

HYPERGEOMETRIC GENERATING FUNCTION OF L -FUNCTION, SLATER'S IDENTITIES, AND QUANTUM INVARIANT

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Dedicated to Ludwig Dmitrievich Faddeev on the occasion of his seventieth birthday

ABSTRACT. Relationships between the quantum invariants of the torus knots $\mathcal{T}_{3,2k}$ and some q -series identities are studied. In particular, new generalizations of the Slater identities (83) and (86) are obtained.

§1. INTRODUCTION

Recent studies revealed an intimate relationship between the quantum invariants for 3-manifolds and modular forms. This remarkable observation originates from [18, 19]. In a slightly different context, in [26] it was observed that the generating function of Stoimenow's upper bound of the number of the Vassiliev invariants coincides with the half-differential of the Dedekind η -function. From the quantum invariant viewpoint this generating function happens to coincide with Kashaev's invariant [17] for the trefoil. Originally, this quantum knot invariant was defined in terms of the quantum dilogarithm function [10] and is a specific value of the colored Jones polynomial [21]. Motivated by this coincidence, in [11, 13, 14, 15] the present authors discovered that, for the torus knot $\mathcal{T}_{s,t}$ and the torus link $\mathcal{T}_{2,2m}$, Kashaev's invariant coincides with the Eichler integral of the Virasoro character for the minimal model $\mathcal{M}(s,t)$ and the Eichler integral of the $\widehat{su}(2)_{m-2}$ character, respectively.

One of the benefits of the correspondence between the character and the quantum invariant is that we can find new q -series identities. For example, it is well known that the Virasoro character for $\mathcal{M}(2, 2m+1)$ is related to the Gordon–Andrews identity (generalization of the famous Rogers–Ramanujan identity). Motivated by an explicit form of Kashaev's invariant for the torus knot $\mathcal{T}_{2,2m+1}$, we obtained a new q -series that may be regarded as a one-parameter extension of the Gordon–Andrews identity [13]. A case of $m = 1$ corresponds to Zagier's q -series identity [26]. For other generalizations of Zagier's identity, see also [3, 20, 9, 22], where q -hypergeometric-type generating functions of the L -function at the negative integers were constructed. Our purpose in this paper is to propose a q -series identity related to Slater's identities.

We define the Santos polynomial (see [5, 4]) by

$$(1.1a) \quad S_n(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{2k^2} \begin{bmatrix} n \\ 2k \end{bmatrix},$$

$$(1.1b) \quad T_n(q) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} q^{2k(k+1)} \begin{bmatrix} n \\ 2k+1 \end{bmatrix}.$$

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(Our notation is summarized in §2.) These polynomials can also be defined recursively by the formula

$$(1.2) \quad \begin{pmatrix} S_{n+1}(q) \\ T_{n+1}(q) \end{pmatrix} = \begin{pmatrix} 1 & q^{n+1} \\ q^n & 1 \end{pmatrix} \begin{pmatrix} S_n(q) \\ T_n(q) \end{pmatrix},$$

with the initial condition

$$\begin{pmatrix} S_0(q) \\ T_0(q) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using the Santos polynomials, we can define formal q -series $X^{(0)}(q)$ and $X^{(1)}(q)$ by

$$(1.3a) \quad X^{(0)}(q) = \sum_{n=0}^{\infty} (q)_n (T_n(q) + T_{n+1}(q)),$$

$$(1.3b) \quad X^{(1)}(q) = \sum_{n=0}^{\infty} (q)_n (S_n(q) + S_{n+1}(q)).$$

To state one of our main theorems, we introduce the following periodic functions:

$$(1.4a) \quad \begin{array}{c|cccc} n \pmod{24} & 5 & 11 & 13 & 19 & \text{others} \\ \hline \chi_{24}^{(0)}(n) & 1 & -1 & -1 & 1 & 0 \end{array}$$

$$(1.4b) \quad \begin{array}{c|cccc} n \pmod{24} & 1 & 7 & 17 & 23 & \text{others} \\ \hline \chi_{24}^{(1)}(n) & 1 & -1 & -1 & 1 & 0 \end{array}$$

Theorem 1. *Let $X^{(a)}(q)$ with $a = 0, 1$ be defined by (1.3). The following asymptotic expansion as $z \searrow 0$ is valid:*

$$(1.5a) \quad X^{(0)}(e^{-z}) = e^{25z/48} \sum_{n=0}^{\infty} \frac{t_n^{(0)}}{n!} \left(\frac{z}{48}\right)^n,$$

$$(1.5b) \quad X^{(1)}(e^{-z}) = e^{z/48} \sum_{n=0}^{\infty} \frac{t_n^{(1)}}{n!} \left(\frac{z}{48}\right)^n.$$

Here the t -series is given in terms of the L -function associated with $\chi_{24}^{(a)}(n)$:

$$(1.6) \quad \begin{aligned} t_n^{(a)} &= \frac{1}{2}(-1)^{n+1} L(-2n-1, \chi_{24}^{(a)}) \\ &= \frac{1}{2}(-1)^n \frac{24^{2n+1}}{2n+2} \sum_{k=1}^{24} \chi_{24}^{(a)}(k) B_{2n+2}(k/24), \end{aligned}$$

where $B_n(x)$ is the n th Bernoulli polynomial.

