Алгебра и анализ
Том. 17 (2005), вып. 2

St. Petersburg Math. J.
Vol. 17 (2006), No. 2, Pages 267–283
S 1061-0022(06)00903-4
Article electronically published on February 10, 2006

# ASYMPTOTIC DIMENSION OF A HYPERBOLIC SPACE AND CAPACITY DIMENSION OF ITS BOUNDARY AT INFINITY

S. BUYALO

Abstract. A quasisymmetry invariant of a metric space $Z$ (called the capacity dimension, $\operatorname{cdim} Z$) is introduced. The main result says that the asymptotic dimension of a visual Gromov hyperbolic space $X$ is at most the capacity dimension of its boundary at infinity plus 1, $\operatorname{asdim} X \le \operatorname{cdim} \partial_\infty X + 1$.

## §1. Introduction

The notion of the asymptotic dimension, which is a quasiisometry invariant of metric spaces, was introduced in [Gr]. The present paper arose as an attempt to restore the details of an argument sketched in [Gr, $1.\mathrm{E}_1'$] to show that the asymptotic dimension of a negatively pinched Hadamard manifold $X$ is bounded above by $\dim X$, $\operatorname{asdim} X \le \dim X$. In this way, the author has come to the notion of the capacity dimension of a metric space, cdim, which, as I expect, should play an important role in many questions.

Recall that for every Gromov hyperbolic space $X$ there is a canonical class of metrics on the boundary at infinity $\partial_\infty X$ called *visual* metrics; see §6. The main result of the paper is the following.

**Theorem 1.1.** *Let $X$ be a visual Gromov hyperbolic space. Then*

$$\operatorname{asdim} X \le \operatorname{cdim} \partial_\infty X + 1$$

*for any visual metric on $\partial_\infty X$.*

The notion of a visual hyperbolic space (see [BoS]) is a rough version of the property that, given a base point $o \in X$, for every $x \in X$ there is a geodesic ray in $X$ emanating from $o$ and passing through $x$; see §6.

The inequality of Theorem 1.1 is sharp. It is known that $\operatorname{asdim} \mathrm{H}^n = n$ for the real hyperbolic space $\mathrm{H}^n$, $n \ge 2$. On the other hand, the standard unit sphere metric is a visual metric on the boundary at infinity $\partial_\infty \mathrm{H}^n = S^{n-1}$, and $\operatorname{cdim} S^{n-1} = n-1$; see Corollary 3.5.

The capacity dimension is a bi-Lipschitz invariant by definition. A remarkable fact discovered in [LS] is that a closely related notion of the Assouad–Nagata dimension is a quasisymmetry invariant. It turns out that the capacity dimension is also a quasisymmetry invariant (see §4); in particular, the right-hand side of the inequality in Theorem 1.1 is independent of the choice of a visual metric on $\partial_\infty X$. This is also compatible with the fact that every quasiisometry of hyperbolic geodesic spaces induces a quasisymmetry of their boundaries at infinity.

Now, we briefly describe the structure of the paper. In §2 we collect the notions and facts of the dimension theory needed in the paper. Here we also recall the definition of the

2000 *Mathematics Subject Classification.* Primary 53B99.
*Key words and phrases.* Visual Gromov hyperbolic space.
The author was supported by RFBR (grant no. 02-01-00090).

asymptotic dimension; see Subsection 2.4. In §3 we give three definitions of the capacity dimension (each being useful in appropriate situations) and prove their equivalence. Here we also compare the capacity dimension with the Assouad–Nagata dimension and obtain the monotonicity of the capacity dimension. In §4 we prove that the capacity dimension is a quasisymmetry invariant. The proof is based on some ideas of the paper [LS]. The core of the paper is in §5, where we recall the notion of the hyperbolic cone over a bounded metric space $Z$ and prove the relevant estimate for the asymptotic dimension of the cone in terms of the capacity dimension of the base $Z$. Finally, in §6 we discuss some facts of the hyperbolic spaces theory and prove Theorem 1.1.

**Acknowledgements.** The author thanks Urs Lang and Viktor Schroeder for inspiring discussions.

## §2. Preliminaries

Here we collect some (more or less) familiar notions and facts of dimension theory that are needed in what follows.

Let $Z$ be a metric space. For $U$, $U' \subset Z$ we denote by $\operatorname{dist}(U, U')$ the distance between $U$ and $U'$, $\operatorname{dist}(U, U') = \inf\{|uu'| : u \in U,\ u' \in U'\}$, where $|uu'|$ is the distance between $u$ and $u'$. For $r > 0$, we denote by $B_r(U)$ the open $r$-neighborhood of $U$, $B_r(U) = \{z \in Z : \operatorname{dist}(z, U) < r\}$, and by $\overline{B}_r(U)$ the closed $r$-neighborhood of $U$, $\overline{B}_r(U) = \{z \in Z : \operatorname{dist}(z, U) \le r\}$. We extend this notation to all real $r$ by putting $B_r(U) = U$ for $r = 0$, and defining $B_r(U)$ for $r < 0$ as the complement of the closed $|r|$-neighborhood of $Z \setminus U$, $B_r(U) = Z \setminus \overline{B}_{|r|}(Z \setminus U)$.

**2.1. Coverings.** Given a family $\mathcal{U}$ of subsets in a metric space $Z$, we put $\operatorname{mesh}(\mathcal{U}, z) = \sup\{\operatorname{diam} U : z \in U \in \mathcal{U}\}$ for every $z \in Z$, and $\operatorname{mesh}(\mathcal{U}) = \sup\{\operatorname{diam} U : U \in \mathcal{U}\}$. Clearly, $\operatorname{mesh}(\mathcal{U}) = \sup_{z \in Z} \operatorname{mesh}(\mathcal{U}, z)$. In the case where $D = \operatorname{mesh}(\mathcal{U}) < \infty$ we say that $\mathcal{U}$ is *$D$-bounded*.

The *multiplicity* $m(\mathcal{U})$ of $\mathcal{U}$ is the maximal number of the members of $\mathcal{U}$ with nonempty intersection. For $r > 0$, the *$r$-multiplicity* $m_r(\mathcal{U})$ of $\mathcal{U}$ is the multiplicity of the family $\mathcal{U}_r$ obtained by taking open $r$-neighborhoods of the members of $\mathcal{U}$. So, $m_r(\mathcal{U}) = m(\mathcal{U}_r)$. We say that a family $\mathcal{U}$ is *disjoint* if $m(\mathcal{U}) = 1$.

A family $\mathcal{U}$ is called a *covering* of $Z$ if $\bigcup\{U : U \in \mathcal{U}\} = Z$. A covering $\mathcal{U}$ is said to be *colored* if it is a union of $m \ge 1$ disjoint families, $\mathcal{U} = \bigcup_{a \in A} \mathcal{U}^a$, $|A| = m$. In this case we also say that $\mathcal{U}$ is $m$-colored. Clearly, the multiplicity of an $m$-colored covering is at most $m$.

Let $\mathcal{U}$ be an open covering of a metric space $Z$. Given $z \in Z$, we put $L'(\mathcal{U}, z) = \sup\{\operatorname{dist}(z, Z \setminus U) : U \in \mathcal{U}\}$ and introduce the Lebesgue number of $\mathcal{U}$ at $z$ by the formula

$$L(\mathcal{U}, z) = \min\{L'(\mathcal{U}, z), \operatorname{mesh}(\mathcal{U}, z)\}$$

(the auxiliary quantity $L'(\mathcal{U}, z)$ may be larger than $\operatorname{mesh}(\mathcal{U}, z)$ and even infinite as, e.g., in the case where $Z = U$ for some $U \in \mathcal{U}$). Next, $L(\mathcal{U}) = \inf_{z \in Z} L(\mathcal{U}, z)$ is the *Lebesgue number of* $\mathcal{U}$. We have $L(\mathcal{U}, z) \le \operatorname{mesh}(\mathcal{U}, z)$, $L(\mathcal{U}) \le \operatorname{mesh}(\mathcal{U})$, and for every $z \in Z$ the open ball $B_r(z)$ of radius $r = L(\mathcal{U})$ and centered at $z$ is contained in some member of the covering $\mathcal{U}$.

**Lemma 2.1.** *Let $\mathcal{U}$ be an open covering of $Z$ with $L(\mathcal{U}) > 0$. Then for every $s \in (0, L(\mathcal{U}))$ the family $\mathcal{U}_{-s} = B_{-s}(\mathcal{U})$ is still an open covering of $Z$, and its $s$-multiplicity $m_s(\mathcal{U}_{-s})$ does not exceed $m(\mathcal{U})$.*

*Proof.* For every $z \in Z$, the ball $B_r(z)$, $r = L(\mathcal{U})$, is contained in some $U \in \mathcal{U}$. Then $z \in B_{-s}(U)$ because $s < r$; thus $\mathcal{U}_{-s}$ is an open covering of $Z$. Furthermore, since $B_s(B_{-s}(U)) \subset U$ for every $U \subset Z$, we have $m_s(\mathcal{U}_{-s}) = m(B_s(\mathcal{U}_{-s})) \le m(\mathcal{U})$. $\square$

A covering $\mathcal{U}$ is said to be *locally finite* if for every $z \in Z$ only finitely many of its elements intersect some neighborhood of $z$.

We define the *nerve* of $\mathcal{U}$ as a simplicial polyhedron whose vertex set is $\mathcal{U}$, and a finite collection of vertices spans a simplex if and only if the corresponding covering elements have a nonempty intersection. Thus, its (combinatorial) dimension is $m(\mathcal{U}) - 1$.

**2.2. Uniform polyhedra.** Given an index set $J$, we let $R^J$ be the Euclidean space of functions $J \to \mathbb{R}$ with finite support, i.e., $x \in \mathbb{R}^J$ precisely when only finitely many coordinates $x_j = x(j)$ are nonzero. The distance $|xx'|$ is well defined by

$$|xx'|^2 = \sum_{j \in J} (x_j - x'_j)^2.$$

Let $\Delta^J \subset \mathbb{R}^J$ be the standard simplex, i.e., $x \in \Delta^J$ if and only if $x_j \geq 0$ for all $j \in J$ and $\sum_{j \in J} x_j = 1$.

A metric in a simplicial polyhedron $P$ is said to be *uniform* if $P$ is isometric to a subcomplex of $\Delta^J \subset \mathbb{R}^J$ for some index set $J$. Every simplex $\sigma \subset P$ is then isometric to the standard $k$-simplex $\Delta^k \subset \mathbb{R}^{k+1}$, $k = \dim \sigma$ (so, for a finite $J$ we have $\dim \Delta^J = |J| - 1$). For every simplicial polyhedron $P$, there is a canonical embedding $u : P \to \Delta^J$, where $J$ is the vertex set of $P$, namely, the embedding is affine on every simplex. Its image $P' = u(P)$ is called the *uniformization* of $P$, and $u$ is called the *uniformization map*.

For example, the nerve $\mathcal{N} = \mathcal{N}(\mathcal{U})$ of a covering $\mathcal{U} = \{U_j\}_{j \in J}$ can always be viewed as a subcomplex of $\Delta^J$, $\mathcal{N} \subset \Delta^J$, and, therefore, as a uniform polyhedron.

