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THOMSON'S THEOREM ON MEAN SQUARE POLYNOMIAL APPROXIMATION

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Abstract. In 1991, J. E. Thomson determined completely the structure of $H^2(\mu)$, the closed subspace of $L^2(\mu)$ that is spanned by the polynomials, whenever μ is a compactly supported measure in the complex plane. As a consequence he was able to show that if $H^2(\mu) \neq L^2(\mu)$, then every function $f \in H^2(\mu)$ admits an analytic extension to a fixed open set Ω, thereby confirming in this context a phenomenon noted earlier in various situations by S. N. Bernštein, S. N. Mergelyan, and others. Here we present a new proof of Thomson's results, based on Tolsa's recent work on the semiadditivity of analytic capacity, which gives more information and is applicable to other problems as well.

§1. Introduction

Questions concerning approximation by analytic functions have a long history and have arisen in a variety of disparate settings. Of particular interest here is one such problem having implications well beyond the immediate context in which it is phrased. *Given a positive measure μ of compact support in the complex plane $\mathbb{C}$, are the polynomials dense in $L^2(d\mu)$; if not, why not?* Over the years that question has been the subject of an intense investigation, not only for its own sake, but due also to its connection with the invariant subspace problem for subnormal operators on a Hilbert space.

A bounded linear operator T on an infinite-dimensional Hilbert space H is *subnormal* if it has a normal extension to a larger Hilbert space; or equivalently, if T is the restriction of a normal operator to a closed invariant subspace. The study of such operators was begun by Halmos [17] in 1950, and since then it has become a catalyst for much activity at the interface between operator theory and the theory of analytic functions.

It is a long standing open problem to determine whether or not every bounded linear operator T on a Hilbert space H has a nontrivial closed invariant subspace. Clearly, it can be assumed from the outset that T has a cyclic vector, that is, a vector x for which the linear span of $x, Tx, T^2x, \dots$ is dense in the underlying space H. Otherwise, invariant subspaces abound and there is nothing to prove. If, in addition, T is subnormal, the spectral theorem guarantees that there is a positive measure μ carried on the spectrum of T such that the given operator T is unitarily equivalent to multiplication by the complex identity function z on $H^2(d\mu)$. Here $H^2(d\mu)$ is the closure of the polynomials in $L^2(d\mu)$. Thus, the study of subnormal operators leads readily to questions concerning approximation by polynomials in $L^2(d\mu)$ (cf. Bram [3, pp. 83–86]).

In this setting invariant subspaces can arise in at least one of two ways. If $H^2(d\mu) = L^2(d\mu)$ and if X is any subset of the support of μ with $0 < \mu(X) < \|\mu\|$, then $S = \{f \in H^2(d\mu) : f = 0 \text{ a.e.-}d\mu \text{ on } X\}$ is a nontrivial closed subspace invariant under multiplication by z. On the other hand, if $H^2(d\mu) \neq L^2(d\mu)$ it may happen that there

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is a point $\xi \in \mathbb{C}$ such that the map $P \to P(\xi)$ can be extended from the polynomials to a bounded linear functional on $H^2(d\mu)$; that is,

$$|P(\xi)| \leq C\|P\|_{L^2(d\mu)} \tag{1.1}$$

for every polynomial P and some absolute constant C. Such a point ξ is called a *bounded point evaluation* or BPE for $H^2(d\mu)$. If we let S be the closure in $H^2(d\mu)$ of the set of polynomials vanishing at ξ we again obtain a nontrivial invariant subspace, since $(z-\xi) \in S$ and $1 \notin S$.

In order, therefore, to settle the invariant subspace problem for subnormal operators in the affirmative it is sufficient to prove that either

(1) $H^2(d\mu)$ has a BPE, or
(2) $H^2(d\mu) = L^2(d\mu)$.

Moreover, initial confirmation that the suggested dichotomy is valid for a large class of measures came quickly. In 1955, Wermer [48] verified it for all measures μ whose closed support X has two-dimensional Lebesgue measure (i.e., dA measure) zero. His reasoning made use of a theorem of Hartogs and Rosenthal [18] on uniform rational approximation, but in essence is this: Let g be any function in $L^2(d\mu)$ that is orthogonal to the polynomials in the sense that $\int Pg\,d\mu = 0$ for every polynomial P. By an argument due essentially to Cauchy we find that

$$P(\xi) = \frac{1}{\widehat{g\mu}(\xi)} \int \frac{P(z)g(z)}{z-\xi}\,d\mu(z) \tag{1.2}$$

at every point $\xi \in \mathbb{C}$ where $\widehat{g\mu}(\xi) = \int \frac{g(z)}{z-\xi}\,d\mu(z)$ is defined and $\widehat{g\mu}(\xi) \neq 0$. In particular, if $\xi \in \mathbb{C} \setminus X$ and $\widehat{g\mu}(\xi) \neq 0$, then (1.2) holds and, since the kernel $(z-\xi)^{-1}$ is bounded on X,

$$|P(\xi)| \leq C \int |P|\,|g|\,d\mu$$

for all polynomials P and a suitable constant C. Hence, the inequality (1.1) is also satisfied and $H^2(d\mu)$ has a BPE at ξ. If, therefore, $H^2(d\mu)$ has no BPEs, it follows that $\widehat{g\mu}(\xi) = 0$ in $\mathbb{C} \setminus X$; that is, $\widehat{g\mu} = 0$ a.e.-dA in $\mathbb{C}$, since X has area zero. Thus, $g\mu = 0$ as a measure, and so $H^2(d\mu) = L^2(d\mu)$.

Subsequent efforts to extend this line of reasoning to absolutely continuous measures $w dA$ would yield clues eventually leading to a complete solution of the bounded point evaluation problem. In a very real sense the essential difficulties can all be found here. Building on ideas introduced in [5] in order to validate the BPE alternative for $H^2(wdA)$ when $w \in L^{1+\epsilon}(dA)$, Thomson [41] overcame substantial technical difficulties to achieve a final resolution. His result is the following.

Theorem 1. *For any measure μ of compact support, not concentrated at a single point, $H^2(d\mu) = L^2(d\mu)$ if and only if $H^2(d\mu)$ has no BPEs.*

Thomson's argument is quite complicated, incorporating an array of deep ideas from Brown [7], Davie [11], and Vitushkin [47]. Our goal here is to give a more transparent proof, which is less obscured by peripheral matters, and which can be more easily adapted to the treatment of other problems as well.

Before proceeding it is appropriate to remind readers that Brown [7] was able to establish the existence of invariant subspaces for subnormal operators without actually addressing the BPE issue (cf. also [40]).

§2. The strategy

Our proof of Thomson's theorem will proceed generally along lines introduced in [5] in order to obtain the same result for a large class of absolutely continuous measures. Due to its importance for our overall point of view we first recall here the main points of the earlier approach. For the remainder of this discussion μ will be a fixed positive measure of compact support in the complex plane $\mathbb{C}$, and dA will denote two-dimensional Lebesgue measure. The letters C and K, unless otherwise indicated, will be used to denote various absolute constants which may differ from one another, even within a single string of estimates.

As in the introduction suppose that $g \in L^2(d\mu)$, that $\int Pg\, d\mu = 0$ for every polynomial P, and form the Cauchy transform

$$\widehat{g\mu}(\xi) = \int \frac{g(z)}{z-\xi}\, d\mu(z).$$

Fix a point $\xi_0 \in \mathbb{C}$, and for each $\lambda > 0$ consider the set $E_\lambda = \{z : |\widehat{g\mu}(z)| < \lambda\}$, which is precisely defined up to a set of dA-measure zero. There are two mutually exclusive possibilities: either

(3) $H^2(d\mu)$ has a BPE at ξ_0, or
(4) almost every circle $|z - \xi_0| = r$ meets each E_λ in a set of positive linear measure.

To see that at least one of these must occur assume contrary to (4) that there exists a set X of nonnegative real numbers having positive linear measure, such that $|\widehat{g\mu}| \geq \lambda$ almost everywhere on $|z - \xi_0| = r$ for every $r \in X$. If we let X^* denote the union of all circles $|z - \xi_0| = r$ corresponding to these values of r, then by the mean value theorem

$$P(\xi_0) = C \int_{X^*} P(\xi)\, dA_\xi$$

for a suitable constant C and all polynomials P. Expressing the integrand in accordance with the formula in (1.2) yields the representation

$$\begin{aligned} P(\xi_0) &= C \int_{X^*} \frac{1}{\widehat{g\mu}(\xi)} \left(\int \frac{P(z)g(z)}{(z-\xi)}\, d\mu_z \right) dA_\xi \\ &= C \int P(z)g(z) \left(\int_{X^*} \frac{1}{\widehat{g\mu}(\xi)} \frac{1}{(z-\xi)}\, dA_\xi \right) d\mu_z. \end{aligned}$$

Because $|\widehat{g\mu}| \geq \lambda$ almost everywhere on X^* and $(z-\xi)^{-1}$ is locally integrable,

$$|P(\xi_0)| \leq C \int |P|\,|g|\, d\mu$$

and hence $|P(\xi_0)| \leq C\|P\|_{L^2(d\mu)}$. Thus, if $H^2(d\mu)$ has no BPEs, each of the sets E_λ is relatively massive near every point of $\mathbb{C}$. Our problem is to use this fact to conclude that $\widehat{g\mu} = 0$ a.e.-dA.

