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ON LOCALLY $GQ(s,t)$ GRAPHS WITH STRONGLY REGULAR μ-SUBGRAPHS

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ABSTRACT. The connected locally $GQ(s,t)$ graphs are studied in which every μ-subgraph is a known strongly regular graph (i.e., $K_{m,m}$ for a positive integer m, the Moore graph with parameters $(k^2+1,k,0,1)$, $k=2,3$, or 7, the Clebsch graph, the Gewirtz graph, the Higman–Sims graph, or the second neighborhood (with parameters $(77,16,0,4)$) of a vertex in the Higman–Sims graph). It is proved that if Γ is a strongly regular locally $GQ(s,t)$ graph in which every μ-subgraph is isomorphic to a known strongly regular graph Δ, then one of the following statements is true: (1) $\Delta = K_{t+1,t+1}$ and either $s=1$ and $\Gamma = K_{3\times(t+1)}$, or $s=4, t=1$, and Γ is a quotient of the Johnson graph $\overline{J}(10,5)$, or $s=t=1,2,3,8,13$; (2) Δ is a Petersen graph and Γ is a unique locally $GQ(2,2)$ graph with parameters $(28,15,6,10)$; (3) Δ is the Gewirtz graph and Γ is the McLaughlin graph.

INTRODUCTION

We consider nonoriented graphs with no loops and no multiple edges. For a vertex a of a graph Γ, we denote by $\Gamma_i(a)$ the i-neighborhood of a, i.e., the subgraph induced by Γ on the set of all vertices lying at a distance of i from a. We put $\Gamma(a)=\Gamma_1(a)$ and $a^\perp = \{a\}\cup\Gamma(a)$. If the graph Γ is fixed, then we write $[a]$ instead of $\Gamma(a)$. Unless otherwise specified, by a subgraph we mean an induced subgraph. Largely, we use the same notation as in [1].

Let $\mathcal{F}$ be a class of graphs. We say that a graph Γ is a *locally $\mathcal{F}$ graph* if $[a]$ lies in $\mathcal{F}$ for each vertex a of Γ. If the class $\mathcal{F}$ consists of graphs isomorphic to a graph Δ, then we say that Γ is a *locally Δ graph*.

Let Γ be a graph, and let $a,b\in\Gamma$. The number of vertices in $[a]\cap[b]$ is denoted by $\mu(a,b)$ if the distance between a and b in Γ is 2 and by $\lambda(a,b)$ if a and b are adjacent in Γ. In the sequel, the induced subgraph $[a]\cap[b]$ is called a *μ-subgraph* (respectively, a *λ-subgraph*).

By the *degree of a vertex* we mean the number of vertices in the neighborhood of that vertex. A graph Γ is said to be *regular* of degree k if the degree of every vertex a in Γ is equal to k. A graph Γ is *edge regular* with parameters (v,k,λ) if it has v vertices, is regular of degree k, and each edge of it lies in λ triangles. A graph Γ is *completely regular* with parameters (v,k,λ,μ) if it is edge regular with the corresponding parameters and $[a]\cap[b]$ contains μ vertices for any two vertices a and b that are 2 apart in Γ. A completely regular graph is said to be *strongly regular* if it has diameter 2. By $K_{m_1,\dots,m_n}$ we denote a complete multipartite graph $\{M_1,\dots,M_n\}$ with parts M_i of order m_i. If $m_1=\dots=m_n=m$, then this graph is denoted by $K_{n\times m}$. For a subgraph Δ of Γ, we

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denote by $X_i(\Delta)$ the set of all vertices in $\Gamma - \Delta$ that are adjacent to exactly i vertices of Δ.

A *geometry G of rank* 2 is an incidence system with a set P of points and set $\mathcal{B}$ of blocks, and without multiple blocks. Here, each block can be identified with the set of points incident to it, and incidence becomes the usual inclusion. Two points in P are *collinear* if they lie in the same block. The *point graph* of a geometry G is the graph on the set P in which two points are adjacent if they are distinct and collinear. The *block graph* is defined similarly. We say that a geometry is *connected* if its point graph is connected.

For $a \in P$, we define the *residue* G_a as the geometry with the set of points P_a collinear to a and the set of blocks $\mathcal{B}_a = \{B - \{a\} \mid a \in B \in \mathcal{B}\}$. Suppose $a \in P$ and $B \in \mathcal{B}$. If $a \notin B$ (respectively, $a \in B$), then the pair (a, B) is called an *antiflag* (respectively, a *flag*).

If any two blocks in $\mathcal{B}$ intersect at one point at most, then the set of blocks is called the *set of lines* and is denoted by $\mathcal{L}$, and the geometry $(P, \mathcal{L})$ is called a *partial space of lines*. A partial space of lines has order (s, t) if each line contains $s + 1$ points and each point lies on $t + 1$ lines.

A partial space of lines of order (s, t) is called an *α-partial geometry* if for each antiflag (a, L) there exist exactly α lines passing through a and intersecting L (notation: $pG_\alpha(s, t)$). For $\alpha = 1$, such a geometry is called a *generalized quadrangle* and is denoted by $GQ(s, t)$. In the point graph for $GQ(s, t)$, a coclique consisting of $st + 1$ points is called an *ovoid*. By a *spread* in $GQ(s, t)$ we mean a set of $st+1$ lines forming a partition of the set of points.