**2.3. Barycentric maps.** Let $\mathcal{U} = \{U_j\}_{j \in J}$ be a locally finite open covering of a metric space $Z$ by bounded sets, and let $\mathcal{N} = \mathcal{N}(\mathcal{U}) \subset \Delta^J$ be its nerve. We define the *barycentric map*

$$p : Z \to \mathcal{N}$$

associated with $\mathcal{U}$ as follows. Given $j \in J$, we define $q_j : Z \to \mathbb{R}$ by $q_j(z) = \min\{\operatorname{diam} Z, \operatorname{dist}(z, Z \setminus U_j)\}$. Since $\mathcal{U}$ is open, $\sum_{j \in J} q_j(z) > 0$ for every $z \in Z$. Since $\mathcal{U}$ is locally finite and its elements are bounded, $\sum_{j \in J} q_j(z) < \infty$ for every $z \in Z$. Now, the map $p$ is defined by its coordinate functions $p_j(z) = q_j(z) / \sum_{j \in J} q_j(z)$, $j \in J$. Clearly, its image lands at the nerve, $p(Z) \subset \mathcal{N}$. Assume in addition that $L(\mathcal{U}) \geq d > 0$ and that the multiplicity $m(\mathcal{U}) = m + 1$ is finite. Then it is easy to check (see, for instance, [BD], [BS]) that $p$ is Lipschitz with Lipschitz constant

$$\operatorname{Lip}(p) \leq \frac{(m+2)^2}{d}.$$

Furthermore, for each vertex $v \in \mathcal{N}$ the preimage of its open star, $p^{-1}(\operatorname{st}_v) \subset Z$, coincides with the member of the covering $\mathcal{U}$ corresponding to $v$.

An (open, locally finite) covering $\mathcal{U}'$ is *inscribed* in $\mathcal{U}$ if every one of its elements is contained in some element of $\mathcal{U}$. In this case there is a simplicial map $\rho : \mathcal{N}' \to \mathcal{N}$ of the nerves that takes every vertex $v' \in \mathcal{N}'$ to some vertex $v \in \mathcal{N}$ with $v' \subset v$ (as covering elements). It is easily seen that this rule is compatible with the simplicial structures of $\mathcal{N}$, $\mathcal{N}'$ and, moreover, for every $z \in Z$ the point $\rho \circ p'(z)$ lies in a face of the minimal simplex containing $p(z) \in \mathcal{N}$.

Note that if $\operatorname{mesh}(\mathcal{U}') < L(\mathcal{U})$, then $\mathcal{U}'$ is inscribed in $\mathcal{U}$.

**2.4. Asymptotic dimension.** The asymptotic dimension is a quasiisometry invariant of a metric space; it was introduced in [Gr]. There are several equivalent definitions; see [Gr], [BD]. We shall use the following one. The *asymptotic dimension* $\operatorname{asdim} X$ of a metric space $X$ is the minimal $n$ such that for every $\lambda > 0$ there is a $\lambda$-Lipschitz map $f : X \to P$ into a uniform simplicial polyhedron $P$ of dimension at most $n$ for which the preimages $f^{-1}(\sigma) \subset X$ of all simplexes $\sigma \subset P$ are uniformly bounded. We say that $f$ is *uniformly cobounded* if the last property is satisfied.

## §3. Capacity dimension

We give three equivalent definitions of the capacity dimension. Each of them is useful in appropriate situations.

Let $\mathcal{U}$ be an open covering of a metric space $Z$. We define the *capacity* of $\mathcal{U}$ by

$$\operatorname{cap}(\mathcal{U}) = \frac{L(\mathcal{U})}{\operatorname{mesh}(\mathcal{U})} \in [0,1];$$

if $\operatorname{mesh}(\mathcal{U}) = 0$ or $L(\mathcal{U}) = \operatorname{mesh}(\mathcal{U}) = \infty$, we put $\operatorname{cap}(\mathcal{U}) = 1$ by definition.

**3.1. First definition.** For $\tau > 0$, $\delta \in (0,1)$, and an integer $m \geq 0$, we put

$$c_{1,\tau}(Z,m,\delta) = \sup_{\mathcal{U}} \operatorname{cap}(\mathcal{U}),$$

where the supremum is taken over all open $(m+1)$-colored coverings $\mathcal{U}$ of $Z$ with $\delta\tau \leq \operatorname{mesh}(\mathcal{U}) \leq \tau$.

Next, we put

$$c_1(Z,m,\delta) = \liminf_{\tau\to 0} c_{1,\tau}(Z,m,\delta).$$

The function $c_1(Z,m,\delta)$ is monotone in $\delta$: $c_1(Z,m,\delta') \geq c_1(Z,m,\delta)$ for $\delta' < \delta$. Hence, there exists a limit $c_1(Z,m) = \lim_{\delta\to 0} c_1(Z,m,\delta)$. Now, we define the *capacity dimension* of $Z$ as

$$\operatorname{cdim}_1(Z) = \inf\{m : c_1(Z,m) > 0\}.$$

**3.2. Second definition.** For $\tau > 0$, $\delta \in (0,1)$, and an integer $m \geq 0$, we put

$$c_{2,\tau}(Z,m,\delta) = \sup_{\mathcal{U}} \operatorname{cap}(\mathcal{U}),$$

where the supremum is taken over all open coverings $\mathcal{U}$ of $Z$ with multiplicity not exceeding $m+1$ and $\delta\tau \leq \operatorname{mesh}(\mathcal{U}) \leq \tau$. Now, we proceed as above, putting

$$c_2(Z,m,\delta) = \liminf_{\tau\to 0} c_{2,\tau}(Z,m,\delta),$$

$c_2(Z,m) = \lim_{\delta\to 0} c_2(Z,m,\delta)$, and finally

$$\operatorname{cdim}_2(Z) = \inf\{m : c_2(Z,m) > 0\}.$$

**3.3. Third definition.** Let $f : Z \to P$ be a map into an $m$-dimensional uniform polyhedron $P$. We define $\operatorname{mesh}(f)$ as the supremum of $\operatorname{diam} f^{-1}(\operatorname{st}_v)$ over open stars $\operatorname{st}_v \subset P$ of vertices $v \in P$. Next, we introduce the *capacity* of $f$ as

$$\operatorname{cap}(f) = (\operatorname{Lip}(f) \cdot \operatorname{mesh}(f))^{-1},$$

and for $\tau > 0$, $\delta \in (0,1)$, and an integer $m \geq 0$ we define $c_{3,\tau}(Z,m,\delta) = \sup_f \operatorname{cap}(f)$, where the supremum is taken over all Lipschitz maps $f : Z \to P$ into $m$-dimensional uniform polyhedra $P$ with $\delta\tau \leq \operatorname{mesh}(f) \leq \tau$. Then, as above, we define $c_3(Z,m,\delta)$, $c_3(Z,m)$, and

$$\operatorname{cdim}_3(Z) = \inf\{m : c_3(Z,m) > 0\}.$$

The basic motivation for the introduction of the capacity dimension is that sometimes we need to control the Lebesgue number $L(\mathcal{U})$ of coverings involved in the definition of

a dimension, e.g., we may need that the capacity $\operatorname{cap}(\mathcal{U})$ stay separated from zero for $\mathcal{U}$'s chosen appropriately. In general, there is no reason for that. However, if we allow coverings with larger multiplicity, we can typically gain control over $L(\mathcal{U})$, and it may happen that $\dim Z < \operatorname{cdim} Z$.

Another feature of the definitions is that they involve the auxiliary variable $\delta$ and the functions $c_{i,\tau}(Z, m, \delta)$, $i = 1, 2, 3$. This is done for a technical reason, to enable "Čech approximations" $\mathcal{U}_k$ for which $\operatorname{mesh}(\mathcal{U}_{k+1})$ is not extremely small compared with $\operatorname{mesh}(\mathcal{U}_k)$ for every $k$.

**3.4. Equivalence of the definitions.** The proof of the statement that the three capacity dimensions coincide is standard; cf. [BD, BS, LS].

**Proposition 3.1.** *All three capacity dimensions coincide,*

$$\operatorname{cdim}_1 = \operatorname{cdim}_2 = \operatorname{cdim}_3 .$$

*Proof.* The multiplicity of every $(m+1)$-colored covering is at most $m+1$. Thus, $c_{1,\tau}(Z, m, \delta) \le c_{2,\tau}(Z, m, \delta)$ for all $\tau > 0$, $\delta \in (0, 1)$, and an integer $m \ge 0$, and hence $\operatorname{cdim}_2(Z) \le \operatorname{cdim}_1(Z)$.

For every integer $m \ge 0$, every open covering $\mathcal{U}$ of $Z$ with multiplicity at most $m+1$ is locally finite. If, moreover, its Lebesgue number is positive, $L(\mathcal{U}) > 0$, then the barycentric map $p : Z \to \mathcal{N}$ ($\mathcal{N} = \mathcal{N}(\mathcal{U})$ is the nerve, $\dim \mathcal{N} \le m$) is Lipschitz with $\operatorname{Lip}(p) \le (m+2)^2/L(\mathcal{U})$. Since $p^{-1}(\operatorname{st}_v) = U$ for the vertex $v \in \mathcal{N}$ corresponding to $U \in \mathcal{U}$, we have $\operatorname{mesh}(p) = \operatorname{mesh}(\mathcal{U})$. Thus, for capacities we have $(m+2)^2 \operatorname{cap}(p) \ge \operatorname{cap}(\mathcal{U})$, and $c_{2,\tau}(Z, m, \delta) \le (m+2)^2 c_{3,\tau}(Z, m, \delta)$ for all $\tau > 0$, $\delta \in (0, 1)$, and an integer $m \ge 0$. Hence, $\operatorname{cdim}_3(Z) \le \operatorname{cdim}_2(Z)$.

Finally, we can assume that $m = \operatorname{cdim}_3(Z) < \infty$. Then $c_0 = \frac{1}{8} c_3(Z, m) > 0$ and $c_3(Z, m, \delta) \ge 4c_0$ for all sufficiently small $\delta > 0$, $c_{3,\tau}(Z, m, \delta) \ge 2c_0$ for all sufficiently small $\tau > 0$. Thus, there is a Lipschitz map $f : Z \to P$ into an $m$-dimensional uniform polyhedron $P$ with $\delta\tau \le \operatorname{mesh}(f) \le \tau$ and $\operatorname{cap}(f) \ge c_0$. Every $a$-dimensional simplex $\sigma \subset P$ is labeled by its barycenter, which is the vertex $v_\sigma$ in the first barycentric subdivision $\operatorname{ba} P$ of $P$. We view $A = \{0, \dots, m\}$ as the color set, and for $a \in A$ we let $\mathcal{P}^a$ be the family of the open stars $\operatorname{st}_\sigma$ in $\operatorname{ba} P$ for all vertices $v_\sigma$ with $\dim \sigma = a$, $\mathcal{P}^a = \{\operatorname{st}_\sigma : \sigma \subset P, \dim \sigma = a\}$. The family $\mathcal{P}^a$ is disjoint, $\operatorname{st}_\sigma \cap \operatorname{st}_{\sigma'} = \varnothing$ for $\sigma \ne \sigma'$, and $\mathcal{P} = \bigcup_{a \in A} \mathcal{P}^a$ is a covering of the polyhedron $P$. Thus, the open covering $\mathcal{P}$ is $(m+1)$-colored.