In case $d\mu = w dA$ with $w \in L^{1+\epsilon}(dA)$ this can be done by appealing to essentially real variable methods. Since, in addition to the restriction on w, the function g belongs to $L^2(wdA)$ it follows that the Cauchy transform $\widehat{gw}$ $(= \widehat{gw\, dA})$ lies in a Sobolev space W_1^q for some $q > 1$ (cf. [5, p. 408]). As such, $\widehat{gw}$ enjoys a certain degree of continuity which is best described in terms of capacity, and, for the purpose of this study it can be assumed that $q \leq 2$, since otherwise every W_1^q function is actually Hölder continuous or, more precisely, has a Hölder continuous representative. Nothing in the way of generality is lost if we further assume that $q < 2$.

If $1 \le q < 2$ the *q-capacity* of a compact planar set E is by definition

$$\Gamma_q(E) = \inf \int |\nabla u|^q \, dA,$$

the infimum being taken over all infinitely differentiable functions u of compact support with $u \equiv 1$ on E. The q-capacity of an arbitrary set X is defined to be

$$\Gamma_q(X) = \sup \Gamma_q(E),$$

where the supremum is taken over all compact sets $E \subseteq X$. If X is a Borel set, it can be shown that

$$\Gamma_q(X) = \inf \Gamma_q(G),$$

where now the infimum is taken over all open sets $G \supseteq X$. A property is said to hold *q-quasieverywhere* if the set where it fails has q-capacity zero.

It is often convenient to have a different (but equivalent) notion of capacity. If ν is a positive Borel measure in $\mathbb{C} = \mathbb{R}^2$, let

$$U^\nu(x) = \int |x-y|^{-1} \, d\nu(y)$$

be the corresponding Newtonian potential; if $d\nu = f dA$, write $U^f(x)$ instead. For an arbitrary set E define

$$C_q(E) = \inf \int f^q \, dA,$$

the infimum being extended over all nonnegative functions $f \in L^q(\mathbb{R}^2, dA)$ such that $U^f(x) \ge 1$ on E. If E is a Borel set and if $1 < q < 2$, it can be shown that

$$C_q(E)^{1/q} = \sup_\nu \nu(E),$$

with the supremum being over all positive measures ν concentrated on E for which $\|U^\nu\|_{L^p(dA)} \le 1$. It is also known that the two capacities Γ_q and C_q are equivalent in the sense that there exists a constant $K > 0$ such that

$$K^{-1} C_q(E) \le \Gamma_q(E) \le K C_q(E)$$

for every E. We will denote this (and similar equivalences) by writing $C_q \approx \Gamma_q$. For additional information and background material on capacity the reader is referred to the two books [1] and [24] where, in particular, proofs of the following can be found (cf. also [37] and [5, p. 411]):

(i) if Φ is a contraction, $C_q(\Phi E) \le C_q(E)$;
(ii) $C_q(B_r) \approx C_q(\operatorname{diam} B_r) \approx r^{2-q}$ for any disk B_r of radius r.

If $k \in L^q(dA)$ and $\int Pk \, dA = 0$ for every polynomial P, then $\hat{k}$ has compact support and, by the Calderón–Zygmund theorem on the continuity of singular integral operators (cf. [39, p. 35]),

$$\|\nabla \hat{k}\|_q \le C \left\| \frac{\partial \hat{k}}{\partial \bar{z}} \right\|_q = C\pi \|k\|_q$$

provided that $q > 1$, and so in this case $\hat{k} \in W_1^q$. Here the norms are taken with respect to area, that is, in $L^q(dA)$. For any $\lambda > 0$ we also have a weak-type inequality

$$\Gamma_q\{z \in \mathbb{C} : |\hat{k}(z)| > \lambda\} \le \frac{C}{\lambda^q} \int |\nabla \hat{k}|^q \, dA,$$

and this is the key to ascertaining the degree of continuity enjoyed by $\hat{k}$. If $\hat{k}_j = \hat{k} * \chi_j$ is a sequence of mollifiers obtained by convolving $\hat{k}$ with a C^∞ approximate identity χ_j, $j = 1, 2, 3, \ldots$, it is well known that

$$\|\hat{k}_j - \hat{k}\|_q \to 0 \quad \text{and} \quad \|\nabla\hat{k}_j - \nabla\hat{k}\|_q \to 0.$$

A careful analysis of the situation now reveals that, by passing to a subsequence if necessary, we can arrange for $\hat{k}_j \to \hat{k}$ uniformly off open sets of arbitrary small q-capacity (cf. [12, p. 354] and [50, p. 124]). The unavoidable conclusion is this: Given any $\epsilon > 0$ there exists an open set U such that $\Gamma_q(U) < \epsilon$ and $\hat{k}$ is continuous in the complement of U. Functions which have this property are said to be *q-quasicontinuous*. It is a fact that every W_1^q function agrees almost everywhere dA with one that is quasicontinuous. If $q > 2$, then $\hat{k}$ is actually continuous and we have a corresponding assertion about W_1^q.

In addition to quasicontinuity there is a much subtler pointwise notion of continuity associated with functions in W_1^q, called *fine continuity*. A function h that is defined q-quasieverywhere is said to be *q-finely continuous* at a point x_0 if there exists a set E that is thin, or sparse, in a potential-theoretic sense at x_0 and

$$\lim_{\substack{x\to x_0\\ x\in\mathbb{C}\setminus E}} h(x) = h(x_0).$$

The precise sense in which E is understood to be thin is this: for $1 < q \le 2$, a set E is *q-thin* at x_0 if and only if

$$\int_0 \left(\frac{\Gamma_q(E\cap B_r)}{r^{2-q}}\right)^{p-1} \frac{dr}{r} < \infty. \tag{2.1}$$

Here $B_r = B(x_0, r)$ is the disk of radius r with center at x_0 and $p = q/(q-1)$ is the index conjugate to q. If E is not thin at x_0 it is said to be *thick* at that point. It is a fact that every q-quasicontinuous function is q-finely continuous q-q.e. ([1, p. 177]). Because C_q is countably subadditive ([1, p. 26]) and $\Gamma_q \approx C_q$ it follows as a corollary to (2.1) that if

$$\limsup_{r\to 0} \frac{\Gamma_q(E\cap B_r)}{r^{2-q}} > 0, \tag{2.2}$$

then E is thick at x_0.

Let us now return to the situation described earlier where $gw \in L^q(dA)$, $q > 1$, and for every $\lambda > 0$ and $\xi_0 \in \mathbb{C}$ almost every circle $|z - \xi_0| = r$ meets $E_\lambda = \{z : |\widehat{gw}(z)| < \lambda\}$ in a set of positive linear measure. Since circular projection about any point ξ_0 onto a radial segment is a contraction, it can be inferred from (i) and (ii) that $\Gamma_q(E_\lambda\cap B_r) \approx r^{2-q}$ in all cases, and from (2.2), E_λ is everywhere thick. Hence, by fine continuity, $|\widehat{gw}| \le \lambda$ q-q.e., and therefore a.e.-dA. Since this holds for every $\lambda > 0$ we have shown that $\widehat{gw} = 0$ a.e.-dA, from which the bounded point evaluation assertion for $H^2(wdA)$ follows whenever $w \in L^{1+\epsilon}(dA)$.

A similar argument (cf. [5, 23]) yields the same conclusion under the weaker assumption that

$$\int w(\log^+ w)^2\, dA < \infty.$$

Under that restriction on w, if $g \in L^2(wdA)$ and $k = gw$, then

$$\int |k|(1 + \log^+ |k|)\, dA < \infty$$

and so, by a theorem of Calderón and Zygmund, $\nabla\hat{k} \in L^1(dA)$ (cf. [39, p. 48]). That is, $\hat{k} \in BV$, the space consisting of all functions having bounded variation in the sense of DeGiorgi. Should g happen to be orthogonal to the polynomials in $L^2(wdA)$, then, in the absence of BPEs, $\hat{k} = \widehat{gw}$ necessarily enjoys property (4) above. By once again considering projections of the corresponding sets E_λ onto linear segments, this is easily seen to be incompatible with $\hat{k} \in BV$, unless $\hat{k} = 0$ a.e.-dA. The argument here can also be recast in terms of a capacity introduced by Fleming [13], and designed to more accurately describe the exceptional sets associated with BV functions.

If we assume only that $w \in L^1(dA)$ the line of reasoning pursued above fails in at least one important respect. In this case no useful estimate for thickness can be obtained by projecting E_λ onto a line segment, since the 1-capacity of any such segment is zero. On the other hand, nowhere have we made use heretofore of the fact that almost every circle $|z - \xi_0| = r$ meets E_λ in a set of *positive linear measure*. In order to establish the general theorem it will be essential to bring this phenomenon to bear on the problem.

§3. Analytic capacity

The concept of analytic capacity was introduced by Ahlfors [2] in 1947 in connection with the problem of characterizing sets of removable singularities for bounded analytic functions, otherwise known as the *Painlevé problem*. In the ensuing years others, and Vitushkin in particular, further developed the concept and used it to settle a number of questions concerning uniform approximation by rational functions on compact subsets of the plane. An extensive and in-depth discussion of analytic capacity and its many applications covering the period from 1950 to the turn of the century can be found collectively in the expository articles of Vitushkin [47], Mel′nikov and Sinanyan [33], and Khavinson [20] and in the monographs of Gamelin [14], Garnett [16], and Zalcman [49] (cf. also Conway [10] and Davie [11]).

