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OPEN MAP THEOREM FOR METRIC SPACES

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ABSTRACT. An open map theorem for metric spaces is proved and some applications are discussed. The result on the existence of gradient flows of semiconcave functions is generalized to a large class of spaces.

§1. INTRODUCTION

This paper is a continuation of [Lytb]. In [Lytb] we discussed the possibility of differentiation of Lipschitz maps between metric spaces; here we prove analogs of the usual submersion and immersion theorems from analysis. In other words, we show that some properties of the infinitesimal portions of a locally Lipschitz map imply the same properties for the map itself. We formulate the theorems in a way not involving the concept of the differential. However, the most interesting applications, such as those to the gradient flow of a semiconcave function, require the existence of special tangent spaces and differentials. As in [Lytb], we denote by $f_x^{(t_i)} : X_x^{(t_i)} \to Y_{f(x)}^{(t_i)}$ the blowup at the scale (t_i) of a Lipschitz function $f : X \to Y$ at the point x (see Subsection 2.8 for the definition). The immersion theorem cannot work everywhere (see §10); however, it is valid at many points.

Proposition 1.1. *Suppose $f : X \to Y$ is a locally Lipschitz map and X is locally complete. Assume that, for some fixed $\rho > 0$, at each point $x \in X$ each blowup $f_x^{(t_i)} : X_x^{(t_i)} \to Y_y^{(t_i)}$ satisfies $\rho d(0, v) \le d(0, f_x^{(t_i)}(v))$ for all $v \in X_x^{(t_i)}$. Then X contains an open dense subset on which f is a locally bi-Lipschitz embedding.*

Remark 1.1. In Proposition 1.1 it is necessary to require that the conditions are satisfied for each blowup at each point; compare Subsection 3.1.

If we want to have the conclusion at each point, we need to decompose X in some countable disjoint union of locally closed subsets. This decomposition is particularly useful as far as rectifiability is concerned and does not require any assumptions on X (see Lemma 3.1 for the precise statement). Our submersion theorem (Theorem 1.2) is deeper and has a much better form, since it is valid at all points.

Theorem 1.2. *Suppose $f : X \to Y$ is a locally Lipschitz map, Y is locally geodesic, X is locally complete, and $C > 0$. If for each $x \in X$ there exists a blowup $f_x^{(t_i)} : X_x^{(t_i)} \to Y_{f(x)}^{(t_i)}$ such that for each $w \in Y_{f(x)}^{(t_i)}$ there is $v \in X_x^{(t_i)}$ with $f_x^{(t_i)}(v) = w$ and $Cd(0, v) \le d(0, w)$, then f is locally $\bar{C}$-open for all $\bar{C} < C$.*

Remark 1.2. The word "locally" can be erased in the proposition above.

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Under some additional conditions, from the fact that all blowups are bi-Lipschitz it can be deduced that the map itself is locally bi-Lipschitz (some assumptions on X and Y are necessary; compare §10). Namely, the following statement is true.

Proposition 1.3. *Let X be a locally compact space, and let Y be a Lipschitz manifold. Assume that the metric on X is locally bi-Lipschitz with respect to the inner metric. Let $f : X \to Y$ be a locally Lipschitz map such that for some $C > 0$ each blowup $f_x^{(t_i)} : X_x^{(t_i)} \to Y_{f(x)}^{(t_i)}$ is C-bi-Lipschitz. Then $f : X \to Y$ is locally bi-Lipschitz.*

Remark 1.3. As the proof shows, Proposition 1.3 remains valid if Y is only assumed to be locally bi-Lipschitz with respect to a $CAT(\kappa)$ space.

The last two results were proved and used in several particular situations in [BGP92, Nag02, Per94] and some other papers. The following are the rigid versions of the above results.

Corollary 1.4. *Let X and Y be locally compact and locally geodesic (respectively, proper and geodesic) spaces, and let $f : X \to Y$ be a locally Lipschitz map such that at each point $x \in X$ some blowup $f_x^{(t_i)} : X_x^{(t_i)} \to Y_y^{(t_i)}$ is a submetry. Then f is a local submetry (respectively, a submetry).*

Corollary 1.5. *Suppose $f : X \to Y$ is as in Corollary 1.4 and each blowup at each point is an isometry. If, moreover, Y is locally bi-Lipschitz with respect to some $CAT(\kappa)$ space, then f is a local isometry.*

The proof of Theorem 1.2 requires the notion of the absolute gradient of a function, together with the result stating that if $f : X \to \mathbb{R}$ grows at each point (infinitesimally) at least with velocity $c > 0$, then f is semiopen in a sense (see Lemma 4.1). On the other hand, this semiopenness result (essentially contained in [PP94]) can be used as the main ingredient in the construction of gradient curves of functions with semicontinuous gradients. Recall that a gradient curve of a function is a curve that grows as fast as f does at (almost) each point (see Definition 6.1 and Definition 6.2). The existence of gradient curves can be shown in all locally compact spaces along the lines of [PP94]. The following result can be regarded as a Peano theorem for metric spaces.

Proposition 1.6. *Let X be locally compact, and let $f : X \to \mathbb{R}$ be a map with lower semicontinuous absolute gradients. Then a maximal gradient curve $\eta : [0, a) \to X$ of f starts at each point $x \in X$. If X is complete and f is bounded or admits a uniform Lipschitz constant, then each η is complete, i.e., $a = \infty$.*

However, in order to get uniqueness and a reasonable gradient flow, we need some assumptions on the differentiable structure of the space. We say that a space X is *appropriate* if it is locally compact and locally geodesic, the upper and the lower angle between each pair of geodesics starting at the same point coincide, and each blowup $X_x^{(t_i)}$ is naturally isometric to the geodesic cone C_x (see Definition 8.1). Each space that is infinitesimally cone-like in the sense of [Lytb] is appropriate. In particular all (proper, finite-dimensional) Aleksandrov spaces, all manifolds with Hölder continuous Riemannian metrics, all sets of positive reach in a smooth Riemannian manifold with the inner metric, and all surfaces with an integral curvature bound [Res93] are appropriate. Moreover, the products and open subsets of appropriate spaces are appropriate.

Theorem 1.7. *Suppose X is either an appropriate space or an open subset of a (not necessarily locally compact) space with one-sided curvature bound. Let $f : X \to \mathbb{R}$ be a semiconcave function. Then, for each $x \in X$, a unique maximal gradient curve starts at x. Moreover, the locally defined gradient flow is locally Lipschitz. The flow is complete if*

X is complete and either f is bounded from above, or f has a global Lipschitz constant. The flow is 1-Lipschitz if f is concave and X is complete and geodesic.

This result was proved in [Sha77] in the case of Riemannian manifolds and was used to study the souls of nonnegatively curved manifolds. In the case of Alexandrov spaces with lower curvature bound, the same fact was proved in [PP94]; it served as a main tool in the study of such spaces. In [Pet99] it was applied (without proof) to spaces with an upper curvature bound. In the case of proper spaces we get the following soul theorem.

Corollary 1.8. *Suppose X is proper, geodesic, and appropriate. Let $f : X \to \mathbb{R}$ be a concave function that attains its maximum on a set S. Then the gradient flow $b^t : X \to X$ is 1-Lipschitz and converges to a 1-Lipschitz retraction $b^\infty : X \to S$.*

The soul theorem is related to the following more general statement concerning regular sublevel sets (see Definition 7.3). In [Lyta] this statement was developed further in a more general context.

Corollary 1.9. *Let X be as in Theorem 1.7, let $f : X \to \mathbb{R}$ be a semiconcave function, and let $U_t = f^{-1}[t, \infty)$ be a regular sublevel set of f. Then for some $r, K > 0$ and all $x_0, x_1 \in U_t$ with $d(x_0, x_1) = s < r$ there is a point $m \in U_t$ satisfying $d(m, x_i) \le \frac{s}{2}(1 + Ks^2)$.*

We proceed as follows. After the preliminaries in §3 we discuss the case of injective differentials and prove Proposition 1.1. In §§4, 5 we prove Theorem 1.2 and Proposition 1.3 and present the differential characterizations of submetries and isometries. In §6 we prove Proposition 1.6. §§7 and 9 are devoted to semiconcave functions. In §8 we recall some results about tangent cones and differentials in metric spaces, to be used in §9. Finally, in §10 we give some examples showing that our assumptions in the first three propositions are essential.