Now, $\mathcal{U} = p^{-1}(\mathcal{P})$ is an $(m+1)$-colored open covering of $Z$, $\mathcal{U} = \bigcup_{a \in A} \mathcal{U}^a$, where $\mathcal{U}^a = p^{-1}(\mathcal{P}^a)$. Since the stars $\operatorname{st}_\sigma$ of $\operatorname{ba} P$ are contained in appropriate open stars of $P$, we have $\operatorname{mesh}(\mathcal{U}) \le \operatorname{mesh}(f) \le \tau$. Since the polyhedron $P$ is uniform, there is a lower bound $l_m > 0$ for the Lebesgue number of the covering $\mathcal{P}$. Therefore, for the Lebesgue number of $\mathcal{U}$ we have $L(\mathcal{U}) \ge l_m / \operatorname{Lip}(f)$. This implies $\operatorname{cap}(\mathcal{U}) \ge l_m \operatorname{cap}(f) \ge l_m c_0$ and

$$\operatorname{mesh}(\mathcal{U}) \ge L(\mathcal{U}) \ge l_m \operatorname{cap}(f) \operatorname{mesh}(f) \ge l_m c_0 \delta\tau.$$

Putting everything together, we obtain $c_{1,\tau}(Z, m, l_m c_0 \delta) \ge c_0 > 0$ for every sufficiently small positive $\tau$, $\delta$. Thus, $\operatorname{cdim}_1(Z) \le m = \operatorname{cdim}_3(Z)$. □

From now on, we denote by $\operatorname{cdim} Z$ the common value of the capacity dimensions of $Z$. Clearly, the capacity dimension dominates the topological dimension, $\dim Z \le \operatorname{cdim} Z$. The condition for coverings to be open in the first and second definitions is inessential, and one can define $\operatorname{cdim} Z$ using coverings by arbitrary sets.

The following characterization of the capacity dimension allows us to compare it with the Assouad–Nagata dimension; see [As], [LS].

**Proposition 3.2.** *The capacity dimension of a metric space $Z$ is the infimum of all integers $m$ with the following property: there exists a constant $c > 0$ such that, for all sufficiently small $s > 0$, $Z$ has a $cs$-bounded covering with $s$-multiplicity at most $m + 1$.*

*Proof.* We must prove that $\operatorname{cdim} Z = \operatorname{cdim}' Z$, where $\operatorname{cdim}' Z$ is defined in accordance with the proposition,

$$\operatorname{cdim}' Z + 1 = \lim_{c\to\infty} \limsup_{s\to 0} \inf\{m_s(\mathcal{U}) : \operatorname{mesh}(\mathcal{U}) \le cs\}.$$

Let $m' = \operatorname{cdim}' Z$. Then there are positive numbers $c$ and $s_0$ such that for every $s \in (0, s_0]$ there is a covering $\mathcal{U}$ of $Z$ with $m_s(\mathcal{U}) \le m' + 1$ and $\operatorname{mesh}(\mathcal{U}) \le cs$. Given a covering $\mathcal{U}$ of $Z$ with $\operatorname{mesh}(\mathcal{U}) \le cs$, note that the covering $\mathcal{U}_s = B_s(\mathcal{U})$ is open, $m(\mathcal{U}_s) = m_s(\mathcal{U})$, and

$$s \le L(\mathcal{U}_s) \le \operatorname{mesh}(\mathcal{U}_s) \le \operatorname{mesh}(\mathcal{U}) + 2s \le (c + 2)s.$$

Thus, for the capacity of $\mathcal{U}_s$ we have $\operatorname{cap}(\mathcal{U}_s) \ge 1/(c+2)$. It follows that $c_{2,\tau}(Z, m', \delta) \ge 1/(c + 2)$ for $\tau = (c + 2)s$ and $\delta \in (0, 1/(c + 2))$. Hence, $c_2(Z, m') \ge 1/(c + 2) > 0$ and, therefore, $\operatorname{cdim} Z \le m'$.

Conversely, put $m = \operatorname{cdim} Z$. Then $c_0 = \frac{1}{8}c_2(Z, m) > 0$, $c_2(Z, m, \delta) \ge 4c_0$ for all sufficiently small $\delta > 0$, and $c_{2,\tau}(Z, m, \delta) \ge 2c_0$ for all $\tau$, $0 < \tau \le \tau_0$. This means that there is an open covering $\mathcal{U}$ of $Z$ with $\delta\tau \le \operatorname{mesh}(\mathcal{U}) \le \tau$ and $L(\mathcal{U}) \ge c_0 \operatorname{mesh}(\mathcal{U})$, and with multiplicity at most $m + 1$. Then $s = c_0\delta\tau/2 < L(\mathcal{U})$. By Lemma 2.1, the family $\mathcal{U}_{-s} = B_{-s}(\mathcal{U})$ is a covering of $Z$ with $\operatorname{mesh}(\mathcal{U}_{-s}) \le \operatorname{mesh}(\mathcal{U}) \le \frac{2}{c_0\delta}s$, and for its $s$-multiplicity we have $m_s(\mathcal{U}_{-s}) \le m(\mathcal{U}) \le m + 1$. Fixing a sufficiently small $\delta$ as above and letting $\tau \to 0$, we obtain $\operatorname{cdim}' Z \le m$. □

This characterization immediately implies the following.

**Corollary 3.3.** *The capacity dimension is monotone*: $\operatorname{cdim} Y \le \operatorname{cdim} Z$ *for every* $Y \subset Z$. □

If we omit "sufficiently small" from the statement of Proposition 3.2, then we come up with the Assouad–Nagata dimension of $Z$, $\operatorname{ANdim} Z$. Thus we obtain the following corollary.

**Corollary 3.4.** *For every metric space $Z$ we have* $\operatorname{cdim} Z \le \operatorname{ANdim} Z$. □

The next corollary is a consequence of [LS].

**Corollary 3.5.** *Every compact Riemannian manifold $M$ satisfies* $\operatorname{cdim} M = \dim M$.

Conversely, the Assouad–Nagata dimension can be characterized by the formula

$$\operatorname{ANdim} Z = \inf\{m \ge 0 : c_2'(Z, m) > 0\},$$

where

$$c_2'(Z, m) = \lim_{\delta\to 0} \inf_{\tau>0} c_{2,\tau}(Z, m, \delta).$$

Loosely speaking, the Assouad–Nagata dimension takes into account all scales, while the capacity dimension only all sufficiently small scales, like the topological dimension does.

## §4. Quasisymmetry invariance

The capacity dimension (as well as the Assouad–Nagata dimension) is obviously a bi-Lipschitz invariant. The striking fact discovered in [LS] is that the Assouad–Nagata dimension is a quasisymmetry invariant.

A map $f : X \to Y$ between metric spaces is said to be *quasisymmetric* if it is not constant and there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that from $|xa| \le t|xb|$ it follows that $|f(x)f(a)| \le \eta(t)|f(x)f(b)|$ for any $a$, $b$, $x \in X$ and all $t \ge 0$. In

this case, we say that $f$ is *$\eta$-quasisymmetric.* It is easily seen that $f$ is injective and continuous, and $f^{-1} : f(X) \to X$ is $\eta'$-quasisymmetric with $\eta'(t) = 1/\eta^{-1}(t^{-1})$ for $t > 0$. Furthermore, if $f : X \to Y$ and $g : Y \to Z$ are $\eta_f$- and $\eta_g$-quasisymmetric, respectively, then $g \circ f : X \to Z$ is $(\eta_g \circ \eta_f)$-quasisymmetric. A quasisymmetric homeomorphism is called a *quasisymmetry.* For more details on quasisymmetric maps, see [He].

For example, the transformation $d \mapsto d^p$ of a metric $d$, where $0 < p < 1$, is a quasisymmetry called a *snow-flake transformation* (see, e.g., [BoS]). Such a transformation can be far from being bi-Lipschitz because the nontrivial rectifiable paths with respect to the metric $d$ are nonrectifiable with respect to the metric $d^p$.

**Theorem 4.1.** *The capacity dimension is a quasisymmetry invariant of metric spaces.*

Combining this with Corollary 3.3, we obtain the following.

**Corollary 4.2.** *Assume that there is a quasisymmetric $f : X \to Y$. Then* $\operatorname{cdim} X \le \operatorname{cdim} Y$. □

Using Proposition 3.2, we can refer to [LS, Theorem 1.2] for the proof of Theorem 4.1 because, actually, the same argument works in our case. However, we give a different proof of Theorem 4.1, based on ideas from [LS], as an attempt to understand very nice and concise arguments in [LS].

A key ingredient of the proof of Theorem 4.1 is the existence of a sequence of coverings established in Proposition 4.4 below.

We say that a family $\mathcal{U}$ of sets in a space $X$ is *separated* if different members $U, U' \in \mathcal{U}$ are either disjoint, $U \cap U' = \varnothing$, or one of them is contained in the other. Note that if $\mathcal{U}$ is separated, then $B_{-s}(\mathcal{U})$ is separated for every $s \ge 0$.

Let $\mathcal{U}$, $\mathcal{U}'$ be families of sets in $X$. We denote by $\mathcal{U} * \mathcal{U}'$ the family obtained by taking for every $U \in \mathcal{U}$ the union $V$ of $U$ and all members $U' \in \mathcal{U}'$ that intersect $U$, $\mathcal{U} * \mathcal{U}' = \{V : U \in \mathcal{U}\}$. It is straightforward to check that the following is true.

**Lemma 4.3.** *Let $\mathcal{U}, \widehat{\mathcal{U}}$ be separated families in $X$ such that no member of $\widehat{\mathcal{U}}$ intersects disjoint members of $\mathcal{U}$. Then the family $\mathcal{V} = \mathcal{U} * \widehat{\mathcal{U}} \cup \widehat{\mathcal{U}}$ is separated; moreover, if $\mathcal{U}$ is disjoint, then $\mathcal{U} * \widehat{\mathcal{U}}$ is also disjoint.* □

**Proposition 4.4.** *Suppose that $X$ is a metric space with finite capacity dimension,* $\operatorname{cdim} X \le n$. *Then there are positive constants $c_0$, $\delta$ such that for every sufficiently small $r > 0$ there exists a sequence of open coverings $\mathcal{U}_j$ of $X$, $j \in \mathbb{N}$, with the following properties*:

(i) *for every $j \in \mathbb{N}$, the covering $\mathcal{U}_j$ is $(n+1)$-colored by one and the same color set $A$, $\mathcal{U}_j = \bigcup_{a \in A} \mathcal{U}_j^a$, $|A| = n+1$;*
(ii) *for every $j \in \mathbb{N}$, we have*
$$\delta r^j \le \operatorname{mesh}(\mathcal{U}_j) \le r^j \quad \text{and} \quad L(\mathcal{U}_j) \ge c_0 \operatorname{mesh}(\mathcal{U}_j);$$
(iii) *for every $i > j$, the covering $\mathcal{U}_i$ is inscribed in the covering $\mathcal{U}_j$;*
(iv) *for every $a \in A$, the union $\mathcal{U}^a = \bigcup_{j \in \mathbb{N}} \mathcal{U}_j^a$ is separated.*

*Proof.* The most important property is (iv). The existence of a sequence of coverings $\widehat{\mathcal{U}}_j$, $j \in \mathbb{N}$, of $X$ satisfying (i)–(iii) is an easy consequence of the condition $\operatorname{cdim} X \le n$ and the definition of cdim. Namely, as in the proof of Proposition 3.2, we have $c_0' = \frac{1}{8} c_1(X, n) > 0$, $c_1(X, n, \delta') \ge 4c_0'$ for all sufficiently small $\delta' > 0$. We fix such a number $\delta'$, and note that $c_{1,\tau}(X, n, \delta') \ge 2c_0'$ for all $\tau$, $0 < \tau \le \tau_0$. This means that for every $\tau \in (0, \tau_0]$ there is an open $(n+1)$-colored covering $\mathcal{U}_\tau$ of $X$ with $\delta'\tau \le \operatorname{mesh}(\mathcal{U}_\tau) \le \tau$ and the capacity arbitrarily close to $c_{1,\tau}(X, n, \delta')$, in particular, $L(\mathcal{U}_\tau) \ge c_0' \operatorname{mesh}(\mathcal{U}_\tau)$.