In keeping with established custom the letters X and K will be used throughout this section to represent compact subsets of the complex plane $\mathbb{C}$; and $\hat{\mathbb{C}}$ will stand for the extended plane or Riemann sphere. The *analytic capacity* of a compact set X, denoted $\gamma(X)$, is defined as follows:

$$\gamma(X) = \sup |f'(\infty)|,$$

where the supremum is extended over all functions f analytic in $\hat{\mathbb{C}} \setminus X$ and normalized so that

(a) $\|f\|_\infty = \sup_{\hat{\mathbb{C}} \setminus X} |f| \leq 1$,

(b) $f(\infty) = 0$.

In this case there exists a unique admissible f with $f'(\infty) = \gamma(X)$. For an arbitrary planar set E we let $\gamma(E) = \sup \gamma(X)$, the supremum now being taken over all compact sets $X \subseteq E$. For a more thorough discussion of analytic capacity and its properties see [14] where, for instance, it is shown that

(i) $\gamma(B_r) = r$ for every disk B_r of radius r;

(ii) $\gamma(X) \approx \operatorname{diam}(X)$ whenever X is compact and connected; in particular, $\gamma(X) \leq \operatorname{diam}(X) \leq 4\gamma(X)$.

Although there are similarities between analytic capacity and the potential-theoretic capacities of Section 2, there are also some significant differences. Two differences that make it impossible to directly extend our earlier reasoning on the bounded point evaluation problem to the general case are these: unlike q-capacity C_q,

(iii) γ is *not* known to be subadditive;

(iv) if Φ is a contraction, it may *not* happen that $\gamma(\Phi E) \leq \gamma(E)$.

The phenomenon referred to in the second assertion was first observed by Vitushkin [46] in 1959. In that paper he constructed a Cantor set E with the property that $\gamma(E) = 0$, and so that its orthogonal projection ΦE onto some line is a nondegenerate interval; that is, $\gamma(\Phi E) > 0$. Hence, there is in general no constant $C > 0$ such that

$\gamma(\Phi E) \leq C\gamma(E)$. Vitushkin's example was eventually simplified by Garnett [15] and Ivanov [22, pp. 346–348].

The situation with regard to subadditivity is not quite as serious, however. Tolsa [44] has recently shown that analytic capacity γ is *semiadditive* in the sense that

$$\gamma(E \cup F) \leq C(\gamma(E) + \gamma(F))$$

for all compact sets $E, F \subseteq \mathbb{C}$ and some absolute constant C. The key consists in showing that γ is equivalent to another capacity γ^+, which is known to be semiadditive, and which for our purpose is more directly linked to the Cauchy integral. For a compact set X we define

$$\gamma^+(X) = \sup_{\nu} \nu(X),$$

where the supremum is taken over all *positive* measures ν supported on X such that $\hat{\nu} \in L^\infty(\mathbb{C})$ and $\|\hat{\nu}\|_\infty \leq 1$. Since $\hat{\nu}$ is analytic in $\mathbb{C} \setminus X$ and $\hat{\nu}'(\infty) = \nu(X)$, the function $\hat{\nu}$ is admissible in the definition of γ and thus

$$\gamma^+(X) \leq \gamma(X).$$

As before, if E is an arbitrary planar set we let

$$\gamma^+(E) = \sup_{X} \gamma^+(X),$$

where X is compact and $X \subseteq E$.

Tolsa's theorem (cf. [44]), which provides a solution to the Painlevé problem and affirms an old conjecture of Vitushkin, has its roots in the work of Mattila, Mel′nikov and Verdera [27, 31, 34] and is as follows.

Theorem 2. *There exists an absolute constant $C > 0$ so that*

$$\gamma^+(E) \leq \gamma(E) \leq C\gamma^+(E)$$

for all sets $E \subseteq \mathbb{C}$; that is, $\gamma \approx \gamma^+$.

Since Tolsa [42] had previously shown that γ^+ is in fact countably semiadditive, the same is true for γ:

(1) If E_n, $n = 1, 2, 3, \ldots$ are Borel sets, then

$$\gamma\Big(\bigcup_n E_n\Big) \leq C \sum_n \gamma(E_n), \tag{3.1}$$

where C is again an absolute constant.

There is also a weak-type estimate for the Cauchy integral which is suggestive of a kind of pointwise continuity similar to that enjoyed by ordinary potentials, but a corresponding phenomenon has yet to be established in this context (cf. [43]):

(2) If μ is a complex measure and $\hat{\mu}(x)$ is taken in the principal value sense, then for any $\lambda > 0$,

$$\gamma\{x \in \mathbb{C} : |\hat{\mu}(x)| > \lambda\} < \frac{C}{\lambda}\|\mu\|,$$

where $\|\mu\|$ is the total variation of the measure μ.

An exposition of many of the details associated with these results and a discussion of the Painlevé problem in general can be found in the Lecture Notes of Pajot [38]. See also the expository articles [9, 26, 32, 45].

In order to settle the bounded point evaluation question along the lines indicated earlier in Section 2 we need a substitute for the notion of fine continuity, a concept which is applicable to potentials but is, so far, lacking in the case of the Cauchy integral. A substitute is provided by the following two lemmas. Since they are similar in nature we shall present a detailed proof of the first and simply indicate the necessary modifications in the proof of the second. Each can be viewed as a variation of a prior result of the author (cf. [4, Lemma 2]), and each was suggested by an idea of Carleson (cf. [8, Lemma 1]) used to give a short proof of Mergelyan's theorem on uniform polynomial approximation.

Lemma 1. *Let ν be a finite positive Borel measure of compact support in the complex plane $\mathbb{C}$ with the property that $|\hat{\nu}(z)| \le C$ a.e.-dA for some constant C. Then $|\hat{\nu}(x_0)| \le C$ at every point x_0 where $U^\nu(x_0) = \int \frac{d\nu(\zeta)}{|\zeta - x_0|} < \infty$.*

Lemma 2. *Let μ be a finite complex, compactly supported, Borel measure in $\mathbb{C}$, and let x_0 be any point where $U^{|\mu|}(x_0) < \infty$. For each $r > 0$ let $B_r = B(x_0, r)$ be the disk with center at x_0 and radius r, and let E be a set with the property that for every $r > 0$ there is a relatively large subset $E_r \subseteq (E \cap B_r)$ on which $U^{|\mu|}$ is bounded; that is,*

(1) *$U^{|\mu|} \le M_r < \infty$ on E_r,*
(2) *$\gamma(E_r) \ge \epsilon\gamma(E \cap B_r)$ for some absolute constant ϵ.*

If, moreover, E is thick at x_0 in the sense that

$$\limsup_{r\to 0} \frac{\gamma(E \cap B_r)}{r} > 0, \tag{3.2}$$

then $|\hat{\mu}(x_0)| \le \limsup_{z\to x_0, z\in E} |\hat{\mu}(z)|$.

Proof of Lemma 1. We may assume that $x_0 = 0$ so that $U^\nu(0) = \int \frac{d\nu(z)}{|z|} < \infty$. For each $r > 0$ let $B_r = B(0, r)$ be the disk of radius r with center at the origin, let χ_r be the characteristic function of B_r, and define a probability measure σ_r on B_r by setting $d\sigma_r = \frac{1}{\pi r^2}\chi_r\, dA$. We shall verify that

$$\lim_{r\to 0} \int \hat{\nu}(\zeta)\, d\sigma_r(\zeta) = \hat{\nu}(0), \tag{3.3}$$

from which it follows that

$$|\hat{\nu}(0)| \le \limsup_{r\to 0} \int |\hat{\nu}|\, d\sigma_r \le C,$$

since $|\hat{\nu}| \le C$ a.e.-$d\sigma_r$ for each r.

In order to establish (3.3) we begin with an interchange in the order of integration which gives

$$\int \hat{\nu}(\zeta)\, d\sigma_r(\zeta) = \int_{|z|<2r} + \int_{|z|\ge 2r} \left\{ \int \frac{d\sigma_r(\zeta)}{z - \zeta} \right\} d\nu(z).$$

The interchange is justified by virtue of the fact that $\int \frac{d\sigma_r(\zeta)}{|z-\zeta|} \le \frac{2}{r}$ for all $z \in \mathbb{C}$, and so the iterated integral on the right converges absolutely. It is then an easy matter to check that

(i) $\int \frac{d\sigma_r(\zeta)}{z-\zeta} \to \frac{1}{z}$ for every $z \ne 0$ as $r \to 0$,
(ii) $\int \frac{d\sigma_r(\zeta)}{|z-\zeta|} \le \frac{2}{|z|}$ for $|z| \ge 2r$.

By assumption $\int \frac{d\nu(z)}{|z|} < \infty$, and so the dominated convergence theorem implies that

$$\lim_{r\to 0} \int_{|z|\ge 2r} \left\{ \int \frac{d\sigma_r(\zeta)}{z - \zeta} \right\} d\nu(z) = \int \frac{d\nu(z)}{z} = \hat{\nu}(0).$$

On the other hand,

$$\left| \int_{|z|<2r} \left\{ \int \frac{d\sigma_r(\zeta)}{z - \zeta} \right\} d\nu(z) \right| \le \int_{|z|<2r} \frac{2}{r}\, d\nu(z) \le \int_{|z|<2r} \frac{4}{|z|}\, d\nu(z),$$

and the last integral tends to zero as $r \to 0$ by our initial assumption. This establishes (3.3) and the lemma as well. □

Proof of Lemma 2. Assume as above that $x_0 = 0$. For some sequence of $r \to 0$ we shall construct a corresponding sequence of probability measures ν_r such that

(a) ν_r is carried by $E \cap B_r$,
(b) $\lim_{r\to 0} \int \hat{\mu}\, d\nu_r = \hat{\mu}(0)$.

