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THE YONEDA ALGEBRAS OF SYMMETRIC SPECIAL BISERIAL ALGEBRAS ARE FINITELY GENERATED

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Abstract. By using the Benson–Carlson diagrammatic method, a detailed combinatorial description is given for the syzygies of simple modules over special biserial algebras. With the help of this description, it is proved that the Yoneda algebras of the algebras mentioned above are finitely generated.

§1. Introduction

In the representation theory of finite groups and finite-dimensional algebras, the special biserial algebras arise quite naturally. Being the simplest from the representation theory point of view, they also constitute a sufficiently large class in terms of combinatorial description. In particular, among them we find the special biserial algebras of finite representation type, also known as "Brauer tree algebras", which are studied most thoroughly due to their simplicity and importance (see, e.g., [1, 2]).

Therefore, any attempts to extend the Golod–Evens–Venkov theorem (see [3, 4, 5]) and the more general Friedlander–Suslin theorem (the latter says that the cohomology rings of the co-commutative Hopf algebras are finitely generated) to some other classes of self-injective finite-dimensional algebras lead naturally to considering the class of symmetric special biserial algebras.

In [7] it was proved that any special biserial algebra has either finite or tame representation type. On the other hand, Erdmann classified the tame blocks of group algebras (all of them are symmetric algebras) up to the Morita equivalence (see [8]). In particular, this implies classification of the special biserial algebras that occur as blocks of group algebras.

We prove that the Yoneda algebra for an arbitrary symmetric special biserial algebra is finitely generated.

The special biserial algebras are defined by describing a quiver with relations. In §2 we specify this description in the case of symmetric algebras. In particular, this allows us to give quite a detailed description for the projective resolution of an arbitrary simple module (more precisely, for the modules of syzygies of a simple module); the proof of the main result is given after that.

For working with syzygy modules, we use the diagrammatic method, which was introduced by Benson and Carlson in [9] and then developed and adapted to the cases of our interest in [10, 11]. It should be noted that a classification of the indecomposable modules over special biserial algebras can also be obtained from the results of [12].

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§2. Quivers of symmetric special biserial algebras

Let K be an algebraically closed field. In this paper we consider path algebras of certain quivers with relations over K. The path composition in these algebras will be written from right to left. We recall the following definition.

Definition. A finite-dimensional basic algebra Λ is said to be *special biserial* if its quiver Q and the ideal of relations I satisfy the following conditions.

1) At most two arrows may start and may end at every vertex of Q.

2) For any arrow there exists at most one arrow such that its product from the left by the given arrow is in I, and there exists at most one arrow such that its product from the right by the given arrow is in I.

We restrict ourselves to only two-side-indecomposable special biserial algebras, because an arbitrary special biserial algebra is a product of indecomposable algebras. An algebra is indecomposable if and only if its quiver is connected; this condition will be assumed throughout.

Definition. The *degree* of a vertex in a quiver is the total number of arrows that start or end at this vertex.

We consider only left modules.

Let S_i denote the simple Λ-module corresponding to the vertex i in Q. Let $\mathcal{S}$ be the set of all isomorphism classes of simple Λ-modules. The projective cover of a module M is denoted by $P(M)$, the radical of M is denoted by $\operatorname{Rad} M$, and the socle of M by $\operatorname{Soc} M$. We write $\operatorname{Top} M = M/\operatorname{Rad} M$. If w is a path in Q, we denote its length by $l(w)$.

We often identify vertices of a quiver with the corresponding simple modules. In particular, the projective module corresponding to a simple module of degree two (four) is chain (respectively, nonchain).

Definition. An algebra Λ is said to be *symmetric* if there exists a linear map $\varphi : \Lambda \to K$ such that its kernel contains no nonzero ideals and $\varphi(ab) = \varphi(ba)$ for any $a, b \in \Lambda$; this map is called a *symmetric form*.

Lemma 2.1. *Let Γ be a path in the quiver of a symmetric algebra Λ. If Γ is not contained in the ideal of relations, then there exists a path Γ_1 such that $\Gamma\Gamma_1$ is also not contained in the ideal of relations, and $\Gamma\Gamma_1$ starts and ends at one and the same point.*

Proof. Consider a symmetric form $\varphi : \Lambda \to K$ and the right ideal $\Gamma\Lambda$ of Λ. This (nonzero) ideal is not contained in the kernel φ; therefore, there exists a path Γ_1 with $\varphi(\Gamma\Gamma_1) \neq 0$ (in particular, $\Gamma\Gamma_1 \neq 0$). Since φ is symmetric, we see that $\varphi(\Gamma_1\Gamma) \neq 0$ and $\Gamma_1\Gamma \neq 0$. Certainly in this case the endpoint of Γ coincides with the starting point of Γ_1. $\square$

The lemma implies that for any arrow that is not a loop, there exists an arrow such that its product from the left by the given arrow is in I (the definition of a special biserial algebra shows that this arrow is unique). Similarly, there exists a unique arrow such that its product from the right by the given arrow is in I. We show that this is also true for any loop (with one exception).

Suppose α is a loop at a vertex s and $\beta\alpha \in I$ for each arrow β in the quiver Λ. Then the element of Λ corresponding to α lies in the socle of the corresponding indecomposable projective module P_s. If α is a unique arrow starting at s, then s is a unique vertex in the quiver (by the lemma and the fact that the quiver is connected); furthermore, $\alpha\alpha = 0$ by assumption, and we have $\Lambda \simeq K[X]/\langle X^2\rangle$; this latter algebra will be excluded from all the considerations below (the main result of the paper is obviously true for this algebra). Otherwise, the maximal path of the form $x\gamma$ not contained in I (here $\gamma \neq \alpha$

is an arrow starting at s, and x is a path) also (like α) corresponds to an element in $\mathrm{Soc}(P_s)$. Then the existence of the relation $\alpha - cx\gamma \in I, c \in K^* = K \setminus \{0\}$ contradicts the fact that any relation is contained in the square of the radical of the path algebra of Q; the nonexistence of such a relation contradicts the fact that the socle of a projective indecomposable module over an arbitrary quasi-Frobenius algebra is one-dimensional. Therefore, no loop with the property in question can exist (in the nonexceptional case).

Suppose that consecutive distinct arrows $a_1, a_2, \dots, a_n$ form a cycle and satisfy

$$a_1 a_2 \cdots a_n \notin I.$$

Let k be the maximal natural number such that $(a_1 a_2 \cdots a_n)^k \notin I$. Since

$$(a_1 a_2 \cdots a_n)^k a_1 = 0$$

by Lemma 2.1, $(a_1 a_2 \cdots a_n)^k$ lies in the socle of Λ. Also, we have

$$(a_i a_{i+1} \cdots a_n a_1 \cdots a_{i-1})^{k+1} = 0$$

for $2 \le i \le n$. Moreover, $(a_1 a_2 \cdots a_n)^k \neq 0$ implies $(a_i a_{i+1} \cdots a_n a_1 \cdots a_{i-1})^{k-1} a_i \neq 0$, whence, again by Lemma 2.1,

$$(a_i a_{i+1} \cdots a_n a_1 \cdots a_{i-1})^k \neq 0.$$

Thus, the number k does not depend on the choice of the starting point of a given cycle.