We note that the generating functions of the t -series look like this:

$$\begin{aligned} \frac{\sinh(3x) \sinh(4x)}{\sinh(12x)} &= \frac{1}{2} \sum_{n=0}^{\infty} \chi_{24}^{(0)}(n) e^{-nx} = \sum_{n=0}^{\infty} (-1)^n \frac{t_n^{(0)}}{(2n+1)!} x^{2n+1}, \\ \frac{\sinh(3x) \sinh(8x)}{\sinh(12x)} &= \frac{1}{2} \sum_{n=0}^{\infty} \chi_{24}^{(1)}(n) e^{-nx} = \sum_{n=0}^{\infty} (-1)^n \frac{t_n^{(1)}}{(2n+1)!} x^{2n+1}. \end{aligned}$$

Explicitly, some of the t -series are given as follows:

n	0	1	2	3	4
$t_n^{(0)}$	1	119	37201	23181479	24453497761
$t_n^{(1)}$	2	142	38882	23439022	24521135042
$t_n^{(2)}$	2	184	53792	32965504	34630287872
n	5			6	
$t_n^{(0)}$	39286795847639			89443016674567921	
$t_n^{(1)}$	39313934084302			89458458867741602	
$t_n^{(2)}$	55579108685824			126502446478794752	

See (4.8) for the definition of the t -series $t_n^{(2)}$.

This paper is organized as follows. In §2 we collect notation and identities for q -series to be used in this paper. See, e.g., [1]. In §3 we prove Theorem 1. In §4 we study a (nearly) modular property of our q -series. We show that $X^{(a)}(q)$ can be regarded as the Eichler integral of the modular form with weight $1/2$, which corresponds to the character of the Virasoro minimal model $\mathcal{M}(3, 4)$, and discuss the relationship with the quantum knot invariant for the torus knot $\mathcal{T}_{3,4}$. In §5 we present several q -hypergeometric-type expressions for the character of the Virasoro minimal model $\mathcal{M}(3, k)$.

§2. NOTATION AND IDENTITIES

For convenience, we give a list of notation and useful identities (see, e.g., [1]):

- the q -product and the q -binomial coefficient (the Gaussian polynomial):

$$(a)_n = (a; q)_n = \prod_{k=1}^n (1 - aq^{k-1}),$$

$$(a_1, a_2, \dots, a_k; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n,$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_k (q)_{n-k}} & \text{for } n \geq k \geq 0, \\ 0, & \text{otherwise;} \end{cases}$$

- the q -binomial formula:

(2.1)
$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_{\infty}}{(z)_{\infty}};$$

- the q -expansion:

(2.2)
$$(z)_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-z)^k q^{k(k-1)/2},$$

(2.3)
$$\frac{1}{(z)_n} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} z^k;$$

- the Euler identity (the limit of (2.2)) as $n \rightarrow \infty$:

(2.4)
$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q)_n} z^n = (-z)_{\infty};$$

- the Jacobi triple product identity:

(2.5)
$$\sum_{k \in \mathbb{Z}} (-1)^k q^{k^2/2} x^k = (q, x^{-1}q^{\frac{1}{2}}, xq^{\frac{1}{2}}; q)_{\infty};$$

- the Watson quintuple identity:

$$(2.6) \quad \sum_{k \in \mathbb{Z}} q^{k(3k-1)/2} x^{3k} (1 - xq^k) = (q, x, qx^{-1}; q)_\infty (qx^2, qx^{-2}; q^2)_\infty.$$

We note that the Watson identity can be proved with the help of (2.5) (see, e.g., [6]).

In what follows, we use some properties of the L -function associated with a periodic function.

Lemma 2. *Let $C_f(n)$ be a periodic function with mean value 0 and modulus f . Then we have the following asymptotic expansion as $t \searrow 0$:*

$$(2.7) \quad \sum_{n=0}^{\infty} n C_f(n) e^{-n^2 t} \simeq \sum_{k=0}^{\infty} L(-2k-1, C_f) \frac{(-t)^k}{k!},$$

where $L(k, C_f)$ is the L -function associated with $C_f(n)$ and given by

$$L(-k, C_f) = -\frac{f^k}{k+1} \sum_{n=1}^f C_f(n) B_{k+1}\left(\frac{n}{f}\right).$$

Proof. This is a standard result obtained by using the Mellin transformation. See, e.g., [19]. □

§3. PROOF OF THEOREM 1

We define functions $H^{(a)}(x) \equiv H^{(a)}(x; q)$ by the formulas

$$(3.1a) \quad \begin{aligned} H^{(0)}(x) &= \sum_{n=0}^{\infty} \chi_{24}^{(0)}(n) q^{\frac{n^2-25}{48}} x^{\frac{n-5}{2}} \\ &= 1 - q^2 x^3 - q^3 x^4 + q^7 x^7 + q^{17} x^{12} - q^{25} x^{15} - q^{28} x^{16} + \dots, \\ H^{(1)}(x) &= \sum_{n=0}^{\infty} \chi_{24}^{(1)}(n) q^{\frac{n^2-1}{48}} x^{\frac{n-1}{2}} \\ &= 1 - qx^3 - q^6 x^8 + q^{11} x^{11} + q^{13} x^{12} - q^{20} x^{15} - q^{35} x^{20} + \dots, \end{aligned}$$

where $\chi_{24}^{(a)}(n)$ is the periodic function defined in (1.4). It is easily seen that these q -series solve the q -difference equations

$$(3.2) \quad \begin{aligned} H^{(0)}(x) &= 1 - q^2 x^3 - q^3 x^4 H^{(1)}(qx), \\ H^{(1)}(x) &= 1 - qx^3 - q^6 x^8 H^{(0)}(qx). \end{aligned}$$

