A KINK IN A FUNNY PLACE

R. JACKIW

Abstract. When the 3-dimensional gravitational Chern–Simons term is reduced to two dimensions, a dilation-like gravity theory emerges. Its solutions involve kinks, which therefore describe 3-dimensional, conformally flat spaces.

Two topics—among many—which I have discussed with Ludwig Faddeev over the twenty-five years that I have known him are first, gravity theory, and second, topological entities, both in mathematical settings such as characteristic classes, and in physical realizations such as kink profiles on a line. So, on the occasion of his significant birthday I present to Ludwig an investigation which unites these diverse elements.

Let me begin with lineal kinks. Consider the field equation

\[ \Box \varphi - C \varphi + \varphi^3 = 0, \]

where \( C \) is a positive constant. When we look for a lineal kink solution, we take \( \varphi \) to depend on a single spatial variable. Then (1) reduces to

\[ -\varphi'' - C \varphi + \varphi^3 = 0 \]

and has the well-known kink solution,

\[ \varphi_k(x) = \sqrt{C} \tanh \sqrt{\frac{C}{2}} x, \]

which interpolates between the “vacuum” solutions \( \varphi_0 = \pm \sqrt{C} \) (see, e.g., [1]). The kink has interesting roles in condensed matter physics, where it triggers fermion fractionization [2]. Other kinks in other models give rise to completely solvable field theories, both in classical and quantal frameworks. These stories do not belong here. But I shall return to the above kink later.

Next let me consider the Chern–Simons characteristic class. In non-Abelian gauge theory it is constructed from a matrix gauge connection \((A_\alpha)^\mu_{\nu}\) as

\[ W(A) = \frac{1}{4\pi^2} \int d^3 x \varepsilon^{\alpha\beta\gamma} \text{tr} \left( \frac{1}{2} A_\alpha \partial_\beta A_\gamma + \frac{1}{3} A_\alpha A_\beta A_\gamma \right). \]

This gauge-theoretic entity finds physical application in the quantum Hall regime, perhaps also in high \( T \) superconductivity. When added with strength \( m \) to the usual Yang–Mills action, the Chern–Simons term gives rise to massive, yet gauge-invariant excitations in (2 + 1)-dimensional space-time. Also, for consistency in the quantized version of a non-Abelian theory, \( m \) must be an integer multiple of \( 2\pi \). This is a precise field-theoretic analog of Dirac’s celebrated quantization of magnetic monopole strength. Finally, in an important mathematical application, the Chern–Simons term gives a functional integral formula for knot invariants.

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The gauge-theoretic Chern–Simons term (4) can be translated into a 3-dimensional geometric quantity by replacing the matrix gauge connection \((A_\alpha)_{\mu\nu}^\sigma\) with the Christoffel connection \(\Gamma_{\mu\alpha\nu}^\sigma\) [3],

\[
W(\Gamma) = \frac{1}{4\pi^2} \int d^3x \varepsilon^{\alpha\beta\gamma} \left( \frac{1}{2} \Gamma_{\alpha\sigma}^\rho \partial_\beta \Gamma_{\gamma\rho}^\sigma + \frac{1}{3} \Gamma_{\alpha\sigma}^\rho \Gamma_{\beta\tau}^\sigma \Gamma_{\gamma\rho}^\tau \right).
\]

But it is to be remembered that \(\Gamma_{\mu\alpha\nu}^\sigma\) is constructed in the usual way from the metric tensor, \(g_{\mu\nu}\), which is taken as the fundamental, independent variable. When \(W(\Gamma)\) is varied with respect to \(g_{\mu\nu}\), there emerges the Cotton tensor, which has an important role in 3-dimensional geometry:

\[
\delta W(\Gamma) = -\frac{1}{4\pi^2} \int d^3x \delta g_{\mu\nu} \sqrt{g} C^{\mu\nu},
\]

\[
C^{\mu\nu} \equiv \frac{1}{2\sqrt{g}} \left( \varepsilon^{\mu\alpha\beta} D_\alpha R_{\beta\nu}^\sigma + \varepsilon^{\nu\alpha\beta} D_\alpha R_{\mu\beta}^\sigma \right).
\]

