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ON GRAPH APPROXIMATIONS OF SURFACES WITH SMALL AREA

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Abstract. It is shown that, for every closed oriented surface M of genus g with an arbitrary Riemann metric, there exists a metric graph of genus at most g such that the Gromov–Hausdorff distance between M and Γ does not exceed $C\sqrt{\operatorname{Vol} M}$, where C depends only on g.

Introduction

Our purpose in this paper is to prove the following theorem.

Theorem 1. *Let M be a closed oriented two-dimensional manifold of genus g. There exists a metric graph Γ of genus at most g such that the Gromov–Hausdorff distance between M and Γ (with the corresponding length metrics) does not exceed $C(g)\sqrt{\operatorname{Vol} M}$, where $C(g)$ is some constant depending only on the genus of M.*

Here by a *metric graph* we mean a finite graph with prescribed lengths for its edges. The distance between two points in the graph is set to be equal to the length of the shortest path joining them.

By the *genus of a graph* we mean its one-dimensional Betti number β_1. Note that $\beta_1 = 1 - \chi$, where χ is the Euler characteristic of the graph. For our purposes it is convenient to define the genus of a graph as the maximal number g such that the graph remains connected after deleting g points different from vertices of degree one. It is easily seen that these definitions are equivalent. A graph has genus g if and only if one can delete g of its edges and get a tree, i.e., a connected graph without cycles.

As motivation for Theorem 1, we mention the following statements.

It is known (see [BI]) that every compact length space X can be obtained as a Gromov–Hausdorff limit of finite graphs. However, usually, the genus of these graphs tends to infinity.

Let $\{x_i\}$ be four points in a metric space X. There are three different ways to divide them into two pairs, i.e., (x_1, x_2) and (x_3, x_4), (x_1, x_3) and (x_2, x_4), (x_1, x_4) and (x_2, x_3). Let $d(x, y)$ denote the distance between points x and y in X. If X is a tree, it is easily seen that the first two biggest sums $d(x_i, x_j) + d(x_k, x_m)$ always coincide. (This property is called the 0-*hyperbolicity*.) On a standard sphere of radius r there exist four points (for instance, the points that divide the equator into four equal parts) such that the biggest sum differs from the other sums by πr.

It follows that for any map of a tree to the sphere S^2 that distorts distances by less than ε, we must have $4\varepsilon \geq \pi r$. Hence, the Gromov–Hausdorff distance between the sphere and a tree cannot be less than $\frac{\sqrt{\pi}}{16}\sqrt{\operatorname{Vol} S^2}$.

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Similarly, a thin torus (the product of a very small circle and a circle with fixed radius R) can have an area as small as we want, but it cannot be approximated by a tree with an error less than $\frac{R}{100}$. We can produce examples of larger genus by taking a graph of genus g, embedding it in the three-dimensional space, and taking the boundary of its small neighborhood as the target manifold. Thus, it is hopeless to try to approximate surfaces of genus g by graphs of smaller genus.

The proof of Theorem 1 is split into several steps.

In §1 a metric graph Γ is constructed along with a continuous surjective map $f : M \to \Gamma$. In the subsequent sections this map will give us the necessary estimate of the Gromov–Hausdorff distance between X and Γ. Simple topological and metric properties of the construction are stated in §1.

In §2 we derive an estimate for the diameter of the preimage of a point under the map f.

In §3 we prove the inequality $\operatorname{dis} f \le C(g)\sqrt{\operatorname{Vol} M}$. Using this estimate, we derive Theorem 1 (the quantity dis is introduced in Proposition 3.1).

A closely related question was considered by Gromov in [Gr, Appendix I]). The paper [Gr] also contains a brief description of a method and an estimate similar to those described in §§1 and 2 of the present paper.

