

GYSIN HOMOMORPHISM IN GENERALIZED COHOMOLOGY THEORIES

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ABSTRACT. Detailed proofs of three formulas announced by Panin and Smirnov are elaborated upon.

§1. INTRODUCTION

The present paper is devoted to the proof of three formulas announced by I. A. Panin and A. L. Smirnov in [6] for oriented cohomology theories on algebraic varieties.

These are the self-intersection formula, the Grothendieck type formula for the highest Chern class, and the excess formula. We systematically use the construction of deformation to the normal cone. The proofs are based only on the axioms stated in [6]. Therefore, the formulas mentioned above are valid for the cohomology theories determined by an appropriate T -spectrum in the \mathbb{A}^1 -homotopy category of Voevodskiĭ, e.g., for the motivic cohomologies, for the algebraic K -theory, and for algebraic cobordisms.

This paper is divided into three sections. In §1 we collect the definitions and statements to be used in the paper. After recalling the basic definitions, we describe the construction of the Gysin homomorphism, which is the main object of our study. We end this section with the proof of the self-intersection formula.

§2 is devoted entirely to the Grothendieck type formula for the highest Chern class. The proof of this formula splits into four parts. First, we consider the case of universal bundles over projective spaces. This case is easy to handle because we know how to calculate the cohomology of projective spaces. Then the formula is proved for a line bundle over an affine variety. After that, we consider the case of an arbitrary line bundle. Finally, we use the splitting principle to reduce the general case to the case of a line bundle.

In §3, we state and prove the excess formula. First, we prove a general lemma (Lemma 8), and then reduce the proof to the case of the normal bundle and adjust our case to the conditions of Lemma 8.

I want to thank I. A. Panin and A. L. Smirnov for the statement of the problem.

1.1. Basic definitions and necessary results. Here we recall the basic definitions of [6, 7, 12].

Let k be a field. In this paper, by a variety we mean an arbitrary quasiprojective variety over k . As a rule, by a subvariety we mean a closed subvariety.

In the sequel, by a *pair* we mean a variety together with an open subset in it. We denote by $\mathcal{S}m$ the category of smooth varieties and by $\mathcal{S}mOp$ the category of pairs (X, U) , where X is a smooth variety and U is an open set in X ; the morphisms in $\mathcal{S}mOp$ are morphisms of pairs. Since the empty set \emptyset is open, the map $X \mapsto (X, \emptyset)$ induces a

2000 *Mathematics Subject Classification.* Primary 14F20.

Key words and phrases. Grothendieck type formula, highest Chern class, self-intersection formula, excess formula.

natural embedding of categories $\mathcal{S}m \subset \mathcal{S}mOp$. We denote by $\mathcal{A}b$ the category of Abelian groups.

We recall that a cohomology theory is a contravariant functor

$$\mathcal{A} : \mathcal{S}mOp \longrightarrow \mathcal{A}b$$

and a morphism of functors

$$\{\partial_{X,U} : \mathcal{A}(U) \longrightarrow \mathcal{A}(X,U)\}$$

satisfying the following properties.

1. (Homotopy axiom) The natural projection $p : \mathbb{A}^1 \times X \longrightarrow X$ induces an isomorphism $p^{\mathcal{A}} : \mathcal{A}(X) \longrightarrow \mathcal{A}(\mathbb{A}^1 \times X)$ in cohomology.
2. (Localization sequence) For every pair (X,U) , the sequence

$$\dots \xrightarrow{i^{\mathcal{A}}} \mathcal{A}(X) \xrightarrow{i^{\mathcal{A}}} \mathcal{A}(U) \xrightarrow{\partial_{X,U}} \mathcal{A}(X,U) \xrightarrow{i^{\mathcal{A}}} \mathcal{A}(X) \xrightarrow{i^{\mathcal{A}}} \dots$$

is exact. Here i is an inclusion (in what follows, we use the letter i for an inclusion and p for a projection; sometimes, to avoid ambiguity, we use subscripts indicating the domain and the target of an embedding or a projection).

3. (Excision) An *étale* morphism $e : (X',U') \longrightarrow (X,U)$ induces an isomorphism $e^{\mathcal{A}} : \mathcal{A}(X,U) \longrightarrow \mathcal{A}(X',U')$ if $e^{-1}(X \setminus U) = X' \setminus U'$ and $e : X' \setminus U' \longrightarrow X \setminus U$ is an isomorphism.

Nearly always, $\mathcal{A}(X,U)$ is a graded Abelian group, and the homomorphism ∂ raises the degree of each element by 1.

Definition 1. A cohomology theory \mathcal{A} is said to be a *ring cohomology theory* if, in addition to the above operations, a bilinear operation

$$\times : \mathcal{A}(X,U) \times \mathcal{A}(Y,V) \longrightarrow \mathcal{A}((X,U) \times (Y,V)) = \mathcal{A}(X \times Y, U \times V \cup X \times V)$$

is defined (in graded theories, the degrees must be additive). The following conditions are assumed:

- 1) \times is functorial in each argument;
- 2) \times is associative;
- 3) there exists $1 \in \mathcal{A}(pt)$;
- 4) $\partial(a \times b) = \partial a \times b$.

