Алгебра и анализ
Том. 17 (2005), вып. 3

St. Petersburg Math. J.
Vol. 17 (2006), No. 3, Pages 453–463
S 1061-0022(06)00914-9
Article electronically published on March 9, 2006

INTEGRAL MEANS SPECTRUM AND THE MODIFIED BESSEL FUNCTION OF ZERO ORDER

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Abstract. A new characteristic $\beta_f^*(t)$ of a conformal mapping f of the disk $\mathbb{D}$ onto a simply connected domain is introduced and its relationship with the so-called integral means spectrum $\beta_f(t)$ is studied. The Brennan conjecture (saying that $\beta_f(-2) \le 1$) is confirmed in the case where the Taylor series of $\log f'(z)$ is Hadamard lacunary with sufficiently large lacunarity exponent.

Introduction

Let Ω be a proper simply connected domain on the plane, and let f be a conformal mapping of the disk $D = \{|z| < 1\}$ onto Ω. In the modern theory of the boundary behavior of conformal mappings, an important part is played by the integral means spectrum

$$\beta_f(t) = \limsup_{r\to 1} \frac{\log \int_{-\pi}^{\pi} |f'(re^{i\theta})|^t \, d\theta}{\log \frac{1}{1-r}}$$

(see [6, 16, 18]). For good domains, the function $\beta_f(t)$ is piecewise linear in t. If the boundary of Ω is fractal, the behavior of the integral means spectrum is far from trivial.

In this paper, we introduce a new quantity closely related to $\beta_f(t)$. Suppose f is a locally univalent and analytic function on $D = \{|z| < 1\}$. Then $\log f'$ is also analytic in D and can be expanded in a series $\sum_{k=0}^{\infty} a_k z^k$ converging uniformly on the compact subsets of D. The quantity

$$\beta_f^*(t) = \limsup_{r\to 1} \frac{\sum_{k=0}^{\infty} \log I_0(t|a_k|r^k)}{\log \frac{1}{1-r}}$$

is called the $*$-*spectrum of integral means*. Here

$$I_0(x) = \sum_{\nu=0}^{\infty} \left(\frac{x^2}{4}\right)^{\nu} \frac{1}{\nu!^2}$$

is the modified Bessel function of zero order. Since $\log I_0(x) \le x^2/4$, we see that $\beta_f^*(t)$ is a finite function of t if

$$\limsup_{r\to 1} \frac{\sum_{k=0}^{\infty} |a_k|^2 r^{2k}}{\log \frac{1}{1-r}} < \infty.$$

This relation is fulfilled for every function f analytic and univalent in D.

In the study of the function $\beta_f(t)$, the modified Bessel function I_0 was first used by Rohde (see [18, p. 19] and [19]). He showed that if $\log f'(z) = a \sum z^{q^k}$, then $\beta_f(t) \ge \log I_0(at)/\log q$.

2000 *Mathematics Subject Classification.* Primary 30C35.
Key words and phrases. Conformal mapping, $*$-spectrum of integral means, modified Bessel function.
This article was supported in part by RFBR (grants no. 05–01–00523 and 03-01-00015), and by the NIOKR AN RT foundation.

In what follows, we study the relationship between $\beta_f(t)$ and $\beta_f^*(t)$. We shall prove that $\beta_f^*(t)$ is a continuous convex function of t (like $\beta_f(t)$). Next, we shall show that $\beta_f(t)$ can be obtained from $\beta_f^*(t)$ by "rotation" of the coefficients of $\log f'(z)$. Also, we shall see that if the Taylor coefficients a_k of $\log f'(z)$ are positive, then $\beta_f(t) \geq \beta_f^*(t)$.

Moreover, the well-known Brennan conjecture saying that $\beta_f(-2) \leq 1$ will be proved in the particular case where $\log f'(z)$ expands in a Hadamard lacunary series with an exponent $q \geq 15$.

Main results

It is well known (see [18, p. 176]) that the spectrum of integral means is a convex function. The $*$-spectrum of integral means has the same property.