§1. The construction of the graph Γ

Let M be a compact oriented two-dimensional Riemannian manifold of genus g. We fix a point $O \in M$ and choose some positive numbers δ, ε such that $\varepsilon < \frac{\sqrt{v}}{100(g+1)}$, where $v = \operatorname{Vol} M$, $\delta < \frac{\varepsilon}{\operatorname{diam}(M)}$, and $\delta < 0.01$. Consider the function $g(x) = d(x, O)$ on M. This is a 1-Lipschitz function with Lipschitz constant equal to 1.

We approximate g by a smooth $\left(1 + \frac{\delta}{2}\right)$-Lipschitz function (for instance, by mollification). Then we approximate the latter function by a Morse function h so that h uniformly approximates g with precision $\frac{\varepsilon}{2}$ and remains $(1 + \delta)$-Lipschitz. This is possible because the Morse functions are dense in the space $C^1(M)$.

It is known (see [BZ, Chapter 3, 5.1(5)]) that every continuous map $h : M \to \mathbb{R}$ admits a unique decomposition

$$M \xrightarrow{f} \Gamma \xrightarrow{p} \mathbb{R},$$

where the elements of Γ are the connected components of the sets $h^{-1}(x)$, $x \in \mathbb{R}$, and the topology on Γ is induced from M. Here the map f is "monotone" in the sense that for any $x \in \Gamma$ the set $f^{-1}(x)$ is connected, and the map p is "pure" in the sense that each connected component of $p^{-1}(x)$ for $x \in \mathbb{R}$ consists of only one point.

When h is a Morse function, Γ is a topological graph. Its vertices correspond to the critical points of the function h: the leaf vertices (i.e., vertices of degree 1) correspond to the local minima and maxima of h (i.e., to the critical points of index 0 and 2), and the vertices of degree 3 correspond to the critical points of index 1. This can be seen directly from the Morse lemma, which says that for some neighborhood of a critical point of index 1 there exists a coordinate chart on M such that the function h can be written as $x^2 - y^2 + h(0)$.

Note that the function p is strictly monotone inside the edges of the graph Γ. (Otherwise, h has an extremum at some point in the f-preimage of the interior of the edge, which is impossible because all the points in that preimage are regular.)

We introduce the length metric on Γ by setting the length of a segment of an edge to be equal to the absolute value of the difference of p at the endpoints of the segments. Then the restriction of p to any edge is an isometry onto its image.

Proposition 1.1. *The function f is $(1+\delta)$-Lipschitz, i.e., for any $x, y \in M$ we have $d_\Gamma(f(x), f(y)) \le (1+\delta) d_M(x, y)$.*

By the choice of δ, it follows that $d_\Gamma(f(x), f(y)) \le d_M(x, y) + \varepsilon$.

Proof. Since we are working with length metrics and the map f is continuous, it suffices to consider the case where x and y belong to one and the same edge. But then the statement follows from the fact that h is $(1+\delta)$-Lipschitz. □

We also note that the distances between the point O and the other points of M are distorted by the map f by at most ε.

These two properties of the metric on Γ are the only properties to be used in what follows.

Proposition 1.2. *Let M and Γ be compact metric spaces, and let $f : M \to \Gamma$ be a continuous map such that the preimage of any point is connected. Then the preimage of any connected subset of Γ is connected.*

Proof. Suppose the contrary. Then there exists a connected set $Y \subset \Gamma$ such that $f^{-1}(Y) = X_1 \sqcup X_2$, where the disjoint subsets X_1, X_2 are closed in M and, hence, are compact because M is compact. Their images $f(X_1)$, $f(X_2)$ must intersect; otherwise, Y is represented as a union of two disjoint nonempty compact (hence closed) sets, which contradicts the fact that Y is connected.

For any point $q \in f(X_1) \cap f(X_2)$ the set $f^{-1}(q)$ is closed and can be represented as $Z_1 \sqcup Z_2$, where the $Z_i = X_i \cap f^{-1}(q)$ are some nonempty closed sets (they are finite intersections of closed sets). Hence, $f^{-1}(q)$ is not connected, a contradiction. □

Corollary 1.3. *The preimage of any path in Γ is a connected subset of M; hence the preimage of any linearly connected subset of Γ is a connected subset of M.*

From the above it easily follows that the number of connected components of any closed or open subset of Γ is equal to the number of connected components of its preimage in M.