Keeping the above notation, we call a path of the form $(a_1 a_2 \cdots a_n)^k$ an *A-cycle*, and the path $a_1 a_2 \cdots a_n$ is the *main subcycle* of this A-cycle. The parameter k is the *multiplicity* of this A-cycle. An A-cycle of multiplicity 1 is called a *degenerate* A-cycle. For any subpath w in an A-cycle $(a_1 a_2 \cdots a_n)^k$, we use the symbol $\Delta(w)$ to denote the path complementary to w in $(a_1 a_2 \cdots a_n)^k$, i.e., the path with the property that $w\Delta(w)$ (and, consequently, $\Delta(w)w$) is the same A-cycle (possibly, with a different starting point).

Furthermore, since the socle of any indecomposable projective Λ-module is one-dimensional, for an arbitrary starting point v_0 of a pair of arrows and the corresponding pair of A-cycles (which may happen to be one cycle passing through v_0 twice), we have $(x_1 x_2 \cdots x_n)^{k_1} = \lambda (y_1 y_2 \cdots y_m)^{k_2}$, where $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_m)$ are the corresponding main subcycles, k_1 and k_2 are the corresponding multiplicities, and $\lambda \in K^*$.

Proposition 2.2. *Let Λ be a symmetric special biserial algebra and (Q, I) its quiver with relations. Then the ideal I is generated by the set $A \cup B \cup C$, where*

1) *A is the set of paths of the form $\alpha\beta$, where α, β are not successive arrows of any A-cycle*;

2) *B is the set of paths of the form $(a_1 a_2 \cdots a_n)^k a_1$, where a_1 is an arbitrary arrow and $(a_1 a_2 \cdots a_n)^k$ is an A-cycle containing this arrow*;

3) *C is the set of elements of the form $r_v = (x_1 x_2 \cdots x_n)^{k_1} - \lambda_v (y_1 y_2 \cdots y_m)^{k_2}$ (one such element for each vertex v of degree 4 in Q), where $(x_1 x_2 \cdots x_n)^{k_1}$ and $(y_1 y_2 \cdots y_m)^{k_2}$ are A-cycles passing through v (they may happen to be one cycle passing through v twice), and $\lambda_v \in K^*$.*

Proof. As has already been shown, for an appropriate choice of the elements λ_v the set $A \cup B \cup C$ is contained in I. We prove that any element of I belongs to the ideal generated by $A \cup B \cup C$. Suppose that $\lambda_1 \Gamma_1 + \cdots + \lambda_k \Gamma_k \in I$, where the Γ_i are different paths in Q, $\lambda_i \in K^*$. We may assume that each Γ_i is a product of several successive arrows contained in some A-cycle (the other summands necessarily belong to the ideal generated by A).

Suppose that the image of one of these paths does not lie in the socle of the algebra; let x be the last arrow in this path. Among the Γ_i that have x as the last arrow, we choose a path Γ_s of minimal length and consider the "complementary" path Γ such that the

image of $\Gamma\Gamma_s$ is a nonzero element in the socle of Λ; multiplying the initial combination of Γ_i's by Γ (from the left), we obtain $\lambda_s\Gamma\Gamma_s \in I$; this contradicts our assumptions. Thus, all Γ_i are either maximal nonzero paths (i.e., A-cycles), or products (lying in I) of some successive arrows in an A-cycle. The definition of an A-cycle shows that the paths of the last kind are generated by elements of B. Removing the summands of this form, we obtain a linear combination of A-cycles that still lies in I. Now we multiply this combination by the idempotents corresponding to the endpoints of Γ_i, obtaining several elements $k_v r_v$ the sum of which coincides with the latter combination. □

§3. Ext-algebras are finitely generated

Let Λ be an arbitrary symmetric special biserial algebra, $J(\Lambda)$ its Jacobson radical, $\overline{\Lambda} = \Lambda/J(\Lambda)$ the corresponding quotient algebra, and

$$\mathcal{E}(\overline{\Lambda}) = \bigoplus_{m\geq 0} \operatorname{Ext}^m(\overline{\Lambda}, \overline{\Lambda})$$

the Yoneda algebra of Λ; the latter is a graded K-algebra. Now we formulate our main statement.

Theorem 3.1. 1. *$\mathcal{E}(\overline{\Lambda})$ is a finitely generated K-algebra.*

2. *The* Ext*-algebra of any simple Λ-module is finitely generated.*

Before proving this theorem, we give some remarks. First, clearly, it suffices to restrict ourselves to indecomposable algebras, because the Yoneda algebra of the product of algebras is isomorphic to the direct product of the corresponding Yoneda algebras, and the Ext-algebra of a simple module over the direct product coincides with its Ext-algebra over the corresponding indecomposable algebra. Moreover, our statement is obvious for the algebra $K[X]/\langle X^2\rangle$ (see §2). So, we assume that our algebra possesses the properties described in §2.

We shall prove the first statement only. The proof of the second is much the same.

For us it is convenient to modify somewhat the description of our algebra in terms of a quiver with relations.

Definition. Let Λ be a symmetric special biserial algebra, and let (Q, I) be its quiver with relations. We introduce the *extended quiver* with relations (Q_r, I_r) as follows.

1. The vertices of the quiver Q_r are the same as the vertices of Q.

2. Let $X_1, X_2, \dots, X_n$ all be vertices of degree 2 in Q. The set of arrows of Q_r is obtained from the set of arrows of Q by addition of loops $w_1, w_2, \dots, w_n$ that start (and end) at the vertices $X_1, X_2, \dots, X_n$, respectively. These loops will be called *formal* loops.

3. For each formal loop w_i, we introduce a relation $w_i = (x_1x_2\cdots x_n)^k$, where $(x_1x_2\cdots x_n)^k$ is an A-cycle of multiplicity k passing through the vertex X_i. Thus, we associate the arrow w_i with the corresponding element of Λ. Adding these n relations to I, we obtain the ideal I_r.

Observe that in the extended quiver all vertices have degree 4; this quiver is not a quiver with relations (in the usual sense) for Λ because the elements corresponding to the formal loops lie in $\operatorname{rad}^2\Lambda$. However, this quiver satisfies all the other conditions in the definition of a quiver with relations of an algebra. Moreover, it agrees with the description (given in §2) of a quiver with relations for a symmetric special biserial algebra.

In a natural way, the notion of an A-cycle is carried over to extended quivers. In particular, each formal loop is an A-cycle of length 1 and of multiplicity 1.

Remark. Thus, an arbitrary indecomposable symmetric special biserial algebra over an algebraically closed field is uniquely determined by the following data:

1) a connected quiver in which every vertex is the starting point for exactly two arrows and the endpoint for exactly two arrows;

2) a fixed partition into cycles of the quiver's set of arrows;

3) a function on the set of these cycles with values in $\mathbb{N}$;

4) a function on the set of vertices with values in K^*.

Concerning the last item, it should be noted that different functions can determine isomorphic algebras (provided the other data are the same). Moreover, some of such functions can yield nonsymmetric algebras. We shall not discuss these details here, because they are not needed in what follows.

We recall some definitions concerning the diagrammatic method.

A diagram D is called a *zigzag* if its vertices $\{x_i\}_{i=1}^n$ can be enumerated in such a way that for each $i = 1, \dots, n-1$ there exists either an edge $e(x_i, x_{i+1})$, or an edge $e(x_{i+1}, x_i)$, and there are no other edges in D. If D is a zigzag (with edges labeled with the generators of $K[Q]$ corresponding to the arrows of Q), then we shall replace the maximal subsequences of equally oriented edges by one link labeled with the corresponding word (moving downwards, we write the corresponding word from right to left).