We take a positive $r < \min\{c_0'\delta'/4, \tau_0\}$, and for every $j \in \mathbb{N}$ consider the covering $\widehat{\mathcal{U}}_j = \mathcal{U}_{\tau_j}$, where $\tau_j = r^j$. Then the sequence $\widehat{\mathcal{U}}_j$, $j \in \mathbb{N}$, of open coverings of $X$ satisfies conditions (i) and (ii) (with $\delta = \delta'$ and $c_0 = c_0'$). Since $L(\widehat{\mathcal{U}}_j) \geq c_0'\delta' r^j > r^{j+1} \geq \operatorname{mesh}(\widehat{\mathcal{U}}_{j+1})$, the covering $\widehat{\mathcal{U}}_{j+1}$ is inscribed in $\widehat{\mathcal{U}}_j$, so that (iii) is also satisfied.

Now, we modify the sequence $\{\widehat{\mathcal{U}}_j\}$ so as to keep properties (i)–(iii) preserved and to obtain (iv). This can be done, e.g., as in [LS, Proposition 4.1]. We take another track to show a different possibility.

For $k \in \mathbb{N}$, we put $s_k = c_0'\delta' r^k/4$. Fix a color $a \in A$ and define $\mathcal{V}_1^a = \mathcal{U}_{1,1}^a := \widehat{\mathcal{U}}_1^a$. Assume that for $k \geq 1$ the family $\mathcal{V}_k^a$ is already defined, it is separated, and $\mathcal{V}_k^a = \bigcup_{j=1}^k \mathcal{U}_{j,k}^a$, where each family $\mathcal{U}_{j,k}^a$ is disjoint. Now, we define

$$\mathcal{V}_{k+1}^a := B_{-s_k}(\mathcal{V}_k^a) * \widehat{\mathcal{U}}_{k+1}^a \cup \widehat{\mathcal{U}}_{k+1}^a.$$

Then $\mathcal{V}_{k+1}^a = \bigcup_{j=1}^{k+1} \mathcal{U}_{j,k+1}^a$, where $\mathcal{U}_{j,k+1}^a = B_{-s_k}(\mathcal{U}_{j,k}^a) * \widehat{\mathcal{U}}_{k+1}^a$ for $1 \leq j \leq k$ and $\mathcal{U}_{k+1,k+1}^a = \widehat{\mathcal{U}}_{k+1}^a$. Since $\operatorname{mesh}(\widehat{\mathcal{U}}_{k+1}^a) \leq \operatorname{mesh}(\widehat{\mathcal{U}}_{k+1}) \leq r^{k+1} < s_k$, no member of $\widehat{\mathcal{U}}_{k+1}^a$ intersects disjoint members of $B_{-s_k}(\mathcal{V}_k^a)$. Then, by Lemma 4.3, the family $\mathcal{V}_{k+1}^a$ is separated and the family $\mathcal{U}_{j,k+1}^a$ is disjoint for every $1 \leq j \leq k+1$.

From the definition of $\mathcal{U}_{j,k+1}^a = B_{-s_k}(\mathcal{U}_{j,k}^a) * \widehat{\mathcal{U}}_{k+1}^a$ and the fact that $r^{k+1} < s_k$ it follows that, for every $k \geq 1$, $1 \leq j \leq k$, every member $U \in \mathcal{U}_{j,k+1}^a$ is contained in a unique member $U' \in \mathcal{U}_{j,k}^a$. In this sense, for every $j \in \mathbb{N}$ the sequence of families $\mathcal{U}_{j,k}^a$, $k \geq j$, is monotone, $\mathcal{U}_{j,k}^a \supset \mathcal{U}_{j,k+1}^a$, and the intersection $\bigcap_{k \geq j} \mathcal{U}_{j,k}^a$ is defined in an obvious sense.

We put $\widehat{s}_j = \sum_{k \geq j} s_k = c_0'\delta' r^j/4(1-r)$. Then $\widehat{s}_j < c_0'\delta' r^j \leq L(\widehat{\mathcal{U}}_j)$ for every $j \geq 1$. By Lemma 2.1, the family $\widehat{\mathcal{U}}_j' = B_{-\widehat{s}_j}(\widehat{\mathcal{U}}_j)$ is still an open covering of $X$, and $B_{-\widehat{s}_j}(\widehat{\mathcal{U}}_j^a) \subset \bigcap_{k \geq j} \mathcal{U}_{j,k}^a$ for every $a \in A$, $j \in \mathbb{N}$. Now, for the interior $\mathcal{U}_j^a = \operatorname{Int}(\bigcap_{k \geq j} \mathcal{U}_{j,k}^a)$, the family $\mathcal{U}_j = \bigcup_{a \in A} \mathcal{U}_j^a$ is an open $(n+1)$-colored covering of $X$ inscribed in $\widehat{\mathcal{U}}_j$ for every $j \in \mathbb{N}$. Then $\operatorname{mesh}(\mathcal{U}_j) \leq \operatorname{mesh}(\widehat{\mathcal{U}}_j) \leq r^j$.

Since $r < 1/2$, we have $\widehat{s}_j = c_0'\delta' r^j/4(1-r) \leq c_0'\delta' r^j/2$ for every $j \in \mathbb{N}$. The covering $\widehat{\mathcal{U}}_j'$ is inscribed in $\mathcal{U}_j$; thus $L(\mathcal{U}_j) \geq L(\widehat{\mathcal{U}}_j') \geq c_0'\delta' r^j - \widehat{s}_j \geq c_0'\delta' r^j/2$. We put $\delta := c_0'\delta'/2 =: c_0$. Then $\operatorname{mesh}(\mathcal{U}_j) \geq L(\mathcal{U}_j) \geq \delta r^j$ and $L(\mathcal{U}_j) \geq c_0 \operatorname{mesh}(\mathcal{U}_j)$. Since $L(\mathcal{U}_j) > r^{j+1} \geq \operatorname{mesh}(\mathcal{U}_{j+1})$, the covering $\mathcal{U}_{j+1}$ is inscribed in $\mathcal{U}_j$ for every $j \in \mathbb{N}$. Therefore, the sequence of open coverings $\mathcal{U}_j$, $j \in \mathbb{N}$, satisfies (i)–(iii). For every $a \in A$, $k \geq 1$, the family $\mathcal{V}_k^a$ is separated. It follows that the family $\mathcal{U}^a = \bigcup_{j \in \mathbb{N}} \mathcal{U}_j^a$ is also separated, so that (iv) is also true. □

We define the *local capacity* of an open covering $\mathcal{U}$ of a metric space $Z$ by

$$\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}) = \inf_{z \in Z} \frac{L(\mathcal{U}, z)}{\operatorname{mesh}(\mathcal{U}, z)}.$$

Clearly, $1 \geq \operatorname{cap}_{\mathrm{loc}}(\mathcal{U}) \geq \operatorname{cap}(\mathcal{U})$. The advantage of the local capacity over the capacity is that its positivity is preserved under quasisymmetries quantitatively; see Lemma 4.5. This implies that a dimension defined exactly as the capacity dimension but with replacement of the capacity of coverings by the local capacity is a quasisymmetry invariant. However, that invariant is not as good as the capacity dimension for applications. We use the local capacity of coverings only as an auxiliary tool to prove quasisymmetry invariance for the capacity dimension.

**Lemma 4.5.** *Let $\mathcal{U}$ be an open covering of a metric space $Z$, $f : Z \to Z'$ an $\eta$-quasisymmetry, $\mathcal{U}' = f(\mathcal{U})$ the image of $\mathcal{U}$. Then for the local capacities of $\mathcal{U}$ and*

*$\mathcal{U}'$ we have*

$$\frac{1}{\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}')} \le 16\eta\left(\frac{2}{\operatorname{cap}_{\mathrm{loc}}(\mathcal{U})}\right).$$

*Proof.* We may assume that $\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}) > 0$. We fix $z \in Z$ and consider $U \in \mathcal{U}$ for which $z \in U$ and $\operatorname{dist}(z, Z \setminus U) \ge L(\mathcal{U}, z)/2$. For $z' = f(z)$ and $U' = f(U)$, there is $a' \in Z' \setminus U'$ with $|z'a'| \le 2\operatorname{dist}(z', Z' \setminus U')$. Then $|z'a'| \le 2L(\mathcal{U}', z')$, and for $a = f^{-1}(a')$ we have $|za| \ge \operatorname{dist}(z, Z \setminus U) \ge L(\mathcal{U}, z)/2$.

Similarly, consider $V' \in \mathcal{U}'$ with $z' \in V'$ and $\operatorname{diam} V' \ge \operatorname{mesh}(\mathcal{U}', z')/2$. There is $b' \in V'$ with $|z'b'| \ge \operatorname{diam} V'/4$. Then $|z'b'| \ge \operatorname{mesh}(\mathcal{U}', z')/8$, and for $b := f^{-1}(b')$ we have $|zb| \le \operatorname{mesh}(\mathcal{U}, z)$. Therefore,

$$\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}) \le \frac{L(\mathcal{U}, z)}{\operatorname{mesh}(\mathcal{U}, z)} \le \frac{2|za|}{|zb|} \quad \text{and} \quad |zb| \le t|za|$$

with $t = 2/\operatorname{cap}_{\mathrm{loc}}(\mathcal{U})$. It follows that $|z'b'| \le \eta(t)|z'a'|$ and

$$\frac{L(\mathcal{U}', z')}{\operatorname{mesh}(\mathcal{U}', z')} \ge \frac{|z'a'|}{16|z'b'|} \ge (16\eta(t))^{-1}$$

for every $z' \in Z'$. Then $\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}') \ge \left(16\eta\left(\frac{2}{\operatorname{cap}_{\mathrm{loc}}(\mathcal{U})}\right)\right)^{-1}$. □

A covering $\mathcal{U}$ of $Z$ is said to be *$c$-balanced*, $c > 0$, if $\inf\{\operatorname{diam}(U) : U \in \mathcal{U}\} \ge c \cdot \operatorname{mesh}(\mathcal{U})$. The notion of a balanced covering combined with the local capacity allows us to estimate the capacity of a covering from below as follows.