Once this has been accomplished the desired conclusion is immediate:

$$|\hat{\mu}(0)| \le \limsup_{z\to 0, z\in E} |\hat{\mu}(z)|.$$

In order to obtain the measures ν_r we make use of the fact that $\gamma \approx \gamma^+$, and so each of the hypotheses in the lemma remains valid for γ^+ with possibly different constants. In particular, according to (3.2),

$$\limsup_{r\to 0} \frac{\gamma^+(E \cap B_r)}{r} > 0.$$

Thus, there is a constant $C > 0$ and a sequence of $r \to 0$ such that $\gamma^+(E_r) > Cr$ for each corresponding r. Consistent with the definition of γ^+, we can then select a positive measure σ_r on E_r with

(c) $\|\sigma_r\| = \sigma_r(E_r) \ge Cr$,
(d) $|\hat{\sigma}_r| \le 1$ a.e.-dA.

Setting $\nu_r = \frac{\sigma_r}{\|\sigma_r\|}$ we obtain a probability measure on $E_r \subseteq (E \cap B_r)$, and $|\hat{\nu}_r| \le \frac{C}{r}$ a.e.-dA for some absolute constant C. As in the proof of the preceding lemma,

(i) $\int \frac{d\nu_r(\zeta)}{z-\zeta} \to \frac{1}{z}$ for every $z \ne 0$ as $r \to 0$,
(ii) $\int \frac{d\nu_r(\zeta)}{|z-\zeta|} \le \frac{2}{|z|}$ for $|z| \ge 2r$.

Because $U^{|\mu|} \le M_r$ on E_r, it follows from Fubini's theorem that

$$\int \left(\int \frac{d\nu_r(\zeta)}{|z-\zeta|} \right) d|\mu|(z) = \int \left(\int \frac{d|\mu|(z)}{|z-\zeta|} \right) d\nu_r(\zeta) \le M_r,$$

and hence $U^{\nu_r} < \infty$ a.e.-$d|\mu|$. Therefore, by Lemma 1, $|\hat{\nu}_r(z)| \le \frac{C}{r}$ a.e.-$d|\mu|$.

We can now proceed exactly as in the proof of Lemma 1, writing

$$\int \hat{\mu}(\zeta)\, d\nu_r(\zeta) = \int_{|z|<2r} + \int_{|z|\ge 2r} \left\{ \int \frac{d\nu_r(\zeta)}{z-\zeta} \right\} d\mu(z).$$

As a consequence of (i) and (ii) we have

$$\lim_{r\to 0} \int_{|z|\ge 2r} \left\{ \int \frac{d\nu_r(\zeta)}{z-\zeta} \right\} d\mu(z) = \int \frac{d\mu(z)}{z} = \hat{\mu}(0).$$

For the remaining integral over $|z| < 2r$ we have the estimate

$$\left| \int_{|z|<2r} \left\{ \int \frac{d\nu_r(\zeta)}{z-\zeta} \right\} d\mu(z) \right| \le \int_{|z|<2r} \frac{C}{r}\, d|\mu|(z) \le 2C \int_{|z|<2r} \frac{d|\mu|(z)}{|z|},$$

and the last integral tends to zero as $r \to 0$ by our assumption that $U^{|\mu|}(0) < \infty$. This establishes property (b), and the lemma follows. □

In some instances, especially in those of particular interest here, verification of the density criterion (3.2) requires the introduction of a third auxiliary capacity $\tilde{\gamma}$ defined as follows: if $X \subseteq \mathbb{C}$ is compact,

$$\tilde{\gamma} = \sup |\mu(X)| = \sup |f'(\infty)|,$$

where the supremum is taken over all absolutely continuous measures $d\mu = g dA$ supported on X with $f = \hat{g}$ and $\|f\|_\infty \le 1$. For an arbitrary measurable set E we define

$$\tilde{\gamma}(E) = \sup_X \tilde{\gamma}(X),$$

where X is compact and $X \subseteq E$. It is an immediate consequence of the definition that

$$\gamma(E) \ge \tilde{\gamma}(E)$$

for all measurable sets E, and consideration of the function $f(z) = \int_E \frac{dA_\zeta}{\zeta - z}$ yields the estimate

$$\tilde{\gamma}(E) \ge \sqrt{\frac{A(E)}{4\pi}}. \tag{3.4}$$

For a proof of the latter, see [14, p. 200]. Capacities similar to $\tilde{\gamma}$ have been studied for many years, beginning with Khavinson [19] (cf. [20] for a brief survey of results prior to 1999).

Consider now an application of the Vitushkin approximation scheme due to Mel′nikov [30]. For each positive integer n form a grid in the plane consisting of lines parallel to the coordinate axes, intersecting at those points whose coordinates are both integral multiples of 2^{-n}. The resulting collection of squares $\mathcal{G}_n = \{S_{nj}\}_{j=1}^\infty$ of side lengths 2^{-n} is an edge-to-edge tiling of the entire plane; its members will be referred to as squares of the n-th *generation*. Of chief concern here are *tilings* made up of squares from several generations. Beginning with a fixed positive integer k, let Π_k be a polygon formed by taking the union of finitely many squares from $\mathcal{G}_k$, and denote its boundary by Γ_k. Next, adjoin to Π_k finitely many additional squares from the next generation $\mathcal{G}_{k+1}$ extending outward from Π_k and separating Γ_k from ∞ to obtain a second polygon Π_{k+1} bounded by Γ_{k+1}. Continuing in this way we construct a finite sequence of polygons

$$\Pi_k \subseteq \Pi_{k+1} \subseteq \cdots \subseteq \Pi_{k+l}, \tag{3.5}$$

and we shall assume at the final stage that Π_{k+l} extends to ∞ in all directions. Collectively, the squares in any such sequence, and hence those at the top of the chain, can be viewed as a kind of tiling of the plane. Enlarging each square S_r occurring in (3.5) by a factor of $5/4$ we obtain an open covering of the plane such that no point $z \in \mathbb{C}$ lies in more than 4 of the corresponding enlarged squares Q_r. Except for minor modifications, the next lemma is due to Mel′nikov (cf. [30, Assertion 1]). For ease of notation we shall henceforth denote the area of a set E by $|E|$ in place of the earlier designation $A(E)$.

Lemma 3. *Suppose that a nested sequence of polygons*

$$\Pi_k \subseteq \Pi_{k+1} \subseteq \cdots \subseteq \Pi_{k+l}$$

has been constructed in the manner described above. Let $K = \bigcup K_n$ be a compact set where for each n,

(1) *$K_n = \bigcup S_{nj}$ is a finite union of closed squares S_{nj} in $\Pi_n \setminus \Pi_{n-1}$, and*
(2) *$n^2 2^{-n} \le \operatorname{dist}(K_{n+1}, \Pi_n) \le 3n^2 2^{-n}$.*

If $E \subseteq K$ is a measurable set with the property that $|E \cap S_{nj}| > \epsilon |S_{nj}|$ for some fixed constant $\epsilon > 0$ and all $S_{nj} \subseteq K$, then

$$\tilde{\gamma}(E) \ge C\epsilon\gamma(K),$$

where C is an absolute constant.

Proof. Let f be the Ahlfors function for K; that is, f is analytic outside of K, $\|f\|_\infty \le 1$, $f(\infty) = 0$, and $f'(\infty) = \gamma(K)$. By modifying f in a small neighborhood of K we can assume that f is continuous on $\hat{\mathbb{C}}$.

The collection of all open squares Q_{nj} obtained by enlarging those in the tiling $\prod_{k+l}$ is an open cover of the plane with no point $z \in \mathbb{C}$ lying in more than 4 squares. Here, each Q_{nj} has side length $\frac{5}{4}\delta_n$ with $\delta_n = 2^{-n}$ being the side length of S_{nj}. We can therefore choose continuously differentiable functions g_{nj} such that

(i) g_{nj} is supported in Q_{nj},
(ii) $\sum g_{nj}(z) = 1$ for all $z \in \mathbb{C}$,
(iii) $\|\operatorname{grad} g_{nj}\|_\infty \le \frac{100}{\delta_n}$

(cf., for example, [11, p. 417]). Setting

$$\begin{aligned} G_{nj}(z) &= \frac{1}{\pi}\int \frac{f(\zeta)-f(z)}{\zeta - z}\frac{\partial g_{nj}}{\partial\bar\zeta}\,dA_\zeta \\ &= f(z)g_{nj}(z) + \frac{1}{\pi}\int \frac{f(\zeta)}{\zeta - z}\frac{\partial g_{nj}}{\partial\bar\zeta}\,dA_\zeta \end{aligned}$$

we obtain functions G_{nj} which are analytic off Q_{nj} and analytic everywhere that f is analytic. It follows from (ii) that $f = \sum G_{nj}$, and the sum is finite since $G_{nj} = 0$ except for those indices for which Q_{nj} meets K.