Note that if a module M over a biserial algebra has a zigzag diagram, then its syzygy module $\Omega(M)$ also has a zigzag diagram. Therefore, the diagrams for the modules $\Omega^n(M)$ satisfy the so-called *D-uniqueness* condition (see [9, p. 68]). This can be shown as in [9, Lemma 11.1].

In particular, if S is a simple module, all its syzygies $\Omega^n(S)$ have zigzag diagrams. We analyze the transformations of a zigzag under passage from $\Omega^n(S)$ to $\Omega^{n+1}(S)$; we only trace the changes at one of the ends of the zigzag.

Type I. A zigzag representing the module $\Omega^n(S)$ begins with a maximal element corresponding to a simple module of degree 4 (i.e., $P(X)$ is nonchain):

$$\begin{array}{lccccccc}
\Omega^n(S): & & X & & & & \dots & \\
 & & & {}_{w_1}\diagdown & & \diagup_{w_2} & & \\
 & & & & Y & & & \\
\Omega^{n+1}(S): & X' & & & Y & & & \\
 & {}_{\Delta(\varphi)}\diagdown & & \diagup_{\Delta(w_1)} & & \diagdown_{\Delta(w_2)} & & \\
 & & X & & & & \dots &
\end{array}.$$

This passage is done by "cutting" the arrow $X \xrightarrow{\varphi} X'$ in the diagram of $P(X)$:

$$\begin{array}{ccccc}
 & & X & & \\
 & \diagup_{\varphi} & & {}_{w_1}\diagdown & \\
X' & & & & Y \\
 & {}_{\Delta(\varphi)}\diagdown & & \diagup_{\Delta(w_1)} & \\
 & & X & &
\end{array}.$$

Type II. The zigzag representing the module $\Omega^n(S)$ begins with a maximal element of the zigzag, and this element corresponds to a simple module of degree 2:

$$\begin{array}{lccccc}
\Omega^n(S): & X & & & \dots & \\
 & & {}_{w}\diagdown & & \diagup_{w_1} & \\
 & & & Y & & \\
 & & & Y & & \\
 & & \diagup_{\Delta(w)} & & \diagdown_{\Delta(w_1)} & \\
\Omega^{n+1}(S): & X & & & \dots &
\end{array}.$$

Type III. The zigzag representing the module $\Omega^n(S)$ begins with a minimal element of the zigzag, and the first link of the zigzag is labeled with a word the complement of which (in a corresponding A-cycle) is a word of length greater than one:

$$\begin{array}{lccccccc}
 & & & Y & & & \dots \\
 & & w\diagup & & w_1\diagdown & \diagup & \\
\Omega^n(S): & X & & & Y_1 & & \\
 & & & & & & . \\
\Omega^{n+1}(S): & X_1 & & & Y_1 & & \\
 & & \Delta(\varphi w)\diagdown & & \diagup\Delta(w_1) & \diagdown & \\
 & & & Y & & & \dots
\end{array}$$

In this case the "cutting" of the arrow $X \xrightarrow{\varphi} X_1$ is carried out in the diagram of $P(Y)$.

Type IV. The zigzag representing the module $\Omega^n(S)$ begins with a minimal element of the zigzag, and the first link of the zigzag is labeled with a word the complement of which is an arrow:

$$\begin{array}{lccccccc}
 & & & Y & & & \dots \\
 & & w\diagup & & w_1\diagdown & \diagup & \\
\Omega^n(S): & X & & & Y_1 & & \\
 & & & & Y_1 & & . \\
 & & & & \diagup\Delta(w_1) & \diagdown & \\
\Omega^{n+1}(S): & & & Y & & & \dots
\end{array} \tag{3.1}$$

In this case the arrow $X \xrightarrow{\varphi} Y$ is "cut". We say that a *collapse* of the diagram $P(Y)$ occurs in the case of a transformation of type IV. The reason for this is that, in the notation of (3.1), after a transformation of this type, the diagram of the module $P(Y_1)$ (instead of $P(Y)$) turns out to be the first part of the zigzag (i.e., the subsequent transformations will be performed with use of the diagram of the module $P(Y_1)$).

Note that, in the above pictures, the zigzags corresponding to the modules $\Omega^n(S)$ and $\Omega^{n+1}(S)$ can have only one link (or even only one vertex); in this case in transformations of types III, IV we allow the module $P(Y)$ to be a chain.

Definition. A *G-cycle* in the extended quiver (Q_r, I_r) is a sequence of (vertices and) arrows $(\alpha_1, \alpha_2, \dots, \alpha_t)$, $X_i \xrightarrow{\alpha_i} X_{i+1}$, $t \geq 1$, such that

1) $X_{t+1} = X_1$;
2) $\alpha_{i+1}\alpha_i \in I_r$ for each i (we assume that $\alpha_{t+1} = \alpha_1$).

Proposition 3.2. *The set of arrows in the extended quiver of a symmetric special biserial algebra is split into mutually disjoint G-cycles.*

Proof. We fix an arbitrary arrow $X_1 \xrightarrow{\alpha_1} X_2$ and consider a sequence of arrows

$$X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{i-1}} X_i \xrightarrow{\alpha_i} X_{i+1} \xrightarrow{\alpha_{i+1}} \cdots$$

such that $\alpha_{i+1}\alpha_i \in I_r$ for each natural i. Since the quiver Q is finite, there exist numbers $t, i \in \mathbb{N}$ with $\alpha_{i+t} = \alpha_i$; choose the minimal i with this property. Suppose $i > 1$. Then $\alpha_i\alpha_{i-1} \in I_r$, $\alpha_i\alpha_{i+t-1} \in I_r$ by construction, but this is impossible (see Lemma 2.1). Thus, we obtain a G-cycle containing the arrow α_1.

Now, suppose that α_1 belongs to two different G cycles:

$$(\alpha_1, \alpha_2, \dots, \alpha_t), \quad (\alpha_1 = \beta_1, \beta_2, \dots, \beta_s).$$

Then there exists a minimal natural number $i > 1$ such that $\alpha_i \neq \beta_i$. Consequently, $\beta_i\alpha_{i-1}, \alpha_i\alpha_{i-1} \in I_r$, which is impossible. $\square$

It is important to note that the sequence of the "leftmost" modules in the diagrams of $\Omega^n(S), n \geq 1$, coincides with the sequence of simple modules arising when we go around one of the G-cycles corresponding to S.

Let

$$\mathcal{G}\colon X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_r} X_{r+1} = X_1$$

be a G-cycle. We shall modify this cycle by the following transformations.

1. If the cycle contains a subpath

$$X_{i_1} \xrightarrow{\alpha_{i_1}} X_{i_2} \xrightarrow{\alpha_{i_2}} \cdots \xrightarrow{\alpha_{i_r}} X_{i_{r+1}} = X_{i_1}$$

coinciding with a degenerate A-cycle, we contract this subpath so that only the vertex X_{i_1} remains and mark this vertex. Proceeding with such operations as far as possible, we obtain a cycle with no subpaths as above, and some vertices of this cycle are marked. In particular, all the remaining vertices of degree 2 are marked because any formal loop is a degenerate A-cycle. Since the sets of arrows of different A-cycles are disjoint, the result of our transformations is well defined. The resulting cycle is said to be *semireduced*. When crossing a marked vertex of such a cycle, we pass along two successive arrows of some A-cycle, because the other pair of arrows that had an end at this vertex was contracted. It should be noted that only marked vertices disappeared in the process of contraction, since no two (different) successive arrows in a G-cycle are successive arrows of some A-cycle. Therefore, all arrows in the extended quiver that have one of their ends at a disappearing vertex belong to the initial G-cycle (and they disappeared either together with that vertex, or at the moment when this vertex was marked).