The bilinear operation introduced in Definition 1 is called the *exterior product* or \times -product (cross-product). The cross-product gives rise to an interior product or \cup -product (cup-product)

$$\cup : \mathcal{A}(X,U_1) \times \mathcal{A}(X,U_2) \longrightarrow \mathcal{A}(X,U_1 \cup U_2),$$

which is defined by the formula

$$a \cup b = \Delta^{\mathcal{A}}(a \times b),$$

where $\Delta : X \longrightarrow X \times X$ is the diagonal. We have:

- 1) \cup is functorial;
- 2) \cup is associative;
- 3) there exists $1 \in \mathcal{A}(X)$;
- 4) $\partial(a \cup b) = \partial a \cup b$.

Thus, the group $\mathcal{A}(X)$ is an associative ring with 1 relative to the interior product.

Conversely, if a \cup -product is defined, then a \times -product can be defined by the formula

$$a \times b = p_1^{\mathcal{A}} a \cup p_2^{\mathcal{A}} b,$$

where p_1 and p_2 are the corresponding projections.

In the sequel, we need the following general lemma concerning \cup -products.

Lemma 1. *Let*

$$i_1 : (Y, C) \hookrightarrow (X, A), \quad i_2 : (Y, D) \hookrightarrow (X, B), \quad \text{and} \quad i : (Y, C \cup D) \hookrightarrow (X, A \cup B)$$

be inclusions displayed in the diagram

$$\begin{array}{ccc} (Y, C) & \xrightarrow{i_1} & (X, A) \\ \downarrow & & \downarrow \\ (Y, C \cup D) & \xrightarrow{i} & (X, A \cup B) \\ \uparrow & & \uparrow \\ (Y, D) & \xrightarrow{i_2} & (X, B). \end{array}$$

Suppose $x \in \mathcal{A}(X, A), y \in \mathcal{A}(X, B)$, *and* $x \cup y \in \mathcal{A}(X, A \cup B)$. *Then*

$$i^{\mathcal{A}}(x \cup y) = i_1^{\mathcal{A}}x \cup i_2^{\mathcal{A}}y.$$

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathcal{A}(X, A) \otimes \mathcal{A}(X, B) & \xrightarrow{\times} & \mathcal{A}((X, A) \times (X, B)) & \xrightarrow{\Delta^{\mathcal{A}}} & \mathcal{A}(X, A \cup B) \\ i_1^{\mathcal{A}} \otimes i_2^{\mathcal{A}} \downarrow & & (i_1 \times i_2)^{\mathcal{A}} \downarrow & & i^{\mathcal{A}} \downarrow \\ \mathcal{A}(Y, C) \otimes \mathcal{A}(Y, D) & \xrightarrow{\times} & \mathcal{A}((Y, C) \times (Y, D)) & \xrightarrow{\Delta^{\mathcal{A}}} & \mathcal{A}(Y, C \cup D), \end{array}$$

where Δ is the diagonal embedding. This diagram is commutative: the left square is commutative by the functorial property of the \times -product, and the right square is commutative because it is induced by a diagram of embeddings. Therefore,

$$i^{\mathcal{A}}(x \cup y) = (i^{\mathcal{A}} \circ \Delta^{\mathcal{A}})(x \times y) = \Delta^{\mathcal{A}}(i_1^{\mathcal{A}}x \times i_2^{\mathcal{A}}y) = i_1^{\mathcal{A}}x \cup i_2^{\mathcal{A}}y. \quad \square$$

Definition 2. Let (\mathcal{A}, \cup) be a ring cohomology theory. We say that (\mathcal{A}, \cup) is equipped with a Chern structure if, with each line bundle L/X , is associated an element $c(L) \in \mathcal{A}(X)$ with the following properties.

- 1) (Functoriality) If f is a mapping, then $f^{\mathcal{A}}(c(L)) = c(f^*(L))$.
- 2) (Vanishing) The Chern class of a trivial bundle is zero.
- 3) If L is the tautological bundle over the projective line \mathbb{P}^1 , then for every variety X we can consider the element

$$\xi_X = p^{\mathcal{A}}(c(L)) \in \mathcal{A}(X \times \mathbb{P}^1),$$

which yields an isomorphism of Abelian groups

$$(\cup 1, \cup \xi_X) : \mathcal{A}(X) \oplus \mathcal{A}(X) \longrightarrow \mathcal{A}(X \times \mathbb{P}^1).$$

- 4) $c(L)$ is a universally central element, i.e., for every mapping f the element $f^{\mathcal{A}}(c(L))$ is central.

The following theorem can be found in [6, 7, 12].

Theorem 1 (Projective bundle theorem). *Let* E *be a vector bundle of rank* d *over* X , *let* $\mathbb{P}(E)$ *be the corresponding projective bundle, and let* L_E *be the tautological bundle over* $\mathbb{P}(E)$. *Let* $\xi_E = c(L_E) \in \mathcal{A}(\mathbb{P}(E))$. *We define* $\xi_E^k : \mathcal{A}(X) \longrightarrow \mathcal{A}(\mathbb{P}(E))$ *as follows:*

$$\mathcal{A}(X) \xrightarrow{p^{\mathcal{A}}} \mathcal{A}(\mathbb{P}(E)) \xrightarrow{\cup \xi_E^k} \mathcal{A}(\mathbb{P}(E)).$$

Then the mapping

$$(\cup 1, \cup \xi_E, \dots, \cup \xi_E^{d-1}) : \bigoplus_0^{d-1} \mathcal{A}(X) \longrightarrow \mathcal{A}(\mathbb{P}(E))$$

is an isomorphism.