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§2. Preliminaries and notation

2.1. Notation. By $\mathbb{R}^n$ we denote the Euclidean space of dimension n, and $\mathbb{R}^+$ stands for the set of positive real numbers. By d we shall always denote the metric in a metric space, without an extra reference to the space. The closed metric ball of radius r around x will be denoted by $B_r(x)$. A subset of a topological space is *locally closed* if it is the intersection of an open and a closed set. A metric space X is said to be locally complete if each point has a complete neighborhood in X. Each locally compact space is locally complete, and each locally closed subset of a locally complete space is locally complete.

From the Baire theorem for closed subsets we get the following.

Lemma 2.1. *If a locally complete space X is a countable union of locally closed subsets X_n, then the union of the sets U_n of inner points of X_n is dense in X.*

A metric space X is *proper* if all closed bounded subsets of X are compact.

2.2. Lipschitz maps. A map $f : X \to Y$ between metric spaces is *C-Lipschitz* if $d(f(x), f(z)) \le Cd(x, z)$ for all $x, z \in X$. The map f is called a *C-bi-Lipschitz embedding* if, moreover, $d(f(x), f(z)) \ge \frac{1}{C}d(x, z)$. By definition, a *locally bi-Lipschitz map* is a topologically open map that is a locally bi-Lipschitz embedding.

2.3. Open maps. Let $C, r > 0$; we say that a locally Lipschitz map $f : X \to Y$ is (C, r)-*open at a point* $x \in X$ if for all $\bar{x} \in B_r(x)$ and all $\bar{r} \leq r - d(x, \bar{x})$ we have $B_{C\bar{r}}(f(\bar{x})) \subset f(B_{\bar{r}}(\bar{x}))$. Observe that in this case f is $(C, r - \bar{r})$-open at each $\bar{x}$ with $\bar{r} = d(x, \bar{x}) < r$. We say that f is *locally* C*-open* if for each $x \in X$ there is $r > 0$ such that f is (C, r)-open at x, and f is C*-open* if the above condition is satisfied for all $x \in X$ and all $r > 0$.

Note that each locally C-open map is topologically open. A locally C-bi-Lipschitz map is locally C-open. A locally C-open map is locally bi-Lipschitz if and only if it is locally injective.

2.4. Submetries. A map $f : X \to Y$ is called a (*local*) *submetry* if it is (locally) 1-Lipschitz and (locally) 1-open. If a (local) submetry is (locally) injective, then it is a (local) isometry.

2.5. Curves. Let $\gamma : [t, a) \to X$ be a locally Lipschitz curve, and let $L(\gamma)$ denote its length. If the (nonnegative) number $\lim_{\epsilon \to 0} \frac{d(\gamma(t), \gamma(t+\epsilon))}{\epsilon}$ exists, we denote it by mD_t^+ and call it the metric differential of γ at t. In fact, this metric differential exists at almost every point $s \in [t, a)$, and we have the area formula $L(\gamma) = \int_t^a (mD_s^+ \gamma)\, ds$ (see [BBI01, p. 57]).

2.6. Geodesics. A geodesic in X is an isometric embedding of an interval into X. The space X is said to be *geodesic* if for all points $x \neq z$ in X the set $\Gamma_{x,z}$ of all geodesics connecting x and z is not empty. The space X is *locally geodesic* if it is covered by open subsets U_j such that all points $z, \bar{z}$ in U_j are joined in X by a geodesic. In a connected locally geodesic space every two points are connected by a Lipschitz curve.

Example 2.1. Each geodesic space is locally geodesic. Each open subset of a locally geodesic space is locally geodesic. By the Arzela–Ascoli theorem, each locally compact inner metric space is locally geodesic.

2.7. Curvature bounds. A space X is called a $CAT(\kappa)$ space (respectively, a space with curvature at least κ) if it is complete and geodesic and the triangles in X are not thicker (respectively, not thinner) than the triangles in the two-dimensional simply connected manifold M_κ^2 of constant curvature κ. We refer the reader to [BBI01] for the theory of such spaces.

2.8. Ultralimits. We choose and fix a nonprincipal ultrafilter ω on the set of natural numbers $\mathbb{N}$ (see [BH99, pp. 77–80]). By $\lim_\omega (X_i, x_i)$ we denote the ultralimit of a sequence of pointed metric spaces (see [BH99] and [Lytb]). For a sequence of Lipschitz maps $f_i : (X_i, x_i) \to (Y_i, y_i)$, we denote by $\lim_\omega f_i$ the ultralimit $f : \lim_\omega (X_i, x_i) \to \lim_\omega (Y_i, y_i)$. For a sequence $\epsilon_j \to 0$, let $X_x^{(\epsilon_j)}$ denote the blowup $X_x^{(\epsilon_j)} = \lim_\omega (\frac{1}{\epsilon_j} X, x)$. The blowups $X_x^{(t_i)}$ are spaces with a distinguished origin $0 = (x, x, x...)$. In blowups, we shall denote by $|v|$ the distance $|v| = d(v, 0)$. For each zero sequence (t_i), a locally Lipschitz map $f : (X, x) \to (Y, y)$ determines a blowup $f_x^{(t_i)} : (X_x^{(t_i)}, 0) \to (Y_y^{(t_i)}, 0)$.

Example 2.2. Suppose $f : X \to Y$ is a locally Lipschitz map. Let $x_i \neq x$ be a sequence of points converging to x, with $f(x_i) = f(x) = y$. Then, for $t_i = d(x, x_i)$, the point $v = (x_i) \in X_x^{(t_i)}$ satisfies $|v| = 1$ and $f_x^{(t_i)}(v) = 0$.

Example 2.3. Let γ be a Lipschitz curve in X, and let $f : X \to Y$ be a locally Lipschitz map. Assume that, for some fixed $C > 0$, for each $x \in X$ there is a zero sequence (t_i) such that $|f_x^{(t_i)}(v)| \leq C|v|$ (respectively, $|f_x^{(t_i)}(v)| \geq C|v|$) for all $v \in X_x^{(t_i)}$. Then from the area formula we get $L(f(\gamma)) \leq CL(\gamma)$ (respectively, $L(f(\gamma)) \geq CL(\gamma)$) [BBI01, p. 57]. In the first case we see that if X is (locally) geodesic, then f is (locally) C-Lipschitz.

The ultralimit $\lim_\omega(X,x)$ of the constant sequence (X,x) is called the *ultraproduct* of X and will be denoted by X^ω. The space X is isometrically embedded in X^ω, and $X = X^\omega$ if and only if X is a proper space.

§3. Injective blowups

3.1. Decomposing a map. We start with a decomposition of a Lipschitz map into pieces with respect to its differential behavior. Let $f : X \to Y$ be a locally Lipschitz map. We consider the Borel function $\bar{f} : X \to [0,\infty)$ defined by $\bar{f}(x) = \liminf_{\bar{x}\to x} \frac{d(f(x),f(\bar{x}))}{d(x,\bar{x})}$.

Observe that $\bar{f}(x) \geq \rho$ if and only if for all zero sequences (t_i) and all $v \in X_x^{(t_i)}$ we have $d(0, f_x^{(t_i)}(v)) \geq \rho d(0,v)$. Moreover, $\bar{f}(x) = 0$ if and only if there is a zero sequence (t_i) and some $v \in X_x^{(t_i)}$ with $|v| = d(0,v) = 1$ and $f_x^{(t_i)}(v) = 0 \in Y_{f(x)}^{(t_i)}$.