2. Now, in the resulting cycle we replace any subpath of the form

$$X_{i_1} \xrightarrow{\alpha_{i_1}} X_{i_2} \xrightarrow{\alpha_{i_2}} \cdots \xrightarrow{\alpha_{i_r}} X_{i_{r+1}},$$

where the vertices $X_{i_2}, X_{i_3}, \ldots, X_{i_r}$ are marked and X_{i_1} and $X_{i_{r+1}}$ are not, with a single arrow

$$X_{i_1} \xrightarrow{\alpha_{i_r} \cdots \alpha_{i_2} \alpha_{i_1}} X_{i_{r+1}}.$$

If all the vertices of the cycle are marked, we replace it with an empty cycle (without arrows and vertices). In the latter case the corresponding semireduced cycle coincides with some (nondegenerate) A-cycle, and the initial G-cycle coincides with a connected component of the extended quiver (see the end of the preceding paragraph). Therefore, due to connectedness, in this case the initial G-cycle coincides with the entire extended quiver.

Thus, we obtain a cycle without marked vertices; the arrows of this cycle are labeled with products of arrows of the extended quiver. This cycle is called the *reduced G-cycle* corresponding to $\mathcal{G}$.

We represent a reduced G-cycle as follows:

$$\overline{\mathcal{G}}\colon X_1 \xrightarrow{u_1} X_2 \xrightarrow{u_2} \cdots \xrightarrow{u_r} X_{r+1} = X_1,$$

where the u_i are the words corresponding to subpaths of the initial G-cycle.

Let $S = S_\omega$ be a simple Λ-module. Consider the two G-cycles $\mathcal{G}$ and $\mathcal{G}'$ passing through the vertex ω (they may coincide, but have different starting points). We denote their lengths by s_1 and t_1, respectively; let $d = \gcd(s_1, t_1)$.

Suppose that ω belongs to both corresponding reduced G-cycles

$$\begin{aligned} &\overline{\mathcal{G}}\colon X_1 = S \xrightarrow{u_1} X_2 \xrightarrow{u_2} \cdots \xrightarrow{u_s} X_{s+1} = X_1, \\ &\overline{\mathcal{G}'}\colon X'_1 = S \xrightarrow{u'_1} X'_2 \xrightarrow{u'_2} \cdots \xrightarrow{u'_t} X'_{t+1} = X'_1. \end{aligned} \tag{3.2}$$

As before, let $\Delta(w)$ denote the complement to the path w of an A-cycle. We regard Δ as an "operator" partially defined on the words corresponding to these paths (i.e., on

the elements of the path algebra); moreover, we consider iterations of this "operator", assuming that $\Delta^2 = \Delta \circ \Delta$ acts identically.

With the reduced G-cycle $\overline{\mathcal{G}}$ (see (3.2)) we associate the following sequence $\mathcal{V}$ of words:

$$\mathcal{V} = \Big(\Delta(u_s), \Delta^2(u_{s-1}), \dots, \Delta^{s-1}(u_2), \Delta^s(u_1)\Big).$$

Let $\Delta(\mathcal{V})$ denote the sequence obtained by applying Δ to the words in $\mathcal{V}$:

$$\Delta(\mathcal{V}) = \Big(u_s, \Delta(u_{s-1}), \dots, \Delta^{s-2}(u_2), \Delta^{s-1}(u_1)\Big).$$

Finally, we define a sequence $\mathcal{W}_l(S)$ as follows. If s is even, then $\mathcal{W}_l(S)$ is the sequence $\mathcal{V}$ repeated t_1/d times:

$$\begin{aligned}\mathcal{W}_l(S) =& (\mathcal{V}, \mathcal{V}, \dots, \mathcal{V})\\ =& \Big(\Delta(u_s), \Delta^2(u_{s-1}), \dots, \Delta^r(u_1), \Delta(u_s), \dots, \Delta^r(u_1), \dots, \Delta(u_s), \dots, \Delta^r(u_1)\Big).\end{aligned}$$

If s is odd, we put

$$\mathcal{W}_l(S) = \Big(\mathcal{V}, \Delta(\mathcal{V}), \Delta^2(\mathcal{V}), \dots, \Delta^{t_1/d-1}(\mathcal{V})\Big).$$

For the second reduced G-cycle $\overline{\mathcal{G}}'$ in (3.2), we introduce similar sequences (but with "opposite" ordering of words):

$$\mathcal{V}' = \Big(\Delta^t(u_1'), \Delta^{t-1}(u_2'), \dots, \Delta^2(u_{t-1}'), \Delta(u_t')\Big);$$

$$\mathcal{W}_r(S) = \begin{cases} \Big(\underbrace{\mathcal{V}', \dots, \mathcal{V}'}_{s_1/d \text{ times}}\Big) & \text{for } t \text{ even},\\ \Big(\Delta^{s_1/d-1}(\mathcal{V}), \dots, \Delta(\mathcal{V}), \mathcal{V}\Big) & \text{for } t \text{ odd}.\end{cases}$$

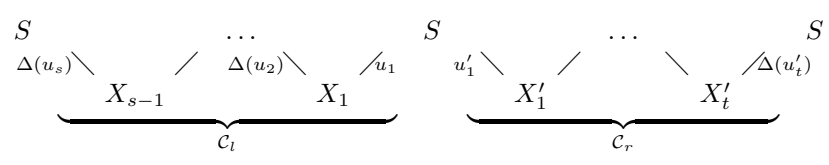
Let $n_S = \text{l.c.m.}(s_1, t_1)$.

Proposition 3.3. *Let S be a simple Λ-module, let $\mathcal{G}, \mathcal{G}'$ be G-cycles containing the vertex corresponding to S, and let $\overline{\mathcal{G}}, \overline{\mathcal{G}}'$ be the corresponding reduced G-cycles.*

1) *If at least one of the reduced G-cycles corresponding to S does not contain S, then, in the above notation, S is an Ω-periodic module, $\mathcal{G}$ and $\mathcal{G}'$ differ from each other only by starting points, and $n_S = s_1 = t_1$.*

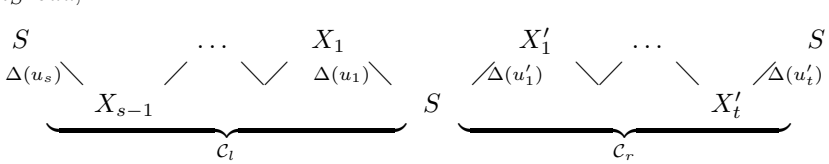
2) *If S belongs to both reduced G-cycles $\overline{\mathcal{G}}$ and $\overline{\mathcal{G}'}$, then the diagram of the module $\Omega^{n_S}(S)$ looks like this:*

a) *for n_S even,*

(3.3)


;

b) *for n_S odd,*

(3.4)


.