Theorem 1. *Suppose f is analytic and univalent in the disk D. Then the $*$-spectrum $\beta_f^*(t)$ of integral means is a continuous convex function of t on the real axis.*

Proof. For $r \in (0,1)$ and $t \in (-\infty, \infty)$, put

$$g(t,r) = \frac{\sum_{k=0}^{\infty} \log I_0(t|a_k|r^k)}{|\log(1-r)|}.$$

We show that $g(t,r)$ is a convex function of t for every fixed $r \in (0,1)$. For this, it suffices to prove that every summand $\log I_0(t|a_k|r^k)$ is a convex function of t. To check the latter, we use the well-known identity

$$I_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(x\cos\theta)\, \mathrm{d}\theta$$

(see [18, p. 191]). Let t_1, t_2 be arbitrary reals, and let $\lambda \in [0,1]$. Then

$$\begin{aligned} \log I_0(\lambda t_1 + (1-\lambda)t_2) &= \log\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(\lambda t_1 + (1-\lambda)t_2)\cos\theta}\, \mathrm{d}\theta\right] \\ &\leq \log\left[\frac{1}{2\pi}\left(\int_{-\pi}^{\pi} e^{t_1\cos\theta}\,\mathrm{d}\theta\right)^{\lambda}\left(\int_{-\pi}^{\pi} e^{t_2\cos\theta}\,\mathrm{d}\theta\right)^{1-\lambda}\right] \\ &= \lambda \log I_0(t_1) + (1-\lambda)\log I_0(t_2) \end{aligned}$$

(we have used the Hölder inequality). Thus, the function $\log I_0(x)$ is convex and, with it, so is $g(t,r)$ as a function of t. Taking again arbitrary reals t_1, t_2 and $\lambda \in [0,1]$, we can write

$$\begin{aligned} \beta_f^*(\lambda t_1 + (1-\lambda)t_2) &= \limsup_{r\to 1} g(\lambda t_1 + (1-\lambda)t_2, r) \\ &\leq \limsup_{r\to 1}[\lambda g(t_1,r) + (1-\lambda)g(t_2,r)] \\ &\leq \lambda \limsup_{r\to 1} g(t_1,r) + (1-\lambda)\limsup_{r\to 1} g(t_2,r) = \lambda\beta_f^*(t_1) + (1-\lambda)\beta_f^*(t_2). \end{aligned}$$ □

If the Taylor coefficients of $\log f'(z)$ are nonnegative, then the $*$-spectrum of integral means does not exceed the spectrum of integral means. Here is the precise statement.

Theorem 2. *Suppose $t \geq 0$, $a_k \geq 0$, and $\sum_{k=0}^{\infty} a_k z^k = \log f'(z)$. Then $\beta_f(t) \geq \beta_f^*(t)$.*

Proof. We proceed by induction. The inductive hypothesis is formulated as follows: there exist numbers $\theta_1(r), \theta_2(r), \ldots, \theta_n(r)$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left[t \operatorname{Re}\sum_{k=1}^{n} a_k r^k e^{i(k\theta+\theta_k)}\right]\mathrm{d}\theta = \prod_{k=1}^{n} I_0(t a_k r^k). \tag{1}$$

This statement is obvious for $n = 1$. Suppose it is true for some natural number n. Consider the continuous 2π-periodic function

$$\varphi(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left[t\,\mathrm{Re}\sum_{k=1}^{n} a_k r^k e^{i(k\theta+\theta_k)} + t\,\mathrm{Re}\left(a_{n+1}r^{n+1}e^{i((n+1)\theta+x)}\right)\right]\mathrm{d}\theta.$$

The Fubini theorem yields

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(x)\,\mathrm{d}x &= I_0(ta_{n+1}r^{n+1})\frac{1}{2\pi}\int_0^{2\pi}\exp\left[t\,\mathrm{Re}\sum_{k=1}^{n} a_k r^k e^{i(k\theta+\theta_k)}\right]\mathrm{d}\theta\\ &= \prod_{k=1}^{n+1} I_0(ta_k r^k).\end{aligned}$$

By the mean value theorem for continuous functions, this implies the existence of a number $\theta_{n+1}(r)$ such that

$$\varphi(\theta_{n+1}) = \prod_{k=1}^{n+1} I_0(tr^k a_k).$$

So, we have constructed the numerical sequence θ_k. Passing to the limit as $n \to \infty$ in (1) and using the fact that the series on the left converges, we conclude that there exists a sequence $\{\theta_k\}$ of real numbers with

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[t\,\mathrm{Re}\sum_{k=1}^{\infty} a_k r^k e^{i(k\theta+\theta_k)}\right]\mathrm{d}\theta = \prod_{k=1}^{\infty} I_0(ta_k r^k).$$