The point graph of the geometry $pG_\alpha(s, t)$ is strongly regular, with $v = (s + 1)(1 + st/\alpha)$, $k = s(t + 1)$, $\lambda = s - 1 + t(\alpha - 1)$, and $\mu = \alpha(t + 1)$. Any strongly regular graph with these parameters is called a *pseudo-geometric graph* for $pG_\alpha(s, t)$.

A geometry G is called an *extension* of α-partial geometries if it is connected and the residue at each point coincides with $pG_\alpha(s, t)$ for appropriate (s, t) (notation: EpG_α). By connectedness, the order (s, t) of a geometry does not depend on the choice of a residue, and such an extension is denoted by $EpG_\alpha(s, t)$. The geometry EpG_α is said to be *triangular* if arbitrary three pairwise collinear points lie in the same block (which is necessarily unique). The study of triangular extensions of geometries G of class pG_α is based on the study of locally connected graphs $\Gamma(G_a)$ (and is equivalent to such a study if $\alpha = 1$).

A subset Λ of a generalized triangle is called a *hyperoval* if each line intersects Λ at 0 or 2 points. In other words, a hyperoval in $GQ(s, t)$ is a regular subgraph of degree $t+1$, without triangles, and having an even number of vertices. It is well known [2] that the μ-subgraphs in locally $GQ(s, t)$ graphs are hyperovals.

If a regular graph of degree k and diameter d has v vertices, then we have the following inequality proved by Moore:

$$v \leq 1 + k + k(k - 1) + \cdots + k(k - 1)^{d-1}.$$

A graph for which this inequality becomes equality is called a *Moore graph*. A simple example of a Moore graph is a $(2d + 1)$-gon. Damerell [3] proved that a Moore graph of degree $k \geq 3$ has diameter 2. In this case, we have $v = k^2 + 1$, the graph is strongly regular with $\lambda = 0$ and $\mu = 1$, and the degree k can be equal to 3 (the Petersen graph), to 7 (the Hoffman–Singleton graph), or to 57. Actually, it is not known whether a Moore graph of degree $k = 57$ really exists.

The Clebsch graph is defined on the set of vectors of the four-dimensional linear space V over the field of two elements; in this case, two vectors are adjacent if the Hamming distance between them is 1 or 4. This is the only strongly regular graph with parameters

$(16, 5, 0, 2)$. The Gewirtz, Higman–Sims, and McLaughlin graphs are the graphs of rank 3 of the group $L_3(4)$, the Higman–Sims group, and the McLaughlin group on 56, 100, and 275 vertices, respectively.

The Johnson graph $J(n, m)$ is defined on the set of m-element subsets of a given n-element set X, and two m-element subsets a and b are adjacent if $|a \cap b| = m - 1$. A quotient Johnson graph $J^\sigma(2m, m)$ is the factor-graph the vertices of which are pairs $(a^\sigma, X - a)$, where σ is a permutation on X of order at most 2 that has at least 8 fixed points. If $\sigma = 1$, then the quotient is said to be *standard* and is denoted by $\overline{J}(2m, m)$.

Let Γ be a connected locally $GQ(s,t)$ graph, and let u and w be its vertices with $d(u, w) = 2$. It is well known that $\mu(u, w)$ is even and $\max\{2(t+1), (s+1)(t+2-s)\} \leq \mu(u, w) \leq 2(st+1)$.

In the present paper, we study the connected locally $GQ(s,t)$ graphs in which the μ-subgraphs are known strongly regular graphs. Let Δ be a known strongly regular graph without triangles. Then one of the following possibilities is realized:

(1) $\Delta = K_{m,m}$ for some positive integer m;

(2) Δ is a Moore graph with parameters $(k^2+1, k, 0, 1)$, $k = 2, 3$, or 7;

(3) Δ is the Clebsch graph with parameters $(16, 5, 0, 2)$ or the Gewirtz graph with parameters $(56, 10, 0, 2)$;

(4) Δ is the Higman graph with parameters $(100, 22, 0, 6)$;

(5) Δ is the second neighborhood (with parameters $(77, 16, 0, 4)$) of a vertex in a Higman–Sims graph.

For each collection of parameters among those listed above, there exists a unique strongly regular graph with these parameters.