Thus, X can be split into disjoint Borel subsets $X = \bigcup X_n \cup S$ in such a way that $\bar{f}(x) \geq \frac{1}{n}$ for each $x \in X_n$, and $S = \{x \in X \mid \bar{f}(x) = 0\}$. Next, note that S admits a disjoint Borel decomposition $S = S_1 \cup S_2$, where S_1 is the set of all $x \in X$ such that for each(!) zero sequence (t_i) there is a point $v \in X_x^{(t_i)}$ with $d(v,0) = 1$ and $f_x^{(t_i)} = 0$.

Remark 3.1. If a point x is in S_2, then the blowups of f at x at different scales are essentially different. In particular, at such points the map f cannot be metrically differentiable (cf. [Lytb]).

In general, it is difficult to say something about the behavior of f on S_1 and S_2, as well as about the size of $f(S)$. For example, it may happen that $X = S_2$ even for quite tame spaces (see the example of [KM03]). However, by the theory of [Kir94], for each measurable subset X of $\mathbb{R}^n$ and a locally Lipschitz map $f : X \to Y$ to an arbitrary metric space Y, the n-dimensional Hausdorff measure of S_2 (hence, also of $f(S_2)$) and of the image $f(S_1)$ vanishes.

3.2. The injective part. On the subsets X_n defined above, the map f is not far from being a bi-Lipschitz embedding. In the case of $X \subset \mathbb{R}^n$, this result is the starting point of the geometric measure theory of [Kir94].

Lemma 3.1. *Let $f : X \to Y$ be a locally Lipschitz map. Assume that $\bar{f} \geq \rho$ on X, i.e., we are under the assumptions of Proposition* 1.1. *Then for each $\epsilon > 0$ the space X has a decomposition $X = \bigcup_{1\leq m\leq\infty} X^m$ in a countable disjoint union of locally closed subsets X^m such that if $x,z \in X^m$ and $d(x,z) < \frac{1}{m}$, then $d(f(x),f(z)) \geq (\rho-\epsilon)d(x,z)$. In particular, the restriction $f : X^m \to Y$ is a locally bi-Lipschitz embedding.*

Proof. Let Z_n be the set of all points $x \in X$ such that if $z \in X$ and $d(x,z) < \frac{1}{n}$, then $d(f(x),f(z)) \geq (\rho-\epsilon)d(x,z)$. The sets Z_n are closed, and the assumption $\bar{f} \geq \rho$ implies $X = \bigcup Z_n$. Now it suffices to consider the subsets $X^m = Z_m \setminus Z_{m-1}$. □

Using Lemma 2.1, we immediately get Proposition 1.1. Also, from Lemma 3.1 we obtain the following rectifiability result (see [Fed70] for the definition).

Corollary 3.2. *Let X be a separable metric space. Assume that for each point $x \in X$ there is a neighborhood U and a Lipschitz map $f : U \to \mathbb{R}^n$ such that the assumptions of Lemma* 3.1 *are satisfied with some $\rho = \rho(U) > 0$. Then X is n-dimensionally countably rectifiable.*

§4. Open map theorem

Our study of open maps starts with the following definition (see [Pla02, p. 862]).

Definition 4.1. Let $f : X \to \mathbb{R}$ be a locally Lipschitz function, and let $x \in X$. We denote by $|\nabla_x f|$ the nonnegative number $\max\{0, \limsup_{\bar{x}\to x} \frac{f(\bar{x})-f(x)}{d(\bar{x},x)}\}$; we call $|\nabla_x f|$ the absolute gradient of f at the point x.

Note that the absolute gradient $|\nabla_x f|$ is nonnegative and is bounded from above by the Lipschitz constant of f near x. Actually, $|\nabla_x f|$ is the supremum of all $C \geq 0$ such that for some zero sequence (t_i) there is a point $v \in X_x^{(t_i)}$ with $|v| = 1$ and $f_x^{(t_i)}(v) \geq C$.

Example 4.1. If f is differentiable at x (see [Lytb]), then $|\nabla_x f| = 0$ if and only if $D_x f \leq 0$ on $T_x X$, and $|\nabla_x f| = \sup\{D_x f(v) \mid v \in T_x X, |v| = 1\}$ otherwise.

Example 4.2. Let $x \in X$ be a point. Consider the distance function $d_x : X \to \mathbb{R}$ and set $f = -d_x$. If X is a geodesic space, then $|\nabla_z f| = 1$ for each $z \neq x$. Actually, for each zero sequence t_i we have $f_z^{(t_i)}(v) = 1$, where $v = (\gamma(t_i)) \in X_z^{(t_i)}$ is the starting direction of a geodesic γ from z to x. From the next lemma it is easy to deduce that if X is proper and $|\nabla_z f| = 1$ for each $z \neq x$, then each point $z \in X$ is connected with x by a geodesic.

Lemma 4.1. *Suppose $f : X \to \mathbb{R}$ is a locally Lipschitz map, $x \in X$, and $f(x) = 0$. For $C, r > 0$, assume that the ball $B_r(x)$ is complete and that $|\nabla_z f| \geq C$ for each z with $d(z,x) < r$ and $f(z) \geq f(x)$. Then for each $0 < \bar{C} < C$ there is a point $z \in X$ with $d(z,x) \leq r$ and $f(z) = \bar{C}r$.*

Proof. Let $l : A \to X$ (where A is a subset of the interval $[0, \bar{C}r]$ containing 0) be a $\frac{1}{\bar{C}}$-Lipschitz map such that $f \circ l(t) = t$ for all $t \in A$ and $l(0) = x$. By the Zorn lemma, there is a maximal subset $A \subset [0, \bar{C}r]$ on which such a map l can be defined. By the completeness of $B_r(x)$, this maximal set A must be closed. Let t be the maximum of A. We are done if $t = \bar{C}r$. Assuming that $t < \bar{C}r$, consider the point $z = l(t)$. We have $d(z,x) < r$. Since $|\nabla_z f| \geq C > \bar{C}$, we can find a point z_0 close to z with $f(z_0) - f(z) \geq \bar{C}d(z, z_0)$. Then the extension of l to $\bar{l} : A \cup \{f(z_0)\} \to X$ given by $\bar{l}(f(z_0)) := z_0$ is $\frac{1}{\bar{C}}$-Lipschitz. This contradicts the maximality of A. □

This lemma allows us to study the sublevel sets $U_t = f^{-1}[t, \infty)$. Namely, we can apply Lemma 4.1 to all points $\bar{x} \in B_r(x)$ instead of x. Letting $\bar{C}$ converge to C, we get the following.

Corollary 4.2. *Under the assumptions of Lemma 4.1, for each $\bar{x} \in B_r(x)$ and each $0 \leq \bar{r} \leq r - d(x, \bar{x})$ we have $d(\bar{x}, U_{f(\bar{x})+C\bar{r}}) \leq \bar{r}$.*

Remark 4.3. If $B_r(x)$ is compact, it is easy to show (arguing as in Lemma 4.4 below) that $\bar{x}$ and $U_{f(\bar{x})+C\bar{r}}$ are connected by a curve of length not exceeding $\bar{r}$.

Proposition 4.3. *Suppose $f : X \to Y$ is a locally Lipschitz map, $x \in X$, and $C, r > 0$. Assume that the ball $B_r(x)$ is complete, and that for each $z \in B_r(x)$ there exists a zero sequence (t_i) such that for each $w \in Y_{f(z)}^{(t_i)}$ there is $v \in X_z^{(t_i)}$ with $f_z^{(t_i)}(v) = w$ and $C|v| \leq |w|$.*

If Y is geodesic, then f is $(\bar{C}, r)$-open at x for each $0 < \bar{C} < C$.

Proof. We set $y = f(x)$ and fix $\bar{C} < C$. It suffices to prove that $B_{\bar{C}r}(y) \subset f(B_r(x))$. Fix a point $\bar{y}$ in $B_{\bar{C}r}(y)$, assume that $\bar{y} \notin f(B_r(x))$, and consider the distance function $d_{\bar{y}} : Y \to \mathbb{R}$ and the composition $h = -d_{\bar{y}} \circ f$. Since Y is geodesic, we can refer to Example 4.2 to show that $|\nabla_z h| \geq C$ for each $z \in B_r(x)$.