Consider the integral

$$\begin{aligned}\int_{-\pi}^{\pi}|f'(re^{i\theta})|^t\,\mathrm{d}\theta &= \int_{-\pi}^{\pi}\left|\exp\left[t\sum_{k=0}^{\infty} a_k r^k\cos(k\theta)\right]\right|\mathrm{d}\theta\\ &= \int_{-\pi}^{\pi}\left|\exp\left[\frac{t}{2}\sum_{k=0}^{\infty} a_k r^k e^{ik\theta}\right]\right|^2\mathrm{d}\theta.\end{aligned}$$

Since the Taylor coefficients of $\exp(tz/2)$ are nonnegative, by the Parseval identity we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\exp\left[\frac{t}{2}\sum_{k=0}^{\infty} a_k r^k e^{ik\theta}\right]\right|^2\mathrm{d}\theta &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\exp\left[\frac{t}{2}\sum_{k=0}^{\infty} a_k e^{i\theta_k} r^k e^{ik\theta}\right]\right|^2\mathrm{d}\theta\\ = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[t\,\mathrm{Re}\sum_{k=1}^{\infty} a_k r^k e^{i(k\theta+\theta_k)}\right]\mathrm{d}\theta &= \prod_{k=1}^{\infty} I_0(ta_k r^k).\end{aligned}$$

This implies the relation

$$\beta_f(t) = \limsup_{r\to 1}\frac{\log\int_0^{2\pi}|f'(re^{i\theta})|^t\,\mathrm{d}\theta}{|\log(1-r)|} \geq \limsup_{r\to 1}\frac{\log\prod_{k=0}^{\infty} I_0(ta_k r^k)}{|\log(1-r)|} = \beta_f^*(t). \qquad \square$$

Theorem 3. *Suppose f is analytic and univalent in D. Then the $*$-spectrum of integral means can be calculated with the help of the formula*

$$\beta_f^*(t) = \limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\log I_0(t|a_k|),$$

where the a_k are the Taylor coefficients of $\log f'(z)$.

Proof. In [1], it was shown that the Taylor coefficients a_k of $\log f'$ satisfy the inequality

$$\sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k} \le \frac{36}{(1-r^2)^2}.$$

Putting $r^2 = 1 - 1/n$, we arrive at

$$\sum_{k=n}^{2n} |a_k|^2 \le 144, \quad n \ge 2. \tag{2}$$

Since

$$\sum_{k=1}^{2^l} |a_k|^2 = \sum_{j=0}^{l-1} \sum_{k=2^j}^{2^{j+1}-1} |a_k|^2,$$

by (2) we obtain

$$\sum_{k=1}^{n} |a_k|^2 \le 288 \log_2 n, \quad n \ge 2. \tag{3}$$

Now, we fix $\varepsilon > 0$ and put $r = 1 - \varepsilon/n$. Consider the difference

$$\frac{\sum_{k=1}^{n} \log I_0(t|a_k|)}{\log n} - \frac{\sum_{k=0}^{\infty} \log I_0(t|a_k|r^k)}{|\log(1-r)|} = \frac{\sum_{k=1}^{n} \log I_0(t|a_k|)}{\log n} - \frac{\sum_{k=0}^{\infty} \log I_0(t|a_k|r^k)}{\log n - \log \varepsilon}.$$

To estimate the right-hand side, first we treat the quantity

$$\sum_{k=1}^{n} \log I_0(t|a_k|) - \sum_{k=0}^{\infty} \log I_0(t|a_k|r^k).$$

We show that

$$|\log I_0(x) - \log I_0(y)| \le \frac{1}{4}|x^2 - y^2|, \quad x, y \in \mathbb{R}.$$

To this end, it suffices to verify that the function $u(x) = \exp(-x^2/4) I_0(x)$ is monotone decreasing on $\mathbb{R}$. Differentiating u, we obtain

$$\begin{aligned} e^{x^2/4} u'(x) = I_0'(x) - \frac{x}{2} I_0(x) &= \sum_{k=0}^{\infty} \frac{2k x^{2k-1}}{4^k k!^2} - \frac{1}{2} \sum_{k=0}^{\infty} \frac{x^{2k+1}}{4^k k!^2} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{x^{2k+1}}{4^k k!(k+1)!} - \frac{1}{2} \sum_{k=0}^{\infty} \frac{x^{2k+1}}{4^k k!^2} \le 0. \end{aligned}$$