By assumption, and in view of (3.4), for these indices we have the lower estimate

$$\tilde\gamma(E \cap S_{nj}) > \sqrt{\frac{\epsilon}{4\pi}}\delta_n.$$

Indeed, for each square S_{nj} choose a compact set $E_{nj} \subseteq (E \cap S_{nj})$ with $|E_{nj}| > \epsilon\delta_n^2$, and let

$$\psi_{nj}(z) = \sqrt{\frac{\epsilon}{4\pi}}\delta_n \frac{1}{|E_{nj}|}\int_{E_{nj}} \frac{dA_\zeta}{\zeta - z}.$$

Thus, ψ_{nj} is analytic off E_{nj}, $\psi_{nj}(\infty) = 0, \|\psi_{nj}\|_\infty \le 1$, and $|\psi'_{nj}(\infty)| = \sqrt{\frac{\epsilon}{4\pi}}\delta_n$. Expanding ψ_{nj} in a Laurent series about z_{nj}, the center of S_{nj}, we have

$$\psi_{nj}(z) = \frac{a_1}{(z - z_{nj})} + \frac{a_2}{(z-z_{nj})^2} + \cdots$$

valid everywhere outside the enlarged square Q_{nj}. Here, $|a_1| = \sqrt{\frac{\epsilon}{4\pi}}\delta_n$, and it is easily checked that $|a_2| \le \sqrt{\frac{\epsilon}{8\pi}}\delta_n^2$. The bounds on $|a_1|$ and $|a_2|$ allow us to construct functions H_{nj} in such a way that

(iv) $H_{nj}(z) = \int_{E_{nj}} \frac{h_{nj}(\zeta)}{\zeta - z}\,dA_\zeta$, $h_{nj} \in L^\infty$;
(v) $\|H_{nj}\|_\infty \le \frac{C}{\epsilon}$, C an absolute constant;
(vi) $\lim_{z\to\infty} z^2[G_{nj}(z) - H_{nj}(z)] = 0$.

We have only to express G_{nj} in a Laurent series

$$G_{nj}(z) = \frac{b_1}{(z - z_{nj})} + \frac{b_2}{(z-z_{nj})^2} + \cdots$$

and proceed as in Vitushkin [47, p. 153], setting

$$H_{nj} = \frac{b_1}{a_1}\psi_{nj} + \frac{b_2}{a_1^2}\psi_{nj}^2 - \frac{b_1 a_2}{a_1^3}\psi_{nj}^2.$$

Since $H_{nj}(\infty) = 0$, property (iv) is satisfied with $h_{nj} = \bar\partial H_{nj}$, and (vi) is guaranteed by construction. Since $|b_1| \le C_1\delta_n$ and $|b_2| \le C_2\delta_n^2$ with C_1 and C_2 independent of n, condition (v) is also fulfilled. Hence,

$$\|G_{nj} - H_{nj}\|_\infty \le \frac{C}{\epsilon}$$

by virtue of the fact that the G_{nj} are uniformly bounded by a constant depending only on f.

Setting $\phi = \sum H_{nj}$ and appealing to the fact that $H'_{nj}(\infty) = G'_{nj}(\infty)$ we conclude that $\phi'(\infty) = f'(\infty) = \gamma(K)$. If it can be shown that $\|\phi\|_\infty \le \frac{C_3}{\epsilon}$ for some absolute constant C_3, the desired inequality

$$\tilde{\gamma}(E) \ge C\epsilon\gamma(K)$$

follows, since $\phi = \hat{h}$ where $h = \sum h_{nj}$ is in L^∞ with support in E.

Toward this end note that according to (vi) the function $(z - z_{nj})^3(G_{nj} - H_{nj})$ is analytic off Q_{nj} and so by the maximum principle

$$|z - z_{nj}|^3|G_{nj} - H_{nj}| \le \frac{C}{\epsilon}\left(\frac{5\sqrt{2}}{8}\right)^3 \delta_n^3$$

everywhere outside and on the boundary of Q_{nj}. It follows that

$$|G_{nj}(z) - H_{nj}(z)| \le \frac{C_4}{\epsilon}\min\left\{1, \frac{\delta_n^3}{|z - z_{nj}|^3}\right\}$$

for all $z \in \mathbb{C}$. In order to verify that $\|\phi\|_\infty \le \frac{C_3}{\epsilon}$ it is enough, therefore, to show that

$$\sum_{n,j}\min\left\{1, \frac{\delta_n^3}{|z - z_{nj}|^3}\right\} \le C_5 \tag{3.6}$$

for some constant C_5 independent of z. We shall argue as in Davie [11, p. 417].

Fix a point $z \in \mathbb{C}$, and consider initially only those terms in the sum (3.6) corresponding to the centers z_{nj} of squares from the n-th generation; that is, to the centers of the squares comprising K_n. Summing over the associated indices it can be shown that

$$\sum_{j}\min\left\{1, \frac{\delta_n^3}{|z - z_{nj}|^3}\right\} \le C_6\min\left\{1, \frac{2^{-n}}{\operatorname{dist}(z, \bigcup_j Q_{nj})}\right\}, \tag{3.7}$$

where C_6 is a constant independent of n. To see this let $d = \operatorname{dist}(z, \bigcup_j Q_{nj})$, and form the sequence of concentric annuli $A_s = \{w : sd < |w - z| < (s+1)d\}$, $s = 0, 1, 2, \ldots$. The proof of (3.7) is as follows.

By a generous estimate there are no more than 25 squares Q_{nj} for which $\operatorname{dist}(z, Q_{nj}) \le 2^{-n}$; for all others $|z - z_{nj}| > d > 2^{-n}$. Since each point is contained in at most 4 of the Q_{nj},

$$\sum_{j}\min\left\{1, \frac{\delta_n^3}{|z - z_{nj}|^3}\right\} \le 100 + \sum_{|z - z_{nj}| > 2^{-n}} \frac{2^{-3n}}{|z - z_{nj}|^3}. \tag{3.8}$$

Moreover, each square Q_{nj} corresponding to a term in the sum on the right meets at least two and at most three of the annuli under consideration. Thus, if $N(s)$ denotes the number of squares Q_{nj} meeting A_s, then

$$N(s)\left(\frac{5}{4}\right)^2 2^{-2n} \le 4|A_s \cup A_{s+1} \cup A_{s+2}| = 4\pi d^2(6s + 9).$$

Hence, $N(s) \le 200\frac{d^2}{2^{-2n}}s$ for all $s = 1, 2, 3,$ and it follows that

$$\sum_{|z - z_{nj}| > 2^{-n}} \frac{2^{-3n}}{|z - z_{nj}|^3} \le \left(200\sum_{s=1}^{\infty}\frac{1}{s^2}\right)\frac{2^{-n}}{d}. \tag{3.9}$$

Together (3.8) and (3.9) yield the desired estimate in (3.7).

If $z \in \Pi_q \setminus \Pi_{q-1}$ for some q, then summing over all generations,

$$\sum_r \min\left\{1, \frac{\delta_r^3}{|z-z_r|^3}\right\} \le 3C_6 + C_6 \sum_{|n-q|\ge 2} \frac{2^{-n}}{\operatorname{dist}(z, \bigcup_j Q_{nj})} \le 3C_6 + C_7 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

since $\operatorname{dist}(z, \bigcup_j Q_{nj}) \approx \operatorname{dist}(z, K_n)$ and, by assumption, $\operatorname{dist}(z, K_n) \ge n^2 2^{-n}$ whenever $|n-q| \ge 2$. This establishes (3.6) and completes the proof of the lemma. □

§4. A scheme for L^2-approximation

We are now at a point where we can begin to address the bounded point evaluation problem in its most general context. Throughout the discussion μ will be a positive compactly supported measure in $\mathbb{C}$, and $g \in L^2(d\mu)$ will be an annihilator for $H^2(d\mu)$; that is,

$$\int Pg\, d\mu = 0$$

for all polynomials P. By convention, $\nu = g\mu$. The question of the existence of bounded point evaluations (BPEs) for $H^2(d\mu)$ is closely linked to an analogous question in the $L^1(|\hat{\nu}| dA)$ norm. This fact played a key role in our discussion of the BPE problem for absolutely continuous measures in Section 2. Our treatment of the general problem is based on the following elementary fact.

Lemma 4. *If there is a BPE for the polynomials in the $L^1(|\hat{\nu}|dA)$ norm at a point x_0, then $H^2(d\mu)$ also has a BPE at x_0.*

Proof. By assumption there exists a function $h \in L^\infty(dA)$ such that $P(x_0) = \int Ph|\hat{\nu}|\, dA$ for every polynomial P. Letting $k = h\frac{|\hat{\nu}|}{\hat{\nu}}$ when $\hat{\nu} \ne 0$ we have $\|k\|_\infty = \|h\|_\infty$ and, by an interchange in the order of integration,

$$P(x_0) = \int Pk\hat{\nu}\, dA = -\int \widehat{Pk}\, d\nu.$$

On the other hand, $\widehat{Pk} = P\hat{k} + F$, where F is entire. Since ν annihilates the polynomials, and so also F (that is, $\int F\, d\nu = 0$), we can infer that

$$P(x_0) = -\int P\hat{k}\, d\nu = -\int P\hat{k} g\, d\mu.$$

By a straightforward application of Schwarz's inequality it follows that

$$|P(x_0)| \le K\|P\|_{L^2(d\mu)}$$

for all polynomials P, and so $H^2(d\mu)$ has a BPE at x_0. □

Our immediate task is to develop a procedure which, in the absence of BPEs for $H^2(d\mu)$, will allow us to conclude that $H^2(d\mu) = L^2(d\mu)$ along lines similar to those employed in Section 2. The scheme that we shall adopt is due, in broad outline, to Thomson [41], and has its roots in the work of Mel′nikov [30]. As in the special cases considered earlier, the sets $E_\lambda = \{z : |\hat{\nu}(z)| < \lambda\}$ will play a critical role.