Theorem. *Let Γ be a completely regular locally $GQ(s,t)$ graph in which every μ-subgraph is isomorphic to a known strongly regular graph Δ. Then one of the following statements is valid*:

(1) $\Delta = K_{t+1,t+1}$ *and* $t+1$ *divides* $s^2(s^2-1)$;

(2) Δ *is a Petersen graph, and* Γ *is a unique locally* $GQ(2,2)$ *graph with parameters* $(28, 15, 6, 10)$;

(3) Δ *is a Hoffman–Singleton graph,* $t = 6$, *and* $s = 9, 14, 15, 24, 29,$ *or* 30;

(4) Δ *is a Clebsch graph,* $t = 4$, *and* $s = 2, 4, 6, 8, 11, 12,$ *or* 16;

(5) Δ *is a Gewirtz graph,* $t = 9$, *and* $s = 3, 7, 27, 31,$ *or* 63;

(6) Δ *is a graph with parameters* $(77, 16, 0, 4)$, $t = 15$, *and* $s = 21, 41, 55, 153,$ *or* 195;

(7) Δ *is a Higman–Sims graph,* $t = 21$, *and* $s = 19, 24, 34, 35, 39, 45, 49, 69, 84, 89, 99, 105, 115, 119, 144, 159, 175, 189, 199, 210, 214, 259, 294, 309, 339, 364, 375, 399, 419,$ *or* 420.

For $t = 1$, the Johnson graphs $J(2m, m)$ and their quotients (for $m > 4$) are locally $GQ(m-1, 1)$ graphs in which every μ-subgraph is isomorphic to a quadrangle. For $t > 1$, it is not known whether there exists a graph as in the conclusion of the theorem and having diameter greater than 2.

Corollary. *Let Γ be a strongly regular locally $GQ(s,t)$ graph in which every μ-subgraph is isomorphic to a known strongly regular graph Δ. Then one of the following statements is valid*:

(1) $\Delta = K_{t+1,t+1}$ *and either* $s = 1$ *and* $\Gamma = K_{3\times(t+1)}$, *or* $s = 4, t = 1$, *and* Γ *is a quotient of the Johnson graph* $\overline{J}(10, 5)$, *or else* $s = t = 1, 2, 3, 8,$ *or* 13;

(2) Δ *is a Petersen graph and* Γ *is a unique locally* $GQ(2,2)$ *graph with parameters* $(28, 15, 6, 10)$;

(3) Δ *is a Gewirtz graph and* Γ *is a McLaughlin graph.*

The existence and uniqueness are known for strongly regular locally $GQ(t,t)$ graphs with μ-subgraphs isomorphic to $K_{t+1,t+1}$ for $t = 1$ (Γ is a quotient of the Johnson graph

$\overline{J}(10,5)$), $t = 2$ (Γ is a unique locally $GQ(2,2)$ graph with parameters $(28, 15, 6, 10)$), and $t = 3$ (Γ is a graph of rank 3 with parameters $(176, 40, 12, 8)$ for the group $U_4(2)$).

We note that the completely regular locally $GQ(4,2)$ graphs were classified in [6]. In the paper [7], the connected locally $GQ(3,t)$ graphs are described.

In §1 we present some auxiliary results. In §2, we consider the cases where Δ is one of the Moore graphs. In §3 we assume that Δ is a Clebsch graph or a Gewirtz graph. In §4, we consider the case where Δ is a graph with parameters (77,16,0,4) or a Higman–Sims graph.

§1. Auxiliary results

Lemma 1.1. *Let Γ be a locally $GQ(s,t)$ graph. Then the maximal cliques in Γ consist of $s+2$ points* (*such cliques will be called blocks*), *each point lies in $(t+1)(st+1)$ blocks, any two points lie in $t+1$ common blocks, and any two blocks intersect at two points at most.*

Proof. All the statements follow from the definition of the extension and the properties of GQ [2]. □

Lemma 1.2. *Let Λ be a hyperoval of a generalized quadrangle $GQ(s,t)$, and let $\mu = |\Lambda|$. Then μ is even and $\mu_* \le \mu \le \mu^*$, where $\mu_* = \max\{2(t+1), (s+1)(t+2-s)\}$ and $\mu^* = 2(st+1)$. If $\mu = (s+1)(t+2-s)$* (*respectively, $\mu = \mu^*$*), *then, for each point $a \notin \Lambda$, exactly $(t+2-s)/2$ lines* (*respectively, exactly $t+1$ lines*) *containing the point a intersect Λ at two points.*

Proof. The estimates for μ and the fact that μ is even follow from Lemmas 3.9 and 3.11 in [2]. If $\mu = (s+1)(t+2-s)$, then from the proof of Lemma 3.11 in [2] it follows that, for $a \notin \Lambda$, the number of lines in $a^{\perp}$ that do not intersect Λ is equal to $(s+t)/2$.

If $\mu = \mu^*$, then, by Lemma 3.9 (b) in [2], each line containing a intersects Λ (obviously, at two points). □

Lemma 1.3. *Let Γ be a strongly regular graph with parameters (v, k, λ, μ). Then*:

(1) $k(k-\lambda-1) = \mu(v-k-1)$ (*rectangular relation*);

(2) *either Γ has parameters $(4\mu+1, 2\mu, \mu-1, \mu)$* (*the half case*), *or the secondary eigenvalues $n-m$ and $-m$ of the graph Γ are integers, where $n^2 = (\lambda-\mu)^2 + 4(k-\mu)$, $n - \lambda + \mu = 2m$, and the multiplicity of the eigenvalue $n-m$ is equal to $\frac{k(m-1)(k+m)}{\mu n}$*;