So,

$$\begin{aligned} &\sum_{k=1}^{n} \log I_0(t|a_k|) - \sum_{k=0}^{\infty} \log I_0(t|a_k|r^k) \\ &\quad = \sum_{k=1}^{n} (\log I_0(t|a_k|) - \log I_0(t|a_k|r^k)) + \sum_{k=n+1}^{\infty} \log I_0(t|a_k|r^k) \\ &\quad \le \frac{t^2}{4} \sum_{k=1}^{n} (|a_k|^2 - |a_k|^2 r^{2k}) + \frac{t^2}{4} \sum_{k=n+1}^{\infty} |a_k|^2 r^{2k} \\ &\quad \le \frac{t^2}{4} (1 - r^{2n}) \sum_{k=1}^{n} |a_k|^2 + \frac{t^2}{4} \sum_{k=n+1}^{\infty} |a_k|^2 r^{2k}. \end{aligned}$$

Putting $r = 1 - \varepsilon/n$ in the last expression, we see that it is asymptotically equivalent to

$$\frac{t^2}{4}(1 - e^{-2\varepsilon}) \sum_{k=1}^{n} |a_k|^2 + \frac{t^2}{4} \sum_{k=n+1}^{\infty} |a_k|^2 e^{-2\varepsilon k/n}. \tag{4}$$

By (3), we arrive at the inequality

$$\frac{t^2}{4}(1 - e^{-2\varepsilon}) \sum_{k=1}^{n} |a_k|^2 \le 72t^2(1 - e^{-2\varepsilon}) \log_2 n.$$

Now, we estimate the second series in (4). We have

$$\begin{aligned}\sum_{k=n+1}^{\infty} |a_k|^2 e^{-2\varepsilon k/n} &= \sum_{j=0}^{\infty} \sum_{k=2^j(n+1)}^{2^{j+1}(n+1)-1} |a_k|^2 e^{-2\varepsilon k/n} \\ &\le \sum_{j=0}^{\infty} e^{-2\varepsilon j} \sum_{k=2^j(n+1)}^{2^{j+1}(n+1)-1} |a_k|^2 \le \frac{144}{1 - e^{-2\varepsilon}}\end{aligned}$$

(we have used (2) at the last stage). Thus,

$$\begin{aligned}&\left| \frac{\sum_{k=1}^{n} \log I_0(t|a_k|)}{\log n} - \frac{\sum_{k=0}^{\infty} \log I_0(t|a_k|r^k)}{\log n} \right| \\ &\qquad \le \frac{72t^2(1 - e^{-2\varepsilon}) \log_2 n + 144(1 - e^{-2\varepsilon})^{-1}}{\log n}.\end{aligned}$$

Passing to the upper limit as $r \to 1$ $(n \to \infty)$, we obtain

$$\left| \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \log I_0(t|a_k|) - \beta_f^*(t) \right| \le A(1 - e^{-2\varepsilon}),$$

where A is independent of n. To complete the proof of Theorem 3, it suffices to pass to the limit as $\varepsilon \to 0$. □

A similar approximation of Abel means by partial sums is possible also for the spectrum of integral means under a certain additional assumption about the Taylor coefficients a_k for $\log f'(z)$.

Lemma 1. *Suppose* $\log f'(z) = \sum_{k=1}^{\infty} a_k z^k$. *If*

$$\sup_n \sum_{k=n}^{2n} |a_k| < +\infty, \tag{5}$$

then

$$\beta_f(t) = \limsup_{n\to\infty} \frac{1}{\log n} \log \int_0^{2\pi} \exp \operatorname{Re} \left\{ t \sum_{k=1}^{n} a_k e^{ik\theta} \right\} \mathrm{d}\theta.$$

Proof. As in the proof of Theorem 3, we fix $\varepsilon > 0$ and put $r = 1 - \varepsilon/n$. Consider the difference

$$\begin{aligned}&\frac{1}{\log n} \log \int_0^{2\pi} \exp \operatorname{Re} \left\{ t \sum_{k=1}^{n} a_k e^{ik\theta} \right\} \mathrm{d}\theta \\ &\qquad - \frac{1}{|\log(1-r)|} \log \int_0^{2\pi} \exp \operatorname{Re} \left\{ t \sum_{k=1}^{\infty} a_k r^k e^{ik\theta} \right\} \mathrm{d}\theta.\end{aligned}$$