Hence, by Lemma 4.1, there is a point $z \in B_r(x)$ with $\bar{C}r = h(z) - h(x) = d(\bar{y}, f(x)) - d(\bar{y}, f(z))$. Since $d(\bar{y}, f(x)) \leq \bar{C}r$, we get $\bar{y} = f(z)$, which contradicts the fact that $\bar{y} \notin f(B_r(x))$. □

If we assume Y to be only locally geodesic, then the same argument works if it is known that all points $y_1, y_2 \in B_{\bar{C}r}(y)$ are connected in Y by a geodesic. This finishes the proof of Theorem 1.2.

Under the assumptions of Proposition 4.3, assume that, moreover, $B_r(x)$ is compact. Then f is (C, r)-open at x. Putting $C = 1$ and using Example 2.3, we get a proof of Corollary 1.4.

Example 4.4. Let X be a proper, geodesically complete $CAT(\kappa)$ space. Let Z be a closed subset of X. Assume that for each $x \in X$ with $d(x, Z) < r$ there is only one point $z \in Z$ with $d(x, z) = d(x, Z)$. Then at each point $x \in X$ with $0 < d(x, Z) < r$ the 1-Lipschitz distance function $d_Z : X \to \mathbb{R}^+$ has a 1-open differential, by geodesic completeness and the first variation formula. Therefore, we see that d_Z is locally 1-open, i.e., as above, each point $x \in X$ lies on a geodesic $\gamma : [0, r] \to X$ with $d(\gamma(t), Z) = t$. Thus, Z has positive reach at least r in X (cf. [Lyta]).

We finish this section with the following useful observation.

Lemma 4.4. *Let $f : X \to Y$ be a locally Lipschitz and locally C-open map. Let $\gamma : [0, Cr] \to Y$ be a 1-Lipschitz curve in Y starting at $y = \gamma(0) = f(x)$. If the ball $B_r(x)$ is compact, then there is a $\frac{1}{C}$-Lipschitz curve $\bar{\gamma} : [0, Cr] \to X$ with $\gamma(0) = x$ and $f \circ \bar{\gamma} = \gamma$.*

Proof. For each $n > 0$, we can use Proposition 4.3 and induction on i to find points x_n^i, $0 \le i \le 2^n$, such that $x_n^0 = x$, $f(x_n^i) = \gamma(\frac{i}{2^n} Cr)$, and for $0 \le i \le 2^n - 1$ the inequality $d(x_n^i, x_n^{i+1}) \le \frac{1}{C} d(\gamma(\frac{i}{2^n} Cr), \gamma(\frac{i+1}{2^n} Cr))$ is fulfilled.

Let A_n denote the set of all numbers of the form $\frac{i}{2^n} Cr$, $0 \le i \le 2^n$; we have constructed a $\frac{1}{C}$-Lipschitz map $\eta_n : A_n \to X$ with $\eta_n(0) = x$ and such that $f \circ \eta_n$ is the restriction of γ to A_n.

Since the ball $B_r(x)$ is compact, we can choose a limit map $\bar{\gamma} = \lim_\omega \eta_n : [0, Cr] \to B_r(x)$, obtaining the desired curve. □

§5. A funny construction and bi-Lipschitz maps

We start with a funny and quite general construction that makes a local submetry from a locally Lipschitz and locally open map. First, let $f : X \to Y$ be an L-Lipschitz map; we assume that X and Y are locally compact and locally geodesic and that X is connected. On X we consider a pseudometric $\tilde{d}$ defined by $\tilde{d}(x, z) = \inf L(f(\gamma))$, where the infimum is taken over all Lipschitz curves γ connecting x and z.

Let $\tilde{X}$ denote the corresponding metric space, i.e., $\tilde{X}$ is obtained from X by identifying points with $\tilde{d}(x, z) = 0$. The natural projection $p : X \to \tilde{X}$ is locally L-Lipschitz and surjective. The projection is locally bi-Lipschitz if each point $x \in X$ is contained in an open set U such that for each Lipschitz curve $\gamma \subset U$ we have $L(\gamma) \le CL(f(\gamma))$ for some constant $C = C(U)$. The map f induces a map $\tilde{f} : \tilde{X} \to Y$ with $f = \tilde{f} \circ p$. We observe that the map $\tilde{f}$ is locally 1-Lipschitz.

Example 5.1. Let G be a Carnot group, and let $f : G \to \mathbb{R}^k$ be the canonical projection onto the horizontal part. Then f is a submetry, and $\tilde{G} = G$ because the rectifiable curves in G must be horizontal. On the other hand, if the set G is considered with a Euclidean metric, then $\tilde{G} = \mathbb{R}^k$ and $\tilde{f}$ is an isometry.

Assume now that, moreover, f is locally C-open. Since X is locally compact, we may apply Lemma 4.4 to show that the map $\tilde{f} : \tilde{X} \to Y$ is locally 1-open; hence, it is a local submetry. Now we are ready to prove Proposition 1.3.

So, let X be a connected locally compact space on which the metric is locally bi-Lipschitz with respect to the inner metric. Let $f : X \to Y$ be a locally Lipschitz map onto a Lipschitz manifold, and suppose that each blowup $f_x^{(t_i)}$ is C-bi-Lipschitz.

We can replace the metric on X by the inner metric and assume that X is locally geodesic. Changing the metric on Y by a bi-Lipschitz map, we may assume that $Y = \mathbb{R}^n$. If we replace X by an open relatively compact subset, then the assumptions will survive (with another constant C). By Example 2.3, for each curve γ in X we have $L(\gamma) \leq CL(f \circ \gamma)$. Since $\mathbb{R}^n$ is geodesic, by the above construction we get a local submetry $\tilde{f} : \tilde{X} \to \mathbb{R}^n$. Since the projection $p : \tilde{X} \to X$ is bi-Lipschitz, we may replace X by $\tilde{X}$. Thus, it suffices to prove the following statement: if X is a locally compact locally geodesic space and $f : X \to \mathbb{R}^n$ is a local submetry such that each blowup $f_x^{(t_i)}$ is C-bi-Lipschitz, then $f : X \to \mathbb{R}^n$ is a local isometry.

Each blowup $f_x^{(t_i)}$ is a submetry, and since it is bi-Lipschitz, it must be an isometry. Each fiber must be discrete by Example 2.2. Hence, for each x there is a neighborhood $B_r(x)$ of x such that for all $z \in B_r(x)$ we have $d(x, z) = d(f(x), f(z))$. This implies that each geodesic $\gamma : [a, b] \to X$ is mapped by f onto a curve $\bar{\gamma}$ in Y that consists of finitely many geodesic segments, i.e., for some sequence $a = a_0 < a_1 < ... < a_n = b$ the curve $\bar{\gamma} : [a_i, a_{i+1}] \to \mathbb{R}^n$ is a geodesic. Moreover, the incoming and the outgoing directions of $\bar{\gamma}$ at $\bar{\gamma}(a_i)$ form an angle of π, because $f_x^{(t_i)}$ is an isometry for $x = \gamma(a_i)$. But in $\mathbb{R}^n$ such a curve $\bar{\gamma}$ is a geodesic. In particular, $\gamma(a) \neq \gamma(b)$. This shows that each subset U of X that is geodesic in X contains at most one point of each fiber of f. Consequently, $f : B_r(x) \to \mathbb{R}^n$ is injective for some $r = r(x) > 0$. This implies that f is a local isometry.

This finishes the proof of Proposition 1.3 and Corollary 1.5.

§6. Gradient-like and gradient curves

6.1. Existence. Let X be a space, $f : X \to \mathbb{R}$ a locally Lipschitz function. We shall say that f has (lower) semicontinuous absolute gradients if $\lim_\omega |\nabla_{x_i} f| \geq |\nabla_x f|$ for each sequence $x_i \to x$ in X. We call x a *critical point* of f if $|\nabla_x f| = 0$. The semicontinuity implies that the set U of noncritical points is open.