**Lemma 4.6.** *If an open covering $\mathcal{U}$ of a metric space $Z$ is $c_1$-balanced and its local capacity satisfies $\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}) \ge c_0$, then $\operatorname{cap}(\mathcal{U}) \ge c_0 \cdot c_1$.*

*Proof.* Since $\mathcal{U}$ is $c_1$-balanced, we have $\operatorname{mesh}(\mathcal{U}, z) \ge c_1 \cdot \operatorname{mesh}(\mathcal{U})$ for every $z \in Z$. Since $\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}) \ge c_0$, we have $L(\mathcal{U}, z) \ge c_0 \cdot \operatorname{mesh}(\mathcal{U}, z)$ for every $z \in Z$. Therefore, $L(\mathcal{U}) \ge c_0 c_1 \cdot \operatorname{mesh}(\mathcal{U})$. □

Let $f : X \to Y$ be a quasisymmetry. To prove Theorem 4.1, it suffices to show that $\operatorname{cdim} Y \le \operatorname{cdim} X$. The idea is that, starting with a sequence $\mathcal{U}_j$, $j \in \mathbb{N}$, of coverings of $X$ as in Proposition 4.4, we shall construct a covering $\mathcal{V}$ of $X$ with local capacity uniformly separated from 0 (see Lemma 4.9) and such that its image $f(\mathcal{V})$ has an arbitrarily small mesh and is balanced (see Lemma 4.10). Then, by Lemma 4.6 combined with Lemma 4.5, the capacity of the covering $f(\mathcal{V})$ of $Y$ is bounded away from zero, which implies $\operatorname{cdim} Y \le \operatorname{cdim} X$.

Fix a sufficiently small $r > 0$ and consider the sequence of open coverings $\mathcal{U}_j$ of $X$ as in Proposition 4.4. We can assume additionally that $\operatorname{diam} U \ge L(\mathcal{U}_j)$ for every $U \in \mathcal{U}_j$, because if $\operatorname{diam} U < L(\mathcal{U}_j)$, then $U$ is contained in another member $U' \in \mathcal{U}_j$ and, thus, it can be deleted from $\mathcal{U}_j$ without destroying any property among (i)–(iv).

As in [LS], we put $\mathcal{U} = \bigcup_{j \in \mathbb{N}} \mathcal{U}_j$ and for $s > 0$ consider the family $\mathcal{U}(s) = \{U \in \mathcal{U} : \operatorname{diam} f(U) \le s\}$.

**Lemma 4.7.** *For every $s > 0$ the family $\mathcal{U}(s)$ is a covering of $X$.*

*Proof.* We fix $x \in X$, consider $x' \in X$ different from $x$, and put $y = f(x)$, $y' = f(x')$. For every $j \in \mathbb{N}$ there is $U_j \in \mathcal{U}_j$ containing $x$. Take $y_j \in f(U_j)$ with $\operatorname{diam} f(U_j) \le 4|yy_j|$ and consider $x_j = f^{-1}(y_j)$. Then $|xx_j| \le t_j|xx'|$ with $t_j \to 0$ as $j \to \infty$, because $\operatorname{diam} U_j \le r^j \to 0$. Therefore, $\operatorname{diam} f(U_j) \le 4|yy_j| \le 4\eta(t_j)|yy'| \le s$ for sufficiently large $j$. Hence, $U_j \in \mathcal{U}(s)$. □

A family $\mathcal{V} \subset \mathcal{U}(s)$ is *minimal* if every $U \in \mathcal{U}(s)$ is contained in some $V \in \mathcal{V}$ and no two different $V$, $V' \in \mathcal{V}$ are contained one in the other.

**Lemma 4.8.** *For every $s > 0$ there is a minimal family $\mathcal{V} \subset \mathcal{U}(s)$. Every minimal family $\mathcal{V} \subset \mathcal{U}(s)$ is an $(n+1)$-colored covering of $X$.*

*Proof.* Given $s > 0$, we construct a family $\mathcal{V} \subset \mathcal{U}(s)$ by deleting every $U \in \mathcal{U}(s)$ that is contained in some other $U' \in \mathcal{U}(s)$. Now, $\mathcal{V}$ is what remains. We only need to check that for every $U \in \mathcal{U}(s)$ there is a maximal $U' \in \mathcal{U}(s)$ with $U \subset U'$. From Proposition 4.4(i) it follows that for every $j \in \mathbb{N}$ there are only finitely many $U' \in \mathcal{U}_j$ containing $U$ (because all of them must have different colors). Since $\operatorname{mesh}(\mathcal{U}_j) \to 0$ as $j \to \infty$, there are only finitely many $U' \in \mathcal{U}(s)$ containing $U$ and hence there is a maximal $U' \in \mathcal{U}(s)$ among them.

Let $\mathcal{V} \subset \mathcal{U}(s)$ be a minimal family. By Lemma 4.7, the family $\mathcal{U}(s)$ is a covering of $X$, and the definition of a minimal family implies that $\mathcal{V}$ is also a covering of $X$. Proposition 4.4(iv) shows that different $V$, $V' \in \mathcal{V}$ having one and the same color are disjoint. Thus, $\mathcal{V}$ is $(n+1)$-colored. □

**Lemma 4.9.** *There is a constant $\nu > 0$ depending only on $c_0$, $\delta$, $r$, and $\eta$ and such that for every $s > 0$ every minimal covering $\mathcal{V} \subset \mathcal{U}(s)$ satisfies $\operatorname{cap}_{\mathrm{loc}}(\mathcal{V}) \geq \nu$.*

*Proof.* Let $\mathcal{V} \subset \mathcal{U}(s)$ be a minimal family. Given $x \in X$, we put $j = j(x) = \min\{i \in \mathbb{N} : x \in V \in \mathcal{V} \cap \mathcal{U}_i\}$. Then $\operatorname{mesh}(\mathcal{V}, x) \leq r^j$. We fix $V \in \mathcal{V} \cap \mathcal{U}_j$ containing $x$, $v \in V$ with $4|xv| \geq \operatorname{diam} V$ and note that $\operatorname{diam} V \geq L(\mathcal{U}_j) \geq c_0 \delta r^j$ by our assumptions.

Furthermore, we fix $\mu > 0$ with $4\eta(4\mu/c_0\delta) \leq 1$. Now we check that for $i \in \mathbb{N}$ with $r^{i-j} \leq \mu$ every $U \in \mathcal{U}_i$ containing $x$ is a member of $\mathcal{U}(s)$. There is $u \in U$ with $4|f(x)f(u)| \geq \operatorname{diam} f(U)$. We have $|xu| \leq t|xv|$ for some $t \leq 4 \operatorname{diam} U / \operatorname{diam} V \leq 4r^{i-j}/c_0\delta \leq 4\mu/c_0\delta$. Then $\operatorname{diam} f(U) \leq 4|f(x)f(u)| \leq 4\eta(4\mu/c_0\delta)|f(x)f(v)| \leq \operatorname{diam} V \leq s$; thus $U \in \mathcal{U}(s)$.

Therefore, $L(\mathcal{V}, x) \geq L(\mathcal{U}_i) \geq c_0 \delta r^i$. Assuming that $i$ is taken to be minimal with $r^{i-j} \leq \mu$, we obtain $L(\mathcal{V}, x) \geq c_0 \delta \mu r^{j+1}$. Thus, $\frac{L(\mathcal{V},x)}{\operatorname{mesh}(\mathcal{V},x)} \geq \nu = c_0 \delta \mu r$ for every $x \in X$, and $\operatorname{cap}_{\mathrm{loc}}(\mathcal{V}) \geq \nu$. □

**Lemma 4.10.** *If $\mathcal{V} \subset \mathcal{U}(s)$ is a minimal family, then the $(n+1)$-colored covering $\mathcal{W} = f(\mathcal{V})$ of $Y$ satisfies $\operatorname{diam} W \geq s/4\eta(t)$ for every $W \in \mathcal{W}$, where $t = 4/c_0\delta r$. In particular, $\operatorname{mesh}(\mathcal{W}) \geq s/4\eta(t)$, and $\mathcal{W}$ is $c$-balanced with $c \geq 1/4\eta(t)$.*

*Proof.* Note that $\operatorname{mesh}(\mathcal{W}) \leq s$ by the definition of $\mathcal{U}(s)$. Take any $W \in \mathcal{W}$ and consider $V = f^{-1}(W)$. We may assume that $V \in \mathcal{U}_j$ for some $j \in \mathbb{N}$. Then $\operatorname{diam} V \geq L(\mathcal{U}_j) \geq c_0 \delta r^j$ by our assumption on the sequence $\{\mathcal{U}_j\}$.

For any $U \in \mathcal{U}$ with $V \subset U$, we have $\operatorname{diam} f(U) > s$ because the family $\mathcal{V}$ is minimal. Since the covering $\mathcal{U}_j$ is inscribed in $\mathcal{U}_{j-1}$, there is $U \in \mathcal{U}_{j-1}$ containing $V$; in particular, $\operatorname{diam} f(U) > s$.

Take $y \in W \subset f(U)$. There is $y' \in f(U)$ with $|yy'| \geq \operatorname{diam} f(U)/4 > s/4$. For $x = f^{-1}(y)$, $x' = f^{-1}(y')$ we have $|xx'| \leq \operatorname{diam} U \leq \operatorname{mesh}(\mathcal{U}_{j-1}) \leq r^{j-1}$. There is $v \in V$ with $|xv| \geq \operatorname{diam} V/4 \geq c_0 \delta r^j/4$. Thus, $|xx'| \leq r^{j-1} \leq t|xv|$ for $t = 4/c_0\delta r$. For $w = f(v) \in W$ we obtain $|yy'| \leq \eta(t)|yw| \leq \eta(t) \operatorname{diam} W$. Hence, $\operatorname{diam} W \geq s/4\eta(t)$. □

*Proof of Theorem* 4.1. Let $f : X \to Y$ be an $\eta$-quasisymmetry. We show that $\operatorname{cdim} Y \leq n$ for every $n \geq \operatorname{cdim} X$. Fix a sufficiently small $r > 0$ and consider a sequence $\mathcal{U}_j$, $j \in \mathbb{N}$, of coverings of $X$ as in Proposition 4.4 with positive $c_0$, $\delta$. Then, by Lemmas 4.8 and 4.10, for every $s > 0$ we have an open $(n+1)$-colored covering $\mathcal{W}$ of $Y$ with $s/4\eta(t) \leq \operatorname{mesh}(\mathcal{W}) \leq s$, which is $c$-balanced, $c \geq 1/4\eta(t)$, where $t = 4/c_0\delta r$. Moreover, by Lemmas 4.9 and 4.5, its local capacity $\operatorname{cap}_{\mathrm{loc}}(\mathcal{W})$ is at least $d$, where the constant $d > 0$ depends only on $\eta$, $c_0$, $\delta$, and $r$. Then by Lemma 4.6 we have $\operatorname{cap}(\mathcal{W}) \geq c \cdot d$ independently of $s$.