It is easy to show that the f-preimage of any interior point of the edge is a one-dimensional submanifold of M homeomorphic to a circle, and that the preimage of any segment contained in the interior of an edge is a submanifold homeomorphic to a cylinder with the corresponding boundary.

Proposition 1.4. *After deletion of any $g+1$ edges of Γ, the graph Γ becomes disconnected.*

Proof. Assume there exists a collection of $g+1$ edges such that the graph remains connected if we delete them. We pick a point inside each of those edges. The preimage of this collection of points consists of $g+1$ disjoint submanifolds of M. Each of them is homeomorphic to a circle.

By assumption, the complement of the selected collection of points in Γ is linearly connected. Hence, by Corollary 1.3, the complement of the collections of circles on M is connected. Thus, we have found a collection of $g+1$ disjoint simple closed curves that do not split M. This contradicts the definition of the genus of M. □

Remark 1 (warning). The function $g(x) = d_M(x, O)$ has one local minimum, i.e., the set $M_{g \le t} = g^{-1}([0, t))$ is always connected. This is no longer true for the approximating function h; however, it suffices to consider the connected component of the set $M_{h \le t}$ that contains O (see the proof of Lemma 2.2).

§2. An estimate for the diameter of the connected component of $h^{-1}(t)$

In this section, $h : M \to \mathbb{R}$ is a smooth $(1+\delta)$-Lipschitz function that approximates the distance function to the point O ε-uniformly. Here M is a two-dimensional compact Riemannian manifold of genus g, $\delta < 0.01$, and ε is as in §1, i.e., $\varepsilon < \frac{\sqrt{\operatorname{Vol} M}}{100(g+1)}$.

To estimate the volume, we use the following well-known statement.

Proposition 2.1 (coarea formula)**.** *Let M be a two-dimensional manifold, and let $h : M \to [t_1, t_2] \subset \mathbb{R}$ be a smooth L-Lipschitz map. Then*

$$\operatorname{Vol} h^{-1}([t_1, t_2]) \geq \int_{t_1}^{t_2} \frac{\operatorname{Length} h^{-1}(t)}{L}\, dt.$$

Since h is a smooth function, we know that $h^{-1}(t)$ is a collection of circles for almost all t. For every $B \subset M$, we introduce the quantity μB as the sum of the diameters of the connected components of the set B. It is clear that, by definition, $\operatorname{Length} h^{-1}(t) \geq \mu h^{-1}(t)$.

The next statement about the behavior of $\mu h^{-1}(t)$ allows us to estimate the diameter of a connected component of $h^{-1}(t)$.

Lemma 2.2. *Let $t_0 > h(O)$ be a regular value of the function h, let $t > t_0$, and let A be a connected component of the set $h^{-1}(t)$. Then for $\Delta = t - t_0$ we have*

$$\mu h^{-1}(t_0) \geq \operatorname{diam} A - (g+1)(2\Delta + 4\varepsilon).$$

Proof. The preimage $h^{-1}(t_0)$ separates $h^{-1}(t)$ away from O. Since t_0 is a regular value of h, $h^{-1}(t_0)$ consists of several simple smooth closed curves. Let $B \subset h^{-1}(t_0)$ be the smallest collection of these curves that still separates A away from O. For every curve in B there exists a simple path from A to O that intersects $h^{-1}(t_0)$ only at points of this curve. It is easily seen that B consists of at most $g+1$ curves. Indeed, otherwise there exists a collection of $g+1$ closed curves in B that does not split M (because the set A is connected). This contradicts the definition of g.

It is clear that $\mu h^{-1}(t_0) \geq \mu B$. We prove that

$$\mu B \geq \operatorname{diam} A - (g+1)(2\Delta + 4\varepsilon).$$

Joining every point x in A with the point O via a shortest path, we let $x \in A_i$ if this path intersects the ith connected component B_i of B.