Definition 6.1. Let f be a locally Lipschitz function in X. A curve $\eta : [a_1, a_2) \to X$ avoiding the set of critical points of f is called a *gradient-like curve* of f if η is 1-Lipschitz and for the right-hand side differential we have $(f \circ \eta)^+(t) = |\nabla_{\eta(t)} f|$ for all $t \in [a_1, a_2)$.

Lemma 6.1. *Suppose $f : X \to \mathbb{R}$ is a locally Lipschitz function with semicontinuous absolute gradients on a locally compact space X. Then a gradient-like curve $\eta : [0, a) \to X$ starts at each noncritical point x of f.*

Proof. By the semicontinuity of the absolute gradients, we can find $r > 0$ such that f is L-Lipschitz in $B_r(x)$ and $|\nabla_z f| \geq C > 0$ for each $z \in B_r(x)$. Assuming that $B_r(x)$ is compact, for simplicity we set $f(x) = 0$. We construct a gradient-like curve η of length at least r starting at x.

As in §4, we denote by U_t the closed set $f^{-1}[t, \infty) \subset X$. For each $\rho > 0$ in $B_r(x)$ we choose a maximal sequence $x = x_0, x_1, ..., x_{n(\rho)}$ with $f(x_i) = i\rho$ and $d(x_i, x_{i+1}) = d(x_i, U_{(i+1)\rho})$. By Corollary 4.2, we have $d(x_i, x_{i+1}) \leq \frac{\rho}{C}$. Since f is L-Lipschitz, we have $d(x_i, x_{i+1}) \geq \frac{\rho}{L}$. Moreover, the maximality of the sequence implies that $d(x, x_{n(\rho)}) \geq r - \frac{\rho}{C}$.

Putting $t_0 = 0$, we define by induction $t_{i+1} = t_i + d(x_i, x_{i+1})$, and let A_ρ denote the set $\{t_0, t_1, ..., t_{n(\rho)}\}$. Consider the map $\eta_\rho : A_\rho \to B_r(x)$ given by $\eta_\rho(t_i) := x_i$. By construction, η_ρ is a 1-Lipschitz map defined on a subset of the compact interval $[0, \frac{L}{C} r]$. The map $f_\rho = f \circ \eta_\rho$ satisfies $f_\rho(\bar{t}) - f_\rho(t) \geq C(\bar{t} - t)$ for all $\bar{t} \geq t \in A_\rho$.

If, moreover, for some $T = A_\rho \cap [t_1, t_2]$ the image $\eta_\rho(T)$ is contained in a ball V on which $|\nabla_z f| \geq \bar{C}$, then for all $t, \bar{t} \in T$ we obtain $f_\rho(\bar{t}) - f_\rho(t) \geq \bar{C}(\bar{t} - t)$, again by Corollary 4.2.

Now, let η be a limit map of some sequence η_{ρ_j} with $\rho_j \to 0$. Using ultralimits, we can consider, e.g., the map $\eta = \lim_\omega \eta_{\rho_j} : \lim_\omega A_\rho \to B_r(x)^\omega$. Since $B_r(x)$ is compact, $B_r(x)^\omega = B_r(x)$. The subsets A_ρ of the real line converge to a finite interval I in the Gromov–Hausdorff topology; hence, η is a 1-Lipschitz curve in $B_r(x)$ starting at x and ending at the boundary of $B_r(x)$.

For an arbitrary $s \in I$, consider the point $z = \eta(s)$. For $\epsilon > 0$, we consider a ball V around z of some positive radius $\tilde{r}$ and such that $|\nabla_{\bar{z}} f| \geq |\nabla_z f| - \epsilon$ for each $\bar{z} \in V$. If ρ is so small that η_ρ is close to η, then a part of η_ρ of length $\frac{\tilde{r}}{2}$ is contained in V; hence, $f_\rho(\bar{t}) - f_\rho(t) \geq \bar{C}(\bar{t} - t)$ with $\bar{C} = |\nabla_z f| - \epsilon$ and all $t, \bar{t}$ close to s.

Thus, $\liminf_{t\to 0} \frac{f\circ\eta(s+t) - f\circ\eta(s)}{t} \geq |\nabla_z f|$. Since η is 1-Lipschitz, the very definition of $|\nabla_z f|$ shows that the above lower limit is in fact a limit, and it is equal to $|\nabla_z f|$. Therefore, $(f \circ \eta)^+(t) = |\nabla_z f|$, and η is indeed a gradient-like curve of f starting at x. □

Remark 6.1. The compactness of $B_r(x)$ was used at two steps in the proof above. First, we chose points $x_{i+1} \in U_{(i+1)\rho}$ with $d(x_i, x_{i+1}) = d(x_i, U_{(i+1)\rho})$. However, should we have chosen x_{i+1} such that $d(x_i, x_{i+1}) - d(x_i, U_{(i+1)\rho}) \leq \rho^3$, the same construction would still work. A more important point was the limiting procedure, where the fact that $B_x(r)^\omega = B_x(r)$ was used. At this place the compactness assumption cannot be avoided. See however Subsection 9.3 for a trick.

Observe that the last argument in the proof of Lemma 6.1 shows that for each gradient-like curve $\eta : [0, a) \to X$ the metric differential $mD_t^+\eta$ (see Subsection 2.5) exists and is equal to 1 for all $t \in [0, a)$. In particular, η is parametrized by arclength.

6.2. Reparametrization. In order to continue gradient-like curves to the set of non-critical points and to study the gradient flow of semiconcave functions, we reparametrize the gradient-like curves. For a gradient-like curve η, by the semicontinuity of absolute gradients we have

$$\frac{f\circ\eta(t+\epsilon) - f\circ\eta(t)}{\epsilon} = \frac{1}{\epsilon}\int_t^{t+\epsilon} (f\circ\eta)^+(r)\,dr = \frac{1}{\epsilon}\int_t^{t+\epsilon} |\nabla_{\eta(t+r)} f|\,dr.$$

Therefore,

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_t^{t+\epsilon} |\nabla_{\eta(t+r)} f|\,dr = |\nabla_{\eta(t)} f|.$$

We want to reparametrize a gradient-like curve η to obtain another curve $\tilde{\eta}(t) = \eta(l(t))$ for some monotone Lipschitz map $l : [0, \tilde{a}) \to [0, a)$ such that $mD_t^+\tilde{\eta} = |\nabla_{\tilde{\eta}(t)} f|$, i.e., such that $l^+(t) = |\nabla_{\eta(l(t))} f|$.

The existence of such l follows directly from the next lemma.

Lemma 6.2. *Let* $g : [0, a) \to \mathbb{R}$ *be a locally bounded lower semicontinuous positive function satisfying* $\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_t^{t+\epsilon} g(r)\,dr = g(t)$ *for all* $t \in [0, a)$. *Then for some* $\rho > 0$ *there is a monotone bijective Lipschitz function* $f : [0, \rho) \to [0, a)$ *with* $f^+(t) = g(f(t))$ *for all* $t \in [0, \rho)$.

Proof. The existence of a map f solving the differential equation locally is deduced precisely as in the usual theorem of Peano. (Find a solution of the equation $f(t) = \int_0^t g(f(s))\,ds$ using the Arzela–Ascoli theorem. The assumptions imposed on g imply that $f^+(t) = g(f(t))$ for all t in the interval of definition of f.) *A priori*, a solution is a monotone function; taking a maximal solution (with the help of the Zorn lemma), we get the required result. □

This shows that a gradient curve starts at each noncritical point.

Definition 6.2. Let $f : X \to \mathbb{R}$ be a function with semicontinuous absolute gradients. A curve $\eta : [0, a) \to X$ is called a *gradient curve* of f if for all $t \in [0, a)$ we have $mD_t^+\eta = |\nabla_{\eta(t)} f|$ and $(f \circ \eta)^+ = |\nabla_{\eta(t)} f|^2$.