This shows that $c_{1,s}(Y,n,\delta') \geq c \cdot d$ for every $s > 0$, where $\delta' = 1/4\eta(t)$. Hence, $c_1(Y,n) \geq c \cdot d > 0$ and $\operatorname{cdim} Y \leq n$. $\square$

## §5. Asymptotic dimension of a hyperbolic cone

Let $Z$ be a bounded metric space. Assuming that $\operatorname{diam} Z > 0$, we put $\mu = \pi/\operatorname{diam} Z$ and note that $\mu|zz'| \in [0,\pi]$ for every $z$, $z' \in Z$. Recall that the hyperbolic cone $\operatorname{Co}(Z)$ over $Z$ is the space $Z \times [0,\infty)/Z \times \{0\}$ with metric defined as follows. Given $x = (z,t)$, $x' = (z',t') \in \operatorname{Co}(Z)$, we consider a triangle $\overline{o}\,\overline{x}\,\overline{x}' \subset \mathrm{H}^2$ with $|\overline{o}\,\overline{x}| = t$, $|\overline{o}\,\overline{x}'| = t'$ and with the angle $\angle_{\overline{o}}(\overline{x},\overline{x}') = \mu|zz'|$. Now, we put $|xx'| := |\overline{x}\,\overline{x}'|$. In the degenerate case $Z = \{\mathrm{pt}\}$ we define $\operatorname{Co}(Z) = \{\mathrm{pt}\} \times [0,\infty)$ as the metric product. The point $o = Z \times \{0\} \in \operatorname{Co}(Z)$ is called the *vertex* of $\operatorname{Co}(Z)$.

**Theorem 5.1.** *For every bounded metric space $Z$ we have*

$$\operatorname{asdim} \operatorname{Co}(Z) \leq \operatorname{cdim} Z + 1.$$

The proof occupies Subsections 5.1–5.5. In Subsections 5.3–5.5 our arguments are close to those in [BD, §2].

**5.1. Some estimates from hyperbolic geometry.** We denote by $Z_t$ the metric sphere of radius $t > 0$ around $o$ in $\operatorname{Co}(Z)$. There are natural polar coordinates $x = (z,t)$, $z \in Z$, $t \geq 0$, in $\operatorname{Co}(Z)$. Then $Z_t = \{(z,t) : z \in Z\}$ is the copy of $Z$ at the level $t$. For $t > 0$, we denote by $\pi_t : Z_t \to Z$ the canonical homeomorphism, $\pi_t(z,t) = z$.

Let $\mathcal{U}$ be an open covering of $Z$ with multiplicity $m+1$ and positive Lebesgue number $L(\mathcal{U})$. Let $\mathcal{N} = \mathcal{N}(\mathcal{U})$ be the nerve of $\mathcal{U}$, $p : Z \to \mathcal{N}$ the barycentric map. Then $\operatorname{Lip}(p) \leq \frac{(m+2)^2}{L(\mathcal{U})}$ (see Subsection 2.3). For every $t > 0$ consider the induced covering $\mathcal{U}_t = \pi_t^{-1}(\mathcal{U})$ of $Z_t$ whose nerve is canonically isomorphic to $\mathcal{N}$, and consider the corresponding barycentric map $p_t : \mathcal{U}_t \to \mathcal{N}$.

Given $\lambda > 0$, we want to find $t > 0$ and conditions on $\mathcal{U}$ ensuring that $\operatorname{Lip}(p_t) \leq \lambda$ and that $p_t$ is still uniformly cobounded with respect to the metric induced from $\operatorname{Co}(Z)$. To this end, first we recall the hyperbolic cosine law. For $t > 0$ and $\alpha \in [0,\pi]$, we define $a = a(t,\alpha)$ by

$$\cosh a = \cosh^2(t) - \sinh^2(t)\cos\alpha,$$

i.e., $a$ is the length of the base opposite to the vertex $o$ with angle $\alpha$ of an isosceles triangle in $\mathrm{H}^2$ with sides $t$. Then for $\alpha$ sufficiently small, we have

$$\cosh a = 1 + \frac{1}{2}\sinh^2(t)\alpha^2 + \sinh^2(t) \cdot o(\alpha^3).$$

Assume that small $\lambda$, $\sigma > 0$ are fixed so that $d := \frac{(m+2)^2}{\lambda} - \ln\frac{1}{\sigma} = \frac{(m+2)^2}{2\lambda}$, and

$$\sigma\tau \leq \mu L(\mathcal{U}) \leq \mu \operatorname{mesh}(\mathcal{U}) \leq \tau$$

for sufficiently small $\tau$. We put

$$t_\tau = \ln\frac{2}{\tau} + \frac{2(m+2)^2}{\lambda}.$$

Then $t_\tau - 2d = \ln\frac{2}{\sigma^2\tau}$, and for $t_\tau - 2d \leq t \leq t_\tau$ we have

$$\sinh^2(t)(\sigma\tau)^2 \simeq \frac{1}{4}e^{2t}(\sigma\tau)^2 \geq \frac{1}{\sigma^2} = \exp\left(\frac{(m+2)^2}{\lambda}\right) \gg 1,$$

while

$$\sinh^2(t) \cdot o(\tau^3) \leq o(\tau) \cdot \exp\left(\frac{4(m+2)^2}{\lambda}\right) \ll 1.$$

Observing that $L(\mathcal{U}_t) = a(t, \mu L(\mathcal{U}))$, we obtain

$$\cosh(L(\mathcal{U}_t)) \geq 1 + \frac{1}{2}\sinh^2(t)(\sigma\tau)^2 \geq 1 + \frac{1}{2}\exp\left(\frac{(m+2)^2}{\lambda}\right)$$

up to a negligible error. Hence $L(\mathcal{U}_t) > \frac{(m+2)^2}{\lambda}$ and $\mathrm{Lip}(p_t) \leq \frac{(m+2)^2}{L(\mathcal{U}_t)} < \lambda$.

Similarly, $\mathrm{mesh}(\mathcal{U}_t) = a(t, \mu\,\mathrm{mesh}(\mathcal{U}))$, and, as above, for $t_\tau - 2d \leq t \leq t_\tau$ we obtain

$$\cosh(\mathrm{mesh}(\mathcal{U}_t)) \leq 1 + \frac{1}{2}\sinh^2(t)\tau^2 \simeq 1 + \frac{1}{8}e^{2t}\tau^2 \leq 1 + \frac{1}{2}\exp\left(\frac{4(m+2)^2}{\lambda}\right),$$

which gives an upper bound for $\mathrm{mesh}(\mathcal{U}_t)$ depending only on $\lambda$.

**5.2. Čech approximation.** Here we construct a sequence $\{\mathcal{U}_k\}$ of coverings and associated barycentric maps to be used in the proof of Theorem 5.1. We may assume that the capacity dimension of $Z$ is finite, $m = \mathrm{cdim}\, Z < \infty$. Then $c_0 = \frac{1}{8}c_2(Z, m) > 0$, and $c_2(Z, m, \delta) \geq 4c_0$ for all sufficiently small $\delta > 0$ (see §3). Given $\lambda > 0$, we take $\delta > 0$ so that

$$d := \frac{(m+2)^2}{\lambda} + \ln(c_0\delta) = \frac{(m+2)^2}{2\lambda},$$

assuming that $\lambda$ is sufficiently small to satisfy $c_2(Z, m, \delta) \geq 4c_0$. Then $c_{2,\tau}(Z, m, \delta) \geq 2c_0$ for all $\tau$, $0 < \tau \leq \tau_0$. Consider the sequence $\tau_k$, $k \geq 0$, defined recursively by $\tau_{k+1} = e^{-2d}\tau_k$. Recall that $\mu = \pi/\,\mathrm{diam}\, Z$. Then the second definition of $\mathrm{cdim}\, Z$ shows that for every $k \geq 0$ there is an open covering $\widehat{\mathcal{U}}_k$ of $Z$ with

(i) $m(\widehat{\mathcal{U}}_k) \leq m + 1$;
(ii) $\delta\tau_k \leq \mu\,\mathrm{mesh}(\widehat{\mathcal{U}}_k) \leq \tau_k$ and $L(\widehat{\mathcal{U}}_k) \geq c_0\,\mathrm{mesh}(\widehat{\mathcal{U}}_k)$.

Since $\mu L(\widehat{\mathcal{U}}_k) \geq c_0\delta\tau_k > \tau_{k+1} \geq \mu\,\mathrm{mesh}(\widehat{\mathcal{U}}_{k+1})$, we additionally have

(iii) $\widehat{\mathcal{U}}_{k+1}$ is inscribed in $\widehat{\mathcal{U}}_k$ for every $k$

(cf. Proposition 4.4). We put

$$t_k = \ln\frac{2}{\tau_k} + \frac{2(m+2)^2}{\lambda},$$

$Z_k = Z_{t_k} \subset \mathrm{Co}(Z)$, and for every $t > 0$ and every integer $k \geq 0$ consider the covering $\mathcal{U}_{t,k} = \pi_t^{-1}(\widehat{\mathcal{U}}_k)$ of $Z_t$. Note that its nerve is independent of $t > 0$ and can be identified with $\mathcal{N}_k = \mathcal{N}(\widehat{\mathcal{U}}_k)$. Then $t_k - t_{k-1} = 2d$, and, using the estimates from Subsection 5.1 with $\tau = \tau_k$ and $\sigma = c_0\delta$, for the barycentric map $p_{t,k} : Z_t \to \mathcal{N}_k$ associated with the covering $\mathcal{U}_{t,k}$ we see that $\mathrm{Lip}(p_{t,k}) < \lambda$ for all $t_{k-1} \leq t \leq t_k$ and all $k \geq 1$. We put $\mathcal{U}_k = \mathcal{U}_{t_k,k}$ and $p_k = p_{t_k,k} : Z_k \to \mathcal{N}_k$. Then $\mathrm{Lip}(p_k) < \lambda$. Furthermore, $\mathrm{mesh}(\mathcal{U}_k)$ is bounded above by a constant depending only on $\lambda$, $\mathrm{mesh}(\mathcal{U}_k) \leq \mathrm{const}(\lambda)$, for all $k$. Hence, the preimages of all simplexes in $\mathcal{N}_k$ under $p_k$ have uniformly bounded diameter not exceeding $\mathrm{const}(\lambda)$ independently of $k$.

**5.3. Homotopy between $p_k$ and $\rho_k \circ p_{k+1}$.** By (iii), for every $k$ there is a simplicial map $\rho_k : \mathcal{N}_{k+1} \to \mathcal{N}_k$ such that $\rho_k \circ p_{k+1}(z, t_{k+1})$ lies in a face of the minimal simplex containing $p_k(z, t_k) \in \mathcal{N}_k$ for every $z \in Z$.

**Lemma 5.2.** *For every $k$, the map $\rho_k : \mathcal{N}_{k+1} \to \mathcal{N}_k$ is $c_1$-Lipschitz with $c_1 = c_1(m)$ depending only on $m$.*

*Proof.* Recall that the nerve $\mathcal{N}_k$ is a uniform polyhedron, and that $\rho_k$ is affine on every simplex, sending it either to an isometric copy in $\mathcal{N}_k$ or to a face of it. In either case, $\rho_k$ is $c_1$-Lipschitz on every simplex. If $x$, $x' \in \mathcal{N}_{k+1}$ are in disjoint simplexes, then $|x - x'| \geq \frac{\sqrt{2}}{c_1}$ for some $c_1 = c_1(m) > 0$, and $|\rho_k(x) - \rho_k(x')| \leq \sqrt{2}$. The remaining case,

when $x$, $x'$ are sitting in different simplexes in $\mathcal{N}_{k+1}$ with a common face, is left to the reader as an exercise. $\square$

Consider the annulus $A_k \subset \mathrm{Co}(Z)$ between $Z_k$ and $Z_{t_k+d}$, $A_k = Z \times [t_k, t_k+d]$ in polar coordinates. We put $s = s_k(t) = \frac{1}{d}(t - t_k)$ for $t_k \le t \le t_k + d$ and define a homotopy $h_k : A_k \to \mathcal{N}_k \times [0,1]$ between $p_k$ and $\rho_k \circ p_{k+1}$ by

$$h_k(z,t) = \big((1-s)p_k(z,t_k) + s\rho_k \circ p_{k+1}(z,t_{k+1}), s\big).$$

This is well defined because the points $p_k(z,t_k)$ and $\rho_k \circ p_{k+1}(z,t_{k+1})$, $z \in Z$, can be joined by a segment in an appropriate simplex.