In particular,

$$\mathcal{A}(\mathbb{P}^n) \simeq \mathcal{A}(pt)[c(L_n)]/(c(L_n)^{n+1}).$$

Therefore, $\mathcal{A}(\mathbb{P}(E))$ carries the structure of a free $\mathcal{A}(X)$ -module with basis $1, \xi_E, \dots, \xi_E^{d-1}$. Then ξ_E^d can be represented in terms of this basis:

$$\xi_E^d - c_1(E)\xi_E^{d-1} + \dots + (-1)^d c_d(E) = 0.$$

The classes $c_i(E)$ arising in this representation are called the *Chern classes* of the bundle E .

We list the basic properties of Chern classes.

- 1) $c_0(E) = 1$.
- 2) If L is a line bundle, then $c(L) = c_1(L)$.
- 3) If $E \simeq E'$, then $c_i(E) = c_i(E')$.
- 4) For every bundle E/X and any mapping $f : Y \rightarrow X$, we have

$$f^A(c_i(E)) = c_i(f^*(E)).$$

- 5) (Cartan formula) For every short exact sequence of bundles

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0,$$

the Chern classes of E are calculated by the formula

$$c_k(E) = \sum_{i+j=k} c_i(E_1) \cup c_j(E_2).$$

- 6) $c_i(E) = 0$ if $i < 0$ or $i > \text{rk}(E)$.

1.2. Construction of the Gysin homomorphism. Here we recall the construction of the Thom class (see [6, 7] and [12]) and the Gysin homomorphism (see [6, 7]), which are the basic objects of study in the present paper. For this, we need two lemmas.

Lemma 2. *Let E be a bundle over a variety X (X is identified with the set of zeros of the bundle E). Then:*

- 1) $\mathbb{P}(\mathbf{1}) \subset \mathbb{P}(\mathbf{1} \oplus E)$ and $\mathbb{P}(E) \subset \mathbb{P}(\mathbf{1} \oplus E)$;
- 2) $\mathbb{P}(\mathbf{1} \oplus E) \setminus \mathbb{P}(\mathbf{1}) \simeq \mathbb{P}(E) \cup (E \setminus X)$ and $\mathbb{P}(\mathbf{1} \oplus E) \setminus \mathbb{P}(E) \simeq E$.

Proof. All mappings that will be constructed are canonical, so that we may assume that X is a point. Since $\mathbb{P}(\mathbf{1} \oplus E)$ is the set of lines in the space $\mathbf{1} \oplus E$, which contains the lines lying in $\mathbf{1}$ and in E , the first statement of the lemma is obvious.

We construct the canonical embedding of E in $\mathbb{P}(\mathbf{1} \oplus E)$ as follows. To a vector $e \in E$, we assign the line passing through the point $(1, e) \in \mathbb{P}(\mathbf{1} \oplus E)$. This mapping is injective, and only the lines lying in E are not in its image. Therefore, the second statement is also proved. \square

Lemma 3. *There is a short exact sequence*

$$0 \rightarrow \mathcal{A}(E, E \setminus X) \rightarrow \mathcal{A}(\mathbb{P}(\mathbf{1} \oplus E)) \rightarrow \mathcal{A}(\mathbb{P}(E)) \rightarrow 0.$$

Proof. Consider the localization sequence for the pair $(\mathbb{P}(\mathbf{1} \oplus E), \mathbb{P}(\mathbf{1} \oplus E) \setminus \mathbb{P}(\mathbf{1}))$:

$$\dots \rightarrow \mathcal{A}(\mathbb{P}(\mathbf{1} \oplus E), \mathbb{P}(\mathbf{1} \oplus E) \setminus \mathbb{P}(\mathbf{1})) \rightarrow \mathcal{A}(\mathbb{P}(\mathbf{1} \oplus E)) \rightarrow \mathcal{A}(\mathbb{P}(\mathbf{1} \oplus E) \setminus \mathbb{P}(\mathbf{1})) \rightarrow \dots$$

By Lemma 2, there is an inclusion

$$(E, E \setminus X) \hookrightarrow (\mathbb{P}(\mathbf{1} \oplus E), \mathbb{P}(\mathbf{1} \oplus E) \setminus \mathbb{P}(\mathbf{1})),$$

which, by excision, induces an isomorphism in cohomology. Next, we can consider the inclusion $i : \mathbb{P}(E) \hookrightarrow \mathbb{P}(\mathbf{1} \oplus E) \setminus \mathbb{P}(\mathbf{1})$ and the projection $p : \mathbb{P}(\mathbf{1} \oplus E) \setminus \mathbb{P}(\mathbf{1}) \rightarrow \mathbb{P}(E)$ induced by the projection $\mathbf{1} \oplus E \rightarrow E$. Since p is a line bundle and i is its zero section,

we see that $i^{\mathcal{A}}$ is an isomorphism. It follows that the initial sequence can be replaced by a simpler sequence,

$$\dots \longrightarrow \mathcal{A}(E, E \setminus X) \longrightarrow \mathcal{A}(\mathbb{P}(\mathbf{1} \oplus E)) \longrightarrow \mathcal{A}(\mathbb{P}(E)) \longrightarrow \dots.$$