Let $x_0 \in \mathbb{C}$ be a point where $U^{|\nu|}(x_0) < \infty$, and fix $\lambda > 0$. Beginning with a particular generation, the n-th say, choose a square $S^* \in \mathcal{G}_n$ with $x_0 \in S^*$. Denote by $\mathcal{G}_n^\lambda$ the collection of all squares S in $\mathcal{G}_n$ for which

$$|E_\lambda \cap S| > \frac{1}{100}|S|. \tag{4.1}$$

K_n will stand for the union of those squares in $\mathcal{G}_n^\lambda$ that can be joined to S^* by a finite chain of squares each lying in $\mathcal{G}_n^\lambda$. In the event that K_n is bounded, or perhaps empty, there exists a closed *corridor*, or *barrier*, $Q_n = \bigcup_j S_{nj}$ composed of squares S_{nj} from the n-th generation $\mathcal{G}_n$ abutting $S^* \cup K_n$, separating the latter from ∞, adjacent to one another along their sides, and such that

$$|E_\lambda \cap S_{nj}| \leq \frac{1}{100}|S_{nj}| \tag{4.2}$$

for each j. The polynomially convex hull of Q_n is a polygon Π_n with its boundary Γ_n lying along the sides of squares for which (4.2) is satisfied. In other words, $|\hat{\nu}| \geq \lambda$ on a large percentage of every square S_{nj} meeting Γ_n.

Next, construct a polygon Π_n^* with boundary Γ_n^* such that

(i) $\Pi_n^* \supseteq \Pi_n$,
(ii) $n^2 2^{-n} \leq \text{dist}(\Gamma_n^*, \Gamma_n) \leq 3n^2 2^{-n}$.

This can be done by simply adjoining to Π_n additional squares from $\mathcal{G}_n$. Let K_{n+1} denote the union of all squares in $\mathcal{G}_{n+1}^\lambda$ that can be joined to Π_n^* by a finite chain of squares in $\mathcal{G}_{n+1}^\lambda$. If K_{n+1} is bounded, or empty, there is a second barrier Q_{n+1} abutting $\Pi_n^* \cup K_{n+1}$ such that

$$|E_\lambda \cap S| \leq \frac{1}{100}|S|$$

for every square S in Q_{n+1}. The polygon Π_{n+1} is taken to be the polynomially convex hull of Q_{n+1}, its boundary is Γ_{n+1}, and the process continues.

The result of the construction is a nested sequence of polygons

$$\Pi_n \subseteq \Pi_{n+1} \subseteq \cdots \subseteq \Pi_{n+l} \subseteq \cdots \tag{4.3}$$

and compact sets $K_j \subseteq \Pi_j$, $j \geq n$, for which the hypotheses of Lemma 3 are satisfied; that is,

(i) $K_j \subseteq \Pi_j \setminus \Pi_{j-1}$,
(ii) $j^2 2^{-j} \leq \text{dist}(K_{j+1}, \Pi_j) \leq 3j^2 2^{-j}$.

There are two mutually exclusive possibilities: either the sequence (4.3)

(A) terminates after l steps and $\infty \in \Pi_{n+l}$, or
(B) it continues indefinitely and $\infty \notin \Pi_j$ for any j.

In the second instance there exists an *infinite* sequence of barriers Q_j bounded by polygonal curves Γ_j extending outward from x_0, and accumulating in a finite portion of the plane. As we shall see, this is sufficient to guarantee that $H^2(d\mu)$ has a BPE at x_0, a key idea in Thomson's original paper [41]. An early hint at such a phenomenon can be found in Luecking [25].

Lemma 5. *If there exists an infinite sequence of barriers Q_j, $j = n, n+1, n+2, \ldots$, surrounding a point x_0, then there is a BPE for the polynomials at x_0 in the $L^1(|\hat{\nu}|dA)$-norm. Hence, $H^2(d\mu)$ also has a BPE at x_0.*

Proof. In the case of the initial barrier Q_n it will be convenient to occasionally drop the subscript n and simply write $Q = Q_n$ and $\Gamma = \Gamma_n$. Because Q is a barrier, Γ is the union of certain specified sides of n-th generation squares S such that $|E_\lambda \cap S| \leq \frac{1}{100}|S|$; or, setting $F_\lambda = \{z : |\hat{\nu}(z)| \geq \lambda\}$, squares S for which

$$|F_\lambda \cap S| \geq \frac{99}{100}|S|.$$

We can assume with no loss of generality that $\lambda = 1$, and we again drop the subscript writing $F = F_1$ and $E = E_1$.

The map $L : P \to P(x_0)$ can be viewed as a bounded linear functional on the space of polynomials when the latter is endowed with the norm $\|P\|_{L^\infty(\Gamma)} = \sup_\Gamma |P|$. As such, L can be extended in a norm-preserving way to $C(\Gamma)$, the full space of continuous functions on Γ likewise endowed with the uniform norm. Hence, there exists a measure ω of finite total variation on Γ such that $\|\omega\| = \|L\|$ and

$$P(x_0) = \int_\Gamma P\,d\omega$$

for all polynomials P. The first step in the proof of the lemma is to replace $\int_\Gamma P\,d\omega$ by an area integral over $F \cap Q$ committing only a small error.

Assume for the moment that P is a *fixed* polynomial. Take $\epsilon > 0$ and let $\Gamma = \bigcup I_j$ be the union of finitely many closed intervals I_j with mutually disjoint interiors chosen so that

$$\left| \int_\Gamma P\,d\omega - \sum_j P(\xi_j)\omega_j \right| < \epsilon, \tag{4.4}$$

whenever $\xi_j \in I_j$ and $\omega_j = \omega(I_j)$. This can be done in such a way that each I_j is contained entirely within the side of a single square S in the barrier Q; we need only ensure that the vertices of any square which lie on Γ are also endpoints of some I_j. Because each point of $\Gamma = \Gamma_n$ is a *peak point* for the uniform algebra on $\prod_n$ generated by the polynomials, the measure ω can have no mass concentrated at a single point. Consequently, there is no ambiguity associated with the approximating sums for $\int_\Gamma P\,d\omega$ as presented in (4.4).

Consider now a fixed barrier square $S \subseteq Q$ with one or more of its sides in Γ, let x_S denote its center, and let ξ_j be one of the points in (4.4) situated on ∂S. Since by construction $\operatorname{dist}(\Gamma_n, \Gamma_{n+1}) \geq n^2 2^{-n}$, a variant of Schwarz's lemma implies that

$$|P(\xi_j) - P(x_S)| \leq \frac{2^{n+1}}{n^2} |\xi_j - x_S| \, \|P\|_{L^\infty(\Gamma_{n+1})} \leq \frac{\sqrt{2}}{n^2} \|P\|_{L^\infty(\Gamma_{n+1})}.$$

Summing over only those indices j for which $I_j \subseteq \partial S$ it follows that

$$\left| \sum_{I_j \subseteq \partial S} (P(\xi_j)\omega_j - P(x_S)\omega_j) \right| \leq \frac{\sqrt{2}}{n^2} \|P\|_{L^\infty(\Gamma_{n+1})} \sum_{I_j \subseteq \partial S} |\omega(I_j)|. \tag{4.5}$$

Here we have passed from Γ_n to Γ_{n+1} for the sole purpose of obtaining a factor of n^{-2} on the right side of (4.5). If, in addition, B is the disk inscribed in S with center at x_S, then

$$P(x_S) = \frac{1}{|B|} \int_B P\,dA.$$

Since $|F \cap B| \geq \frac{98}{100}|B|$ or, equivalently, $|E \cap B| \leq \frac{2}{100}|B|$ by virtue of the fact that $|F \cap S| \geq \frac{99}{100}|S|$, we have

$$\left| P(x_S) - \frac{1}{|B|} \int_{F \cap B} P\,dA \right| \leq \frac{1}{|B|} \int_{E \cap B} |P|\,dA \leq \frac{2}{100} \|P\|_{L^\infty(\Gamma_n)}.$$

Multiplying by ω_j and again summing over the indices for which $I_j \subseteq \partial S$ yields the estimate

$$\left| \sum_{I_j \subseteq \partial S} P(x_S)\omega_j - \frac{\omega(S)}{|B|} \int_{F \cap B} P\,dA \right| \leq \frac{2}{100} \|P\|_{L^\infty(\Gamma_n)} \sum_{I_j \subseteq \partial S} |\omega(I_j)|. \tag{4.6}$$

Thus, if $S_1, \ldots, S_k$ represent the totality of squares in Q with sides along Γ and $B_1, \ldots, B_k$ are the corresponding inscribed disks, it can be inferred from (4.4), (4.5), (4.6) and the fact that Γ_n lies entirely inside the region bounded by Γ_{n+1} that

$$\left| P(x_0) - \int_{F \cap Q} P h_n\,dA \right| \leq \epsilon + \left(\frac{\sqrt{2}}{n^2} + \frac{2}{100} \right) \|\omega\| \, \|P\|_{L^\infty(\Gamma_{n+1})},$$

where $h_n = \sum_{j=1}^{k} \frac{\omega(S_j)}{|B_j|} \chi_{F\cap B_j}$. Hence, $h_n = \sum_{j=1}^{k} \frac{4}{\pi} 2^{2n} \omega(S_j) \chi_{F\cap B_j}$ since all squares S_j are from the n-th generation. Since $\epsilon > 0$ is arbitrary it can now be dropped from the inequality, and by choosing n sufficiently large for the initial barrier $Q = Q_n$ of the presumed sequence we can arrange that

$$\left| P(x_0) - \int_{F\cap Q_n} P h_n \, dA \right| \le \frac{3}{100} \|\omega\| \, \|P\|_{L^\infty(\Gamma_{n+1})} \tag{4.7}$$

for *all* polynomials P, and $\|h_n\|_\infty \le \frac{2^{2n+2}}{\pi} \|\omega\|$.