Proposition 3. *Let the functions $H^{(a)}(x)$ be defined by (3.1). Then for $a = 0, 1$ we have*

$$(3.3) \quad \begin{aligned} H^{(1-a)}(x) &= \sum_{n=0}^{\infty} (x)_{n+1} x^{2n} \left(\sum_{k=0}^{\lfloor (n-a)/2 \rfloor} x^{2k-a} q^{2k(k+a)} \begin{bmatrix} n \\ 2k+a \end{bmatrix} \right. \\ &\quad \left. + \sum_{k=0}^{\lfloor (n+1-a)/2 \rfloor} x^{2k+1-a} q^{2k(k+a)} \begin{bmatrix} n+1 \\ 2k+a \end{bmatrix} \right). \end{aligned}$$

Proof. The periodic function (1.4) is related to the Dirichlet character

$$-\chi_{24}^{(0)}(n) + \chi_{24}^{(1)}(n) = \left(\frac{12}{n}\right) \equiv \chi_{12}(n),$$

where we have used the Legendre symbol. Next, we introduce a q -series $H(x; q)$ by

$$(3.4) \quad -q^{\frac{1}{2}}x^2H^{(0)}(x) + H^{(1)}(x) = \sum_{n=0}^{\infty} \chi_{12}(n)q^{\frac{n^2-1}{48}}x^{\frac{n-1}{2}} \equiv H(x; q^{\frac{1}{2}}).$$

Since the functions $H^{(a)}(x)$ have integral powers of q , a q -hypergeometric expression can be obtained if we know that of $H(x; q^{\frac{1}{2}})$. Fortunately, from [1, 26] we know that

$$(3.5) \quad H(x; q^{\frac{1}{2}}) = \sum_{n=0}^{\infty} (x; q^{\frac{1}{2}})_{n+1}x^n.$$

To represent this expression as a sum of $H^{(a)}(x)$, we compute as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} (x; q^{\frac{1}{2}})_{n+1}x^n &= \sum_{n=0}^{\infty} (x)_{n+1}(xq^{\frac{1}{2}})_n x^{2n} + \sum_{n=0}^{\infty} (x)_{n+1}(xq^{\frac{1}{2}})_{n+1}x^{2n+1} \\ &= \sum_{n=0}^{\infty} (x)_{n+1}x^{2n} \left(\sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j x^j q^{j^2/2} + \sum_{j=0}^{n+1} \begin{bmatrix} n+1 \\ j \end{bmatrix} (-1)^j x^{j+1} q^{j^2/2} \right) \\ &= \sum_{n=0}^{\infty} (x)_{n+1}x^{2n} \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \left(\begin{bmatrix} n \\ 2k \end{bmatrix} + x \begin{bmatrix} n+1 \\ 2k \end{bmatrix} \right) x^{2k} q^{2k^2} \\ &\quad - q^{\frac{1}{2}}x^2 \sum_{n=0}^{\infty} (x)_{n+1}x^{2n-1} \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\begin{bmatrix} n \\ 2k+1 \end{bmatrix} + x \begin{bmatrix} n+1 \\ 2k+1 \end{bmatrix} \right) x^{2k} q^{2k(k+1)}; \end{aligned}$$

we have used (2.2) in the second identity and have split the sum into even and odd parts in the first and the last identities. This proves the proposition. \square

Using the q -binomial formula (2.1), we can rewrite (3.3) as

$$(3.6) \quad \begin{aligned} H^{(1-a)}(x) &= (qx)_{\infty} \sum_{n=0}^{\infty} \frac{q^{2n(n+a)}}{(x^2q)_{2n+a}} x^{6n+a-1} \\ &\quad + (1-x) \sum_{n=0}^{\infty} ((qx)_n - (qx)_{\infty}) x^{2n} \\ &\quad \quad \times \sum_{k \geq 0} x^{2k-a} q^{2k(k+a)} \left(\begin{bmatrix} n \\ 2k+a \end{bmatrix} + x \begin{bmatrix} n+1 \\ 2k+a \end{bmatrix} \right). \end{aligned}$$

In the limit $x \rightarrow 1$, the functions $H^{(a)}(x)$ defined by (3.1) can be written in an infinite product form with the help of the Watson quintuple identity (2.6). Using (3.6) and letting $x \rightarrow 1$, we recover the following identities.

Corollary 4 (Slater’s identities; see [24], (83) and (86)).

$$(3.7a) \quad (q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q)_{2n+1}} = (q^3, q^5, q^8; q^8)_{\infty} \cdot (q^2, q^{14}; q^{16})_{\infty},$$

$$(3.7b) \quad (q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q)_{2n}} = (q, q^7, q^8; q^8)_{\infty} \cdot (q^6, q^{10}; q^{16})_{\infty}.$$

The following formulas come from the next order of $x - 1$ in (3.6).