We prove that, actually, this sequence is a short exact sequence. For this, it suffices to prove that the mapping

$$\mathcal{A}(\mathbb{P}(\mathbf{1} \oplus E)) \longrightarrow \mathcal{A}(\mathbb{P}(E))$$

is an epimorphism. Obviously, the element $\xi_{\mathbf{1} \oplus E}$ is taken to ξ_E . At the same time, these elements are multiplicative generators of the free $\mathcal{A}(X)$ -modules $\mathcal{A}(\mathbb{P}(\mathbf{1} \oplus E))$ and $\mathcal{A}(\mathbb{P}(E))$, respectively. Therefore, the mapping in question is an epimorphism. \square

Now, we define the Thom class $\text{th}(E) \in \mathcal{A}(E, E \setminus X)$. The short exact sequence obtained above shows that it suffices to specify an element $\mathcal{A}(\mathbb{P}(\mathbf{1} \oplus E))$ that becomes zero after restriction to $\mathcal{A}(\mathbb{P}(E))$. As such an element we can take $c_n(L_{\mathbf{1} \oplus E}^* \otimes p^*(E))$, where $p: \mathbb{P}(\mathbf{1} \oplus E) \rightarrow X$ is a projective bundle and $L_{\mathbf{1} \oplus E}^*$ is a bundle dual to $L_{\mathbf{1} \oplus E}$. Thus, we have defined $\text{th}(E) \in \mathcal{A}(E, E \setminus X)$. Sometimes, it is reasonable to use a different class

$$\overline{\text{th}(E)} \in \mathcal{A}(\mathbb{P}(\mathbf{1} \oplus E), \mathbb{P}(\mathbf{1} \oplus E) \setminus \mathbb{P}(\mathbf{1})),$$

which goes to $\text{th}(E)$ under an appropriate restriction.

Let X be a variety, Y a subvariety of X , $N_{X/Y}$ a normal bundle, $i: Y \hookrightarrow X$ an inclusion, and $p: N \rightarrow Y$ a projection. Then there exists a variety X_t , a closed embedding $Y \times \mathbb{A}^1 \hookrightarrow X_t$, and inclusions

$$i_0: (N, N \setminus Y) \hookrightarrow (X_t, X_t \setminus Y \times \mathbb{A}^1)$$

and

$$i_1: (X, X \setminus Y) \hookrightarrow (X_t, X_t \setminus Y \times \mathbb{A}^1)$$

such that the induced diagram

$$\mathcal{A}(N, N \setminus Y) \xleftarrow{i_0^{\mathcal{A}}} \mathcal{A}(X_t, X_t \setminus Y \times \mathbb{A}^1) \xrightarrow{i_1^{\mathcal{A}}} \mathcal{A}(X, X \setminus Y)$$

consists of isomorphisms (see [12, Theorem 2.2.8]).

We define the Gysin homomorphism $i_{\text{gys}}: \mathcal{A}(Y) \rightarrow \mathcal{A}(X)$ as the composition

$$\mathcal{A}(Y) \xrightarrow{p^{\mathcal{A}}} \mathcal{A}(N) \xrightarrow{\cup \text{th}(N)} \mathcal{A}(N, N \setminus Y) \xleftarrow{i_0^{\mathcal{A}}} \mathcal{A}(X_t, X_t \setminus Y \times \mathbb{A}^1) \xrightarrow{i_1^{\mathcal{A}}} \mathcal{A}(X, X \setminus Y) \longrightarrow \mathcal{A}(X),$$

where the final homomorphism is induced by the inclusion of X in $(X, X \setminus Y)$.

In particular, the Gysin homomorphism has the following important properties:

- 1) $(i \circ j)_{\text{gys}} = i_{\text{gys}} \circ j_{\text{gys}}$;
- 2) (projection formula) $i_{\text{gys}}(i^{\mathcal{A}}(a) \cup b) = a \cup i_{\text{gys}}(b)$.

We prove yet another general lemma.

Lemma 4. *The following diagram is commutative:*

$$\begin{array}{ccccc} \mathcal{A}(X_t, X_t \setminus Y \times \mathbb{A}^1) & \xrightarrow{\sim} & \mathcal{A}(X, X \setminus Y) & \longrightarrow & \mathcal{A}(X) \\ \simeq \downarrow & & & & \downarrow \\ \mathcal{A}(N, N \setminus Y) & \longrightarrow & \mathcal{A}(N) & \longrightarrow & \mathcal{A}(Y). \end{array}$$

Proof. By the properties of the variety X_t that was constructed in [12], the following diagram of inclusions is commutative:

$$\begin{array}{ccccc} N & \longrightarrow & X_t & \longleftarrow & X \\ \uparrow & & \uparrow & & \uparrow \\ Y & \xrightarrow{i_0} & Y \times \mathbb{A}^1 & \xleftarrow{i_1} & Y, \end{array}$$

where $i_0(y) = (y, 0)$ and $i_1(y) = (y, 1)$. Moreover, the following diagram is homotopy commutative:

$$\begin{array}{ccccc} Y & \xrightarrow{i_0} & Y \times \mathbb{A}^1 & \xleftarrow{i_1} & Y \\ \parallel & & p \downarrow & & \parallel \\ Y & \xlongequal{\quad} & Y & \xlongequal{\quad} & Y. \end{array}$$

The induced diagrams are commutative and, taken together, yield the required diagram. □

