8\pi^2} \int d^2x \sqrt{-g}(fr + f^3).$$

Here  $g = \det g_{\alpha\beta}$ ,  $\partial_\alpha a_\beta - \partial_\beta a_\alpha \equiv f_{\alpha\beta} \equiv \sqrt{-g}\varepsilon_{\alpha\beta}f$ , and  $r$  is the 2-dimensional scalar curvature. The absence of  $\varphi$ -dependence is a consequence of the conformal invariance of the gravitational Chern–Simons term, and this also ensures that the Cotton tensor is traceless. Henceforth we set  $\varphi$  to 1.

The above expressions look like they are describing 2-dimensional dilation gravity, with  $f$  taking the role of a dilation field [5]. However, in fact  $f$  is not a fundamental field; rather it is the curl of the vector potential  $a_\alpha$ . Alternative expressions for the action (10) are  $\int da(r + f^2)$  (where  $da$  is a 2-form; this exposes the topological character of our theory), and  $\int d^2x \Theta \varepsilon^{\alpha\beta} f_{\alpha\beta}$ ,  $\Theta \equiv r + f^2$  (which highlights an axion-like interaction in 2-dimensional space-time).

Variation of  $a_\alpha$  and  $g_{\alpha\beta}$  produces the equations

$$(11) \quad 0 = \varepsilon^{\alpha\beta} \partial_\beta (r + 3f^2),$$

$$(12) \quad 0 = g_{\alpha\beta} (D^2 f - f^3 - \frac{1}{2} r f) - D_\alpha D_\beta f.$$

The first is solved by

$$(13) \quad r + 3f^2 = \text{constant} \equiv C.$$

Eliminating  $r$  in the second equation, and decomposing it into the trace and trace-free parts leaves

$$(14) \quad 0 = D^2 f - Cf + f^3,$$

$$(15) \quad 0 = D_\alpha D_\beta f - \frac{1}{2} g_{\alpha\beta} D^2 f.$$

Note that the equations are invariant against changing the sign of  $f$  (the action then also changes sign).

A homogenous solution that respects the  $f \leftrightarrow -f$  symmetry is

$$(16) \quad f = 0, \quad r = C.$$

However, there is also a “symmetry breaking” solution,

$$(17) \quad f = \pm\sqrt{C}, \quad r = -2C, \quad C > 0.$$

Forms for  $g_{\alpha\beta}$  and  $a_\alpha$  that lead to the above results are

$$(18) \quad \text{(a) } f = 0, r = C > 0: \quad g_{\alpha\beta} = \frac{2}{Ct^2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad a_\alpha = (0, 0),$$

$$(19) \quad \text{(b) } f = 0, r = C < 0: \quad g_{\alpha\beta} = \frac{2}{|C|x^2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad a_\alpha = (0, 0),$$

$$(20) \quad \text{(c) } f = \pm\sqrt{C}, r = -2C < 0: \quad g_{\alpha\beta} = \frac{1}{Cx^2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad a_\alpha = \left( \frac{\pm 1}{\sqrt{Cx}}, 0 \right).$$

In the first case, the 2-dimensional space-time is deSitter; in the last two, it is anti-deSitter.

The 3-dimensional scalar curvature  $R$ , with metric tensor as in our *Ansatz* (9) at  $\varphi = 1$ , is related to the 2-dimensional curvature  $r$  by

$$(21) \quad R = r + \frac{1}{2}f^2.$$

Hence for the three cases, the 3-dimensional curvature and line element read

$$(22) \quad (a) \quad R = C > 0, \quad (ds)^2 = \frac{2}{C} \left[ \left( \frac{dt}{t} \right)^2 - \left( \frac{dx}{t} \right)^2 \right] - (dy)^2,$$

$$(23) \quad (b) \quad R = C < 0, \quad (ds)^2 = \frac{2}{|C|} \left[ \left( \frac{dt}{x} \right)^2 - \left( \frac{dx}{x} \right)^2 \right] - (dy)^2,$$

$$(24) \quad (c) \quad R = -\frac{3}{2}C < 0, \quad (ds)^2 = \pm \frac{2}{\sqrt{C}x} dt dy - \left( \frac{dx}{\sqrt{C}x} \right)^2 - (dy)^2.$$

Although all three solutions carry constant 3-dimensional curvature, the ‘‘symmetry breaking’’ solution, (c) above, possesses greater geometrical symmetry in three dimensions: it supports six Killing vectors that span  $SO(2.1) \times SO(2.1) = SO(2.2)$ , the isometry of 3-dimensional anti-deSitter space. Moreover, one verifies that, as expected,  $R^\mu_\nu = \frac{1}{3}\delta^\mu_\nu R = -\frac{1}{2}\delta^\mu_\nu C$ —the 3-dimensional space-time is maximally symmetric. The ‘‘symmetry preserving’’ solutions, (a) and (b) above, admit only four Killing vectors that span  $SO(2.1) \times SO(2)$ .