Thus, we have covered the set A by at most $g+1$ closed sets.

Every set A_i is compact, being a closed subset of a compact set M. Therefore, its diameter is attained at some pair of points x_i, y_i. We shall show that $\operatorname{dist}_M(x_i, B_i) \leq t - t_0 + 2\varepsilon$. Indeed, by the definition of A_i, there exists a point $b_i \in B_i$ such that

$$d_M(x_i, b_i) = d_M(x_i, O) - d_M(b_i, O) < t + \varepsilon - (t_0 - \varepsilon).$$

(A similar argument will be used in Lemma 3.6.) So, $\operatorname{diam} A_i \leq \operatorname{diam} B_i + 2(t - t_0 + 2\varepsilon)$. Since $\operatorname{diam} A \leq \sum_i \operatorname{diam} A_i$, the statement is proved. □

Now it is easy to deduce the main statement of this section.

Proposition 2.3. *For any $t \in \mathbb{R}$, the diameter of any connected component of the set $h^{-1}(t)$ does not exceed $3\sqrt{(g+1)\operatorname{Vol} M}$.*

Proof. Let L be the diameter of some connected component of the preimage $h^{-1}(t)$. Take $\Delta = \frac{L}{4(g+1)}$ and consider the segment $[t - \Delta, t]$. We may assume that this segment is included in the range of h and that $t - \Delta > h(O)$. (If this fails, we have $t < \varepsilon + \Delta$ because $|h(O)| < \varepsilon$. Therefore, since $h(x)$ approximates $d_M(x, O)$, the distance from O to the points of $h^{-1}(t)$ does not exceed $\Delta + 2\varepsilon$. Then $L \leq \operatorname{diam} h^{-1}(t) \leq 2\Delta + 4\varepsilon$, whence

$L \le 8\varepsilon$, which is much smaller than $3\sqrt{(g+1)\operatorname{Vol} M}$ by the choice of ε.) For almost all points x in this interval we have the estimate obtained in Lemma 2.2:

$$\operatorname{Length} h^{-1}(x) \ge \mu h^{-1}(x) \ge L - (g+1)\Big(2\frac{L}{4(g+1)} + 4\varepsilon\Big) \ge \frac{L}{2} - 4\varepsilon(g+1).$$

Therefore, by the coarea formula,

$$(1+\delta)\operatorname{Vol} h^{-1}([t-\Delta, t]) \ge \frac{L}{4(g+1)}\left(\frac{L}{2} - 4\varepsilon(g+1)\right) = \frac{L^2}{8(g+1)} - \varepsilon L.$$

Then

$$(1+0.01)\operatorname{Vol} M \ge \frac{L^2}{8(g+1)} - \varepsilon L \ge \frac{L^2}{8(g+1)} - \frac{\sqrt{\operatorname{Vol} M}}{100(g+1)} L.$$

Finally, since $100 > 96 = 12 \cdot 8$, we get the quadratic inequality

$$L^2 - \frac{L}{12}\sqrt{\operatorname{Vol} M} - 8.08(g+1)\operatorname{Vol} M \le 0,$$

which implies that

$$L \le 3\sqrt{(g+1)\operatorname{Vol} M}. \qquad \square$$

§3. An estimate for $\operatorname{dis} f$

For a map $f : M \to \Gamma$, consider the number

$$\operatorname{dis} f = \sup_{x,y \in M} |d_M(x,y) - d_\Gamma(f(x), f(y))|.$$

Our aim in this section is to prove that the graph Γ and the map f constructed in §1 obey the following estimate.

Proposition 3.1. $\operatorname{dis} f \le 6(g+1)^{3/2}\sqrt{\operatorname{Vol} M}$.