(3) *if m is an integer greater than 1, then $m-1$ divides $k-\lambda-1$ and*

$$\mu = \lambda + 2 + (m-1) - \frac{k-\lambda-1}{m-1}, \quad n = m - 1 + \frac{k-\lambda-1}{m-1}.$$

Proof. Statements (1) and (2) are well known (see, e.g., Chapter 1 in [1]). Statement (3) is Lemma 3.1 in [4]. □

The requirement that $\frac{k(m-1)(k+m)}{\mu n}$ be an integer is called the *integer condition.*

Lemma 1.4. *Let Γ be a point graph of a generalized quadrangle $GQ(s,t)$. Then Γ is a strongly regular graph with parameters $v = (s+1)(st+1)$, $k = s(t+1)$, $\lambda = s-1$, $\mu = t+1$, and $s+t$ divides $st(s+1)(t+1)$.*

Proof. The relations for the parameters follow from the definition of GQ. The integer condition for Γ takes the following form: $s+t$ divides $st(s+1)(t+1)$. □

Lemma 1.5. *Let Γ be a strongly regular locally $GQ(s,t)$ graph with parameters (v, k, λ, μ). Then*:

(1) $k = (s+1)(st+1)$, $\lambda = s(t+1)$, *and $s+t$ divides $st(s+1)(t+1)$*;

(2) $t \leq m-1 \leq st$, $m-1$ *divides* s^2t, $\mu = s(t+1)+m+1-s^2t/(m-1)$, *and* $n = m-1+s^2t/(m-1)$;

(3) $\mu \cdot n$ *divides* $(m-1)(s+1)(st+1)((s+1)(st+1)+m)$;

(4) $s+2$ *divides* $2t(t+1)(2t-1)(4t+\mu-2)$.

Proof. Statement (1) follows from Lemma 1.4. Statements (2) and (3), except for the inequality $t \leq m-1 \leq st$, follow from the definition of GQ and Lemma 1.3. The inequality itself follows from the fact that $(s+1)(t+2-s) \leq \mu \leq 2(st+1)$.

The number of blocks in Γ is equal to $v(t+1)(st+1)/(s+2)$. This implies statement (4). The lemma is proved. □

§2. The case of Moore graphs

In what follows, Γ is a completely regular locally $GQ(s,t)$ graph in which every μ-subgraph is isomorphic to a known strongly regular graph Δ. Then $k = (s+1)(st+1)$ and $\lambda = s(t+1)$. Let $\Delta = [a] \cap [b]$. We note that each line L of the generalized quadrangle $[a]$ intersects Δ at 0 or 2 points. Indeed, if $c \in L \cap \Delta$, then $M = (L \cup \{a\}) - \{c\}$ is a line in $[c]$, and, by the definition of a generalized quadrangle, $[b]$ contains a unique point d in M. Thus, $\{c,d\} = L \cap \Delta$.

In this section, we consider the case where Δ is a complete bipartite graph or a Moore graph.

Lemma 2.1. *If* $\Delta = K_{t+1,t+1}$, *then* $t+1$ *divides* $s^2(s^2-1)$.

Proof. From the rectangular relation $k(k-\lambda-1) = k_2\mu$, where $k_2 = |\Gamma_2(a)|$ for every vertex a, we conclude that $2(t+1)$ divides $(s+1)(st+1)s^2t$.

Since $t+1$ is prime to t, we have $(t+1, st+1) = (t+1, s-1)$, and $t+1$ divides $s^2(s^2-1)$. The lemma is proved. □

Lemma 2.2. *If* $\Delta = K_{t+1,t+1}$ *and* Γ *is strongly regular, then either* $s=1$, *or* $s=4$ *and* $t=1$, *or else* $s=t=2,3,8$, *or* 13.

Proof. By [8, Theorem 3.1], either the conclusion of the lemma is valid, or $s=t=4$. However, by [9], in the latter case there are no such graphs. □

We note that a locally $GQ(1,t)$ graph coincides with $K_{3\times(t+1)}$. Let $s>1$. For small values of t, some additional information is known. If $t=1$, then the neighborhoods of the vertices in Γ are $(s+1)\times(s+1)$ lattices, and by [5], Γ is a Johnson graph or its quotient. If $t=2$, then $s=2$ or 4. In the case where $s=2$, the graph Γ is a unique locally $GQ(2,2)$ graph with parameters $(36,15,6,6)$. If $s=4$, then, by [6], Γ is a unique distance-regular locally $GQ(4,2)$ graph on 378 vertices with the intersection array $(45,32,12,1;1,6,32,45)$.

If $t=3$, then $s=3,5$, or 9. If $s=3$, then, by [7], Γ is a unique locally $GQ(3,3)$ graph with parameters $(176,40,12,8)$. If $t=4$, then 5 divides $s^2(s^2-1)$, so that $s=4,6,11$, or 16. For $s=4$, the graph Γ is strongly regular and, by a result in [9], does not exist.

Let Δ be a Moore graph. If Δ is a pentagon, then $t=1$ and Γ is locally an $(s+1)\times(s+1)$ lattice. However, in this case the connected components of the μ-subgraphs are cycles of even length, a contradiction.