We have

$$\begin{aligned}
&\log \int_0^{2\pi} \exp \operatorname{Re} \left\{ t \sum_{k=1}^{n} a_k e^{ik\theta} \right\} \mathrm{d}\theta \\
&\quad = \log \int_0^{2\pi} \exp \operatorname{Re} \left\{ t \sum_{k=1}^{n} a_k e^{ik\theta}(1-r^k) - t \sum_{k=n+1}^{\infty} a_k r^k e^{ik\theta} + t \sum_{k=1}^{\infty} a_k r^k e^{ik\theta} \right\} \mathrm{d}\theta \\
&\quad \le |t| \sum_{k=1}^{n} |a_k|(1-r^k) + |t| \sum_{k=n+1}^{\infty} |a_k| r^k + \log \int_0^{2\pi} \exp \operatorname{Re} \left\{ t \sum_{k=1}^{\infty} a_k r^k e^{ik\theta} \right\} \mathrm{d}\theta.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\log \int_0^{2\pi} \exp \operatorname{Re} \left\{ t \sum_{k=1}^{n} a_k e^{ik\theta} \right\} \mathrm{d}\theta \\
&\quad \ge -|t| \sum_{k=1}^{n} |a_k|(1-r^k) - |t| \sum_{k=n+1}^{\infty} |a_k| r^k + \log \int_0^{2\pi} \exp \operatorname{Re} \left\{ t \sum_{k=1}^{\infty} a_k r^k e^{ik\theta} \right\} \mathrm{d}\theta.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{\log n} \left| \log \int_0^{2\pi} \exp \operatorname{Re} \left\{ t \sum_{k=1}^{n} a_k e^{ik\theta} \right\} \mathrm{d}\theta - \log \int_0^{2\pi} \exp \operatorname{Re} \left\{ t \sum_{k=1}^{\infty} a_k r^k e^{ik\theta} \right\} \mathrm{d}\theta \right| \\
&\quad \le \frac{|t|}{\log n} \sum_{k=1}^{n} |a_k|(1-r^k) + \frac{|t|}{\log n} \sum_{k=n+1}^{\infty} |a_k| r^k.
\end{aligned}$$

The remaining arguments are much as in the proof of Theorem 3. Indeed, (5) is an analog of (2), and the following direct consequence of (5) is a counterpart of (3):

$$\sup_n \frac{1}{\log n} \sum_{k=1}^{n} |a_k| < +\infty. \qquad \square$$

Theorem 4. *Suppose* $\log f'(z) = \sum_{k=1}^{\infty} a_k z^k$. *If the* a_k *satisfy* (5), *then there exists a sequence* $\{\theta_k\}$ *of reals such that* $\beta_f^*(t) = \beta_g(t)$, *where* $\log g'(z) = \sum_{k=1}^{\infty} a_k e^{i\theta_k} z^k$.

In other words, under condition (5), the coefficients of the logarithm of the derivative can be "rotated" in such a way that the spectrum of integral means for the resulting function will coincide with the $*$-spectrum of integral means for the initial one.

Proof. In the proof of Theorem 2, it was shown that there exist numbers $\theta_1, \theta_2, \dots, \theta_n$ with

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left[t \operatorname{Re} \sum_{k=1}^{n} a_k e^{i(k\theta + \theta_k)} \right] \mathrm{d}\theta = \prod_{k=1}^{n} I_0(t|a_k|).$$

After this, the proof is finished by application of Theorem 3 and Lemma 1, and by passage to the limit as $n \to \infty$.

We denote by

$$\sigma_f^2 = \limsup_{r \to 1} \frac{\sum_{k=1}^{\infty} |a_k|^2 r^{2k}}{\log \frac{1}{1-r}}$$

the asymptotic variance of the conformal mapping f (see [18, p. 186]). $\square$

Theorem 5. *For every function* f *univalent and analytic in* D, *and for every* t *we have*

$$\beta_f^*(t) \le \sigma_f^2 \frac{t^2}{4}. \tag{6}$$

If $a_k \to 0$ *as* $k \to \infty$, *then we have equality in* (6).

Proof. The inequality follows from the inequality $\log I_0(x) \le x^2/4$ (already proved).

Now, suppose $a_k \to 0$ as $k \to \infty$. We fix $\varepsilon > 0$; then there exists $N = N(\varepsilon)$ such that $|a_k| < \varepsilon$ for all $k \ge N$. It is easily seen that

$$\Big|\log I_0(tx) - \frac{t^2x^2}{4}\Big| \le \Big|\log\Big(1 + \frac{t^2x^2}{4}\Big) - \frac{t^2x^2}{4}\Big| \le t^4x^4.$$

It follows that

$$\Big|\log I_0(t|a_k|r^k) - \frac{t^2|a_k|^2r^{2k}}{4}\Big| \le t^4|a_k|^4r^{4k} \le t^4\varepsilon^2|a_k|^2r^{2k}, \quad k \ge N.$$