1.3. Self-intersection formula. Here, we prove the first of the main results of the present paper.

Theorem 2 (Self-intersection formula). *Let X be a variety, and let Y be a subvariety of X of codimension d . Let $N_{X/Y}$ be a normal bundle of Y in X , let $i : Y \hookrightarrow X$ be an inclusion, and let (\mathcal{A}, \cup, c) be a ring cohomology theory with Chern structure. Then*

$$(i^{\mathcal{A}} \circ i_{\text{gvs}})(x) = x \cup c_d(N).$$

Lemma 5. *The following diagram is commutative:*

$$\begin{array}{ccccc} \mathcal{A}(Y) & \xrightarrow{p^{\mathcal{A}}} & \mathcal{A}(\mathbb{P}(\mathbf{1} \oplus N)) & & \\ p^{\mathcal{A}} \downarrow & (1) & \parallel & & \\ \mathcal{A}(N) & \xleftarrow{i^{\mathcal{A}}} & \mathcal{A}(\mathbb{P}(\mathbf{1} \oplus N)) & = & \mathcal{A}(\mathbb{P}(\mathbf{1} \oplus N)) \\ \cup \text{th}(N) \downarrow & (2) & \overline{\cup \text{th}(N)} \downarrow & (3) & \downarrow \cup c_d(L_{\mathbf{1} \oplus N}^* \otimes p^*(N)) \\ \mathcal{A}(N, N \setminus Y) & \xleftarrow{i^{\mathcal{A}}} & \mathcal{A}(\mathbb{P}(\mathbf{1} \oplus N), \mathbb{P}(\mathbf{1} \oplus N) \setminus \mathbb{P}(\mathbf{1})) & \xrightarrow{i^{\mathcal{A}}} & \mathcal{A}(\mathbb{P}(\mathbf{1} \oplus N)) \\ \parallel & & (4) & & \downarrow i^{\mathcal{A}} \\ \mathcal{A}(N, N \setminus Y) & \longrightarrow & & & \mathcal{A}(N) \\ \simeq \uparrow & & & & \parallel \\ \mathcal{A}(X_t, X_t \setminus Y \times \mathbb{A}^1) & & (5) & & \mathcal{A}(N) \\ \simeq \downarrow & & & & \downarrow \\ \mathcal{A}(X, X \setminus Y) & \longrightarrow & \mathcal{A}(X) & \longrightarrow & \mathcal{A}(Y). \end{array}$$

Proof. Diagrams (1) and (4) are commutative, being induced by commutative diagrams. Diagram (5) is commutative by Lemma 4.

To prove that the square (2) is commutative, we apply Lemma 1 with $X = \mathbb{P}(\mathbf{1} \oplus N)$, $Y = N$, $A = C = \emptyset$, $B = \mathbb{P}(\mathbf{1} \oplus N) \setminus \mathbb{P}(\mathbf{1})$, $D = N \setminus Y$, and $y = \overline{\text{th}(N)}$. As a result, we obtain

$$i^{\mathcal{A}}(x \cup \overline{\text{th}(N)}) = i^{\mathcal{A}}(x) \cup i^{\mathcal{A}}(\overline{\text{th}(N)}) = i^{\mathcal{A}}(x) \cup \text{th}(N).$$

The fact that (3) is commutative can be proved similarly. After an appropriate substitution, we have the relation

$$i^{\mathcal{A}}(x \cup \overline{\text{th}(N)}) = x \cup i^{\mathcal{A}}(\overline{\text{th}(N)}) = x \cup c_d(L_{\mathbf{1} \oplus N}^* \otimes p^*(N)). \quad \square$$

Proof of Theorem 2. Using the relations obtained above, we see that

$$\begin{aligned} (i^{\mathcal{A}} \circ i_{\text{gys}})(x) &= i^{\mathcal{A}}(p^{\mathcal{A}}(x) \cup c_d(L_{\mathbf{1} \oplus N}^* \otimes p^*(N))) \\ &= i^{\mathcal{A}} p^{\mathcal{A}}(x) \cup i^{\mathcal{A}}(c_d(L_{\mathbf{1} \oplus N}^* \otimes p^*(N))) \\ &= x \cup c_d(i^*(L_{\mathbf{1} \oplus N}^*) \otimes i^*(p^*(N))) \\ &= x \cup c_d(\mathbf{1} \otimes N) = x \cup c_d(N), \end{aligned}$$

because the embedding and the projection are mutually inverse, and $L_{\mathbf{1} \oplus N}$ is trivial on Y . Thus, Theorem 2 is proved completely. \square

§2. GROTHENDIECK TYPE FORMULA FOR THE HIGHEST CHERN CLASS

In this section, we state and prove Theorem 3 containing a Grothendieck type formula [8]. The proof is split into four parts and is given below.

Definition 3. Let $E \rightarrow X$ be a vector bundle, let s_1 and s_2 be its sections, and let Y be a subset of X on which the values of s_1 and s_2 coincide. We say that s_1 and s_2 are *transversal* if the following squares are transversal:

$$\begin{array}{ccc} s_1(X) & \longrightarrow & E \\ \uparrow & & \uparrow \\ Y & \longrightarrow & s_2(X) \end{array} \quad \text{and} \quad \begin{array}{ccc} s_2(X) & \longrightarrow & E \\ \uparrow & & \uparrow \\ Y & \longrightarrow & s_1(X). \end{array}$$

In this case, Y is a subvariety of X of codimension equal to the rank of E .