If x is a critical point of f, then the constant curve $\eta(t) = x$ is a gradient curve starting at x. Now, consider a maximal gradient curve $\eta : [0, a) \to X$ starting at an arbitrary point x. First, assume that $a < \infty$. If $L(\eta) = \infty$, then $\int_0^a |\nabla_{\eta(t)} f| \, dt = \infty$, whence $\int_0^a |\nabla_{\eta(t)} f|^2 \, dt = \infty$. We see that in this case f admits no uniform Lipschitz constant, and $\lim_{t \to a} f(\eta(t)) = \infty$. If $L(\eta) < \infty$ and the point $z = \lim \eta(t_i)$ exists for some sequence $t_i \to a$ (which is always the case if X is complete), then $z = \lim_{t \to a} \eta(t)$, and adding to η a gradient curve starting at z, we get a contradiction with the maximality of η. This finishes the proof of Proposition 1.6.

Finally, we note that if for a gradient curve $\eta : [0, \infty) \to X$ of f and some sequence $t_j \to \infty$ the point $z = \lim \eta(t_i)$ exists, then z must be a critical point of f.

§7. Semiconcave functions

7.1. Basics. Recall that a function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is said to be λ-*concave* for a real number λ if the function $f + \lambda t^2$ is concave on I. This is equivalent to the inequality $f'' \leq -2\lambda$ (in the weak sense).

The following lemma is well known.

Lemma 7.1. *A λ-concave function $f : I \to \mathbb{R}$ is locally Lipschitz and differentiable (from both sides) at each point. For all $t, t + \epsilon \in I$ we have $f(t + \epsilon) - f(t) \leq \epsilon f^+(t) - \lambda\epsilon^2$ and $f^+(t) - f^-(t + \epsilon) \geq 2\lambda\epsilon$.*

Definition 7.1. Let λ be a real number. A locally Lipschitz function $f : X \to \mathbb{R}$ on a metric space X is said to be λ-*concave* if the restriction of f to each geodesic is λ-concave. Next, f is *semiconcave* if each point x has a neighborhood U such that the restriction of f to U is $\lambda(x)$-concave for some real number $\lambda(x)$.

The disadvantage of this definition is that it is not stable under limits, as the following example shows.

Example 7.1. Let B be a finite-dimensional Banach space. If the norm of B is uniformly convex, then the geodesics are affine lines, and each function that is semiconcave on the Euclidean space $\mathbb{R}^n$ is also semiconcave on B. If the norm of B is not strongly convex, then a function as above may fail to be semiconcave, because geodesics in B may have corners. Approximating a non-strongly-convex norm on a two-dimensional Banach space by strongly convex norms, and letting f_j be the same function, we see that λ-concavity is not stable under limits.

This motivates the following stable definition.

Definition 7.2. We say that a locally Lipschitz map $f : X \to \mathbb{R}$ is *weakly λ-concave* on a subset Z of X if for all $x_1, x_2 \in Z$ there is at least one geodesic γ in X connecting x_1 and x_2 and such that $f \circ \gamma$ is λ-concave. We call f a *weakly semiconcave* function if X is covered by open sets U such that on each of them f is weakly $\lambda(U)$-concave for some real number $\lambda(U)$.

If X is a (locally) geodesic space (this will be assumed from now on), then each λ-concave function is also (locally) weakly λ-concave. Example 7.1 shows that the converse may fail. However, if geodesics between each two points are unique, then weak semiconcavity implies semiconcavity. For example, this is the case for the general $CAT(\kappa)$ spaces. We indicate another important situation where this is true.

Example 7.2. Let X be a space with curvature bounded from below by κ. Consider an arbitrary geodesic γ in X. Then γ is a unique geodesic between arbitrary inner points of γ. This shows that on X each weakly semiconcave function is also semiconcave.

The advantage of weak concavity is in stability under limits (see the lemma below), which follows directly from the fact that the ultralimits of geodesics are geodesics and that a pointwise limit of a sequence of L-Lipschitz λ_j-concave functions on an interval is $(\lim(\lambda_j))$-concave.

Lemma 7.2. *Let $f_j : (X_j, x_j) \to (\mathbb{R}, t_j)$ be L-Lipschitz functions, and let $f = \lim_\omega f_j : (X, x) \to (\mathbb{R}, t)$ be their ultralimit. If f_j is weakly λ_j-concave on the ball $B_r(x_j)$ and $\lim_{j\to\infty} \lambda_j = \lambda$, then f is weakly λ-concave on the ball $B_r(x)$.*

For example, let $f : X \to \mathbb{R}$ be a fixed weakly semiconcave function. Considering $f = f_j$, we see that the ultraproduct $f^\omega : X^\omega \to \mathbb{R}$ is weakly semiconcave in a neighborhood of $X \subset X^\omega$. Taking $f_j = f : (\frac{1}{t_j}X, x) \to \mathbb{R}$, we get a weakly 0-concave function $f_x^{(t_i)} : X_x^{(t_j)} \to \mathbb{R}$.

The next lemma is also well known (see [Pla02, p. 865]).

Lemma 7.3. *Let $f : CX \to \mathbb{R}$ be a homogeneous weakly 0-concave function of a geodesic Euclidean cone CX. Assume that f is positive for some $x \in CX$. Then there is a unique point $v \in X$ with $f(v) = \sup\{f(x) \mid x \in X\}$. Moreover, for each other point $w \in X$ we have $f(w) \le f(v)\langle v, w\rangle$.*

Proof. We view X as the unit sphere of CX. Let $x_i \in X$ be a sequence such that $f(x_i) \to \sup\{f(x) \mid x \in X\}$. We choose a geodesic γ between x_i and x_j such that $f \circ \gamma$ is concave. For the midpoint v of γ we have $|v| = \sqrt{1 - \frac{1}{4}d(x_i, x_j)^2}$ and $2f(v) \ge f(x_i) + f(x_j)$. This shows that x_i must be a Cauchy sequence, because otherwise $f(\frac{v}{|v|}) = \frac{1}{|v|}f(v)$ is larger than $\lim f(x_j) = \sup\{f(x) \mid x \in X\}$. The last statement of the lemma follows in the same way, by considering a midpoint m of a geodesic γ between tv and w as $t \to \infty$. □

The following semicontinuity result for a single function can be found in [Pla02, p. 864].

Lemma 7.4. *Let $f_j : (X_j, x_j) \to \mathbb{R}$ be as in Lemma 7.2, and let $f : (X, x) \to \mathbb{R}$ be their ultralimit. Then $\lim_\omega |\nabla_{x_j} f_j| \ge |\nabla_x f|$.*

Proof. We may assume that $|\nabla_x f| > 0$. Given a small number $\epsilon > 0$, we choose a point $z = (z_j) \in X$ such that $d(z, x) < \epsilon$ and $f(z) - f(x) \ge (|\nabla_x f| - \epsilon)d(x, z)$. Let $\gamma_j \in \Gamma_{x_j, z_j}$ be a geodesic such that $f_j \circ \gamma_j$ is λ_j-concave. Then $(f_j \circ \gamma_j)^+ \ge |\nabla_x f| - 3\epsilon$ if ϵ is sufficiently small and $f_j(z_j) - f_j(x_j) \ge (|\nabla_x f| - 2\epsilon)d(x_j, z_j)$. This implies the claim. □

7.2. Sublevel sets. In particular, for a single weakly semiconcave function $f : X \to \mathbb{R}$ the absolute gradients vary semicontinuously. Using this semicontinuity, we can study the sublevel sets of f outside the critical locus.

Definition 7.3. Let $f : X \to \mathbb{R}$ be a weakly semiconcave function. We say that the sublevel set $U_t = f^{-1}[t, \infty)$ is (C, r, λ)-*regular* if for all z with $f(z) < t$ and $d(z, U_t) < r$ we have $|\nabla_z f| \ge C$ and f is weakly λ-concave in the ball $B_{2r}(z)$.