The entire process can now be repeated. In view of (4.7) the map

$$L_{n+1} : P \to P(x_0) - \int_{F\cap Q_n} P h_n \, dA$$

can be extended from the space of polynomials and treated as a bounded linear functional on $C(\Gamma_{n+1})$ with $\|L_{n+1}\| \le \frac{3}{100}\|L\|$, where $L = L_n$ and where $\|L\| = \|\omega\|$. Since all squares adjacent to Γ_{n+1} are barrier squares from Q_{n+1}, the argument presented above allows us to conclude that there exists a function h_{n+1} with support in $F \cap Q_{n+1}$ such that

$$\left| P(x_0) - \int P h_n \, dA - \int P h_{n+1} \, dA \right| \le \left(\frac{3}{100} \right)^2 \|L\| \, \|P\|_{L^\infty(\Gamma_{n+2})}$$

for all polynomials P, and $\|h_{n+1}\|_\infty \le \frac{4}{\pi} 2^{2(n+1)} \left(\frac{3}{100}\right) \|L\|$. Continuing in this way we obtain an infinite sequence of functions $h_n, h_{n+1}, \ldots$ such that for any $k > 0$,

$$\left| P(x_0) - \int P(h_n + \cdots + h_{n+k}) \, dA \right| \le \left(\frac{3}{100} \right)^{k+1} \|L\| \, \|P\|_{L^\infty(\Gamma_{n+k+1})}.$$

Moreover, for a given polynomial P the right side tends to zero as $k \to +\infty$, since the curves Γ_{n+k} all lie in a bounded portion of the plane. Therefore, setting $h = \sum_{k=0}^{\infty} h_{n+k}$ it follows that h is supported on $F = \{z : |\hat{\nu}(z)| \ge 1\}$ and

$$P(x_0) = \int P h \, dA$$

for all polynomials P. Also, $h \in L^\infty$ because the individual h_j's have disjoint supports and $\|h_{n+k}\|_\infty \le \|h_n\|_\infty$ for all $k > 0$; hence, $\|h\|_\infty = \|h_n\|_\infty$. Inasmuch as h is supported on the set where $|\hat{\nu}| \ge 1$, it follows that

$$|P(x_0)| \le \|h\|_\infty \int |P| \, |\hat{\nu}| \, dA,$$

from which we conclude that there is a BPE for the polynomials at x_0 in the $L^1(|\hat{\nu}| dA)$-norm, and so also in the $L^2(d\mu)$-norm. □

The main theorem on BPEs can now be established along lines similar to those presented earlier under conditions which allowed the use of purely potential-theoretic methods.

Theorem 1. *If μ is a positive measure of compact support in $\mathbb{C}$, not concentrated at a single point, then $H^2(d\mu) = L^2(d\mu)$ if and only if $H^2(d\mu)$ has no BPEs.*

Proof. Suppose that $H^2(d\mu)$ has no BPEs. Let g be any function in $L^2(d\mu)$ with the property that $\int P g \, d\mu = 0$ for all polynomials P, and set $\nu = g\mu$. Fix a point x_0 at which the potential $U^{|\nu|}(x_0) < \infty$.

For an arbitrary, but fixed, $\lambda > 0$ consider once again the set $E_\lambda = \{z : |\hat{\nu}(z)| < \lambda\}$. By Lemma 4 there are no BPEs in the $L^1(|\hat{\nu}| \, dA)$-norm, and consequently by Lemma 5 there can be no infinite sequence of barriers associated with $\mathbb{C}\setminus E_\lambda$, for any λ. Beginning therefore with any generation, the n-th say, there exists a finite sequence of polygons

$$\Pi_n \subseteq \Pi_{n+1} \subseteq \cdots \subseteq \Pi_{n+l}$$

constructed in the manner of (4.3) with $x_0 \in \Pi_n$ and $\infty \in \Pi_{n+l}$. In particular, for $n < j < n+l$ there exist compact sets $K_j \subseteq \Pi_j \setminus \Pi_{j-1}$, some of which may be empty, such that if $K_j \neq \varnothing$,

(i) K_j is the union of squares in $\mathcal{G}_j$ and connects Γ^*_{j-1} to Q_j,
(ii) $|E_\lambda \cap S| > \frac{1}{100}|S|$ for each square $S \subseteq K_j$,
(iii) $\operatorname{dist}(K_j, \Gamma^*_j) \leq \operatorname{dist}(K_j, \Gamma_j) + \operatorname{dist}(\Gamma_j, \Gamma^*_j) < 2^{-j} + 3j^2 2^{-j} < 4j^2 2^{-j}$.

By assumption K_{n+l} extends to ∞. Here the notation is consistent with that established in the discussion preceding Lemma 5.

As in the earlier description, let S^* be an n-th generation square containing x_0, and now form a chain of squares and rectangles leading from S^* to ∞ as follows. Join S^* to the first nonempty K_j by a narrow rectangle R_0 so that $\operatorname{diam}(R_0) \approx \operatorname{dist}(S^*, \Gamma^*_{j-1})$. Choose a connected component of K_j meeting R_0, and thereby obtain a chain from S^* to the barrier Q_j. According to property (iii) that chain can then be joined to Γ^*_j by another narrow rectangle R_j, this time with $\operatorname{diam}(R_j) < 4j^2 2^{-j}$. At this point the resulting chain either meets K_{j+1} at Γ^*_j, or it does not. If it does we adjoin a connected component of K_{j+1} extending the chain to the next barrier Q_{j+1}; if not, we adjoin a narrow rectangle R_{j+1} extending the chain to Γ^*_{j+1}, taking care to ensure that $\operatorname{diam}(R_{j+1}) < 4(j+1)^2 2^{-(j+1)}$. We continue in this way until the chain of squares and rectangles from S^* escapes to ∞. The result is a connected set X joining S^* to ∞, which is composed of squares satisfying property (ii) and certain narrow rectangles.

Given $r > 0$, let $B_r = B(x_0, r)$ be the disk with center at x_0 and radius r. Fix n large; how large depends on r and will be determined presently. Choose $S^* \in \mathcal{G}_n$ with $x_0 \in S^*$ and let X be formed as above. By discarding certain superfluous pieces we can assume that $X \cap B_r$ is connected and joins S^* to ∂B_r. Hence,

$$\gamma(X \cap B_r) \geq \frac{1}{4}\operatorname{diam}(X \cap B_r) \geq \frac{r}{8}.$$

Denote by K the collection of all squares in $X \cap B_r$ that lie entirely inside B_r. By construction $K = \bigcup K_j$, where each K_j is the union of squares from $\mathcal{G}_j$ that satisfy

(1) $|E_\lambda \cap S| > \frac{1}{100}|S|$ for each square $S \subseteq K$,
(2) $j^2 2^{-j} \leq \operatorname{dist}(K_j, \Gamma^*_j) < 4j^2 2^{-j}$.

It follows from the preceding remarks and the semiadditivity of analytic capacity that

$$\frac{r}{16} \leq \gamma(X \cap B_{r/2}) \leq C[\gamma(K) + \sum_{j=n}^{\infty} j^2 2^{-j}], \tag{4.8}$$

where C is an absolute constant. Since, by assumption, construction of the set X can begin with an arbitrary generation we are free to choose n as large as we please, and (4.8) remains valid with the same constant C. In particular, n can be chosen large enough so that the infinite sum on the right side of (4.8) is negligible, from which we conclude that $\gamma(K) \geq Cr$ for another absolute constant. Inasmuch as properties (1) and (2) are satisfied, Lemma 3 implies that

$$\gamma(E_\lambda \cap B_r) \geq \tilde{\gamma}(E_\lambda \cap B_r) \geq C\epsilon\gamma(K) \geq C\epsilon r,$$

for some constant ϵ independent of r, and therefore E_λ is thick at x_0 in the sense that

$$\limsup_{r\to 0} \frac{\gamma(E_\lambda \cap B_r)}{r} > 0.$$

For a given r, once n has been fixed the set K consists of finitely many squares for which (1) is satisfied. Since $U^{|\nu|} < \infty$ a.e.-dA, we can find a subset $\Omega_r \subseteq (E_\lambda \cap B_r)$ on which $U^{|\nu|}$ is bounded and for which property (1) is preserved, that is, for which $|\Omega_r \cap S| > \frac{1}{100}|S|$ for every square $S \subseteq K$. By the same reasoning presented above in support of the lower estimate for $\gamma(E_\lambda \cap B_r)$ it follows that $\gamma(\Omega_r) \geq C\epsilon r$, and since $\gamma(E_\lambda \cap B_r) \leq r$, that

$$\gamma(\Omega_r) \geq C\epsilon\gamma(E_\lambda \cap B_r).$$

The hypotheses of Lemma 2 are therefore satisfied, and so

$$|\hat{\nu}(x_0)| \leq \limsup_{\substack{z \to x_0 \\ z \in E_\lambda}} |\hat{\nu}(z)| \leq \lambda. \tag{4.9}$$