Lemma 2.3. *If* Δ *is a Petersen graph, then* Γ *is a unique strongly regular locally* $GQ(2,2)$ *graph with parameters* $(28,15,6,8)$.

Proof. If Δ is a Petersen graph, then $t=2$ and $s=1,2$, or 4. For $s=1$, we obtain $\mu=6$, a contradiction. If $s=2$, then the conclusion of the lemma is valid (see, e.g., [2]).

Let $s=4$. Then, by [6], for a completely regular locally $GQ(4,2)$ graph we have $\mu=6$ or $\mu=18$. The lemma is proved. □

For the remainder of this section, we assume that Δ is a Hoffman–Singleton graph. Then $t = 6$. By the integer condition, $s+6$ divides $42s(s+1)$, and $s = 3, 4, 6, 8, 9, 12, 14, 15, 22, 24, 29, 30$, or 36. Since $\mu = 50$ divides $6s^2(s+1)(6s+1)$, we have $s = 4$, 9, 14, 15, 24, 29, or 30.

Lemma 2.4. *Let a and b be vertices in Γ with $d(a,b) = 2$, and let $\Delta = [a] \cap [b]$ be a Hoffman–Singleton graph. Then for $x_i = |X_i(\Delta) \cap [b]|$ we have*:
(1) *$x_i = 0$ for odd i and for $i > 8$*;
(2) *$\sum x_i = (s+1)(6s+1) - 50$ and $\sum ix_i = 50(7s-7)$.*

Proof. If a vertex c in $[b]-[a]$ is adjacent to a vertex in Δ, then $[c]\cap\Delta$ consists of isolated edges. Observe that, for an edge $\{u,w\}$ of Δ, the subgraph $\Delta - ([u] \cup [w])$ consists of three isolated edges. Therefore, $x_i = 0$ for odd i and for $i > 8$. Statement (1) is proved.

Next, $[b] - \Delta$ contains $(s+1)(6s+1) - 50$ vertices, and each vertex of Δ is adjacent to $7s-7$ vertices of $\Gamma - \Delta$. This implies (2). □

Lemma 2.5. *The parameter s is not equal to* 4.

Proof. If $s = 4$, then $\mu = 2(st+1) = 50$, and by Lemma 1.2, the graph Γ is strongly regular with parameters $(366, 125, 28, 50)$. Now, $n^2 = (\lambda-\mu)^2 + 4(k-\mu) = 28^2$, and the secondary eigenvalues of the graph are $n - m = 3$ and $-m = -25$. By the integer condition, $n\mu = 28 \cdot 50$ divides $(m-1)k(k+m) = 24 \cdot 125 \cdot 150$, a contradiction. □

Lemma 2.6. *The graph Γ is not strongly regular.*

Proof. If Γ is a strongly regular graph with the smallest eigenvalue $-m$, then $m-1$ divides $6s^2$ and $\mu = 50 = \lambda + 2 + (m-1) - 6s^2/(m-1)$.

If $s = 9$, then $\lambda = 63$ and $15 + (m-1) = 6 \cdot 9^2/(m-1)$, a contradiction. If $s = 14$, then $\lambda = 98$ and $50 + (m-1) = 6 \cdot 14^2/(m-1)$. If $m-1$ is not divisible by 7, then $50 + (m-1)$ is divisible by 49, a contradiction. If $m-1$ is divisible by 7, then $(m-1)$ is divisible by 49, a contradiction.

If $s = 15$, then $\lambda = 105$ and $57 + (m-1) = 6 \cdot 15^2/(m-1)$. Therefore, $m-1 = 3r$ and $19 + r = 450/r$. If r is divisible by 5, then r is divisible by 25 and $19 + r > 450/r$, a contradiction. Thus, r is not divisible by 5 and $19 + r$ is divisible by 25. Therefore, $r = 6$ and $25 \neq 450/6$, a contradiction.

If $s = 24$, then $\lambda = 168$ and $120 + (m-1) = 6 \cdot 24^2/(m-1)$. Therefore, $m-1 = 3r$ and $40 + r = 2 \cdot 24^2/r$. If r is divisible by 3, then r is divisible by 9 and $40 + r$ is a power of 2, a contradiction. Thus, r is a power of 2 and $40 + r$ is divisible by 9, so that $r = 32$ and $72 \neq 36$, a contradiction.

If $s = 29$, then $\lambda = 203$ and $155 + (m-1) = 6 \cdot 29^2/(m-1)$. If $m-1$ is divisible by 29, then $m-1$ is divisible by 29^2, a contradiction. If $m-1$ is not divisible by 29, then $155 + (m-1) < 6 \cdot 29^2/(m-1)$.

Finally, if $s = 30$, then $\lambda = 210$ and $162 + (m-1) = 6 \cdot 30^2/(m-1)$. Therefore, $m-1 = 3^r l$. If $r \leq 1$, then $162 + (m-1)$ is not divisible by 9 and $6 \cdot 30^2/(m-1)$ is divisible by 9. If $r \geq 2$, then $162 + (m-1)$ is divisible by 9 and $6 \cdot 30^2/(m-1)$ is not divisible by 9. Thus, we obtain a contradiction in all cases. The lemma is proved. □

§3. The case of Clebsch graphs and Gewirtz graphs

In this section, we consider the case where Δ is either a Clebsch graph or a Gewirtz graph.