The sublevel set is said to be *regular* if it is (C, r, λ)-regular for some $\lambda \in \mathbb{R}$ and some $C, r > 0$.

Example 7.3. Let X be a $CAT(1)$ space, let $x \in X$, and let $f = -d_x$. Then the sublevel sets U_t coincide with the balls $B_{-t}(x)$. For $t > -\pi$ the sublevel set U_t of f is regular.

Now we prove that the regular sublevel sets are not too far from being convex; this property plays a crucial role in [Lyta]; see also Corollary 1.9.

Lemma 7.5. *Suppose $f : X \to \mathbb{R}$ is weakly semiconcave and U_t is a (C, r, λ)-regular sublevel set of f. Then for all $x_0, x_1 \in U_t$ with $d(x_0, x_1) < r$ and some midpoint $\bar{m}$ between x_0 and x_1 there is a point $m \in U_t$ satisfying $d(m, \bar{m}) \leq K d(x_0, x_1)^2$ with $K = \max\{0, -\frac{\lambda}{C}\}$.*

Proof. Let γ be a geodesic between x_0 and x_1 such that f is λ-concave on γ. Then for the midpoint m of γ we get $f(m) - t \geq \lambda d(x_0, x_1)^2$. If λ is nonnegative, then U_t is convex and there is nothing to prove. If $\lambda < 0$, then we can apply Corollary 4.2, which shows that $d(m, U_t) \leq -\frac{\lambda}{C} d(x_0, x_1)^2$. □

Remark 7.4. If f is λ-concave, then the above statement remains valid for each midpoint m between x_0 and x_1.

§8. Differentials in metric spaces

We refer to [Lytb] for the details on differentials. Let X be an arbitrary metric space, and let $x \in X$. We denote by Γ_x the set of all geodesics starting at x. On $\Gamma_x \times [0, \infty)$ we define a pseudometric by $d((\gamma_1, t_1), (\gamma_2, t_2)) = \limsup_{s \to 0} \frac{d(\gamma_1(st_1), \gamma_2(st_2))}{s}$. The geodesic cone C_x at x is the completion of the metric space arising from $\Gamma_x \times [0, \infty)$. For each zero sequence (t_i), there is a natural 1-Lipschitz map $\exp_x^{(t_i)} : C_x \to X_x^{(t_i)}$ given by $\exp_x^{(t_i)}((\gamma, s)) = (\gamma(st_i)) \in X_x^{(t_i)}$. The upper angle is equal to the lower angle (see [BBI01, p. 96]) if and only if C_x is a Euclidean cone, and each exponential map $\exp_x^{(t_i)}$ is an isometric embedding. In this case we identify C_x with its image $\exp_x^{(t_i)}(C_x)$ in $X_x^{(t_i)}$.

Definition 8.1. We say that a locally geodesic locally compact space is *appropriate* if for each $x \in X$ the geodesic cone C_x is a Euclidean cone and for each zero sequence (t_i) the exponential map $\exp_x^{(t_i)} : C_x \to X_x^{(t_i)}$ is an isometry.

Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. We say that f is *directionally differentiable* at x if for each geodesic $\gamma \in \Gamma_x$ the composition $f \circ \gamma$ is differentiable from the right at 0. If f is directionally differentiable at x, then f determines a homogeneous Lipschitz map $D_x f : C_x \to \mathbb{R}$ such that $D_x f = f_x^{(t_i)} \circ \exp_x^{(t_i)}$ for each zero sequence (t_i).

A Lipschitz curve $\gamma : [t, a) \to X$ is said to be *strongly differentiable* (from the right) at t with differential $v \in C_{\gamma(t)}$ if for each zero sequence (t_i) the point $(\gamma(t + t_i)) \in X_{\gamma(t)}^{(t_i)}$ coincides with $\exp_x^{(t_i)}(v)$.

Finally, let X be an arbitrary space, and let x, z be two points in X connected by a geodesic γ. Assume that at x and z the upper angles and the lower angles between geodesics coincide. Let μ and ν be two Lipschitz curves starting at x (respectively, at z) and strongly differentiable at 0 with differentials $v \in C_x$ (respectively, $w \in C_z$). Let $\gamma^+ \in C_x$ (respectively, $\gamma^- \in C_z$) be the original (respectively, the terminal) direction of γ. Then we have the following first variation inequality (see [Lytb]).

Lemma 8.1. *If under the above conditions the function $l(t) = d(\mu(t), \nu(t))$ is differentiable at 0 from the right, then $l^+(0) \leq -\langle \gamma^+, v \rangle - \langle \gamma^-, w \rangle$.*

§9. Gradient flow

9.1. Differentiation of semiconcave functions. Suppose $f : X \to \mathbb{R}$ is a semiconcave function defined on a locally geodesic space X. Since the restriction of f to each geodesic is differentiable, at each point $x \in X$ we get a unique homogeneous (directional) differential $D_x f : C_x \to \mathbb{R}$ such that $D_x f = f_x^{(t_i)} \circ \exp_x^{(t_i)}$ for each zero sequence (t_i). If z is a point in X close to x and f is λ-concave on $\gamma \in \Gamma_{x,z}$, then from Lemma 7.1 we deduce that $(f \circ \gamma)^+ \cdot d(x,z) \geq f(z) - f(x) - \lambda d(x,z)^2$. This immediately implies that $|\nabla_x f| = \sup\{D_x f(v) \mid v \in C_x, |v| = 1\}$.

Now, assume that X is an appropriate space in the sense of Definition 8.1. Then $\exp_x^{(t_i)}$ is an isometry; hence, the cone C_x is a geodesic Euclidean cone and $D_x f$ is a weakly concave function on C_x. By Lemma 7.3, either x is a critical point of f and we set $\nabla_x f = 0 \in C_x$ in this case, or $D_x f$ attains its maximum $|\nabla_x f|$ on the unit sphere S_x of C_x at a unique point v, and then we set $\nabla_x f = |\nabla_x f| v$. From Lemma 7.3 we conclude that in both cases the inequality $D_x f(w) \leq \langle \nabla_x f, w \rangle$ is fulfilled.

9.2. Gradient curves. In this and the next subsection, X is an appropriate space and $f : X \to \mathbb{R}$ is a fixed semiconcave function. Since X is assumed to be locally compact, a maximal gradient curve $\eta : [0,a) \to X$ starts at each point (Proposition 1.6). For each $t \in [0,a)$, the curve η is strongly differentiable from the right, and $\eta^+(t) = \nabla_{\eta(t)} f$ by Lemma 7.3, because for each zero sequence (t_i) the point $v = (\eta(t+t_i)) \in X_{\eta(t)}^{(t_i)} = C_{\eta(t)}$ satisfies $|v| = |\nabla_{\eta(t)} f|$ and $D_{\eta(t)} f(v) = |\nabla_{\eta(t)} f|^2$.

Assume that η_1 and η_2 are two gradient curves defined on the same interval $[0,a)$ and contained in a small open set where f is λ-concave. Consider the locally Lipschitz function $l(t) = d(\eta_1(t), \eta_2(t))$. Let $\gamma : [0,s] \to X$ be a geodesic between $x = \gamma(0) = \eta_1(0)$ and $z = \gamma(s) = \eta_2(0)$. By Lemma 8.1, we have $l^+(0) \leq -\langle \gamma^+, \nabla_x f \rangle - \langle \gamma^-, \nabla_z f \rangle$ if $l^+(0)$ exists. Consequently, by Lemma 7.3 and Lemma 7.1, we get

$$\begin{aligned} l^+(0) &\leq -D_x f(\gamma^+) - D_z f(\gamma^-) \\ &= -(f \circ \gamma)^+(0) + (f \circ \gamma)^-(s) \\ &\leq -2\lambda s. \end{aligned}$$

Thus, $l^+(t) \leq -2\lambda l(t)$ for all t where $l^+(t)$ exists. Since l is a locally Lipschitz function, $l^+(t)$ exists for almost every t.

Remark 9.1. The above inequality is equivalent to $(ln \circ l)^+ \leq -2\lambda$.