Here in both cases the links of the left part of $\mathcal{C}_r$ (respectively, of the right part of $\mathcal{C}_r$) are labeled by the words of the sequence $\mathcal{W}_l(S)$ (respectively, $\mathcal{W}_r(S)$).

Proof. The ends of the links of an arbitrary zigzag will be call the *vertices* of the zigzag. For each i, consider the zigzag of the module $\Omega^i(S)$ and its set of vertices; we endow this set with natural enumeration from left to right (including both upper and lower ends). Assume that the enumerations on zigzags agree with each other; namely, the lower vertex of the zigzag of $\Omega^i(S)$ has the same number as the corresponding upper vertex of $\Omega^{i+1}(S)$. The starting point of our enumeration is the diagram of S; in this case the only vertex has number 0, whereas in the diagram of $\Omega^1(S)$ the vertex corresponding to the successor of S in the cycle $\mathcal{G}$ has number -1.

We show that, in the diagram of the module $\Omega^{n_S}(S)$, the piece corresponding to the part of the zigzag with negative numbers of vertices can be written as the sequence $\mathcal{W}_l(S)$ if S belongs to its reduced G-cycle, and consists of the only vertex S otherwise.

1. We prove the following statement. For a G-cycle containing $X_0 = S$ and corresponding to the changes at the left ends of the zigzags of $\Omega^i(S)$, let

$$(*) \qquad T_k \to T_{k+1} \to \cdots \to T_{k+r} = T_k$$

be its proper subpath contracted in the process of reduction. Also, suppose that the leftmost vertex of the zigzag of $\Omega^k(S)$ is a maximal upper vertex and corresponds to the module T_k; let $(r_1, r_2, \ldots, r_l)$ be the sequence that describes the links of the left part of the zigzag. Then:

a) r is odd;

b) the links of the left part of the zigzag $\Omega^{k+r}(S)$ are of the form

$$\Delta(r_1), \Delta(r_2), \ldots, \Delta(r_l).$$

The proof is by induction on r. If $r = 1$, the subpath $(*)$ is a formal loop, and we pass from $\Omega^k(S)$ to $\Omega^{k+1}(S)$ by (one) transformation of type II, so that the claim is obvious in this case.

If $r > 1$, any contracted subpath as above consists, by the definition of a reduced G-cycle, of edges of some degenerate A-cycle and contracted subpaths of shorter length between each pair of successive arrows of this A-cycle. Joining each of these subpaths to the consequent arrow of the corresponding degenerate A-cycle, we see that the entire subpath in question (excluding the first edge) splits into paths of even length (we have used the inductive hypothesis). Consequently, r is odd.

Omitting all the contracted subpaths of smaller length in $(*)$, we obtain a degenerate A-cycle

$$T_k \xrightarrow{a_1} T_{k+l_1} \xrightarrow{a_2} T_{k+l_2} \to \cdots \xrightarrow{a_t} T_{k+l_t} = T_{k+r} = T_k,$$

where $l_1 = 1$ and $T_{k+l_i-1} = T_{k+l_{i-1}}$ for $i = 1, 2, \ldots, t$; here we set $l_0 = 0$.

Now $b = a_t \cdots a_2 a_1$ is the word corresponding to this A-cycle; therefore, when passing from $\Omega^k(S)$ to $\Omega^{k+1}(S)$, we cut the arrow a_1 (transformation of type I), and the left part of the zigzag of $\Omega^{k+1}(S)$ can be written as $r_0, \Delta(r_1), \Delta(r_2), \ldots, \Delta(r_l)$, where $r_0 = a_t \cdots a_2$.

Now we look at the left parts of the diagrams of some $\Omega^i(S)$ for $i > k+1$; we only write words corresponding to the links of these left parts. We have

$$\Omega^{k+l_2-1}(S) : (a_1 = \Delta(r_0), r_1, r_2, \ldots, r_l)$$

by the inductive hypothesis. Observe that the left end of the diagram of $\Omega^{k+1}(S)$ is a maximal vertex, so that the inductive hypothesis applies.

Next, we have

$$\Omega^{k+l_2}(S) : (a_t \cdots a_4 a_3, \Delta(r_1), \Delta(r_2), \ldots, \Delta(r_l))$$

(a type III transformation was applied), and

$$\Omega^{k+l_3-1}(S) : (a_2 a_1, r_1, r_2, \ldots, r_l)$$

by the inductive hypothesis, because after a type III transformation the left end of the diagram $\Omega^{k+l_2}(S)$ remains a maximal vertex.

Repeating this argument, we obtain

$$\Omega^{k+l_t-1}(S) : (a_{t-1} \cdots a_2 a_1, r_1, r_2, \dots, r_l)$$

by the inductive hypothesis, and

$$\Omega^{k+l_t}(S) : (\Delta r_1, \Delta r_2, \dots, \Delta r_l)$$

by a type IV transformation (collapse). Thus, we see that the left part of the zigzag of $\Omega^{k+l_t}(S) = \Omega^{k+r}(S)$ has the required form.

The claim proved above can be used for proving statement 1) of the proposition. Indeed, if some module S disappeared in the process of reducing a G-cycle, then this module had already been marked at some moment, i.e., it occurs in the G-cycle twice, and one of the two subpaths bounded by these two occurrences was contracted at the moment of marking.

Two cases are possible: either this subpath begins when we start moving along the G-cycle, $S = T_0 \to T_1 \to \cdots \to T_k = S$, $0 < k < t_1$, and then we can apply the claim proved above to this subpath and to the empty sequence of words that describes the diagram of S to show that $\Omega^k(S) = S$, or this subpath ends when we finish moving along the G-cycle: $S = T_{t_1-k} \to T_{t_1-k+1} \to \cdots \to T_{t_1} = S$, $0 < k < t_1$. In the latter case, moving back along the cycle (and applying Ω^{-1}-shifts), we can show that $\Omega^{-k}(S) = S$ (and, therefore, $\Omega^k(S) = S$) by a dual argument.

With the knowledge of how syzygy diagrams are transformed in the process of moving along the contracted subpaths, we can trace their modifications while moving along a semireduced cycle.

Let

$$(**) \qquad Y_{k-1} \to Y_k \to Y_{k+1} \to Y_{k+2} \to \cdots \to Y_{k+l} \to Y_{k+l+1}$$

be a subpath of a semireduced G-cycle all vertices of which except the extreme ones are marked. Let n_i denote the number of the edge $Y_i \to Y_{i+1}$ in the G-cycle. The subpath $(**)$ is also a proper subpath of an A-cycle (because otherwise we have $Y_{k-1} = Y_{k+l+1}$, so that this cycle would be degenerate and the entire subpath would have been contracted). Suppose that we enter Y_k with a transformation of type I and that the given A-cycle is written as the word $(a_n \cdots a_2 a_1)^p$, where $p > 1$ is the multiplicity of this cycle and a_1 is the arrow that is cut when passing to $\Omega^{n_{k-1}}(S)$. Then the word describing the leftmost link of the zigzag of $\Omega^{n_k-1}(S)$ is of the form $r_1 = (a_n \cdots a_2 a_1)^{p-1} a_n \cdots a_2$.