Lemma 3.1. *If Δ is a Clebsch graph, then $t = 4$ and $s = 2, 4, 6, 8, 11, 12$, or 16. If $s = 2$, then Γ is a locally $GQ(2,4)$ Taylor graph on 56 vertices.*

Proof. If Δ is a Clebsch graph, then $t=4$ and $s=2,4,6,8,11,12$, or 16. In all cases, 16 divides $4(s+1)(4s+1)s^2$, and the rectangular relation imposes no new restrictions.

If $s=2$, then Γ is a locally $GQ(2,4)$ Taylor graph on 56 vertices (see [2]). □

Lemma 3.2. *Let a and b be vertices in Γ with $d(a,b)=2$, and let $\Delta=[a]\cap[b]$ be a Clebsch graph. We denote by X_i the set of vertices in $[b]-[a]$ adjacent to exactly i vertices in Δ, $x_i=|X_i|$. Then:*

(1) *$x_i=0$ for odd i and for $i>8$;*

(2) *$\sum x_i=(s+1)(4s+1)-16$ and $\sum ix_i=16(5s-5)$;*

(3) *if $s=4$, then the diameter of Γ is equal to 3.*

Proof. If a vertex c of $[b]-[a]$ is adjacent to a vertex in Δ, then $[c]\cap\Delta$ consists of isolated edges. Observe that, for an edge $\{u,w\}$ of Δ, the subgraph $\Delta-([u]\cup[w])$ consists of three isolated edges. Therefore, $x_i=0$ for odd i and for $i>8$. Statement (1) is proved.

Next, $\Gamma-\Delta$ has $(s+1)(4s+1)-16$ vertices, and each vertex of Δ is adjacent to $5s-5$ vertices of $\Gamma-\Delta$. This implies (2).

Let $s=4$. Then $k=85$, $\lambda=20$, and for each vertex a we have $|\Gamma_2(a)|=340$. Also, $x_0+x_2+x_4+x_6+x_8=69$ and $x_2+x_4+3x_6+4x_8=120$.

If Γ is a strongly regular graph, then its parameters are $(426,85,20,16)$. However, in this case we have $n^2=(\lambda-\mu)^2+4(k-\mu)=292$, a contradiction.

If the diameter of Γ is at least 4 and *acbde* is a geodesic path in Γ, then $[b]$ contains some Clebsch subgraphs $\Delta_1=[a]\cap[b]$ and $\Delta_2=[b]\cap[e]$ such that each vertex of Δ_1 is at a distance of 2 from every vertex of Δ_2. We note that the number of lines in $[b]$ is 85, and 40 of them are secants for Δ_1 and do not intersect Δ_2. By symmetry, 40 lines in $[b]$ are secants for Δ_2 and do not intersect Δ_1. Thus, exactly 5 lines $L_1,\dots,L_5$ in $[b]$ do not intersect either Δ_1 or Δ_2.

For a vertex f of $[b]-(\Delta_1\cup\Delta_2)$, we put $\delta_f=|[f]\cap(\Delta_1\cup\Delta_2)|$. Then $\delta_f\le 10$, and for each secant M of Δ_2 we obtain $\sum_{f\in M}|[f]\cap\Delta_2|=12$ and $\sum_{f\in M}|[f]\cap\Delta_1|=16$. Therefore, $\sum_{f\in M}\delta_f=28$. Thus, for two points $g\in M$ we have $\delta_g=10$ (and these points do not lie on the lines L_i), and for one point $f\in M$ we have $\delta_f=8$ (and f belongs to a unique line L_i).

Thus, either there is a vertex z in $[b]$ lying on each of the five lines $L_1,\dots,L_5$, or the lines L_i and L_j are pairwise disjoint. However, in the latter case, the line passing through points lying on L_1 and L_2 is a secant for one of the subgraphs Δ_1 or Δ_2 and contains two points f with $\delta_f=8$, a contradiction.

We put $\Sigma=[b]-(z^\perp\cup\Delta_1\cup\Delta_2)$, $Y_i=X_i(\Delta_1)\cap\Sigma$, and $y_i=|Y_i|$. Then Σ is a regular graph on 32 vertices, Σ is of degree 5 and without triangles, $\sum y_i=32$, and $\sum iy_i=160$.

Now, we describe the types relative to Δ_1 of the lines that do not pass through z. If a line M does not intersect Δ_1, then this line is a secant for Δ_2 and contains one point from each of $X_i(\Delta_1)$, $X_j(\Delta_1)$, and $X_l(\Delta_1)$, so that $i+j+l=16$. We assume that the point in $M\cap X_i(\Delta_1)$ belongs to $[z]$ and $j\le l$. In this case, we say that the line M is of type (i,j,l). The structure of a Clebsch graph shows that $\{i,j,l\}\ne\{4,6,6\}$. Therefore, the lines that do not intersect Δ_1 are of types $(0,8,8)$, $(2,6,8)$, $(4,4,8)$, or $(6,2,8)$. Similarly, a secant L for Δ_1 contains one point from each of $[z]\cap X_i(\Delta_1)$, $X_j(\Delta_1)$, and $X_l(\Delta_1)$, and $j\ge l$ and $i+j+l=12$. We say that the line L is of type (i,j,l).