Summing over k from N to infinity, dividing by $\log[1/(1-r)]$, and passing to the limit as $r \to 1$, we obtain

$$\Big|\beta^*_f(t) - \sigma_f^2\frac{t^2}{4}\Big| \le t^4\varepsilon^2\sigma_f^2.$$

Passage to the limit as $\varepsilon \to 0$ finishes the proof. □

In [17], Pommerenke showed that if the radial limits $\lim_{r\to 1}\log f'(re^{i\theta})$ exist almost everywhere (by the Fatou theorem, this is the case if the boundary of $f(D)$ is rectifiable), then $\sigma_f^2 = 0$. Thus, Theorem 5 readily implies that if $f(D)$ has rectifiable boundary, then $\beta^*_f(t) = 0$ for every real t. Thus, the $*$-spectrum of integral means may have nontrivial behavior only for domains with fractal boundary.

If the logarithm of the derivative is representable by a Hadamard lacunary series, the $*$-spectrum can be estimated in the following way.

Theorem 6. *If f is analytic and univalent on D, and $\log f'(z) = \sum a_k z^{n_k}$, $\lim\limits_{k\to\infty} \frac{n_{k+1}}{n_k} = q > 1$, then*

$$\frac{\log I_0(a^- t)}{\log q} \le \beta^*_f(t) \le \frac{\log I_0(a^+ t)}{\log q},$$

where

$$a^- = \liminf_{k\to\infty} |a_k|, \quad a^+ = \limsup_{k\to\infty} |a_k|.$$

Proof. We verify the upper estimate. For simplicity, we assume that $|a_k| \le a^+$, because the general case can be reduced to this by elementary techniques. We have

$$\begin{aligned}\beta^*_f(t) &= \limsup_{n\to\infty}\frac{1}{\log n}\sum_{k:n_k\le n}\log I_0(|a_k|t) \le \limsup_{n\to\infty}\frac{1}{\log n}\sum_{k:n_k\le n}\log I_0(a^+t)\\ &= \log I_0(a^+t)\limsup_{n\to\infty}\frac{\max\{k : n_k \le n\}}{\log n}.\end{aligned}$$

Now, we show that

$$\lim_{n\to\infty}\frac{\max\{k : n_k \le n\}}{\log n} = \frac{1}{\log q}$$

(this will imply the upper estimate). For this, it suffices to prove the relation $(\frac{1}{k})\log n_k \to \log q$ as $k \to \infty$. But this is a consequence of the relation

$$\frac{\log n_k}{k} = \frac{1}{k}\sum_{j=2}^{k}\log\frac{n_j}{n_{j-1}} + \frac{\log n_1}{k} \to \log q, \quad k \to \infty.$$

We have used the fact that a convergent series is Cesaro convergent, and that $n_j/n_{j-1} \to q$ as $j \to \infty$.

The lower estimate is proved similarly. □

We note yet another link between $\beta_f(t)$ and $\beta^*_f(t)$. The following statement was proved in [11].

Let q be a natural number exceeding 1. *Suppose that* $\log f'(z) = \sum a_k z^{q^k}$. *Then*

$$\beta_f(t) = \beta^*_f(t) + O(t^q) \quad \text{as } t \to 0.$$

Let Ω be a proper simply connected domain on the plane, and let φ be a conformal mapping of Ω onto D. In [5], Brennan conjectured that $\varphi' \in L_p(\Omega)$ for $4/3 < p < 4$, i.e.,

$$\int_\Omega |\varphi'|^p \, \mathrm{d}x\mathrm{d}y < +\infty, \quad 4/3 < p < 4.$$

If Ω is the plane cut along some ray, then, for $p \notin (4/3, 4)$,

$$\int_\Omega |\varphi'|^p \, \mathrm{d}x\mathrm{d}y = +\infty.$$

Let $f : D \to \Omega$ be a conformal mapping of D onto Ω. In [18] it was shown that the Brennan conjecture is equivalent to the relation

$$\int_{-\pi}^{\pi} \frac{1}{|f'(re^{i\theta})|^2} \, \mathrm{d}\theta = O\Big(\frac{1}{1-r}\Big), \quad r \to 1-,$$

which is equivalent to the inequality

$$\beta_f(t) \le |t| - 1, \quad t \le -2, \tag{7}$$

for the spectrum of integral means. Carleson and Makarov [7] proved (7) for sufficiently large $|t|$.

In his thesis [4], Bertilsson studied a local version of the Brennan conjecture for functions close to the Koebe function.