**Lemma 5.3.** *The map $h_k$ is $c_2\lambda$-Lipschitz with respect to the product metric on $\mathcal{N}_k \times [0,1]$ for some constant $c_2 = c_2(m) > 0$ depending only on $m$.*

*Proof.* By the convexity of the distance function in Euclidean space, the distance between $h_k(z,t)$, $h_k(z',t) \in \mathcal{N}_k \times \{s\}$ is bounded above by the maximum of the distances between the endpoints of the vertical segments $z \times [0,1]$, $z' \times [0,1]$; therefore,

$$|h_k(x_t) - h_k(x'_t)| \le \max\{|p_k(x_k) - p_k(x'_k)|, |\rho_k \circ p_{k+1}(x_{k+1}) - \rho_k \circ p_{k+1}(x'_{k+1})|\},$$

where $t_k \le t \le t_k + d$, $x_t = (z,t)$, $x'_t = (z',t)$, $x_k = x_{t_k}$, $x'_k = x'_{t_k}$. On the other hand,

$$|p_k(x_k) - p_k(x'_k)| \le \mathrm{Lip}(p_k)|x_k x'_k| \le \lambda |x_t x'_t|,$$

and by Lemma 5.2 we have

$$\begin{aligned}|\rho_k \circ p_{k+1}(x_{k+1}) - \rho_k \circ p_{k+1}(x'_{k+1})| &= |\rho_k \circ p_{t,k+1}(x_t) - \rho_k \circ p_{t,k+1}(x'_t)| \\ &\le c_1 \,\mathrm{Lip}(p_{t,k+1})|x_t x'_t| \le c_1\lambda|x_t x'_t|\end{aligned}$$

for $t_k \le t \le t_k + d$. Furthermore, since every edge of any standard simplex has length $\sqrt{2}$, we have

$$|h_k(z',t) - h_k(z',t')| \le \sqrt{3}|s - s'| = \frac{\sqrt{3}}{d}|(z',t)(z',t')|,$$

where $s' = s_k(t')$. Since

$$|(z,t)(z',t')| \ge \max\{|x_t x'_t|, |(z',t)(z',t')|\}$$

(if $t' \ge t$) and $d = \frac{(m+2)^2}{2\lambda}$, we see that $\mathrm{Lip}(h_k) \le c_2\lambda$. $\square$

**5.4. Simplicial mapping cylinder of $\rho_k$.** We consider the annulus $B_k \subset \mathrm{Co}(Z)$ between $Z_{t_k+d}$ and $Z_{k+1}$, $B_k = Z \times [t_k + d, t_{k+1}]$ in polar coordinates, and define $g_k : B_k \to \mathcal{N}_{k+1} \times [0,1]$ as follows: $g_k(z, t_k + d) = (p_{k+1}(z,t_{k+1}), 0)$, $g_k(z,t_{k+1}) = (p_{k+1}(z,t_{k+1}), 1)$, and $g_k$ is affine on every segment $z \times [t_k + d, t_{k+1}] \subset B_k$, $z \in Z$.

Since $\frac{1}{d} < \lambda$, the estimates in Subsection 5.2 show immediately that $g_k$ is $\lambda$-Lipschitz (with respect to the product metric on $\mathcal{N}_{k+1} \times [0,1]$).

Next, we recall the notion of the simplicial mapping cylinder for a simplicial map $\rho : K \to L$ of simplicial complexes (see, e.g., [Sp]). Assuming that the vertices of $K$ are linearly ordered, we introduce the mapping cylinder $C_\rho$ of $\rho$ as a simplicial complex whose vertex set is the union of the vertex sets of $K$ and $L$, and whose simplexes are the simplexes of $K$ and $L$ and all subsets of the sets $\{v_0, \dots, v_k, \rho(v_k), \dots, \rho(v_p)\}$, where $\{v_0 < \dots < v_p\}$ is a simplex in $K$.

Now, assuming that a linear order on $\mathcal{N}_{k+1}$ is fixed, we triangulate $\mathcal{N}_{k+1} \times [0,1]$ as the mapping cylinder of the identity map and introduce the canonical simplicial map $\varphi_k : \mathcal{N}_{k+1} \times [0,1] \to C_k = C_{\rho_k}$ that sends $\mathcal{N}_{k+1} \times \{0\}$ onto the subcomplex $\rho_k(\mathcal{N}_{k+1}) \subset C_k$ by $\rho_k$, and $\mathcal{N}_{k+1} \times \{1\}$ onto $\mathcal{N}_{k+1} \subset C_k$ identically.

By [BD, Proposition 3], $\varphi_k$ is $c(m)$-Lipschitz for some constant $c(m) > 0$ depending only on $m \ge \dim \mathcal{N}_{k+1}$, where the cylinder $C_k$ is endowed with the uniform metric (there

is a minor inaccuracy in an argument in [BD], namely, it is claimed that a simplicial map between uniform complexes is always 1-Lipschitz, which is not true as is easily seen for $\Delta^m \to \Delta^1$ with $m \geq 2$: this map is only $c(m)$-Lipschitz; see Lemma 5.2). Finally, the composition $\varphi_k \circ g_k : B_k \to C_k$ is $c_3\lambda$-Lipschitz with $c_3 = c_3(m)$. Note that $h_k$ and $\varphi_k \circ g_k$ coincide on $Z_{t_k+d} = A_k \cap B_k \subset \mathrm{Co}(Z)$ if we identify $\rho_k(\mathcal{N}_{k+1}) \subset C_k$ with a subcomplex in $\mathcal{N}_k \times \{1\}$.

**5.5. Proof of Theorem 5.1.** For every sufficiently small $\lambda > 0$, we must find a uniform polyhedron $P$ with $\dim P \leq \operatorname{cdim} Z + 1$ and a uniformly cobounded $\lambda$-Lipschitz map $f : \mathrm{Co}(Z) \to P$.

Given $\lambda > 0$, we take $\delta > 0$ as in Subsection 5.2. Then we generate the sequence $\{\tau_k\}$ of positive reals, the sequence $\{\mathcal{U}_k\}$ of open coverings of $Z$, $k \geq 0$, and all the machinery around them from Subsections 5.2–5.4.

With this at hand, we define $P$ as the uniformization of the union

$$P' = P_{-1} \bigcup_{k \geq 0} P_k,$$

where $P_k$ is constructed out of the uniformization of $\mathcal{N}_k \times [0,1]$ (triangulated by fixing a linear order on $\mathcal{N}_k$) and the simplicial mapping cylinder $C_k$ by attaching them along the common subcomplex $\rho_k(\mathcal{N}_{k+1}) \subset (\mathcal{N}_k \times \{1\}) \cap C_k$. Furthermore, $P_{k+1}$ is attached to $P_k$ along the common subcomplex $\mathcal{N}_{k+1}$ for every $k \geq 0$. The polyhedron $P_{-1}$ is the cone over $\mathcal{N}_0$ attached to $P_0$ along the base. Then $\dim P \leq m+1$ for $m = \operatorname{cdim} Z$.

The map $f : \mathrm{Co}(Z) \to P$ is obtained by composing the map $f' : \mathrm{Co}(Z) \to P'$ with the uniformization of $P'$, where $f'$ coincides with $h_k$ on $A_k$ and with $\varphi_k \circ g_k$ on $B_k$ for every $k \geq 0$. Finally, $f'$ is affine on every segment $z \times [0, t_0]$, $z \in Z$, and sends $o = Z \times \{0\}$ to the vertex of $P_{-1}$. Lemma 5.3 and Subsection 5.4 imply that $f$ is $c\lambda$-Lipschitz for some $c = c(m) > 0$ on every $A_k$, $B_k$, $k \geq 0$, and on $Z \times [0, t_0] \subset \mathrm{Co}(Z)$.

Since $\operatorname{diam} P \leq \sqrt{2}$ and $\frac{1}{d} < \lambda$, the $c\lambda$-Lipschitz condition is certainly satisfied for the points $(z,t)$, $(z',t') \in \mathrm{Co}(Z)$ separated by some annulus $A_k$ or $B_k$. Thus, we assume that $(z,t)$ and $(z',t')$ are sitting in adjacent annuli. Unfortunately, we cannot directly apply the argument in [BD, Proposition 4], which would be well adapted to our situation if $\mathrm{Co}(Z)$ is geodesic. In general, this is not the case, and we slightly modify the argument as follows.

Assume that $t' > t$. We take $t'' \in (t, t')$ for which $(z', t'')$ is common for the annuli, and note that

$$|(z,t)(z',t')| \geq \max\{|(z,t)(z',t)|, |t-t'| = |t-t''| + |t''-t'|\}$$

by the geometry of $\mathrm{Co}(Z)$. Now, the required Lipschitz condition for the pair $(z,t)$ and $(z',t')$ follows in an obvious way from the conditions for the three pairs: $(z,t)$ and $(z',t)$; $(z',t)$ and $(z',t'')$; $(z',t'')$ and $(z',t')$, each belonging to one annulus.

It remains to check that $f$ is uniformly cobounded. For every simplex $\sigma \subset P_k$, $k \geq 0$, the preimage $f'^{-1}(\sigma) \subset A_k \cup B_k$ is contained in $Z_\sigma \times [t_k, t_{k+1}] \subset \mathrm{Co}(Z)$, where $\operatorname{diam}(Z_\sigma \times \{t_k\}) \leq \mathrm{const}(\lambda)$ by estimates of Subsection 5.2. Thus, $\operatorname{diam} f'^{-1}(\sigma) \leq 4d + \operatorname{diam}(Z_\sigma \times \{t_k\}) \leq \mathrm{const}(\lambda)$, and $f$ is uniformly cobounded. This completes the proof of Theorem 5.1.

## §6. Hyperbolic spaces

**6.1. Basics on hyperbolic spaces.** We briefly recall necessary facts of the hyperbolic spaces theory. For more details, the reader may consult, e.g., [BoS].

Let $X$ be a metric space. Fix a base point $o \in X$ and for $x$, $x' \in X$ put $(x|x')_o = \frac{1}{2}(|xo| + |x'o| - |xx'|)$. The number $(x|x')_o$, which is nonnegative by the triangle inequality,

is called the Gromov product of $x$, $x'$ with respect to $o$. A *$\delta$-triple* is a triple of three real numbers $a$, $b$, $c$ with the property that the two smallest of these numbers differ at most by $\delta$.