Since the map f is surjective, we can use it to construct a relation R between M and Γ such that $\operatorname{dis} R = \operatorname{dis} f$. It is known (see [BI]) that then the Gromov–Hausdorff distance between M and Γ does not exceed $\frac{1}{2}\operatorname{dis} R$. Thus, Proposition 3.1 implies Theorem 1 with the estimate $3(g+1)^{3/2}\sqrt{\operatorname{Vol} M}$.

The rest of this section is devoted to the proof of Proposition 3.1. Let δ and ε be as in §1. First, we briefly review the properties of the construction to be used in the proof. We denote $f(O) \in \Gamma$ by θ.

Proposition 3.2. *For any points $x, y \in M$ we have*

$$d_\Gamma(f(x), f(y)) - d_M(x,y) < \varepsilon.$$

By the choice of δ, this follows from the fact that f is $(1+\delta)$-Lipschitz (see Proposition 1.1).

Proposition 3.3. *For every point $p \in M$ we have*

$$|d_M(p, O) - d_\Gamma(f(p), \theta)| < \varepsilon.$$

This follows immediately from the construction of the metric on Γ and the fact that h approximates the function $d_M(O, p)$.

Recall also that the f-preimage of any path in Γ is a closed connected subset of M (see Corollary 1.3), and that, by construction, the degree of the vertices of Γ is at most 3, while the genus of Γ is at most g.

Proposition 3.4. *For every point $x \in \Gamma$ we have*

$$\operatorname{diam} f^{-1}(x) \le 3\sqrt{(g+1)\operatorname{Vol} M}.$$

Proof. Since $h = p \circ f$, it suffices to apply Proposition 2.3 to $t = p(x)$ and use the connectedness of $f^{-1}(x)$. □

Proposition 3.2 shows that the proof of Proposition 3.1 will be complete if we verify the following estimate.

Proposition 3.5. *For all $x, y \in M$ we have*

$$d_M(x,y) \le d_\Gamma(f(x), f(y)) + 6(g+1)^{3/2}\sqrt{\operatorname{Vol} M}.$$

The proof of this proposition is preceded by a sequence of lemmas. Let A, B, C be some closed subsets of the graph Γ (some of them may be singletons). We say that B *separates* A and C if any path in Γ that joins A and C intersects B. (The sets are not assumed to be disjoint; for instance, B may include A or C.)

Lemma 3.6. *Suppose $P \in M$, $r \in \Gamma$, and there exists a point $R \in f^{-1}(r)$ such that*

$$d_M(P,R) + d_M(R,O) = d_M(P,O).$$

Then

$$\operatorname{dist}_M(P, f^{-1}(r)) \le d_\Gamma(f(P), r) + 2\varepsilon.$$

Proof. Since $f(R) = r$, Proposition 3.3 yields

$$\begin{aligned} d_M(P,R) = d_M(P,O) - d_M(R,O) &\le d_\Gamma(f(P),\theta) + \varepsilon - (d_\Gamma(r,\theta) - \varepsilon) \\ &\le d_\Gamma(f(P), r) + 2\varepsilon. \end{aligned}$$ □

In what follows, $[ab]$ denotes the shortest path that realizes the distance between $a, b \in \Gamma$.

Lemma 3.7. *Suppose $P \in M$, $r \in \Gamma$, $p = f(P)$. If the point r separates p and θ, then*

$$\operatorname{dist}_M(P, f^{-1}(r)) \le d_\Gamma(p, r) + 2\varepsilon.$$

Proof. Let $\gamma = [PO]$. Then $f(\gamma)$ joins p and θ, so that it passes through r. Hence, γ intersects $f^{-1}(r)$, and we are under the conditions of Lemma 3.6. □

Lemma 3.8. *Assume that the set $\Gamma \setminus \{p,q\}$ is not connected for some points p, $q \in \Gamma$. If γ is a path joining p and q, and θ and γ lie in different connected components of $\Gamma \setminus \{p,q\}$ (i.e., $\{p,q\}$ separates γ and θ), then*

$$\operatorname{dist}_M(f^{-1}(p), f^{-1}(q)) \le \operatorname{Length}\gamma + 4\varepsilon.$$