Theorem 3 (Grothendieck type formula). *Let X be a variety and E a vector bundle of rank n over X . Suppose $z : X \rightarrow E$ is the zero section, and $s : X \rightarrow E$ is another section intersecting z transversally. Let $Y = \{x \in X \mid s(x) = z(x)\}$ be the set of zeros of s , and let $i : Y \hookrightarrow X$ be the corresponding inclusion. Finally, let (\mathcal{A}, \cup, c) be a ring cohomology theory with Chern structure. Then*

$$i_{\text{gys}}(1_Y) = c_n(E).$$

To prove Theorem 3, we use the fact that an arbitrary transversal square (see [7, Definition 4.2.2])

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ \uparrow i_{X/Y} & & \uparrow i_{Z/T} \\ Y & \xrightarrow{f|_Y} & T \end{array}$$

induces the following commutative diagram (see [7, Property 4.2.2]):

$$\begin{array}{ccc} \mathcal{A}(X) & \xleftarrow{f^{\mathcal{A}}} & \mathcal{A}(Z) \\ i_{X/Y, \text{gys}} \uparrow & & i_{Z/T, \text{gys}} \uparrow \\ \mathcal{A}(Y) & \xleftarrow{f|_Y^{\mathcal{A}}} & \mathcal{A}(T). \end{array}$$

2.1. The case of a universal bundle. Here, we prove Theorem 3 in the case where

$$X = \mathbb{P}^n = \{[x_0 : x_1 : \cdots : x_n] \mid \exists i : x_i \neq 0\}$$

(x_0, x_1, \dots, x_n are homogeneous coordinates on \mathbb{P}^n);

$$E_n^\vee = \mathbb{P}^{n+1} \setminus \{[1 : 0 : \cdots : 0]\};$$

$$p_n : E_n^\vee \longrightarrow \mathbb{P}^n, \quad p_n([x_0 : x_1 : \cdots : x_{n+1}]) = [x_1 : \cdots : x_{n+1}]$$

is a line bundle;

$$i_n : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n, \quad i_n([x_0 : x_1 : \cdots : x_{n-1}]) = [x_0 : x_1 : \cdots : x_{n-1} : 0]$$

is the standard embedding;

$$z_n : \mathbb{P}^n \longrightarrow E_n^\vee, \quad z_n([x_0 : x_1 : \cdots : x_n]) = [0 : x_0 : x_1 : \cdots : x_n];$$

$$s_n : \mathbb{P}^n \longrightarrow E_n^\vee, \quad s_n([x_0 : x_1 : \cdots : x_n]) = [x_0 : x_1 : \cdots : x_n : 0].$$

In this case, the set of zeros of s coincides with $i_n(\mathbb{P}^{n-1})$. We must prove that $i_{n,\text{gys}}(1) = c_1(E_n^\vee)$.

The case under consideration is easy to analyze because $N_{\mathbb{P}^{n+1}/\mathbb{P}^n} = E_n^\vee$. We want to use Theorem 2. Consider the diagram

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{z_n} & \mathbb{P}^{n+1} \\ i_n \uparrow & & \uparrow i_{n+1} \\ \mathbb{P}^{n-1} & \xrightarrow{z_{n-1}} & \mathbb{P}^n. \end{array}$$

To obtain a transversal square, we should take z rather than i as horizontal arrows. However, this does not cause trouble since i_{n+1} is homotopic to z_n and, consequently, $i_{n+1}^A = z_n^A$ and $i_{n+1,\text{gys}} = z_{n,\text{gys}}$. We prove that $z_n^*(E_n^\vee) = E_{n-1}^\vee$. Indeed,

$$\begin{aligned} z_n^*(E_n^\vee) &= \{([x_0 : x_1 : \cdots : x_{n-1}], [y_0 : y_1 : \cdots : y_{n+1}]) \in \mathbb{P}^{n-1} \times E_n^\vee \\ &\quad \mid [0 : x_0 : x_1 : \cdots : x_{n-1}] = [y_1 : \cdots : y_{n+1}]\} \\ &= \{[y_0 : x_1 : \cdots : x_{n-1}] \mid \exists i : x_i \neq 0\} = E_{n-1}^\vee. \end{aligned}$$

By transversality,

$$z_n^A \circ i_{n+1,\text{gys}} = i_{n,\text{gys}} \circ z_{n-1}^A.$$

Replacing z with the mapping i homotopic to z , we obtain

$$i_{n,\text{gys}}(1) = i_{n,\text{gys}} \circ i_n^A(1) = i_{n+1}^A \circ i_{n+1,\text{gys}}(1) = c_1(E_n^\vee).$$

The latter identity follows from Theorem 2. Thus, the case of a universal bundle is analyzed completely.

2.2. The case of a linear bundle over an affine variety. Let V^n be a vector space of dimension n , let V^* be the dual space, and let $\mathbb{P}(V)$ be the set of lines in V , or, which is the same, the set of hyperplanes in V^* . If W is a hyperplane in V , then there is a natural inclusion

$$i_W : \mathbb{P}(W) \hookrightarrow \mathbb{P}(V).$$

We consider the bundle

$$p : \mathbb{P}(\mathbf{1} \oplus V) \setminus \mathbb{P}(\mathbf{1}) \longrightarrow \mathbb{P}(V),$$

where a line distinct from $\mathbf{1}$ is sent to its projection onto V . The bundle $N_{\mathbb{P}(V)/\mathbb{P}(W)}$ is of the same structure: it is $\mathbb{P}(V) \setminus \mathbb{P}(L)$ with the natural projection onto $\mathbb{P}(W)$, where L is defined so that $W \oplus L = V$.