Since the inequality is valid for all $\lambda > 0$, we can infer that $\hat{\nu}(x_0) = 0$. Hence, $\hat{\nu} = 0$ a.e.-dA, and $\nu = g\mu = 0$ as a measure. Therefore, $H^2(d\mu) = L^2(d\mu)$. □

§5. Some additional corollaries

The bounded point evaluation question, which is the principal focus of this investigation, is a special case of a more general problem, various aspects of which have been extensively examined over many years: *Given a subset $X \subseteq \mathbb{C}$, a Banach space B of functions defined on X and a subfamily $\mathcal{F} \subset B$ of functions analytic in a neighborhood of X, is $\mathcal{F}$ dense in B*? The difficulties can be quite varied depending on the choice of X, the space B and the family $\mathcal{F}$. However, it often happens that either $\mathcal{F}$ is dense in B or the only functions that can be so approximated have a natural analytic extension outside of X. This is especially true in case $\mathcal{F}$ is the set of all complex analytic polynomials (cf. [6]).

Illustrations of the continuation phenomenon associated with polynomial approximation occur in the following familiar settings, when

(1) X is compact and $B = A(X)$ is the space of all functions continuous on X, analytic in its interior, and endowed with the uniform norm;
(2) $X = \Omega$ is a simply connected domain and $B = L^p_a(\Omega, dA)$, $1 \leq p < \infty$, is the set of all functions in $L^p(\Omega, dA)$ that are analytic in Ω.

In the case of uniform approximation suppose, for example, that X is a compact set which separates the plane, so that $\mathbb{C}\setminus X$ has at least one bounded component G. Then any sequence of polynomials that converges uniformly on X necessarily converges uniformly on $X \cup G$ as well. This is an obstacle to uniform polynomial approximation in $A(X)$, since the limit of any such sequence admits an analytic continuation to G. In particular, if $a \in G$, then $(z-a)^{-1} \in A(X)$, but $(z-a)^{-1}$ cannot be uniformly approximated on X by polynomials. On the other hand, by a theorem of Mergelyan (cf. [14, p. 48]) this is the only obstacle to the density of the polynomials in $A(X)$. In other words, the density-continuation dichotomy is valid here for a purely topological reason.

While the polynomial approximation question for $A(X)$ can be settled in this way, the same cannot be said for approximation in the $L^p(\Omega, dA)$-norm. That discovery was made by Keldysh in 1939 (cf. [35, p. 117]). If Ω is a *crescent domain*, that is, a domain topologically equivalent to the region bounded by two internally tangent circles, he showed that the polynomials may or may not be dense in $L^p_a(\Omega, dA)$ depending on the *thickness* of Ω near the multiple boundary point. Again, however, if density fails, then the only functions in $L^p_a(\Omega, dA)$ which admit approximation by polynomials can be continued analytically across $\partial\Omega$ into the bounded component of $\mathbb{C} \setminus \bar{\Omega}$, another instance of the density-continuation dichotomy.

In 1955 Mergelyan [36] conjectured that this is the only way that the polynomials can fail to be dense in $L^p_a(\Omega, dA)$ for any bounded simply connected domain Ω. Subsequently, the author [5], using potential-theoretic ideas introduced by Hedberg [21] and Maz′ya and Havin [28, 29] to study approximation in the mean by analytic functions, verified the conjecture in the more general setting of approximation in $L^p_a(\Omega, wdA)$ for a weighted measure wdA with $w \in L^{1+\epsilon}(dA) \cap C(\Omega)$. The same phenomenon can be observed in connection with the Bernstein problem for weighted polynomial approximation on the entire real line, in both the weighted uniform and weighted L^p metrics (cf. [6]). The following was noted by Thomson [41] and is further evidence of the reciprocal relation between approximation and analytic continuation.

Corollary 1. *Let μ be a positive measure of compact support in the complex plane $\mathbb{C}$. Then, either $H^2(d\mu) = L^2(d\mu)$ or there exists a point x_0 and an open set U containing x_0 such that every $f \in H^2(d\mu)$ admits an analytic continuation to U.*

The corollary is actually valid for approximation in $L^p(d\mu)$, $1 \le p < \infty$, and the proof presented here can easily be adapted to cover the added situations.

Proof. Suppose that $H^2(d\mu) \ne L^2(d\mu)$. Then $H^2(d\mu)$ has a BPE at some point x_0, and so there exists $g \in L^2(d\mu)$ with

$$P(x_0) = \int Pg\, d\mu$$

for all polynomials P. Hence, $\nu = (z - x_0)g\mu$ is an annihilating measure for which
(i) $\hat{\nu}(x_0) = 1$,
(ii) $U^{|\nu|}(x_0) = \int |g|\, d\mu < \infty$.

For some $\lambda > 0$, according to the proof of Theorem 1, the set $F_\lambda = \{z : |\hat{\nu}(z)| \ge \lambda\}$ gives rise to an infinite sequence of barriers Q_j, $j = n, n+1, \ldots$, surrounding x_0 and extending outward from there such that

$$|F_\lambda \cap S| \ge \frac{99}{100}|S|$$

for all barrier squares S in each generation $j = n, n+1, \ldots$. Repeating the argument in the proof of Lemma 5, it follows that each point ξ in the region bounded by the initial barrier Q_n corresponds to a BPE with norm depending only on $\operatorname{dist}(\xi, Q_n)$. Thus, if U is a neighborhood of x_0 with $\operatorname{dist}(U, Q_n) > 0$ there exists a fixed constant $C > 0$ such that

$$|P(\xi)| \le C\|P\|_{L^2(d\mu)}$$

for all $\xi \in U$ and all polynomials P. Therefore, every $f \in H^2(d\mu)$ admits an analytic continuation to U.

Added in translation. To allow for the possibility that μ may contain point masses, let $\nu = g\mu$ be any nonzero annihilator of $H^2(\mu)$. Since $|\hat{\nu}| > 0$ on a set of positive dA measure we can choose a point x_0 where $\hat{\nu}(x_0) \ne 0$ and $U^{|\nu|}(x_0) < \infty$, and the proof proceeds as before. □

The following two well-known results of Lavrentiev and Vitushkin are also immediate consequences of the approximation scheme outlined above and, in particular, consequences of Lemma 2. Given a compact set $X \subset \mathbb{C}$ let $P(X)$ and $R(X)$ denote the closures in $C(X)$ of the polynomials and rational functions, respectively.

Corollary 2 (Lavrentiev)**.** *$P(X) = C(X)$ if and only if X has no interior and the complement $\mathbb{C} \setminus X$ is connected.*

Proof. Let ν be any measure on X with the property that $\int P\,d\nu = 0$ for all polynomials P, and form the Cauchy transform $\hat{\nu}$. Since $\hat{\nu}$ is analytic in the connected region $\mathbb{C} \setminus X$ and $\hat{\nu} = 0$ in a neighborhood of ∞, it follows that $\hat{\nu} \equiv 0$ in $\mathbb{C} \setminus X$. If $|X| = 0$, then $\hat{\nu} = 0$ a.e.-dA, from which we can infer that $\nu = 0$ as a measure. Hence, $P(X) = C(X)$.

We may assume, therefore, that $|X| > 0$. Fix a point $x_0 \in X$ where $U^{|\nu|}(x_0) < \infty$ and let $B_r = B(x_0, r)$ be the disk of radius r with center at x_0. Because $\mathbb{C} \setminus X$ is connected we can find a compact set $E_r \subset B_r \setminus X$, an arc for example, with diam $E_r \geq \frac{r}{2}$, and so $\gamma(E_r) \geq \frac{r}{8}$. Since $U^{|\nu|}$ is bounded on E_r and

$$\limsup_{r \to 0} \frac{\gamma(B_r \setminus X)}{r} > 0,$$

the requirements of Lemma 2 are satisfied and

$$|\hat{\nu}(x_0)| \leq \limsup_{\substack{z \to x_0 \\ z \in B_r \setminus X}} |\hat{\nu}(z)| = 0.$$

Inasmuch as $U^{|\nu|} < \infty$ a.e.-dA on X, if follows that $\hat{\nu} = 0$ a.e.-dA. Again,

$$P(X) = C(X). \qquad \square$$

Corollary 3 (Vitushkin)**.** *$R(X) = C(X)$ if and only if*

$$\limsup_{r \to 0} \frac{\gamma(B(x,r) \setminus X)}{r} > 0,$$

for dA almost every $x \in X$.

Proof. Let ν be a measure on X such that $\int f\,d\nu = 0$ for every rational function f with poles off X. Clearly, $\hat{\nu} \equiv 0$ in $\mathbb{C} \setminus X$ and just as in Corollary 2 the capacitary density assumption implies that $\hat{\nu} = 0$ a.e.-dA on X. Hence, $R(X) = C(X)$. For a proof in the other direction see [14, p. 207]. $\square$

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