Proposition 5. *We have the following q -series identities:*

$$\begin{aligned}
(3.8a) \quad & \frac{1}{2} \sum_{n=0}^{\infty} n \chi_{24}^{(0)}(n) q^{\frac{n^2-25}{48}} \\
& = (q^3, q^5, q^8; q^8)_{\infty} (q^2, q^{14}; q^{16})_{\infty} \left(\sum_{k=1}^{\infty} \frac{-q^k}{1-q^k} \right) \\
& + (q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q)_{2n+1}} \left(6n + \sum_{k=1}^{2n+1} \frac{2q^k}{1-q^k} \right) \\
& - \sum_{n=0}^{\infty} ((q)_n - (q)_{\infty}) (T_n(q) + T_{n+1}(q)),
\end{aligned}$$

$$\begin{aligned}
(3.8b) \quad & \frac{1}{2} \sum_{n=0}^{\infty} n \chi_{24}^{(1)}(n) q^{\frac{n^2-1}{48}} \\
& = (q, q^7, q^8; q^8)_{\infty} (q^6, q^{10}; q^{16})_{\infty} \left(-1 - \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \right) \\
& + (q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q)_{2n}} \left(6n + \sum_{k=1}^{2n} \frac{2q^k}{1-q^k} \right) \\
& - \sum_{n=0}^{\infty} ((q)_n - (q)_{\infty}) (S_n(q) + S_{n+1}(q)).
\end{aligned}$$

Proof. We differentiate both sides of (3.6) with respect to x , and then let $x \rightarrow 1$. \square

Proof of Theorem 1. We plug $q = e^{-z}$ in (3.8). Since the terms that include infinite product terms such as $(q)_{\infty}$ vanish in the limit $z \searrow 0$, we get the formal q -series identities:

$$(3.9a) \quad X^{(0)}(q) = -\frac{1}{2} \sum_{n=0}^{\infty} n \chi_{24}^{(0)}(n) q^{\frac{n^2-25}{48}},$$

$$(3.9b) \quad X^{(1)}(q) = -\frac{1}{2} \sum_{n=0}^{\infty} n \chi_{24}^{(1)}(n) q^{\frac{n^2-1}{48}}.$$

Applying Lemma 2, we arrive at the statement of the theorem. \square

We see that

$$\chi_{24}^{(0)}(n) + \chi_{24}^{(1)}(n) = \left(\frac{24}{n} \right) \equiv \chi_{24}(n),$$

where we have used the Legendre symbol. Using Zagier's result, we can show that the q -series $X(q)$ defined by

$$(3.10) \quad X(q) = \sum_{n=0}^{\infty} (-q^{\frac{1}{2}}; -q^{\frac{1}{2}})_n$$

admits an asymptotic expansion of the form

$$(3.11) \quad X(e^{-z}) = e^{z/48} \sum_{n=0}^{\infty} \frac{t_n}{n!} \left(\frac{z}{48} \right)^n,$$

where

$$t_n = \frac{1}{2} (-1)^{n+1} L(-2n-1, \chi_{24}) = t_n^{(0)} + t_n^{(1)}.$$

§4. MODULARITY AND KNOT INVARIANTS

The q -series that we have studied in the preceding sections is related to the modular forms as follows. We define

$$(4.1) \quad \Phi(\tau) \equiv \begin{pmatrix} \Phi^{(0)}(\tau) \\ \Phi^{(1)}(\tau) \\ \Phi^{(2)}(\tau) \end{pmatrix} = \begin{pmatrix} q^{\frac{25}{48}} H^{(0)}(x=1) \\ q^{\frac{1}{48}} H^{(1)}(x=1) \\ \eta(2\tau) \end{pmatrix}.$$

Here $q = \exp(2\pi i\tau)$ with τ in the upper half-plane \mathbb{H} , and we have used the Dedekind η -function,

$$(4.2) \quad \eta(\tau) = q^{\frac{1}{24}}(q)_\infty = \sum_{n=0}^{\infty} \chi_{12}(n)q^{\frac{n^2}{24}}.$$

The Poisson summation formula allows us to see that the vector $\Phi(\tau)$ is modular with weight $1/2$; the modular S - and T -transformations are written as

$$(4.3) \quad \Phi(\tau) = \sqrt{\frac{i}{\tau}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \cdot \Phi(-1/\tau) \equiv \sqrt{\frac{i}{\tau}} \mathbf{M} \cdot \Phi(-1/\tau),$$

$$(4.4) \quad \Phi(\tau + 1) = \begin{pmatrix} e^{\frac{25}{24}\pi i} & & \\ & e^{\frac{1}{24}\pi i} & \\ & & e^{\frac{1}{6}\pi i} \end{pmatrix} \Phi(\tau),$$

respectively. These modular forms represent the character of the minimal model $\mathcal{M}(3, 4)$, i.e., the Ising model (see, e.g., [16, 23]).

We consider the asymptotic behavior of $X^{(a)}(q)$ when q is near to the N th primitive root of unity; we put $\omega = e^{2\pi i/N}$. We define the Eichler integral of $\Phi(\alpha)$ for $\alpha \in \mathbb{Q}$ by

$$(4.5) \quad \tilde{\Phi}(\alpha) \equiv \begin{pmatrix} \tilde{\Phi}^{(0)}(\alpha) \\ \tilde{\Phi}^{(1)}(\alpha) \\ \tilde{\eta}(2\alpha) \end{pmatrix} = \begin{pmatrix} e^{\frac{25}{24}\pi i\alpha} X^{(0)}(e^{2\pi i\alpha}) \\ e^{\frac{1}{24}\pi i\alpha} X^{(1)}(e^{2\pi i\alpha}) \\ e^{\frac{1}{6}\pi i\alpha} X^{(2)}(e^{2\pi i\alpha}) \end{pmatrix},$$

where $X^{(0)}(q)$ and $X^{(1)}(q)$ are defined as in (1.3), and

$$(4.6) \quad X^{(2)}(q) = 2 \sum_{n=0}^{\infty} (q^2; q^2)_n.$$

Then $\tilde{\Phi}(\alpha)$ converges to a finite value for $\alpha \in \mathbb{Q}$, because the infinite sum terminates at finite order due to $(q)_n$.

The asymptotic behavior of the Eichler integral of the modular form of weight $1/2$ was studied in [15] (see also [19, 26]).