Consider the section

$$s_V : \mathbb{P}(V) \longrightarrow \mathbb{P}(L \oplus V) \setminus \mathbb{P}(L), \quad s(l) = (p_L(l), l),$$

where p_L is the projection onto L . Clearly, the set of zeros of s_V is $\mathbb{P}(W)$. In the preceding subsection it was proved that, in the present notation, we have

$$i_{W, \text{gys}}(1) = c_1(\mathbb{P}(L \oplus V) \setminus \mathbb{P}(L)).$$

Using this fact, we shall show that for an arbitrary linear bundle $p : E \rightarrow X$ over an affine variety X and a section $s : X \rightarrow E$ transversally intersecting the zero section along a subvariety Y of codimension 1, we have $i_{\text{gys}}(1_Y) = c_1(E)$, where $i : Y \hookrightarrow X$ is the natural inclusion.

Let $p : E \rightarrow X$ be a line bundle. Since X is affine, we can cover it by the trivial bundle

$$\tilde{\varphi} : X \times W \rightarrow E.$$

We modify $\tilde{\varphi}$ as follows: let L be a one-dimensional space, let $V = W \oplus L$, and let $\varphi : X \times V \rightarrow E$ act by the formula

$$\varphi(x, w, \alpha a) = \tilde{\varphi}(x, w) + \alpha s(x),$$

where a is a basis vector of the line L .

We consider the mapping

$$f_E : X \rightarrow \mathbb{P}(V^*), \quad f_E(x) = \text{Ker } \varphi(x).$$

($\text{Ker } \varphi(x)$ is a hyperplane in V , i.e., a line in V^* .) Also, we consider the space

$$E^\vee = \mathbb{P}(L \oplus V^*) \setminus \mathbb{P}(L),$$

which is a bundle space over $\mathbb{P}(V^*)$. Let

$$\tilde{E} = f_E^*(E^\vee) = \{(x, P) \in X \times \mathbb{P}(V) \mid \text{Ker } \varphi(x) = p_{\mathbb{P}(V^*)}(P)\},$$

where $p_{\mathbb{P}(V^*)}$ is the projection onto $\mathbb{P}(V^*)$.

Lemma 6. *In the above notation, let $h : E \rightarrow \tilde{E}$ be the mapping defined as follows. For $e \in E$, consider an element (x, v) in $X \times V$ such that $\varphi(x, v) = e$; also, consider the mapping*

$$g : \mathbb{P}(V^*) \times V \rightarrow E^\vee, \quad g(f, v) = [f(v)a : f]$$

(a is a basis vector of L). We put $h(e) = (x, g(f_E(x), v))$. Then h is an isomorphism of the bundles E and \tilde{E} .

Proof. 1. We have $(x, g(f_E(x), v)) \in \tilde{E}$, i.e., $\text{Ker } \varphi(x) = p_{\mathbb{P}(V^*)}(g(f_E(x), v))$, because $\text{Ker } \varphi(x) = f_E(x)$ and

$$\begin{aligned} p_{\mathbb{P}(V^*)}(g(f_E(x), v)) &= p_{\mathbb{P}(V^*)}([f_E(x)(v)a : f_E(x)]) \\ &= p_{\mathbb{P}(V^*)}([\text{Ker } \varphi(x)(v)a : \text{Ker } \varphi(x)]) = \text{Ker } \varphi(x). \end{aligned}$$

2. The fact that h is a well-defined isomorphism follows from the relation $\text{Ker } \varphi(x) = \text{Ker } g(f_E(x))$. Indeed,

$$g(f_E(x), v) = 0 \iff f_E(x)(v) = 0 \iff v \in \text{Ker } \varphi(x) = f_E(x). \quad \square$$

We note that h commutes with sections, namely, $s_V \circ f_E = \tilde{f}_E \circ h \circ s$, where $\tilde{f}_E : \tilde{E} \rightarrow E^\vee$ is induced by f_E . This equation is obtained from the following observation. Let $(x, 0, a) \in X \times W \oplus L$. Then $\varphi(x, 0, a) = s(x)$ (this is the reason why we have modified the trivial bundle on $s(x)$). We have

$$h(s(x)) = (x, g(f_E(x), 0, a)) = (x, [f_E(x)(0, a)a : f_E(x)]).$$

Consequently,

$$\tilde{f}_E(x, [f_E(x)(0, a)a : f_E(x)]) = [f_E(x)(0, a)a : f_E(x)] = s_V(f_E(x)).$$

It is easily seen that $f_E^*(N_{\mathbb{P}(V)/\mathbb{P}(W)}) = N_{X/Y}$. Thus, $i_{\text{gys}} \circ f_E|_Y^A = f_E^A \circ i_{W,\text{gys}}$ and

$$i_{\text{gys}}(1) = i_{\text{gys}}(f|_Y^A(1)) = f_E^A(i_{W,\text{gys}}(1)) = f_E^A(c_1(E^\vee)) = c_1(f_E^*(E^\vee)) = c_1(E),$$

which proves Theorem 3 for line bundles over affine varieties.