If a line is of type (i,j,l) relative to Δ_1, then it is of type $(8-i,10-j,10-l)$ relative to Δ_2; therefore, no secant for Δ_1 can be of type $(4,4,4)$ or $(2,6,4)$. Thus, the secants for Δ_1 are of types $(8,2,2)$, $(6,4,2)$, $(4,6,2)$, or $(2,8,2)$.

Thus, a neighborhood of a point in Y_2 contains 4 points of Y_8 and 1 point of $Y_2\cup Y_4\cup Y_6\cup Y_8$, a neighborhood of a point in Y_4 contains 3 points of Y_8 and 2 points of Y_2, a neighborhood of a point in Y_6 contains 2 points of Y_8 and 3 points of Y_2, and a neighborhood of a point in Y_8 contains 4 points of Y_2 and 1 point of $Y_0 \cup Y_2 \cup Y_4 \cup Y_6$. Now, the number of edges between Y_2 and Y_8 that lie on lines that do not intersect Δ_1 is equal to $4y_2$ and does not exceed y_8. Similarly, the number of edges between Y_2 and Y_8 that lie on lines intersecting Δ_1 is equal to $4y_8$ and does not exceed y_2. Hence, $y_2 = y_8 = 0$. This contradicts the fact that, in this case, we also have $y_4 = y_6 = 0$. □

Lemma 3.3. *If Δ is a Clebsch graph, then Γ is not strongly regular.*

Proof. Let Δ be a Clebsch graph, and let Γ be a strongly regular graph with the smallest eigenvalue $-m$. By Lemma 1.3, the number $m-1$ divides $4s^2$ and $\mu = 16 = \lambda + 2 + (m-1) - 4s^2/(m-1)$.

If $s = 6$, then $\lambda = 30$ and $16 + (m-1) = 4 \cdot 6^2/(m-1)$. Therefore, $m-1$ is a power of 2 and $16 + (m-1)$ is divisible by 9. But $18 \neq 72$, a contradiction.

If $s = 8$, then $\lambda = 40$ and $26 + (m-1) = 4 \cdot 8^2/(m-1)$, a contradiction. If $s = 11$, then $\lambda = 55$ and $41 + (m-1) = 4 \cdot 11^2/(m-1)$. Therefore, $m-1$ is not divisible by 11 and $41 + (m-1)$ is divisible by 11^2, a contradiction.

If $s = 12$, then $\lambda = 60$ and $46 + (m-1) = 4 \cdot 12^2/(m-1)$. If $m-1$ is divisible by 3, then either $m-1 = 9$ and $55 \neq 64$, or $m-1 = 18$ and $64 \neq 32$. Thus, $m-1$ is a power of 2 and $46 + (m-1)$ is divisible by 9. Therefore, $m-1 = 8$ and $54 \neq 72$, a contradiction.

If $s = 16$, then $\lambda = 80$ and $66 + (m-1) = 4 \cdot 16^2/(m-1)$, a contradiction. □

Lemma 3.4. *If Δ is a Gewirtz graph, then $t = 9$ and $s = 3, 7, 27, 31,$ or 63. If $s = 3$, then Γ is a McLaughlin graph.*

Proof. If Δ is a Gewirtz graph, then $t = 9$ and $s = 3$, 6, 7, 9, 11, 15, 18, 21, 27, 31, 36, 39, 45, 51, 63, 71, 72, or 81. Since 56 divides $9s^2(s+1)(9s+1)$, we have $s = 3, 7, 27, 31$, or 63.

If $s = 3$, then, by [7], Γ is a McLaughlin graph. □

Lemma 3.5. *If Δ is a Gewirtz graph and Γ is strongly regular, then Γ is a McLaughlin graph.*

Proof. Let Δ be a Gewirtz graph, and let Γ be strongly regular with the smallest eigenvalue $-m$. Then, by Lemma 1.3, the number $m-1$ divides $9s^2$ and $\mu = 56 = \lambda + 2 + (m-1) - 9s^2/(m-1)$. By Lemma 3.4, we have $s = 3, 7, 27, 31$, or 63.

If $s = 7$, then $\lambda = 70$ and $16 + (m-1) = 9 \cdot 7^2/(m-1)$. If $m-1$ is divisible by 7, then $m-1$ is divisible by 49 and $65 \neq 9$. Thus, $m-1$ is a power of 3 and $16 + (m-1)$ is divisible by 49, a contradiction.

If $s = 27$, then $\lambda = 270$ and $216 + (m-1) = 9 \cdot 27^2/(m-1)$, a contradiction. If $s = 31$, then $\lambda = 310$ and $256 + (m-1) = 9 \cdot 31^2/(m-1)$. Therefore, $m-1$ is not divisible by 31 and $256 + (m-1)$ is divisible by 31^2, a contradiction.