Let $(r_1, r_2, \dots, r_m)$ be the sequence corresponding to the left part of the zigzag of $\Omega^{n_{k-1}}(S)$. As before,

$$\Omega^{n_k-1}(S) : (a_1 = \Delta(r_1), \Delta(r_2), \dots, \Delta(r_m))$$

by the claim proved above,

$$\Omega^{n_k}(S) : ((a_n \cdots a_2 a_1)^{p-1} a_n \cdots a_4 a_3, r_2, r_3, \dots, r_m)$$

(a type III transformation),

$$\Omega^{n_{k+1}-1}(S) : (a_2 a_1, \Delta(r_2), \Delta(r_3), \dots, \Delta(r_m))$$

by the claim proved above, and so on. Finally,

$$\Omega^{n_{k+l+1}-1}(S) : (a_{l+1} \cdots a_2 a_1, \Delta(r_2), \dots, \Delta(r_l))$$

by the same claim, and

$$\Omega^{n_{k+l+1}}(S) : (a_n \cdots a_{l+3} (a_n \cdots a_2 a_1)^{p-1}, r_2, \dots, r_m)$$

(a type III transformation). (It may happen that $l+2=n$, i.e., $a_n \cdots a_{l+3}$ is an empty word.)

Now consider an arbitrary unmarked vertex Y_{k-1} in a semireduced G-cycle. Suppose that there is no collapse at this vertex. Then passage from Y_{k-1} to Y_k is done by a transformation of type I, and there is no collapse again. Moreover, either Y_k is still an unmarked vertex, or we have a sequence of the form $(**)$, with only extreme vertices unmarked. As was shown above, in this case there is no collapse at the vertex Y_{k+l+1} and at the next step we act by a transformation of type I again.

Thus, we have proved that if S survives in its reduced G-cycle, then no collapse occurs in the process of movement along the semireduced G-cycle. In particular, the above arguments are applicable to all the contracted subpaths (these arguments were based on the assumption that no collapse occurs at the starting point of the subpath in question).

These descriptions also imply that, moving along a reduced G-cycle, we necessarily replace any inner word in the sequence corresponding to the left part of the syzygy zigzag with its complement (and the latter remains inner). At the same time, one link is added from the left: this link corresponds either to the complement of an arrow (if this edge is an arrow of the initial G-cycle and the passage through it involves a transformation of type I), or to the complement of the product of arrows (if we deal with a subpath with marked vertices – this case was analyzed above).

Let s_1 be the length of the initial G-cycle. The links of the left part of the zigzag of the module $\Omega^{s_1}(S)$, namely, of the part consisting of links with nonpositive numbers of vertices, is described by the sequence $\mathcal{V}$, and no collapse occurs. When we walk along the cycle the next time, the same links as in the first circuit will be added from the left (and the other links will be "turned over" consecutively). The same transformations occur in the right part of the syzygy diagrams. Thus, after $\mathrm{l.c.m.}(s_1,t_1)/s_1$ circuits over the G-cycle $\mathcal{G}$, we shall cover exactly $\mathrm{l.c.m.}(s_1,t_1)/t_1$ circuits over the other G-cycle $\mathcal{G}'$, obtaining the required form of the diagram of the module $\Omega^{n_S}(S)$. □

Corollary 3.4. *In the notation of Proposition* 3.3, *for any* $k \in \mathbb{N}$ *the diagram of the syzygy* $\Omega^{kn_S}(S)$ *is a zigzag of the following form*:

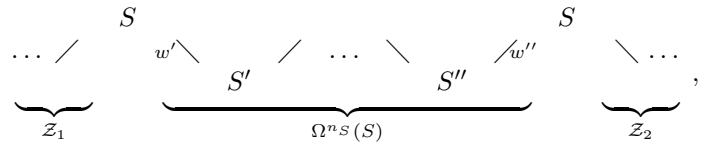
a) *if* n_S *is even, then, when moving from left to right, we have the zigzag* $\mathcal{C}_l$ *repeated* k *times and then the zigzag* $\mathcal{C}_r$ *repeated* k *times*;

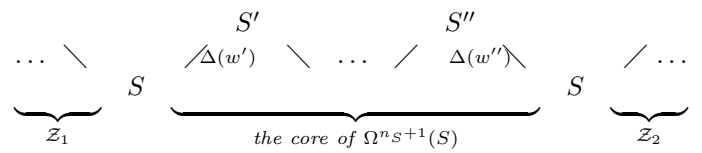
b) *for odd* n_S, *the zigzags* $\mathcal{C}_l$, $\Delta(\mathcal{C}_l)$, $\Delta^2(\mathcal{C}_l)$, $\ldots$, $\Delta^{k-1}(\mathcal{C}_l)$ *and then the zigzags* $\Delta^{k-1}(\mathcal{C}_r)$, $\Delta^{k-2}(\mathcal{C}_r)$, $\ldots$, $\Delta(\mathcal{C}_r)$, $\mathcal{C}_r$ *are adjoined consecutively.*

Proof. This follows directly from the proof of Proposition 3.3. □

The (open) part of the diagram of the module $\Omega^{n_S+1}(S)$ contained between the leftmost and the rightmost lower vertices will be called the *core* of the diagram.

Corollary 3.5. *Let* $S = S_\omega$ *be a simple non-*Ω*-periodic* R*-module. For any* $m > n_S$, *the diagram of the syzygy* $\Omega^m(S)$ *has one of the following forms*:

(3.5)


(3.6)


Moreover, after omitting the middle part of the diagram (this part coincides either with the diagram $\Omega^{n_S}(S)$ or with the core of $\Omega^{n_S+1}(S)$, respectively) and identifying the two vertices marked by S, we obtain the zigzag of the syzygy $\Omega^{m-n_S}(S)$.

Proof. This also follows directly from the proof of Proposition 3.3. □

Proof of Theorem 3.1. For each simple Λ-module S, we fix some K-base $\mathcal{M}_S$ of the space

$$\bigoplus_T \bigoplus_{i=1}^{n_S+1} \operatorname{Ext}^i_R(S,T),$$

where T runs over the set $\mathcal{S}$ of all (pairwise nonisomorphic) simple Λ-modules. We prove that, as a K-algebra, the Yoneda algebra $\mathcal{Y}(\Lambda)$ is generated by the set $\mathcal{M} = \bigcup_S \mathcal{M}_S$.

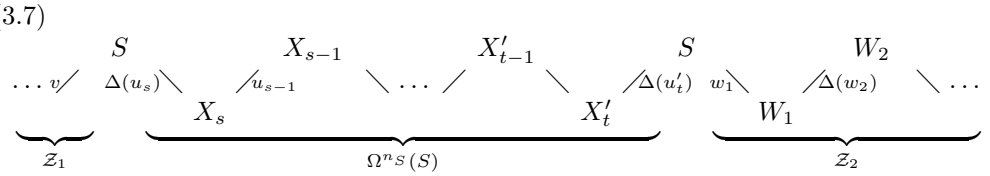
Let S, T be simple Λ-modules, and let

$$f \in \operatorname{Ext}^m_R(S,T) \simeq \operatorname{Hom}_R(\Omega^m(S), T)$$

be a nonzero element with $m > n_S + 1$. By induction on m, we prove that this map is a linear combination of maps obtained from the elements of $\mathcal{M}$ by Ω-shifts and compositions. It suffices to consider the non-Ω-periodic modules S. Let $\overline{\mathcal{G}}$ and $\overline{\mathcal{G}'}$ be reduced G-cycles of the form (3.2) that pass through the vertex ω corresponding to the module S. For arrows α, β of the quiver Q that outgo from ω, we fix the notation so that $\alpha \cdot u_s = 0$, $\beta \cdot u'_t = 0$. Then $\Delta(u_s)$ starts with the arrow β and $\Delta(u'_t)$ starts from the arrow α. Note that f induces a nonzero map $\overline{f}\colon \operatorname{Top}\Omega^m(S) \to T$. Fix some decomposition of the module $\operatorname{Top}\Omega^m(S)$ into a direct sum of simple summands; these summands are in one-to-one correspondence with the maximal elements (upper vertices) of the diagram of the module $\Omega^m(S)$. We may assume that there is a unique summand on which $\overline{f}$ is nonzero or, loosely speaking, f is nonzero on the corresponding upper vertex of the zigzag.