Theorem 6 ([15]). *For $N \in \mathbb{Z}_{>0}$, we have the following asymptotic expansion as $N \rightarrow \infty$:*

$$(4.7) \quad \tilde{\Phi}(1/N) + (-iN)^{\frac{3}{2}} \mathbf{M} \cdot \tilde{\Phi}(-N) \simeq \sum_{n=0}^{\infty} \frac{\mathbf{t}_n}{n!} \left(\frac{\pi}{24iN} \right)^n,$$

where \mathbf{M} is the (3×3) -matrix defined in (4.3). The t -series is

$$\mathbf{t}_n = \begin{pmatrix} t_n^{(0)} \\ t_n^{(1)} \\ t_n^{(2)} \end{pmatrix},$$

where $t_n^{(0)}$ and $t_n^{(1)}$ are as in (1.6), and

$$(4.8) \quad \begin{aligned} t_n^{(2)} &= -(-4)^n L(-2n-1, \chi_{12}) \\ &= \frac{1}{2} (-1)^n \frac{24^{2n+1}}{2n+2} \sum_{k=1}^{12} \chi_{12}(k) B_{2n+2}(k/12). \end{aligned}$$

It should be noted that for $N \in \mathbb{Z}$ the Eichler integrals $\tilde{\Phi}(N)$ on the left-hand side of (4.7) look like this:

$$(4.9) \quad \tilde{\Phi}(N) = \begin{pmatrix} e^{\frac{25}{24}\pi i N} \\ 2e^{\frac{1}{24}\pi i N} \\ 2e^{\frac{1}{6}\pi i N} \end{pmatrix}.$$

Proposition 7 ([15]). *The Kashaev invariant $\langle \mathcal{K} \rangle : N$ for the torus knot $\mathcal{K} = \mathcal{T}_{3,4}$ (see the figure) is proportional to $X^{(0)}(\omega)$:*

$$(4.10) \quad \omega^3 X^{(0)}(\omega) = \langle \mathcal{T}_{3,4} \rangle : N.$$

See also [12] for the relationship with the colored Jones polynomial with generic q , the Alexander polynomial, and the A-polynomial for the torus knot.

In [25], the quantum invariant such as the colored Jones polynomial was identified with the Chern–Simons path integral. The asymptotic behavior of the latter is known to be related to classical topological invariants such as the Reidemeister torsion and the Chern–Simons invariant. By looking at the nearly modular property (4.7) and relation (4.10), the limit value of the Eichler integral (4.9) as $\tau \rightarrow N \in \mathbb{Z}$ can be identified with the Chern–Simons invariant of the torus knot $\mathcal{T}_{3,4}$ (see [15]).

§5. APPLICATIONS

In our earlier paper [15], we demonstrated that the Kashaev invariant for the torus knot $\mathcal{T}_{s,t}$, where s and t are relatively prime positive integers (see, e.g., Figure 1) can be regarded as the Eichler integral of the Virasoro character of the minimal model $\mathcal{M}(s, t)$. Proposition 7 refers to the case of $(s, t) = (3, 4)$. For an irreducible highest weight with conformal weight $\Delta_{n,m}^{s,t} = \frac{(nt-ms)^2 - (s-t)^2}{4st}$, $1 \leq n \leq s-1$, $1 \leq m \leq t-1$, the Virasoro character $\cosh_{n,m}^{s,t}(\tau)$ of $\mathcal{M}(s, t)$ is known (see [23]):

$$(5.1) \quad \cosh_{n,m}^{s,t}(\tau) = \frac{\Phi_{s,t}^{(n,m)}(\tau)}{\eta(\tau)}.$$

Here

$$(5.2) \quad \Phi_{s,t}^{(n,m)}(\tau) = \sum_{k=0}^{\infty} \chi_{2st}^{(n,m)}(k) q^{\frac{k^2}{4st}},$$

with a periodic function

$k \pmod{2st}$	$nt-ms$	$nt+ms$	$2st-(nt+ms)$	$2st-(nt-ms)$	others
$\chi_{2st}^{(n,m)}(k)$	1	-1	-1	1	0

The function $\Phi_{s,t}^{(n,m)}(\tau)$ is modular covariant with weight $1/2$ (see [8, 16]) and spans an $(s-1)(t-1)/2$ -dimensional space due to the symmetry $\cosh_{n,m}^{s,t}(\tau) = \cosh_{s-n, t-m}^{s,t}(\tau)$. Then the Kashaev invariant $\langle \mathcal{K} \rangle_N$ for the torus knot $\mathcal{K} = \mathcal{T}_{s,t}$ is written in terms of the

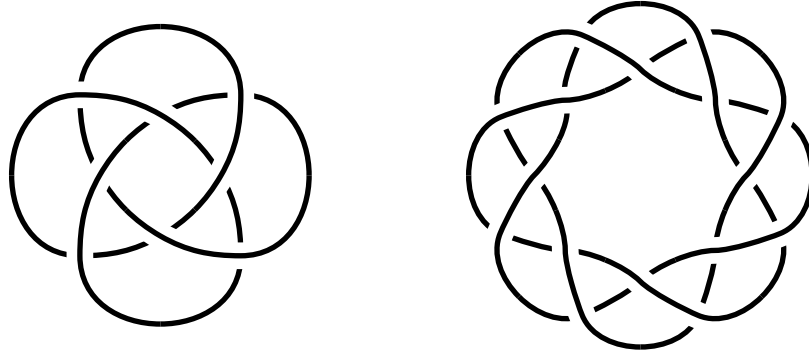


FIGURE 1. The torus knots $\mathcal{T}_{3,4}$ and $\mathcal{T}_{3,8}$.