Since the Cotton tensor vanishes, we expect that the above space-times are (locally) conformally flat. This can be seen explicitly for the ‘‘symmetry preserving’’ solutions. In (22), set  $T = t \cosh \sqrt{\frac{C}{2}}y$ ,  $Y = t \sinh \sqrt{\frac{C}{2}}y$ , and  $X = x$  to find

$$(25) \quad (ds)^2 = \frac{2}{C(T^2 - Y^2)} ((dT)^2 - (dX)^2 - (dY)^2),$$

while in (23) the coordinate transformation

$$X = x \cos \sqrt{\frac{|C|}{2}}y, \quad Y = x \sin \sqrt{\frac{|C|}{2}}y, \quad T = t$$

gives the line element

$$(26) \quad (ds)^2 = \frac{2}{|C|(X^2 + Y^2)} ((dT)^2 - (dX)^2 - (dY)^2).$$

We have not found the relevant coordinate transformation for the ‘‘symmetry breaking’’ solution (c), but we know that the 3-dimensional space-time is indeed conformally flat since it is anti-deSitter.

Equations (14), (15) also possess a kink solution, which interpolates between the ‘‘symmetry breaking’’ solutions (17). One can verify that

$$(27) \quad f(x) = \sqrt{C} \tanh \frac{\sqrt{C}}{2}x,$$

with

$$(28) \quad g_{\alpha\beta} = \begin{pmatrix} 1/\cosh^4 \frac{\sqrt{C}}{2}x & 0 \\ 0 & -1 \end{pmatrix},$$

satisfies the relevant equations. That the solution depends only on one variable (only  $x$ , not both  $t, x$ ) is a general property (provided coordinates are selected properly). Thus in (14), (15) one is dealing with a system of second-order ordinary (not partial) differential equations, whose solution involves two integration constants. One integration constant is the trivial origin of the  $x$ -coordinate (taken to be  $x = 0$  in (27), (28)). The other involves choosing an integration constant in a first integral, so that one achieves a kink: a profile that interpolates between  $\pm\sqrt{C}$  as  $x \rightarrow \pm\infty$ . (Other choices for this second

constant lead to the same local geometry, but to different global properties. This has been thoroughly explained by Grumiller and Kummer [6].)

The 2-dimensional curvature corresponding to (28) is

$$(29) \quad r = -2C + \frac{3C}{\cosh^2(\frac{\sqrt{C}}{2}x)}.$$

Also the 3-dimensional line element for (27), (28) reads

$$(30) \quad (ds)^2 = -(dx)^2 - \frac{2dtdy}{\cosh^2 \frac{\sqrt{C}}{2}x} - (dy)^2,$$

and the 3-dimensional scalar curvature is according to (21), (27) and (28),

$$(31) \quad R = -\frac{3C}{2} + \frac{5C}{2 \cosh^2 \frac{\sqrt{C}}{2}x}.$$

Again, because (30) ensures that the Cotton tensor vanishes, there should exist a coordinate transformation to conformally flat, 3-dimensional coordinates. We have not found it.

This then is the kink in a “funny place”—in conformally flat (2+1)-dimensional space-time. A question remains: can one understand *a priori* that such a kink should exist in that geometry? This question may be posed in a more general setting.

Observe that the flat space kink in equations (1)–(3) possesses the same profile as (27), except for a change in scale. In fact this is a general feature. The following can be proved. If the nonlinear equation in flat space-time

$$(32) \quad \square\varphi + V'(\varphi) = 0$$

possesses a kink solution  $\varphi_k(x) = k(x)$ , then the curved (1+1)-dimensional space-time equations

$$(33) \quad D^2 f + V'(f) = 0,$$

$$(34) \quad D_\alpha D_\beta f - \frac{1}{2}g_{\alpha\beta} D^2 f = 0$$

are solved by

$$(35) \quad f(x) = k(x/\sqrt{2}),$$

with 2-dimensional line element

$$(36) \quad (ds)^2 = V(f)(dt)^2 - (dx)^2,$$

leading to a 2-dimensional curvature

$$(37) \quad r = -V''(F).$$

Perhaps Ludwig Faddeev can illuminate the deeper geometric reasons behind this coincidence.

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Received 4/OCT/2004  
Originally published in English