Case 1. Suppose that the diagram of the syzygy $\Omega^m(S)$ is of the following form:

(3.7)

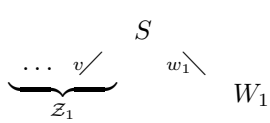


a) Consider the middle part of the zigzag, coinciding with the diagram of $\Omega^{n_S}(S)$ (see (3.7)), and suppose additionally that f is nonzero on some maximal vertex in this part. Observe that there is an epimorphism $\rho\colon \Omega^m(S) \to \Omega^{n_S}(S)$ the kernel of which lies in the direct sum of submodules corresponding to the zigzags $\mathcal{Z}_1$ and $\mathcal{Z}_2$ in the diagram (3.7) with excluded extreme vertices (i.e., vertices common with the middle part). Then we have $f = f' \circ \rho$ for some map $f'\colon \Omega^{n_S}(S) \to T$, and $\rho = \Omega^{n_S}(\tilde{\rho})$ for some $\tilde{\rho}\colon \Omega^{m-n_S}(S) \to S$; therefore, the required statement is obtained by induction on m.

b) Now suppose that f is nonzero on some maximal vertex in the left part $\mathcal{Z}_1$ of the diagram (3.7). Observe that $l(w_1) \le l(\Delta(u_s))$; indeed, since $\beta \cdot u_s \ne 0$, w_1 is a subpath of $\Delta(u_s)$ (it should be noted that if several collapsing vertices belong to a subpath of the G-cycle between S and W_1, then the arrows of the corresponding "collapsing" paths do not belong to the path w_1).

In the case where $l(w_1) < l(\Delta(u_s))$, the module $\Omega^m(S)$ has a quotient module U with the diagram

(3.8)



obtained by a suitable "cutting" of a link in $\Delta(u_s)$; moreover, f factors through this module U, i.e., $f = f' \cdot \rho$, where $\rho\colon \Omega^m(S) \to U$ is a canonical epimorphism and $f'\colon U \to T$ is a map. The diagram (3.8) coincides with the open subdiagram of $\Omega^{m-n_S}(S)$ obtained from the entire diagram by omitting all links to the right of W_1; therefore, the module U is embedded in $\Omega^{m-n_S}(S)$. Consider the commutative square

$$\begin{array}{ccc} U & \xrightarrow{\ i\ } & \Omega^{m-n_S}(S) \\ {\scriptstyle \rho_1}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\rho}} \\ U/\operatorname{Soc} U & \xrightarrow{\ \overline{i}\ } & \Omega^{m-n_S}(S)/\operatorname{Soc}\Omega^{m-n_S}(S), \end{array}$$

where ρ_1 and $\overline{\rho}$ are canonical epimorphisms, i and $\overline{i}$ are embeddings. Consider the part of the zigzag $\mathcal{Z}_2$ (see (3.7)) situated to the left of W_1 (including W_1). Let V be the corresponding submodule. Then $\Omega^{m-n_S}(S) = U + V$. Since $\operatorname{Soc} U \subset \operatorname{Ker} f'$, we have $f' = g \cdot \rho_1$ for some homomorphism $g\colon U/\operatorname{Soc} U \to T$. Since $\Omega^{m-n_S}(S)/\operatorname{Soc}\Omega^{m-n_S}(S) \simeq U/\operatorname{Soc} U \oplus V/\operatorname{Soc} V$, we can define a map $h\colon \Omega^{m-n_S}(S)/\operatorname{Soc}\Omega^{m-n_S}(S) \to T$ obtained from g by putting it to be zero on $V/\operatorname{Soc} V$. We have $f = (h\overline{\rho})(i\rho)$, and $i\rho = \Omega^{m-n_S}(\tilde{\rho})$ for some homomorphism $\tilde{\rho}\colon \Omega^{n_S}(S) \to S$; it remains to apply the inductive hypothesis to $h\overline{\rho}$ and $\tilde{\rho}$.

Finally, if $l(w_1) = l(\Delta(u_s))$ (and consequently, $W_1 = X_s$), then w_1 is not the last link in (3.7), because a simple module T of degree 4 can correspond to a minimal vertex only if $P(T)$ is collapsing. As before, the inequality $l(w_2) \le l(\Delta(u_{s-1}))$ is deduced from the fact that $w_2 \cdot w_1 = 0$ and $w_2 \cdot u_{s-1} \ne 0$. Thus, we have $l(\Delta(w_2)) \ge l(u_{s-1})$ (it should be noted that $w_2, u_{s-1}, \Delta(w_2)$, and $\Delta(u_{s-1})$ lie on a common A-cycle). If $l(\Delta(w_2)) > l(u_{s-1})$, or $l(\Delta(w_2)) = l(u_{s-1})$ and $\Delta(w_2) = u_{s-1}$ is the last link in the zigzag (3.6), then we find a module V with the diagram

$$\underbrace{\cdots\ v}_{\mathcal{Z}_1}\quad S\quad {\scriptstyle w_1}\quad X_s = W_1\quad {\scriptstyle u_{s-1}}\quad X_{s-1}$$

so that V is simultaneously a submodule of $\Omega^{m-n_S}(S)$ and a quotient module of $\Omega^m(S)$. Repeating the same argument as above (with V in place of U), we prove the required statement.

If $l(\Delta(w_2)) = l(u_{s-1})$ and $\Delta(w_2)$ is not the last link of the diagram, we apply similar arguments to the link to the right of it, and so on.

Finally, in the case where f is nonzero at a maximal vertex in the right part $\mathcal{Z}_2$ of (3.7), our statement follows from the above arguments by symmetry.