If γ is the shortest path ($\gamma = [pq]$), then

$$\operatorname{dist}_M(f^{-1}(p), f^{-1}(q)) \le d_\Gamma(p,q) + 4\varepsilon.$$

Proof. Note that $f^{-1}(\gamma)$ is a connected compact set (see Corollary 1.3), and that for any $X \in f^{-1}(\gamma)$ the shortest path $[XO]$ must intersect either $f^{-1}(p)$ or $f^{-1}(q)$. Accordingly, the points of $f^{-1}(\gamma)$ fall into one of two closed sets, which cannot be disjoint because $f^{-1}(\gamma)$ is connected.

Let R be a point in their intersection. Then there exist points $P \in f^{-1}(p)$ and $Q \in f^{-1}(q)$ such that

$$d_M(R,O) = d_M(R,P) + d_M(P,O) = d_M(R,Q) + d_M(Q,O).$$

We put $r = f(R)$. By Lemma 3.6, we have

$$\operatorname{dist}_M(R, f^{-1}(p)) \le d_\Gamma(p,r) + 2\varepsilon,$$
$$\operatorname{dist}_M(R, f^{-1}(q)) \le d_\Gamma(q,r) + 2\varepsilon,$$

whence

$$\operatorname{dist}_M(f^{-1}(p), f^{-1}(q)) \le d_\Gamma(p,r) + d_\Gamma(q,r) + 4\varepsilon.$$

Since $r \in \gamma$, we get $d_\Gamma(p,r) + d_\Gamma(r,q) \le \operatorname{Length}\gamma$. □

Proof of Proposition 3.5. We shall use an appropriate subdivision of the shortest path $\gamma = [f(x)f(y)] = [pq]$ to reduce the general case to Lemmas 3.8 and 3.7.

Lemma 3.9. *Let Γ be a finite topological graph with marked point θ. Assume that the genus of Γ is equal to g, and that the degrees of the vertices of Γ do not exceed* 3.

Then for every path $\gamma = [pq]$ in Γ that does not intersect itself, there exist points $\{r_i\}_1^n$, $n \le 2g+1$, that split γ into segments $[pr_1], [r_ir_{i+1}], [r_nq]$ such that r_1 separates p and θ, r_n separates q and θ, and the sets $\{r_i, r_{i+1}\}$ separate $[r_ir_{i+1}]$ and θ.

Thus, Lemma 3.7 *applies to the segments $[pr_1]$ and $[r_nq]$, and Lemma* 3.8 *to each of the segments $[r_ir_{i+1}]$.*

We postpone the proof and deduce Proposition 3.5. We apply Lemma 3.9 to the path $[pq]$. Let $R_i = f^{-1}(r_i)$. By Lemmas 3.7 and 3.8, we have

$$\begin{aligned} \operatorname{dist}_M(x, R_1) &\le d_\Gamma(p, r_1) + 2\varepsilon, \\ \operatorname{dist}_M(R_i, R_{i+1}) &\le d_\Gamma(r_i, r_{i+1}) + 4\varepsilon, \\ \operatorname{dist}_M(R_n, y) &\le d_\Gamma(r_n, q) + 2\varepsilon. \end{aligned}$$

Let A, R_i, C be closed subsets of M. It is clear that

$$\operatorname{dist}_M(A, C) \le \operatorname{dist}_M(A, R_i) + \operatorname{dist}_M(R_i, C) + \operatorname{diam} R_i.$$

Summing the inequalities obtained, we get the estimate

$$d_M(x,y) \le d_\Gamma(p,q) + n \max\{\operatorname{diam} R_i\} + 4n\varepsilon.$$

Now, the result follows from the inequality $n \le 2g+1$ and the diameter estimate in Proposition 3.4. □

Proof of Lemma 3.9. First, we consider the case of zero genus, when Γ is a tree. Then the path $\gamma = [pq]$ lies in the union of the paths $[p\theta]$ and $[\theta q]$. (Any two points in a tree can be joined by a unique simple path.) Consequently, there exists a (unique) point $r \in [pq]$ (the point "closest" to θ among the points of γ) that separates p and q from θ ($r = \theta$ if γ passes through θ).