The conception of integral means [6, 7, 16] implies that it suffices to verify the Brennan conjecture for domains with fractal boundary. (Heuristically, this is explained as follows: if the boundary is sufficiently "good", then integration over the circle reduces the order of growth of the derivative by 1.) The first step in this direction was made by Barański, Volberg, and Zdunik [3]: they proved the conjecture in the case where Ω is the attraction domain of ∞ for an arbitrary quadratic polynomial p and Ω is simply connected.

Another important class of fractals is provided by the conformal mappings f for which $\log f'$ is representable by a Hadamard lacunary series, i.e.,

$$\log f' = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad \frac{n_{k+1}}{n_k} \ge q > 1.$$

Such mappings have turned out to be quite useful in the proofs of lower estimates for the integral means spectrum; see [8, 10, 12, 14], [18, p. 188], and [19]. On the other hand, it can easily be proved that the existence of a universal constant C such that every function in this class satisfies the inequality

$$\int_{-\pi}^{\pi} \frac{1}{|f'(re^{i\theta})|^2} \, \mathrm{d}\theta \le \frac{C}{1-r}$$

implies Brennan's conjecture in the general case.

We prove the conjecture under the assumption that $q \ge 15$. For this, we need several useful lemmas.

Lemma 2. *Suppose* $0 \le x \le 3\pi/2$ *and* $q \ge 15$. *Then*

$$\Big| x\cos\theta - \log\Big(I_0(x) + 2\sum_{n=1}^{q-1} I_n(x)\cos n\theta\Big)\Big| \le \frac{C}{q!}\Big(\frac{x}{2}\Big)^q, \quad C = 370, \tag{8}$$

where $I_n(x)$ is the modified Bessel function.

Proof. We recall the well-known formula

$$\exp(x\cos\theta) = I_0(x) + 2\sum_{n=1}^{\infty} I_n(x)\cos(n\theta),$$

whence

$$\begin{aligned}\Big|\exp(x\cos\theta) - I_0(x) - 2\sum_{n=1}^{q-1} I_n(x)\cos(n\theta)\Big| &\le 2\sum_{n=q}^{\infty} I_n(x)\\ \le 2\sum_{n=q}^{\infty}\frac{1}{n!}\Big(\frac{x}{2}\Big)^n \exp\Big(\frac{|x|^2}{4(n+1)}\Big)\\ \le \frac{2}{q!}\Big(\frac{x}{2}\Big)^q \exp\Big(\frac{|x|^2}{4(q+1)}\Big)\Big/\Big(1-\frac{x}{2(q+1)}\Big).\end{aligned}$$

By using the Lagrange formula, we obtain

$$\begin{aligned}&\Big|x\cos\theta - \log\Big(I_0(x) + 2\sum_{n=1}^{q-1} I_n(x)\cos n\theta\Big)\Big|\\ &\quad= \Big|\log\exp(x\cos\theta) - \log\Big(I_0(x) + 2\sum_{n=1}^{q-1} I_n(x)\cos n\theta\Big)\Big|\\ &\quad\le \frac{1}{\lambda\exp(x\cos\theta) + (1-\lambda)(I_0(x) + 2\sum_{n=1}^{q-1} I_n(x))}\\ &\qquad\times\Big|\exp(x\cos\theta) - I_0(x) - 2\sum_{n=1}^{q-1} I_n(x)\cos n\theta\Big|,\end{aligned}$$

where λ is some number in $[0,1]$.

Clearly, (8) is fulfilled if

$$\frac{2\exp\Big(\frac{|x|^2}{4(q+1)}\Big)}{\Big[\lambda\exp(x\cos\theta) + (1-\lambda)(I_0(x) + 2\sum_{n=1}^{q-1} I_n(x)\cos n\theta)\Big]\Big(1-\frac{x}{2(q+1)}\Big)} \le 370$$

for $x = 3\pi/2$, $q = 15$. The last inequality is easily verified numerically. □

Lemma 3. *Let $\{\theta_k\}$ be a sequence of reals. Then*

$$\int_{-\pi}^{\pi}\prod_{k=0}^{\infty}\Big[I_0(t|a_k|r^{n_k}) + 2\sum_{s=1}^{q-1} I_s(t|a_k|r^{n_k})\cos(sn_k\theta + s\theta_k)\Big]\,\mathrm{d}\theta = 2\pi\prod_{k=0}^{\infty} I_0(t|a_k|r^{n_k}),$$

and, consequently, the value of this integral does not depend on the sequence $\{\theta_k\}$.