Case 2. Suppose that the diagram of the syzygy $\Omega^m(S)$ has the following form:

(3.9)

$$\underbrace{{\scriptstyle \Delta(z_1)}}_{\mathcal{Z}_1}\quad S\quad \underbrace{{\scriptstyle u_s}\quad X_s\quad {\scriptstyle \Delta(u_{s-1})}\quad {\scriptstyle u_{s-2}}\quad X_{s-2}\quad \cdots\quad X'_t\quad {\scriptstyle u'_t}\quad X_{s-1}}_{\text{the core of } \Omega^{n_S+1}(S)}\quad {\scriptstyle \Delta(w_1)}\quad S\quad \underbrace{{\scriptstyle w_2}\quad W_1\quad W_2\quad \cdots}_{\mathcal{Z}_2}.$$

The diagram of the module Ω^{n_S+1} is obtained from the diagram of the core by attaching the zigzags $\Delta(\alpha)$ and $\Delta(\beta)$ from the left and from the right, respectively. Clearly,

$l(\Delta(\alpha)) \geq l(\Delta(z_1))$ and $l(\Delta(\beta)) \geq l(\Delta(w_1))$. Therefore, there exists a map $\rho\colon \Omega^m(S) \to \Omega^{n_S+1}(S)$ such that the diagram of its image $Y = \operatorname{Im}\rho$ has the form

$$\begin{array}{ccccccc} Z_1 & & X_s & & X'_t & & W_1 \\ {\scriptstyle \Delta(z_1)}\diagdown & & \diagup{\scriptstyle u_s} & & {\scriptstyle u'_t}\diagdown & & \diagup{\scriptstyle \Delta(w_1)} \\ & S & & \cdots & & S & \\ & & \underbrace{\qquad\qquad\qquad\qquad}_{\text{the core of } \Omega^{n_S+1}(S)} & & & & \end{array}.$$

a) If f is nonzero at some maximal vertex in the middle part of (3.9) (this part coincides with the core of $\Omega^{n_S+1}(S)$), then f passes through Y: $f = f' \cdot \rho$, where $\rho\colon \Omega^m(S) \to Y$ is a canonical epimorphism and $f'\colon Y \to T$ is a homomorphism. Since $f'(\operatorname{Soc} Y) = 0$, we have $f' = g \cdot \rho_1$, where $\rho_1\colon Y \to Y/\operatorname{Soc} Y$ is a canonical epimorphism and $g\colon Y/\operatorname{Soc} Y \to T$ is a homomorphism. Note that $Y = \widetilde{Z} + \widetilde{Y} + \widetilde{W}$, where $\widetilde{Y}$ is a submodule with the diagram coinciding with the core of $\Omega^{n_S+1}(S)$, and $\widetilde{Z}$ (respectively, $\widetilde{W}$) is a chain submodule corresponding to the leftmost link of $\Delta(z_1)$ (respectively, to the rightmost link of $\Delta(w_1)$) in (3.9). Consequently,

$$Y/\operatorname{Soc} Y \simeq \widetilde{Z}/\operatorname{Soc}\widetilde{Z} \oplus \widetilde{Y}/\operatorname{Soc}\widetilde{Y} \oplus \widetilde{W}/\operatorname{Soc}\widetilde{W}.$$

Similarly, $\Omega^{n_S+1}(S) = Z + \widetilde{Y} + W$, where Z and W are the chain submodules corresponding to the links $\Delta(\alpha)$ and $\Delta(\beta)$ in the diagram of the module $\Omega^{n_S+1}(S)$; moreover, we have $\widetilde{Z} \subset Z$, $\widetilde{W} \subset W$ and

$$\Omega^{n_S+1}(S)/\operatorname{Soc}\Omega^{n_S+1}(S) = Z/\operatorname{Soc} Z \oplus \widetilde{Y}/\operatorname{Soc}\widetilde{Y} \oplus W/\operatorname{Soc} W.$$

Since g induces the zero map on $\widetilde{Z}/\operatorname{Soc}\widetilde{Z}$ and on $\widetilde{W}/\operatorname{Soc}\widetilde{W}$, we can extend g by zero to obtain a map $h\colon \Omega^{n_S+1}(S)/\operatorname{Soc}\Omega^{n_S+1}(S) \to T$ such that its restriction to $Y/\operatorname{Soc} Y$ coincides with g. Now, an argument similar to that in Case 1 b) finishes our proof.

b) Suppose that f is nonzero at a maximal vertex in the left part $\mathcal{Z}_1$ in (3.9). It is clear that $l(u_s) \leq l(\Delta(w_1))$.

If $l(u_s) < l(\Delta(w_1))$, or $l(u_s) = l(\Delta(w_1))$ and $\Delta(w_1)$ is the last link of the zigzag, we consider the quotient module V of $\Omega^m(S)$ with the following diagram:

$$\begin{array}{ccccc} & & & & X_s \\ \underbrace{\cdots\diagdown}_{\mathcal{Z}_1} & S & & \diagup{\scriptstyle u_s} & \end{array}. \tag{3.10}$$

The module V is also embedded in $\Omega^{m-n_S}(S)$. Then the proof is completed by an argument similar to that in Case 1 b) (with V in place of U).

Finally, in the case where $\Delta(w_1)$ is not the last link of the zigzag (and $l(u_s) = l(\Delta(w_1))$), we repeat similar arguments. After a finite number of steps, we exhaust all the possible cases; thus, the proof of Theorem 3.1 is complete. □

Corollary 3.6. *For each simple Λ-module $S \in \mathcal{S}$, fix some K-base $\mathcal{X}_S$ of the space*

$$\bigoplus_{i=1}^{n_S} \bigoplus_{T \in \mathcal{S}} \operatorname{Ext}^i_R(S, T).$$

Then, as a K-algebra, the Yoneda algebra $\mathcal{Y}(R)$ is generated by the set $\mathcal{X} = \bigcup_{S \in \mathcal{S}} \mathcal{X}_S$.

Proof. The proof of Theorem 3.1 shows that it suffices to check that for any non-Ω-periodic simple Λ-module S the space

$$\bigoplus_{T\in\mathcal{S}} \operatorname{Ext}_R^{n_S+1}(S,T)$$

is contained in the subalgebra generated by the set $\mathcal{X}$.

Again we may assume that $f\colon \Omega^{n_S+1}(S)\to T$ induces a nonzero map at exactly one maximal vertex of $\Omega^{n_S+1}(S)$.

First, suppose that this maximal vertex belongs to the core of $\Omega^{n_S+1}(S)$. Consider the diagram of the module $\Omega^{n_S-1}(S)$:

$$\begin{array}{ccccccccc} & & X_s & & & & & X'_t & \\ & \tilde{u}_s \diagup & & \diagdown\Delta(u_{s-1}) & & \diagup & \cdots & \diagdown & \diagup\Delta(u'_{t-1}) \quad \diagdown\tilde{u}'_t \\ X & & & & X_{s-1} & & & & X'_{t-1} & & & & X' \end{array}.$$

Recall that when passing from $\Omega^{n_S-1}(S)$ to $\Omega^{n_S}(S)$ we "cut" the arrows $X\to S$ and $X'\to S$ (it may happen that $l(\tilde{u}_s)=0$, $l(\tilde{u}'_t)=0$). Since $l(\tilde{u}_s)<l(u_s)$ and $l(\tilde{u}'_t)<l(u'_t)$, there exists an epimorphism $\tilde{\rho}\colon \Omega^{n_S+1}(S)\to\Omega^{n_S-1}(S)$; moreover, f factors through this homomorphism. An argument similar to that in Case 1 b) of Theorem 3.1 completes the proof.

If f is nonzero at the leftmost vertex A in the diagram of $\Omega^{n_S+1}(S)$, then the module $\Omega^{n_S+1}(S)$ has a quotient module V with the diagram

$$\begin{array}{ccccc} A & & & & X_s \\ & \alpha\diagdown & & \diagup u_s & \\ & & S & & \end{array},$$

V is embedded in $\Omega(S)$, and f factors through V, so that we can use an argument similar to that in Case 1 b) in Theorem 3.1.

The case where f is nonzero at the rightmost vertex of the diagram of $\Omega^{n_S+1}(S)$ is treated in the same way. $\square$

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