This shows that at each point at most one gradient curve starts. Hence, we can consider the flow $\phi : D \subset X \times \mathbb{R}$ defined by $\phi^t(x) = \phi(x,t) = \eta(t)$, where η is the gradient curve starting at x. The subset D is the set where the flow ϕ is defined, and D is an open neighborhood of $X \times \{0\} \subset X \times \mathbb{R}$, by Proposition 1.6. We have proved the following statement.

Proposition 9.1. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz semiconcave function. If X is appropriate, then precisely one gradient curve starts at each point $x \in X$. The flow along the gradient curves is locally defined and locally Lipschitz. It is locally* 1*-Lipschitz if the function f is concave.*

We note that the Lipschitz constant of the flow on a set U depends only on the concavity constant of f on U. Using Proposition 1.6, we get Theorem 1.7 in the case of appropriate spaces.

Now, assume that X is a proper, geodesic, and appropriate space. Let $f : X \to \mathbb{R}$ be a concave function. In this case the set S of critical points of f is easily seen to be the set of all points where f attains its global maximum. We assume that S is not empty. Then S is a totally convex subset of X. For each $p \in S$ and each gradient line $\eta : [0,\infty) \to X$ of f, the function $d(p, \eta(t))$ is monotone nonincreasing; therefore, η is contained in a bounded set. Since this set is compact, for some sequence $t_i \to \infty$ the point $z = \lim \eta(t_i)$ exists. This point must be critical, and the fact that $d(z, \eta(t))$ is monotone nonincreasing shows that $z = \lim_{t\to\infty} \eta(t)$. Thus, each gradient line γ of f in $X \setminus S$ has exactly one limit point in S, and we see that the map $\phi^\infty := \lim_{t\to\infty} \phi^t$ is a well-defined 1-Lipschitz retraction of X onto S. This proves Corollary 1.8.

9.3. Semiconcave functions in nonproper spaces. Assume that Z is a geodesic space with one-sided curvature bound, $U \subset Z$ is an open subset, and $f : U \to \mathbb{R}$ is a semiconcave function. Then the ultraproduct Z^ω has the same curvature bound, so that f^ω is a semiconcave function in a neighborhood of U in X^ω. Moreover, for each $z \in U$ the blowup $f_z^{(t_i)} : Z_z^{(t_i)} \to \mathbb{R}$ is a concave function. Suppose z is not a critical point of f. First, assume that Z has curvature bounded from above. Then, by the results of [Nik95] (cf. [Lytb]), C_z is a totally geodesic subset of $Z_z^{(t_i)}$. Therefore, Lemma 7.3 applies to the restriction $D_z f : C_z \to \mathbb{R}$ of the concave function $f_z^{(t_i)} : Z_z^{(t_i)} \to \mathbb{R}$. If Z has curvature bounded from below, then C_z may fail to be a convex subset of $Z_z^{(t_i)}$. However, for arbitrary points $v, w \in C_z$, each geodesic γ between v and w determines together with v and w a Euclidean triangle [Lytb]. Therefore, the proof of Lemma 7.3 applies again.

Let η be a gradient curve of f. Then η is a locally Lipschitz curve; hence, by [Lytb], it is strongly differentiable almost everywhere. Arguing as above, we see that $\eta^+(t) = \nabla_{\eta(t)} f$ almost everywhere. Again arguing as above, we conclude that at most one gradient curve starts at each point and that the gradient flow is locally Lipschitz whenever it is defined. Now, let $x \in Z$ be an arbitrary noncritical point at which no gradient curve and (as a consequence) no gradient-like curve starts. Using the construction of Lemma 6.1, we see that the curves η_{ρ_j} (defined in the proof of Lemma 6.1) do not converge. Consequently, we can choose two different subsequences $t_n, s_n \to 0$ of the sequence ρ_j such that the curves $\eta_1 = \lim_\omega \eta_{t_n}$ and $\eta_2 = \lim_\omega \eta_{s_n}$ in the ultraproduct X^ω are different. As in Lemma 6.1, we see that η_1 and η_2 are gradient-like curves of $f^\omega : X^\omega \to \mathbb{R}$ starting at $x \in X \subset X^\omega$. Hence, we get different gradient curves of f^ω in X^ω starting at x. But X^ω is again a space with one-sided bounded curvature, and we arrive at a contradiction.

Thus, Theorem 1.7 is also valid for the general spaces with one-sided curvature bound.

9.4. Regular sublevel sets. We are going to prove Corollary 1.9. So, let U_t be a (C, r, λ)-regular sublevel set of a semiconcave function $f : X \to \mathbb{R}$. By our assumptions on X, the gradient flow of f exists and is locally Lipschitz. Since the infimum of two λ-concave functions is λ-concave, we may assume that $t = \max f$, replacing f by $\bar{f}(x) = \max\{f(x), t\}$. Also, we may assume that $\lambda < 0$.

Let $x_0, x_1 \in U_t$ be such that $d(x_0, x_1) = s < r$, let γ be a geodesic between x_0 and x_1, and let $\bar{m}$ be its midpoint. Consider the gradient curve η of f starting at $\bar{m}$. Our assumptions and Lemma 7.5 imply that the point $m = \eta(\bar{K}s^2)$ is contained in U_t, for some fixed $\bar{K} > 0$. The gradient curves η_i starting at x_i are constant. For s sufficiently small, the gradient curve η is contained in the ball $B_{2r}(\bar{m})$ where the function f is λ-concave. Therefore, we may apply the considerations of Subsection 9.2 to show that the function $l_i(t) = d(\eta(t), \eta_i(t))$ satisfies $l_i(0) = \frac{s}{2}$ and $(\ln \circ l_i)^+ \le -2\lambda$.

We obtain $\ln(l_i(\bar{K}s^2)) \le \ln(l_i(0)) - 2\lambda\bar{K}s^2$. It follows that $d(x_i, m) \le \frac{s}{2} e^{-2\lambda\bar{K}s^2}$. Using the Taylor expansion of the exponential function, for sufficiently small s we can estimate the right-hand side from above by $\frac{s}{2}(1 - 4\lambda\bar{K}s^2)$. This finishes the proof of Corollary 1.9.

§10. Some easy counterexamples

In $\mathbb{R}^2$, consider the graph Γ of the function $y = x^2$. Let $\Gamma_1 \subset \mathbb{R}^3$ be obtained by rotation of Γ around the x-axis. The tangent space of Γ_1 at the origin coincides with the x-axis, and at each other point the tangent space is 2-dimensional. Let Γ_2 denote the intersection of Γ_1 with the (x, y)-plane, i.e., Γ_2 consists of two copies of Γ attached to each other at the origin.

Example 10.1. The projection of Γ_2 to the x-axis has an almost isometric differential at each point close to 0, but the projection is not locally injective.

Example 10.2. On Γ_1 the inner metric is locally bi-Lipschitz with respect to the induced metric. Consider the map $f : \Gamma_1 \to \Gamma_1$ that sends a point (x, v) with $v = (y, z)$ to $(x, O(v))$, where O is a map of the (y, z)-plane to itself given by $O(v) = \|v\|\tilde{O}(\frac{v}{\|v\|})$, and $\tilde{O} : S^1 \to S^1$ is defined by $\tilde{O}(z) = z^2$ (in the language of complex numbers). Observe that the map f is differentiable at each point, and the differential at each point is bi-Lipschitz. However, f is not locally injective at the origin.

Example 10.3. Consider the interval $I = [-1, 1]$ on the x-axis in $\mathbb{R}^3$. To each point $x_n = (\frac{1}{n}, 0)$ in I we attach a $\mathcal{C}^\infty$-loop γ_n of length $\frac{1}{n^2}$ that starts and ends orthogonally to I. Consider the curve γ starting at $(1, 0)$ that runs through I and all the loops γ_n in the natural order. Parametrized by arclength, γ satisfies the conditions of Proposition 1.1, but it is not locally injective at the point x with $\gamma(x) = 0$.

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