Proof. We expand the product under the integral sign and integrate the summands separately. Since $\cos\alpha\cos\beta = 2^{-1}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$, we have

$$\prod_{j=0}^{m}\cos(s_j n_{p_j}\theta + s_j\theta_{p_j}) = \frac{1}{2^m}\sum\cos\Big[\theta\sum_{j=0}^{m}(\pm s_j n_{p_j}) + \delta\Big]. \tag{9}$$

There is no loss of generality in assuming that the sequence $\{n_{p_j}\}$ is monotone increasing. Clearly, integration of (9) in θ gives 0 because

$$\sum_{j=0}^{m-1} s_j n_{p_j} \le (q-1)\sum_{j=0}^{m-1} n_{p_j} < n_{p_m}.$$ □

Theorem 7. *Suppose f is univalent and analytic in D, and*

$$\log f' = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad n_{k+1}/n_k \geq q > 1.$$

If $q \geq 15$, then

$$\beta_f(-2) \leq 1.$$

Proof. First, we show that

$$\beta_f(-2) \leq \beta_f(2) + \frac{2Ca^q}{q! \log q}, \tag{10}$$

where C is the same as in (8) and $a = 3\pi/4$.

Consider the integral

$$\int_{-\pi}^{\pi} |f'|^{-2}\, \mathrm{d}\theta = \int_{-\pi}^{\pi} \prod_{k=0}^{\infty} \exp\left[2|a_k| r^{n_k} \cos(n_k\theta + \theta_k)\right] \mathrm{d}\theta. \tag{11}$$

By Lemma 2, this integral does not exceed the quantity

$$\int_{-\pi}^{\pi} \prod_{k=0}^{\infty} \Big[I_0(2|a_k| r^{n_k}) + 2 \sum_{s=1}^{q-1} I_s(2|a_k| r^{n_k}) \cos(s n_k \theta + s\theta_k)\Big] \exp\Big(\frac{C}{q!} |a_k|^q r^{q n_k}\Big)\, \mathrm{d}\theta.$$

It is well known (see [18, p. 189]) that

$$\sum_{k=0}^{\infty} r^{n_k} \leq \frac{\log \frac{1}{1-r}}{\log q} + O(1), \quad r \to 1.$$

From this and Lemma 3 we deduce that

$$\beta_f(-2) \leq \limsup_{r\to 1} \frac{\log \prod_{k=0}^{\infty} I_0(2|a_k| r^{n_k})}{\log \frac{1}{1-r}} + \frac{Ca^q}{q! \log q}.$$

For the same reason,

$$\beta_f(2) \geq \limsup_{r\to 1} \frac{\log \prod_{k=0}^{\infty} I_0(2|a_k| r^{n_k})}{\log \frac{1}{1-r}} - \frac{Ca^q}{q! \log q}.$$

Together with Lemma 3, these inequalities imply (10).

To complete the proof, we use a result by Carleson and Jones [6]. They showed that if f is analytic and univalent on D, and $|f'(z)| \leq \mathrm{const}(1-|z|)^{\gamma-1}$, then

$$\beta_f(2) \leq 1 - \frac{\gamma^3}{8\pi}. \tag{12}$$

In [13] it was shown that if f is analytic and univalent on D, and $\log f'(z)$ has an Hadamard lacunary Taylor series with lacunarity exponent greater than 2, then, asymptotically, the Taylor coefficients of $\log f'(z)$ do not exceed $3\pi/4$ in absolute value. Therefore, $\gamma \leq 1 - 3\pi/(4\log q)$. So,

$$\begin{aligned}\beta_f(-2) &\leq \beta_f(2) + \frac{2C}{q!\log q}\Big(\frac{3\pi}{4}\Big)^q \\ &\leq 1 - \frac{1}{8\pi}\Big(1 - \frac{3\pi}{4\log q}\Big)^3 + \frac{2C}{q!\log q}\Big(\frac{3\pi}{4}\Big)^q.\end{aligned}$$

Direct calculations show that the latter quantity does not exceed 1 if $q \geq 15$. □

It is interesting that (12) can be improved. Specifically, Jones and Makarov showed in [9] that

$$\beta_f(2-\varepsilon) \le 1 - \varepsilon + A\varepsilon^2$$

for all bounded univalent functions, where $A > 0$ is a certain universal constant. Consequently,

$$\beta_f(2) \le 1 - \frac{\gamma^2}{4A},$$

which is stronger asymptotically than (12). Unfortunately, we cannot use this estimate because no numerical upper bound for A is known presently.

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Received 15/JUN/2004

Translated by S. V. KISLYAKOV