A metric space $X$ is (*Gromov*) *hyperbolic* if for some $\delta \geq 0$, some base point $o \in X$, and all $x$, $x'$, $x'' \in X$ the numbers $(x|x')_o$, $(x'|x'')_o$, and $(x|x'')_o$ form a $\delta$-triple. This condition is equivalent to the *$\delta$-inequality*

$$(x|x'')_o \geq \min\{(x|x')_o, (x'|x'')_o\} - \delta.$$

Let $X$ be a hyperbolic space, and let $o \in X$ be a base point. A sequence of points $\{x_i\} \subset X$ *converges to infinity* if

$$\lim_{i,j\to\infty} (x_i|x_j)_o = \infty.$$

Two sequences $\{x_i\}$, $\{x_i'\}$ that converge to infinity are *equivalent* if

$$\lim_{i\to\infty} (x_i|x_i')_o = \infty.$$

The *boundary at infinity* $\partial_\infty X$ of $X$ is defined as the set of equivalence classes of sequences converging to infinity. The Gromov product extends to $X \cup \partial_\infty X$ as follows. For points $\xi$, $\xi' \in \partial_\infty X$, the Gromov product is defined by

$$(\xi|\xi')_o = \inf \liminf_{i\to\infty} (x_i|x_i')_o,$$

where the infimum is taken over all sequences $\{x_i\} \in \xi$, $\{x_i'\} \in \xi'$. Note that $(\xi|\xi')_o$ takes values in $[0, \infty]$, and that $(\xi|\xi')_o = \infty$ if and only if $\xi = \xi'$. Furthermore, for $\xi$, $\xi'$, $\xi'' \in \partial_\infty X$ the following is true; see [BoS, §3]:

(1) for sequences $\{y_i\} \in \xi$, $\{y_i'\} \in \xi'$ we have

$$(\xi|\xi')_o \leq \liminf_{i\to\infty} (y_i|y_i')_o \leq \limsup_{i\to\infty} (y_i|y_i')_o \leq (\xi|\xi')_o + 2\delta;$$

(2) $(\xi|\xi')_o$, $(\xi'|\xi'')_o$, $(\xi'|\xi'')_o$ is a $\delta$-triple.

Similarly, the Gromov product

$$(x|\xi)_o = \inf \liminf_{i\to\infty} (x|x_i)_o$$

is defined for any $x \in X$, $\xi \in \partial_\infty X$, where the infimum is taken over all sequences $\{x_i\} \in \xi$, and the $\delta$-inequality holds for any three points in $X \cup \partial_\infty X$.

A metric $d$ on the boundary at infinity $\partial_\infty X$ of $X$ is said to be *visual* if there is $o \in X$, $a > 1$, and positive constants $c_1$, $c_2$ such that

$$c_1 a^{-(\xi|\xi')_o} \leq d(\xi, \xi') \leq c_2 a^{-(\xi|\xi')_o}$$

for all $\xi$, $\xi' \in \partial_\infty X$. In this case we say that $d$ is a visual metric with respect to the base point $o$ and the parameter $a$.

### 6.2. The hyperbolic cone.

**Proposition 6.1.** *Let $Z$ be a bounded metric space. Then the hyperbolic cone $Y = \mathrm{Co}(Z)$ is a $\delta$-hyperbolic space with $\delta = \delta(\mathrm{H}^2)$, there is a canonical inclusion $Z \subset \partial_\infty Y$, and the metric of $Z$ is visual. If, moreover, $Z$ is complete, then $\partial_\infty Y = Z$.*

*Proof.* We may assume that $(y|y')_o \leq (y|y'')_o \leq (y''|y')_o$ for $y$, $y'$, $y'' \in Y$. We show that $(y|y'')_o \leq (y|y')_o + \delta$. For this, consider triangles $\overline{o}\,\overline{y}\,\overline{y}''$ and $\overline{o}\,\overline{y}''\,\overline{y}'$ in $\mathrm{H}^2$ with common side $\overline{o}\,\overline{y}''$ separating them such that $|\overline{o}\,\overline{y}| = |oy|$, $|\overline{o}\,\overline{y}'| = |oy'|$, $|\overline{o}\,\overline{y}''| = |oy''|$, and $|\overline{y}\,\overline{y}''| = |yy''|$, $|\overline{y}''\,\overline{y}'| = |y''y'|$. Then $|yy'| \leq |\overline{y}\,\overline{y}'|$ by the triangle inequality in $Z$. It follows that $(\overline{y}|\overline{y}'')_{\overline{o}} = (y|y'')_o$, $(\overline{y}''|\overline{y}')_{\overline{o}} = (y''|y')_o$ and $(\overline{y}|\overline{y}')_{\overline{o}} \leq (y|y')_o$. Therefore, $(y|y'')_o - (y|y')_o \leq (\overline{y}|\overline{y}'')_{\overline{o}} - (\overline{y}|\overline{y}')_{\overline{o}} \leq \delta$ because $\mathrm{H}^2$ is $\delta$-hyperbolic.

For every $z \in Z$ the ray $\{z\} \times [0, \infty) \subset Y$ represents a point in $\partial_\infty Y$, which we identify with $z$. This yields an inclusion $Z \subset \partial_\infty Y$. The last assertion of the proposition can easily be checked.

It remains to show that the metric of $Z$ is visual. Given $z$, $z' \in Z$, consider the geodesic rays $\gamma(t) = (z, t)$, $\gamma'(t) = (z', t)$ in $\mathrm{Co}(Z)$. Then $\gamma \in z$, $\gamma' \in z'$ viewed as points at infinity, and for $(\gamma|\gamma')_o = \lim_{t\to\infty}(\gamma(t)|\gamma'(t))_o$ (it is easily seen that the Gromov product $(\gamma(t)|\gamma'(t))_o$ is monotone) we have

$$(z|z')_o \le (\gamma|\gamma')_o \le (z|z')_o + 2\delta.$$

For the comparison geodesic rays $\overline{\gamma}, \overline{\gamma}' \subset \mathrm{H}^2$ with common vertex $\overline{o}$ and $\angle_{\overline{o}}(\overline{\gamma}, \overline{\gamma}') = \mu|zz'|$ (recall that $\mu = \pi/\operatorname{diam} Z$) we have $(\overline{\gamma}|\overline{\gamma}')_{\overline{o}} = (\gamma|\gamma')_o$ and $(\overline{\gamma}|\overline{\gamma}')_{\overline{o}} \le d \le (\overline{\gamma}|\overline{\gamma}')_{\overline{o}} + \delta$, where $d = \operatorname{dist}(\overline{o}, \overline{z}\,\overline{z}')$ and $\overline{z}\,\overline{z}' \subset \mathrm{H}^2$ is the infinite geodesic with the endpoints at infinity, $\overline{z} = \overline{\gamma}(\infty)$, $\overline{z}' = \overline{\gamma}'(\infty)$. By the angle of parallelism formula from the geometry of $\mathrm{H}^2$, we have $\tan\frac{\mu|zz'|}{4} = e^{-d}$; therefore, we conclude that

$$e^{-3\delta} e^{-(z|z')_o} \le \tan\frac{\mu|zz'|}{4} \le e^{-(z|z')_o}$$

for every $z$, $z' \in Z$. The function $s \mapsto \frac{1}{s}\tan\frac{\mu s}{4}$ is uniformly bounded and bounded away from zero on $[0, \operatorname{diam} Z]$. It follows that the metric of $Z \subset \partial_\infty Y$ is visual with respect to the vertex $o \in Y$ and the parameter $a = e$. □

Let $X$ be a hyperbolic space, $x_0 \in X$ a base point. For $x \in X$ we denote $|x| = |xx_0|$. Also, we omit the subscript $x_0$ in the notation for Gromov products with respect to $x_0$. The space $X$ is said to be *visual* if for some base point $x_0 \in X$ there is a positive constant $D$ such that for every $x \in X$ there is $\xi \in \partial_\infty X$ with $|x| \le (x|\xi) + D$ (it is easily seen that this property is independent of the choice of $x_0$). This definition is due to Schroeder; cf. [BoS, §5].

**Proposition 6.2.** *Every visual hyperbolic space $X$ is roughly similar to a subspace of the hyperbolic cone $\mathrm{Co}(\partial_\infty X)$ over the boundary at infinity, where $\partial_\infty X$ is taken with a visual metric.*

*Proof.* We fix a visual metric on $\partial_\infty X$ with respect to $x_0 \in X$ and a parameter $a > 1$. Replacing $X$ by $\lambda X$ with $\lambda = 1/\ln a$, we can assume that $a = e$. Since $X$ is visual, there is a constant $D > 0$ such that for every $x \in X$ there is $\xi = \xi(x) \in \partial_\infty X$ with $|x| \le (x|\xi) + D$. We define $F : X \to Y$, $Y = \mathrm{Co}(\partial_\infty X)$ by $F(x) = (\xi(x), |x|) \in Y$. Note that $F(x_0) = o$.

From Proposition 6.1 it follows that the Gromov product $(\xi|\xi')$ in $X$ coincides with the Gromov product $(\xi|\xi')_o$ in $Y$ up to a uniformly bounded error for every $\xi$, $\xi' \in \partial_\infty X$. Since $|F(x)| = |x|$ for every $x \in X$, by [BoS, Lemma 5.1] we have

$$|xx'| \doteq |x| + |x'| - 2\min\{(\xi(x)|\xi(x')), |x|, |x'|\} \doteq |F(x)F(x')|$$

up to a uniformly bounded error for every $x$, $x' \in X$. Hence, $F$ is roughly isometric, and $X$ is roughly similar to a subspace of $Y$. □

**6.3. Proof of Theorem 1.1.** Since the asymptotic dimension is a quasiisometry invariant, we have $\operatorname{asdim} X \le \operatorname{asdim} \mathrm{Co}(\partial_\infty X)$ by Proposition 6.2. By Theorem 5.1, we obtain $\operatorname{asdim} X \le \operatorname{cdim}(\partial_\infty X) + 1$. □

*Remark* 6.3. We come back to Gromov's proof (see the Introduction) of the fact that $\operatorname{asdim} X \le n$ for every negatively pinched Hadamard manifold $X$ of dimension $n$. In my opinion, to complete the proof, one must show that $\operatorname{cdim} \partial_\infty X = n - 1$, where $\partial_\infty X = S^{n-1}$ is taken with a visual metric. However, for the time being this is only known for $X = \mathrm{H}^n$.

## References


[As] P. Assouad, *Sur la distance de Nagata*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 1, 31–34. MR0651069 (83b:54034)

[BD] G. Bell and A. Dranishnikov, *On asymptotic dimension of groups acting on trees*, arXiv:math.GR/0111087.

[BoS] M. Bonk and O. Schramm, *Embeddings of Gromov hyperbolic spaces*, Geom. Funct. Anal. **10** (2000), no. 2, 266–306. MR1771428 (2001g:53077)

[BS] S. Buyalo and V. Schroeder, *Hyperbolic dimension of metric spaces*, arXiv:math.GT/0404525.

[Gr] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, Vol. 2 (Sussex, 1991) (G. A. Niblo, M. A. Roller, eds.), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR1253544 (95m:20041)

[He] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001. MR1800917 (2002c:30028)

[LS] U. Lang and T. Schlichenmaier, *Nagata dimension, quasisymmetric embeddings and Lipschitz extensions*, arXiv:math.MG/0410048.

[Sp] E. Spanier, *Algebraic topology*, Corrected reprint, Springer-Verlag, New York–Berlin, 1981. MR0666554 (83i:55001)



St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023, Russia

*E-mail address*: sbuyalo@pdmi.ras.ru



Received 1/NOV/2004

Translated by THE AUTHOR