Eichler integral $\tilde{\Phi}_{s,t}^{(1,t-1)}(1/N)$, which is defined by the half-differential of the modular form

$$(5.3) \quad \tilde{\Phi}_{s,t}^{(n,m)}(\tau) = -\frac{1}{2} \sum_{k=0}^{\infty} k \chi_{2st}^{(n,m)}(k) q^{\frac{k^2}{4st}},$$

where $\tau \in \mathbb{H}$. The limit value as $\tau \rightarrow 1/N \in \mathbb{Q}$ looks like this:

$$(5.4) \quad \tilde{\Phi}_{s,t}^{(n,m)}(1/N) = \frac{stN}{2} \sum_{k=1}^{2stN} \chi_{2st}^{(n,m)}(k) e^{\frac{k^2}{2stN} \pi i} B_2\left(\frac{k}{2stN}\right).$$

We consider the q -series identities associated with (the Eichler integral of) the minimal model $\mathcal{M}(3, k)$. As has already been mentioned, the character of the minimal model $\mathcal{M}(3, 4)$ is related to Slater's identities, and we can expect that such identities could be constructed. In fact, there are multivariable generalizations of Slater's identities due to G. Andrews [2]. These q -series identities may be interpreted as fermionic formulas for the characters of the Virasoro algebra. The q -series identities that come from the computation of the quantum invariants of the corresponding torus knots $\mathcal{T}_{3,k}$ are rather messy.

To study the q -series identities associated with the character of the minimal model $\mathcal{M}(s, t)$, it will be convenient to introduce the function [15, 12]

$$(5.5) \quad H_{s,t}^{(n,m)}(x) \equiv H_{s,t}^{(n,m)}(x; q) = \sum_{k=0}^{\infty} \chi_{2st}^{(n,m)}(k) q^{\frac{k^2 - (nt - ms)^2}{4st}} x^{\frac{k - |nt - ms|}{2}}.$$

It is easily seen that

$$\Phi_{s,t}^{(n,m)}(\tau) = q^{\frac{(nt - ms)^2}{4st}} H_{s,t}^{(n,m)}(1),$$

and that for $a = 0, 1$ the function $H^{(a)}(x)$ defined in (3.1) is none other than $H_{s,t}^{(n,m)}(x)$ with $(s, t) = (3, 4)$ and $(n, m) = (1, 3), (1, 1)$, respectively. Below we show how our function (5.5) works for the study of the q -hypergeometric-type functions associated with the minimal Virasoro characters. It should be noted that linear combinations of characters for the minimal Virasoro model were studied in [7] by a different method.

5.1. $\mathcal{M}(3, 2^p)$. First, we consider the case where $(s, t) = (3, 2k)$. The periodic function satisfies

$$(5.6) \quad \chi_{12}(2k + 3) \chi_{12}(n) = \sum_{a=0}^{k-1} (-1)^{a-1} \chi_{12k}^{(1, 2a+1)}(n).$$

This identity leads to the following proposition.

Proposition 8. *Let $H(x; q)$ be the function defined by (3.4). We have*

$$(5.7) \quad \chi_{12}(2k+3)H(x; q^{\frac{1}{k}}) = \sum_{a=0}^{k-1} (-1)^{a-1} q^{\ell_k(a)} x^{\frac{|2k-6a-3|-1}{2}} H_{3,2k}^{(1,2a+1)}(x; q),$$

where

$$\ell_k(a) = \frac{(k-3a-1)(k-3a-2)}{6k}.$$

If $k = 2^p$, we see that $\ell_k(a) - \ell_k(b) \notin \mathbb{Z}$ for all a and b satisfying $0 < a \neq b \leq 2k-1$. Thus, in the case of $\mathcal{M}(3, 2^{p+1})$ we can extract $H_{3,2^{p+1}}^{(1,2a+1)}(x; q)$ from the above lemma, by the method of Proposition 3.

For instance, we take $\mathcal{M}(3, 8)$. We have

$$(5.8a) \quad H(x; q^{\frac{1}{4}}) = H_{3,8}^{(1,3)}(x) - q^{\frac{1}{4}}x^2H_{3,8}^{(1,1)}(x) - q^{\frac{1}{2}}x^3H_{3,8}^{(1,5)}(x) + q^{\frac{7}{4}}x^6H_{3,8}^{(1,7)}(x),$$

$$(5.8b) \quad H_{3,8}^{(1,2)}(x) = H^{(1)}(x^2; q^2),$$

$$(5.8c) \quad H_{3,8}^{(1,6)}(x) = H^{(0)}(x^2; q^2),$$

$$(5.8d) \quad H_{3,8}^{(1,4)}(x) = H(x^4; q^4).$$

From the result of §3 we already know a q -hypergeometric expression for $H_{3,8}^{(1,a)}(x)$ for a even. For a odd, q -hypergeometric functions can be obtained from (5.8a) by the same method as in the preceding section. We have

$$\begin{aligned} H_{3,8}^{(1,1)}(x) &= -q^{-\frac{1}{4}}x^{-2}I^{(1)}(x), \\ H_{3,8}^{(1,3)}(x) &= I^{(0)}(x), \\ H_{3,8}^{(1,5)}(x) &= -q^{-\frac{1}{2}}x^{-3}I^{(2)}(x), \\ H_{3,8}^{(1,7)}(x) &= q^{-\frac{7}{4}}x^{-6}I^{(3)}(x), \end{aligned}$$