(In 7 one may freely replace the Ricci tensor \(R_{\mu\nu}\) by the Einstein tensor \(G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} R\).) The Cotton tensor is like the covariant curl of the Ricci or Einstein tensor. It is manifestly symmetric; it is covariantly conserved and traceless because it is the variation of the diffeomorphism and conformally invariant \(W(\Gamma)\). Furthermore, the Cotton tensor replaces the Weyl tensor, which is absent in three dimensions, as a template for conformal flatness: \(C^{\mu\nu}\) vanishes if and only if the space is conformally flat, i.e.,

\[
C^{\mu\nu} = 0 \iff \text{conformally flat space}.
\]

The absence of the 3-dimensional Weyl tensor has the consequence that 3-dimensional geometries satisfying Einstein’s equation carry nonvanishing curvature only in regions where there are sources. Therefore, there are no propagating excitations. However, upon extending Einstein’s gravity equation by adding \(\frac{1}{m} C^{\mu\nu}\) to the Einstein tensor (equivalently, adding \(\frac{4\pi^2}{m} W(\Gamma)\) to the Einstein–Hilbert action) the theory acquires a propagating mode with mass \(m\), all the while preserving diffeomorphism invariance! Here we have another perspective on the absence of propagating modes in 3-dimensional Einstein theory: to regain the Einstein equations from the modified equations, we must pass \(m\) to infinity, whereupon the supermassive propagating mode decouples.

This is a well-known story, with which I do not concern myself now. Rather, I consider the opposite limit of the extended theory, where only the Cotton tensor survives, and the equation that I shall examine demands its vanishing, i.e., equation (8). But as indicated previously, that equation is not sufficiently restrictive to be interesting: any conformally flat space-time (coordinates \((t, x, y)\)) is a solution. So I shall place a further restriction: the solution that I seek should be independent of the \(y\)-coordinate in a Kaluza–Klein dimensional reduction from \((2+1)\) to \((1+1)\) dimensions of the gravitational Chern–Simons term \(W(\Gamma)\) in (5) and of the Cotton tensor \(C^{\mu\nu}\) in (7) (see [4]).

To effect the dimensional reduction, we begin by making a Kaluza–Klein Ansatz for the 3-dimensional metric tensor. It is taken in the form

\[
3\text{-d metric tensor} = \varphi \begin{pmatrix} g_{\alpha\beta} - a_\alpha a_\beta, & -a_\alpha \\ -a_\beta, & -1 \end{pmatrix},
\]

where the 2-dimensional metric tensor \(g_{\alpha\beta}\), vector \(a_\alpha\), and scalar \(\varphi\) depend only on \(t\) and \(x\). (Henceforth, Greek letters from the beginning of the alphabet \((\alpha, \beta, \gamma, \ldots)\) index 2-dimensional \((t, x)\)-dependent geometric entities, which are written with lower case letters; in three space-time dimensions geometric entities are capitalized (save the metric tensor) and are indexed by middle Greek alphabet letters \((\mu, \nu, p, \ldots)\).)
It is easy to show that under infinitesimal diffeomorphisms that leave the $y$-coordinate unchanged, $g_{\alpha\beta}$, $a_{\alpha}$ and $\varphi$ transform as a 2-dimensional coordinate tensor, a vector and a scalar, respectively, and, moreover, $a_{\alpha}$ undergoes a gauge transformation.

With the above Ansatz for the 3-dimensional metric, the Chern–Simons action becomes

$$(10) \quad CS = -\frac{1}{8\pi^2} \int d^2 x \sqrt{-g} (f_r + f^3).$$

Here $g = \det g_{\alpha\beta}$, $\partial_\alpha a_\beta - \partial_\beta a_\alpha \equiv f_{\alpha\beta} \equiv \sqrt{-g} \varepsilon_{\alpha\beta} f$, and $r$ is the 2-dimensional scalar curvature. The absence of $\varphi$-dependence is a consequence of the conformal invariance of the gravitational Chern–Simons term, and this also ensures that the Cotton tensor is traceless. Henceforth we set $\varphi$ to 1.