Now we pass to the general case. We know that the degrees of the vertices of Γ do not exceed 3. We contract Γ consecutively by removing edges that start at leaf vertices. As a result, we get a graph $\Gamma' \subset \Gamma$ whose vertices have degrees either 2 or 3.

Let E be the number of the edges in Γ', and let V_i denote the number of vertices of Γ' that have degree i. We can evaluate the number of the vertices of degree 3:

$$\begin{aligned} 3V_3 + 2V_2 &= 2E, \\ V_3 + V_2 &= E - g + 1 \end{aligned}$$

(we get a tree after removing g edges). Hence (multiply by two and subtract), we have $V_3 = 2g-2$. Let $v_1, \dots, v_{2g-2}$ be the vertices of degree 3.

Note that for every point in $\Gamma \setminus \Gamma'$ there exists a unique shortest path joining it with Γ' and intersecting Γ' at only one point (this path realizes the distance to Γ' with respect to every intrinsic metric on Γ).

We put $\gamma' = \gamma \cap \Gamma'$. If the set γ' is not empty, it is an internal segment of the path γ (the path γ meets every point of the graph at most once; therefore, if it enters the contractible part of the graph, i.e., the set $\Gamma \setminus \Gamma'$, it cannot leave this set afterwards).

If $\theta \notin \Gamma'$, then θ belongs to the contractible part of Γ and there exists a unique simple path $[\theta x]$ connecting θ with Γ'. If $\theta \in \Gamma'$, we take $x = \theta$ for convenience. We consider two special cases where there exists a point $r \in \gamma$ separating p, q and θ.

Case 1: *The segment* $[\theta x]$ *intersects* γ. Clearly, the intersection $[\theta x] \cap \gamma$ is some segment $[cd]$. Let $[\theta x] = [\theta c] \cup [cd] \cup [dx]$. Then we can take $r = c$, because there exists a unique simple path joining c and θ.

Case 2: *The set* γ' *is empty.* We may assume that the intersection $[\theta x] \cap \gamma$ is empty. Then there exists a point $a \in \Gamma$ that separates γ and $\Gamma' \cup \{\theta\}$. Consider the closure of the connected component of $\Gamma \setminus \{a\}$ that contains γ (in this case taking closure means putting a back). This closure is simply connected, i.e., it is a tree. Hence (as in the case of zero genus), we can separate γ and a. This suffices, because any path joining p or q with θ goes through a.

In the sequel it is assumed that γ' is nonempty and that none of the points of γ mentioned above coincides with θ. Thus, $\gamma' = [ab]$, and we have represented γ as $[pq] = [pa] \cup [ab] \cup [bq]$. Moreover, $[ab] \subset \Gamma'$, the paths $[pa]$, $[bq]$ lie in contractible parts of the graph, the point a separates p and θ, and the point b separates q and θ.

Since $[\theta x] \cap \gamma$ is empty, the vertices $\{v_i : v_i \in \gamma\}$ split γ' into parts to which Lemma 3.8 applies. The resulting subdivision $\{r_i\} = \{a, b\} \cup \{v_i : v_i \in \gamma\}$ consists of at most $2g - 2 + 2 = 2g$ points. $\square$

Proposition 3.1 follows from the inequalities obtained in Propositions 3.2 and 3.5. This completes the proof of Theorem 1.

Remark 2. It can be shown that it suffices to have $g - 1$ points to subdivide the path $[ab]$, i.e., multiplication by two can be lifted in the estimate of Proposition 3.5.

It remains unclear whether the distance between the surface and the graph really depends on the genus of the surface. Probably, the estimate obtained is not optimal in the sense of the dependence on the genus.

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