where

$$\begin{aligned} I^{(a)}(x) &= \sum_{k=0}^{\infty} (x)_{k+1} x^{4k} \sum_{1 \geq \epsilon_1 \geq \epsilon_2 \geq \epsilon_3 \geq 0} x^{\epsilon_1 + \epsilon_2 + \epsilon_3} \\ &\times \sum_{\substack{\ell_1, \ell_2, \ell_3 \geq 0 \\ \ell_1 + 2\ell_2 + 3\ell_3 = a \pmod{4b}}} (-x)^{\ell_1 + \ell_2 + \ell_3} q^{\frac{\ell_1 + 2\ell_2 + 3\ell_3}{4}} \left(\prod_{j=1}^3 \begin{bmatrix} k + \epsilon_j \\ \ell_j \end{bmatrix} q^{\frac{1}{2}\ell_j(\ell_j-1)} \right). \end{aligned}$$

5.2. $\mathcal{M}(3, 5)$. We list some formulas concerning $H_{s,t}^{(n,m)}(x)$ (as defined in (5.5)) for $(s, t) = (3, 5)$. In this case we have four independent functions for $(n, m) = (1, 1), (1, 2), (1, 3)$, and $(1, 4)$, and they satisfy the difference equations

$$\begin{aligned} H_{3,5}^{(1,4)}(x) &= 1 - q^2x^3 - q^4x^5 \cdot H_{3,5}^{(1,1)}(qx), \\ H_{3,5}^{(1,3)}(x) &= 1 - q^3x^5 - q^4x^6 \cdot H_{3,5}^{(1,2)}(qx), \\ H_{3,5}^{(1,2)}(x) &= 1 - q^2x^5 - q^6x^9 \cdot H_{3,5}^{(1,3)}(qx), \\ H_{3,5}^{(1,1)}(x) &= 1 - qx^3 - q^8x^{10} \cdot H_{3,5}^{(1,4)}(qx). \end{aligned}$$

We observe that

$$\begin{aligned} -q^{\frac{3}{4}}x^{\frac{5}{2}}H_{3,5}^{(1,4)}(x) + H_{3,5}^{(1,1)}(x) &= H_{3,5}^{(1,3)}(x^{\frac{1}{2}}; q^{\frac{1}{4}}), \\ -q^{\frac{1}{4}}x^{\frac{3}{2}}H_{3,5}^{(1,3)}(x) + H_{3,5}^{(1,2)}(x) &= H_{3,5}^{(1,1)}(x^{\frac{1}{2}}; q^{\frac{1}{4}}), \end{aligned}$$

and that the Watson quintuple identity (2.6) yields

$$\begin{aligned} H_{3,5}^{(1,4)}(1) &= (q^2, q^{18}; q^{20})_\infty (q^4, q^6, q^{10}; q^{10})_\infty, \\ H_{3,5}^{(1,3)}(1) &= (q^4, q^{16}; q^{20})_\infty (q^3, q^7, q^{10}; q^{10})_\infty, \\ H_{3,5}^{(1,2)}(1) &= (q^6, q^{14}; q^{20})_\infty (q^2, q^8, q^{10}; q^{10})_\infty, \\ H_{3,5}^{(1,1)}(1) &= (q^8, q^{12}; q^{20})_\infty (q, q^9, q^{10}; q^{10})_\infty. \end{aligned}$$

Though we obtain no q -series identity among Slater's 130 identities, we do obtain the following formulas.

Proposition 9.

$$\begin{aligned} &H_{3,5}^{(1,4)}(x) \\ (5.9) \quad &= \sum_{n=0}^{\infty} (-q^4 x^5; q^{10})_n (-q^4 x^5)^n - q^2 x^3 \sum_{n=0}^{\infty} (-q^6 x^5; q^{10})_n (-q^6 x^5)^n \\ (5.10) \quad &= \sum_{n=0}^{\infty} (-q^2 x^3; q^6)_n (-q^2 x^3)^{n+2c} \sum_{c=0}^n q^{6c^2} \begin{bmatrix} n \\ c \end{bmatrix}_{q^6} \\ &\quad - q^4 x^5 \sum_{n=0}^{\infty} (-q^4 x^3; q^6)_n (-q^4 x^3)^{n+2c} \sum_{c=0}^n q^{6c^2} \begin{bmatrix} n \\ c \end{bmatrix}_{q^6} \\ (5.11) \quad &= \sum_{n,c=0}^{\infty} (-1)^n q^{\frac{5}{2}n(n+1)+5c(c+1)-n-c} x^{5(n+c)} (1 - q^{2(n+c+1)} x^3) \begin{bmatrix} n \\ c \end{bmatrix}_{q^5}, \end{aligned}$$

$$\begin{aligned} &H_{3,5}^{(1,3)}(x) \\ (5.12) \quad &= -q^3 x^5 \sum_{n,c=0}^{\infty} (-qx^3; q^6)_{n+1} (-qx^3)^{n+2c} q^{6c(c+1)} \begin{bmatrix} n \\ c \end{bmatrix}_{q^6} \\ &\quad + \sum_{n,c=0}^{\infty} (-q^{-1}x^3; q^6)_{n+1} (-q^{-1}x^3)^{n+2c} q^{6c(c+1)} \begin{bmatrix} n \\ c \end{bmatrix}_{q^6} \\ (5.13) \quad &= \sum_{n,c=0}^{\infty} (-1)^n q^{\frac{5}{2}n(n+1)+5c(c+1)-2(n+c)} x^{5(n+c)} (1 - q^{4(n+c+1)} x^6) \begin{bmatrix} n \\ c \end{bmatrix}_{q^5}, \end{aligned}$$