The above expressions look like they are describing 2-dimensional dilation gravity, with $f$ taking the role of a dilation field $[5]$. However, in fact $f$ is not a fundamental field; rather it is the curl of the vector potential $a_\alpha$. Alternative expressions for the action (10) are $\int d a (r + f^2)$ (where $d a$ is a 2-form; this exposes the topological character of our theory), and $\int d^2 x \Theta \varepsilon^{\alpha\beta} f_{\alpha\beta}, \Theta \equiv r + f^2$ (which highlights an axion-like interaction in 2-dimensional space-time).

Variation of $a_\alpha$ and $g_{\alpha\beta}$ produces the equations

$$0 = \varepsilon^{\alpha\beta} \partial_\beta (r + 3f^2),$$
$$0 = g_{\alpha\beta} (D^2 f - f^3 - f r f) - D_\alpha D_\beta f. \quad (12)$$

The first is solved by

$$r + 3f^2 = \text{constant} \equiv C. \quad (13)$$

Eliminating $r$ in the second equation, and decomposing it into the trace and trace-free parts leaves

$$0 = D^2 f - C f + f^3, \quad (14)$$
$$0 = D_\alpha D_\beta f - \frac{1}{2} g_{\alpha\beta} D^2 f. \quad (15)$$

Note that the equations are invariant against changing the sign of $f$ (the action then also changes sign).

A homogenous solution that respects the $f \leftrightarrow -f$ symmetry is

$$f = 0, \quad r = C. \quad (16)$$

However, there is also a “symmetry breaking” solution,

$$f = \pm \sqrt{C}, \quad r = -2C, \quad C > 0. \quad (17)$$

Forms for $g_{\alpha\beta}$ and $a_\alpha$ that lead to the above results are

$$(a) \quad f = 0, r = C > 0: \quad g_{\alpha\beta} = \frac{2}{C^2} \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right], \quad a_\alpha = (0, 0), \quad (18)$$
$$(b) \quad f = 0, r = C < 0: \quad g_{\alpha\beta} = \frac{2}{|C|^2} \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right], \quad a_\alpha = (0, 0), \quad (19)$$
$$(c) \quad f = \pm \sqrt{C}, r = -2C < 0: \quad g_{\alpha\beta} = \frac{1}{C^2} \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right], \quad a_\alpha = \left( \frac{\pm 1}{\sqrt{C}}, 0 \right). \quad (20)$$

In the first case, the 2-dimensional space-time is deSitter; in the last two, it is anti-deSitter.
The 3-dimensional scalar curvature \( R \), with metric tensor as in our \textit{Ansatz} \((9)\) at \( \varphi = 1 \), is related to the 2-dimensional curvature \( r \) by

\[
R = r + \frac{1}{2} f^2. 
\]

Hence for the three cases, the 3-dimensional curvature and line element read

\begin{align}
(22) \quad & R = C > 0, \quad (ds)^2 = \frac{2}{C}\left[\left(\frac{dt}{C}\right)^2 - \left(\frac{dx}{t}\right)^2\right] - (dy)^2, \\
(23) \quad & R = C < 0, \quad (ds)^2 = \frac{2}{|C|}\left[\left(\frac{dt}{C}\right)^2 - \left(\frac{dx}{x}\right)^2\right] - (dy)^2, \\
(24) \quad & R = -\frac{3}{2}C < 0, \quad (ds)^2 = \pm \frac{2}{\sqrt{C}x} dt dy - \left(\frac{dx}{\sqrt{C}x}\right)^2 - (dy)^2. 
\end{align}

Although all three solutions carry constant 3-dimensional curvature, the “symmetry breaking” solution, \((c)\) above, possesses greater geometrical symmetry in three dimensions: it supports six Killing vectors that span \( SO(2.1) \times SO(2.1) = SO(2.2) \), the isometry of 3-dimensional anti-deSitter space. Moreover, one verifies that, as expected, \( R^\mu_\nu = \delta^\mu_\nu R = -\frac{3}{4} \delta^\mu_\nu C \)—the 3-dimensional space-time is maximally symmetric. The “symmetry preserving” solutions, \((a)\) and \((b)\) above, admit only four Killing vectors that span \( SO(2.1) \times SO(2) \).