If $s = 63$, then $\lambda = 630$ and $576 + (m-1) = 9 \cdot 63^2/(m-1)$. Therefore, $m-1 = 9r$ and $64 + r = 63^2/r$. If r is divisible by 7, then r is divisible by 49 and $64 + 49 \neq 81$. If r is not divisible by 7, then r is a power of 3 and $64 + r$ is divisible by 49, a contradiction. The lemma is proved. □

§4. The case where Δ is a graph with $\mu > 2$

In this section, we consider the cases in which Δ is a graph with $\mu > 2$.

Lemma 4.1. *Let Δ be a strongly regular graph with parameters $(77, 16, 0, 4)$. Then $t = 15$ and $s = 21, 41, 55, 154,$ or 195. Moreover, Γ is not strongly regular.*

Proof. Let Δ be a strongly regular graph with parameters $(77, 16, 0, 4)$. Then $t = 15$, $s+15$ divides $15 \cdot 16s(s+1)$, and 77 divides $15s^2(s+1)(15s+1)$. Hence, $s = 21, 41, 55, 154$, or 195.

If Γ is strongly regular with the smallest eigenvalue $-m$, then, by Lemma 1.3, $m-1$ divides $15s^2$ and $\mu = 77 = \lambda + 2 + (m-1) - 15s^2/(m-1)$.

If $s = 21$, then $\lambda = 336$ and $261 + (m-1) = 15 \cdot 21^2/(m-1)$. Therefore, $15 \cdot 21^2/(m-1) - (m-1)$ is divisible by 9, a contradiction.

If $s = 41$, then $\lambda = 656$ and $581 + (m-1) = 15 \cdot 41^2/(m-1)$. Therefore, $m-1$ is not divisible by 41 and $581 + (m-1)$ is divisible by 41^2, a contradiction.

If $s = 55$, then $\lambda = 880$ and $803 + (m-1) = 15 \cdot 55^2/(m-1)$. Therefore, $m-1 = 11r$ and $71 + r = 3 \cdot 5^3/r$. We see that r is not divisible by 5 and $71 + r$ is divisible by 125, a contradiction.

If $s = 154$, then $\lambda = 2464$ and $2387 + (m-1) = 15 \cdot 154^2/(m-1)$. Therefore, $m-1 = 11r$ and $217 + r = 15 \cdot 14^2/r$. We have $r = 7l$ and $31 + l = 60/l$, a contradiction.

If $s = 195$, then $\lambda = 3120$ and $3043 + (m-1) = 15 \cdot 195^2/(m-1) = 3^3 \cdot 5^3 \cdot 13/(m-1)$. If $m-1$ is divisible by 5, then $m-1$ is divisible by 5^3 and $3043 + (m-1) > 15 \cdot 195^2/(m-1)$. If $m-1$ is divisible by 3, then $m-1$ is divisible by 3^3 and again we have $3043 + (m-1) > 15 \cdot 195^2/(m-1)$. Thus, $m-1$ is equal to 1 or 13, and $3043 + (m-1)$ is not divisible by 5, a contradiction. □

Lemma 4.2. *Let* Δ *be a Higman–Sims graph. Then* $t = 21$, $s = 19$, 24, 34, 35, 39, 45, 49, 69, 84, 89, 99, 105, 115, 119, 144, 159, 175, 189, 199, 210, 214, 259, 294, 309, 339, 364, 375, 399, 419, *or* 420. *Moreover,* Γ *cannot be strongly regular.*

Proof. Let Δ be a Higman–Sims graph with parameters $(100, 22, 0, 6)$. Then $t = 21$, $s + 21$ divides $21 \cdot 22s(s+1)$ and 100 divides $21s^2(s+1)(21s+1)$. Thus, s or $s+1$ is divisible by 5 and $s + 21$ divides $2^3 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11$. Hence, $s = 9$, 14, 15, 19, 24, 34, 35, 39, 45, 49, 69, 84, 89, 99, 105, 115, 119, 144, 159, 175, 189, 199, 210, 214, 259, 294, 309, 339, 364, 375, 399, 419, or 420.

We recall that if $t > s$, then $\mu \geq (s+1)(t+2-s)$, so that $s \geq 19$. If Γ is strongly regular and $-m$ is its smallest eigenvalue, then, by Lemma 1.3, $m-1$ divides $21s^2$ and $\mu = 100 = \lambda + 2 + (m-1) - 21s^2/(m-1)$.

Next, $n^2 = (\lambda - \mu)^2 + 4(k - \mu) = (22s - 100)^2 + 4((s+1)(21s+1) - 100)$. Therefore, $x = (11s - 50)^2 + (s+1)(21s+1) - 100$ is a square. If s is odd, then $x \equiv 1 \pmod 8$ and $s + 1 \equiv 2 \pmod 4$. If s is even, then again $x \equiv 1 \pmod 8$ and $s \equiv 0 \pmod 4$. Consequently, $s = 24, 45, 49, 69, 84, 89, 105, 144, 189, 309, 364$, or 420.

Substituting these values of s, we see that in all cases $(\lambda - \mu)^2 + 4(k - \mu)$ is not a square. □

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