$$\begin{aligned} &H_{3,5}^{(1,2)}(x) \\ (5.14) \quad &= \sum_{n=0}^{\infty} (-q^2 x^5; q^{10})_n (-q^2 x^5)^n - q^6 x^9 \sum_{n=0}^{\infty} (-q^8 x^5; q^{10})_n (-q^8 x^5)^n \end{aligned}$$

$$(5.15) \quad = \sum_{n,c=0}^{\infty} (-1)^n q^{\frac{5}{2}n(n+1)+5c(c+1)-3(n+c)} x^{5(n+c)} (1 - q^{6(n+c+1)} x^9) \begin{bmatrix} n \\ c \end{bmatrix}_{q^5},$$

$$\begin{aligned} &H_{3,5}^{(1,1)}(x) \\ (5.16) \quad &= 1 - x^{-2} + x^{-2} \sum_{n,c=0}^{\infty} (-1)^n q^{\frac{5}{2}n(n+1)+5c(c+1)-4(n+c)} x^{5(n+c)} \\ &\quad \times (1 - q^{8(n+c+1)} x^{12}) \begin{bmatrix} n \\ c \end{bmatrix}_{q^5} \end{aligned}$$

$$(5.17) \quad = -qx^3 \sum_{n=0}^{\infty} (-qx^5; q^{10})_{n+1} (-qx^5)^n + \sum_{n=0}^{\infty} (-q^{-1}x^5; q^{10})_{n+1} (-q^{-1}x^5)^n.$$

Proof. We recall some results of [26, 13]:

$$(5.18) \quad \sum_{n=0}^{\infty} \chi_6(n) q^{\frac{n^2-1}{24}} x^{\frac{n-1}{2}} = \sum_{n=0}^{\infty} (-x)_{n+1} (-x)^n,$$

$$(5.19) \quad \sum_{n=0}^{\infty} \chi_{10}^{(0)}(n) q^{\frac{n^2-9}{40}} x^{\frac{n-3}{2}} = \sum_{n=0}^{\infty} (-x)_{n+1} (-x)^n \sum_{c=0}^n q^{c(c+1)} x^{2c} \begin{bmatrix} n \\ c \end{bmatrix},$$

$$(5.20) \quad \sum_{n=0}^{\infty} \chi_{10}^{(1)}(n) q^{\frac{n^2-1}{40}} x^{\frac{n-1}{2}} = \sum_{n=0}^{\infty} (-x)_{n+1} (-x)^n \sum_{c=0}^{n+1} q^{c^2} x^{2c} \begin{bmatrix} n+1 \\ c \end{bmatrix}.$$

Here the periodic functions involved are as follows:

$n \pmod{6}$	1	5	others
$\chi_6(n)$	1	-1	0
$n \pmod{10}$	3	7	others
$\chi_{10}^{(0)}(n)$	1	-1	0
$n \pmod{10}$	1	9	others
$\chi_{10}^{(1)}(n)$	1	-1	0

Using (5.18) and the relations $\chi_{30}^{(1,1)}(n) = -\chi_6(\frac{n-3}{5}) + \chi_6(\frac{n+3}{5})$ and $\chi_{30}^{(1,2)}(n) = -\chi_6(\frac{n-6}{5}) + \chi_6(\frac{n+6}{5})$, we obtain (5.16) and (5.14). Identity (5.9) is deduced in a similar way. Identities (5.10) and (5.12) follow from (5.19), (5.20), and the relations $\chi_{30}^{(1,3)}(n) = -\chi_{10}^{(0)}(\frac{n-5}{3}) + \chi_{10}^{(0)}(\frac{n+5}{3})$, $\chi_{30}^{(1,4)}(n) = -\chi_{10}^{(1)}(\frac{n-10}{3}) + \chi_{10}^{(1)}(\frac{n+10}{3})$.

To prove the remaining identities, we use a formula proved in [14], namely,

$$(5.21) \quad \sum_{n=0}^{\infty} \tilde{\chi}_6(n) q^{\frac{n^2-4}{12}} x^{\frac{n-2}{2}} = \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n+1)} \sum_{c=0}^n x^{n+c} q^{c(c+1)} \begin{bmatrix} n \\ c \end{bmatrix}$$

with

$n \pmod{6}$	2	4	others
$\tilde{\chi}_6(n)$	1	-1	0

Combining this with the fact that $\chi_{30}^{(1,5-a)}(n) = -\tilde{\chi}_6(\frac{n-3a}{5}) + \tilde{\chi}_6(\frac{n+3a}{5})$ for $a = 1, 2, 3$, we complete the proof. \square

It should be noted that identity (5.21) implies that

$$(5.22) \quad H_{3,t}^{(1,m)}(x) = \begin{cases} I_t^{(m)}(x) & \text{if } t - 3m < 0, \\ 1 - x^{3m-t} + x^{3m-t} I_t^{(m)}(x) & \text{if } t - 3m > 0, \end{cases}$$

where

$$I_t^{(m)}(x) = \sum_{n,c=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n+1)+tc(c+1)-(t-m)(n+c)} x^{t(n+c)} \\ \times (1 - q^{2(t-m)(c+n+1)} x^{3(t-m)}) \begin{bmatrix} n \\ c \end{bmatrix}_{q^t}.$$

Here $t > 0$ and $1 \leq m < t$ are such that $(3, t) = 1$, and $H_{3,t}^{(1,m)}(x = 1)$ is a $(t-1)$ -dimensional representation of the modular group.

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