Since the Cotton tensor vanishes, we expect that the above space-times are (locally) conformally flat. This can be seen explicitly for the “symmetry preserving” solutions. In \( (22) \), set \( T = t \cosh \sqrt{\frac{C}{2}} y, \ Y = t \sinh \sqrt{\frac{C}{2}} y, \) and \( X = x \) to find

\[
(ds)^2 = \frac{2}{C(T^2 - Y^2)} ((dT)^2 - (dX)^2 - (dY)^2),
\]

while in \( (23) \) the coordinate transformation

\[
X = x \cos \sqrt{\frac{|C|}{2}} y, \quad Y = x \sin \sqrt{\frac{|C|}{2}} y, \quad T = t
\]
gives the line element

\[
(ds)^2 = \frac{2}{|C|(X^2 + Y^2)} ((dT)^2 - (dX)^2 - (dY)^2).
\]

We have not found the relevant coordinate transformation for the “symmetry breaking” solution \((c)\), but we know that the 3-dimensional space-time is indeed conformally flat since it is anti-deSitter.

Equations \((14), (15)\) also possess a kink solution, which interpolates between the “symmetry breaking” solutions \((17)\). One can verify that

\[
f(x) = \sqrt{C} \tanh \frac{\sqrt{C}}{2} x,
\]

with

\[
g_{\alpha\beta} = \begin{pmatrix} 1/\cosh^4 \frac{\sqrt{C}}{2} x & 0 \\ 0 & -1 \end{pmatrix},
\]

satisfies the relevant equations. That the solution depends only on one variable (only \( x \), not both \( t, x \)) is a general property (provided coordinates are selected properly). Thus in \((14), (15)\) one is dealing with a system of second-order ordinary (not partial) differential equations, whose solution involves two integration constants. One integration constant is the trivial origin of the \( x \)-coordinate (taken to be \( x = 0 \) in \((27), (28))\). The other involves choosing an integration constant in a first integral, so that one achieves a kink: a profile that interpolates between \( \pm \sqrt{C} \) as \( x \to \pm \infty \). (Other choices for this second
constant lead to the same local geometry, but to different global properties. This has been thoroughly explained by Grumiller and Kummer [6].

The 2-dimensional curvature corresponding to (28) is

\[ r = -2C + \frac{3C}{\cosh^2 \left( \frac{\sqrt{C} x}{2} \right)}. \]  

Also the 3-dimensional line element for (27), (28) reads

\[ (ds)^2 = -(dx)^2 - \frac{2dt dy}{\cosh^2 \frac{\sqrt{C} x}{2}} - (dy)^2, \]

and the 3-dimensional scalar curvature is according to (21), (27) and (28),

\[ R = -\frac{3C}{2} + \frac{5C}{2 \cosh^2 \frac{\sqrt{C} x}{2}}. \]

Again, because (30) ensures that the Cotton tensor vanishes, there should exist a coordinate transformation to conformally flat, 3-dimensional coordinates. We have not found it.

This then is the kink in a “funny place”—in conformally flat (2+1)-dimensional space-time. A question remains: can one understand a priori that such a kink should exist in that geometry? This question may be posed in a more general setting.

Observe that the flat space kink in equations (1)–(3) possesses the same profile as (27), except for a change in scale. In fact this is a general feature. The following can be proved. If the nonlinear equation in flat space-time

\[ \Box \phi + V'(\phi) = 0 \]

possesses a kink solution \( \phi_k(x) = k(x) \), then the curved (1+1)-dimensional space-time equations

\[ D^2 f + V'(f) = 0, \]

\[ D_\alpha D_\beta f - \frac{1}{2} g_{\alpha \beta} D^2 f = 0 \]

are solved by

\[ f(x) = k(x/\sqrt{2}), \]

with 2-dimensional line element

\[ (ds)^2 = V(f)(dt)^2 - (dx)^2, \]

leading to a 2-dimensional curvature

\[ r = -V''(F). \]

Perhaps Ludwig Faddeev can illuminate the deeper geometric reasons behind this coincidence.

REFERENCES


Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307

E-mail address: jackiw@lns.mit.edu

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