2.3. The case of a line bundle over an arbitrary variety. Let $L \rightarrow X$ be a line bundle over an arbitrary variety X , and let $s : X \rightarrow L$ be a section transversal to the zero section.

We use the following result proved in [11]: there exists an affine variety X' and an affine bundle $p : X' \rightarrow X$.

Let $L' = p^*(L) = \{(x, m) \in X' \times L \mid p(x) = m\}$; consider the section $s' : X' \rightarrow L'$, $s'(x) = (s(p(x)), x)$. We recall that Y is the set of zeros of s (see Theorem 3). Since p is an affine bundle, $Y' = p^{-1}(Y)$ has codimension 1 in X' . Obviously, the section s' is transversal to the zero section.

We prove that the square

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ \uparrow & & \uparrow \\ Y' & \xrightarrow{p} & Y \end{array}$$

is transversal. We know that the squares

$$\begin{array}{ccccc} s'(X') & \longrightarrow & L' & \longrightarrow & L \\ \uparrow & & \uparrow & & \uparrow \\ Y' & \longrightarrow & X & \xrightarrow{p} & X \end{array}$$

are transversal (the second square is transversal because $L' = p^*(L)$), so that their composition is transversal. This composition coincides with the composition

$$\begin{array}{ccccc} s'(X') & \xrightarrow{p'} & s(X) & \xrightarrow{j} & L \\ \uparrow & & \uparrow & & \uparrow \\ Y' & \longrightarrow & Y & \longrightarrow & X. \end{array}$$

If the composition of two squares is transversal and one of the squares is transversal, then the other is also transversal because

$$N_{s'(X')/Y'} = (jp')^*N_{L/z(X)} = p^*j^*N_{L/z(X)} = p^*N_{s(X)/Y}.$$

Thus, the square

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ i' \uparrow & & \uparrow i \\ Y' & \xrightarrow{p} & Y \end{array}$$

is transversal.

In the preceding subsection we proved that $i'_{\text{gys}}(1_{Y'}) = c_1(L')$. Since p is an affine bundle, p^A is an isomorphism. Consequently,

$$\begin{aligned} i_{\text{gys}}(1_Y) &= p^{A-1} p^A i_{\text{gys}}(1_{Y'}) = p^{A-1} i'_{\text{gys}} p^A(1_{Y'}) \\ &= p^{A-1} i'_{\text{gys}}(1_{Y'}) = p^{A-1} c_1(L') = c_1(L). \end{aligned}$$

Thus, Theorem 3 is proved for arbitrary line bundles.

2.4. Reduction to the case of a line bundle. The following theorem can be found in [7].

Theorem 4 (Splitting principle). *Let E be a bundle of rank n over a variety X . There exists a variety X' , a bundle E' over X' , and a mapping $f : X' \rightarrow X$ with the following properties:*

- 1) $E' = f^*(E)$;
- 2) there exist line bundles L_1, L_2, \dots, L_n over X' such that $E' \simeq \bigoplus_{i=1}^n L_i$;
- 3) the homomorphism $f^A : \mathcal{A}(X) \rightarrow \mathcal{A}(X')$ is injective.

By Theorem 4, it suffices to consider the case of $E = L_1 \oplus \dots \oplus L_n$, where the L_i are line bundles. The remaining part of the proof follows from Lemma 7 stated below.

We say that the triple (E, s, Y) consisting of a bundle E over X , its section s , and the set Y of zeros of s is *nice* if s intersects the zero section z transversally and $i_{\text{gys}}(1_Y) = c_d(E)$, where $i : Y \hookrightarrow X$ is the inclusion and d is the rank of the bundle E .

Lemma 7. *Let (E_1, s_1, Y_1) and (E_2, s_2, Y_2) be nice triples. Suppose that Y_1 and Y_2 intersect transversally in X . Then $(E_1 \oplus E_2, s_1 \oplus s_2, Y_1 \cap Y_2)$ is a nice bundle.*

Proof. Observe that $Y = Y_1 \cap Y_2$ is the set of zeros of $s_1 \oplus s_2$. Let $i : Y \hookrightarrow X$ be the inclusion. We consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{j_1} & Y_1 \\ j_2 \downarrow & & i_1 \downarrow \\ Y_2 & \xrightarrow{i_2} & X, \end{array}$$

where j_1 and j_2 are the corresponding inclusions. It is well known that $i_{1,\text{gys}}(1) = c_{d_1}(E_1)$ and $i_{2,\text{gys}}(1) = c_{d_2}(E_2)$. Since

$$j_{1,\text{gys}}(1) = c_{d_2}(i_1^* E_2) = i_1^A(c_{d_2}(E_2)),$$

we obtain

$$\begin{aligned} i_{\text{gys}}(1) &= (i_1 \circ j_1)_{\text{gys}}(1) = i_{1,\text{gys}} \circ j_{1,\text{gys}}(1) \\ &= i_{1,\text{gys}} \circ i_1^A(c_{d_2}(E_2)) = c_{d_2}(E_2) \cup i_{1,\text{gys}}(1). \end{aligned}$$

Using the projection formula, we obtain

$$c_{d_2}(E_2) \cup i_{1,\text{gys}}(1) = c_{d_2}(E_2) \cup c_{d_1}(E_1).$$

By the Cartan formula (in the case of the highest Chern classes this formula has only one summand), we have

$$c_{d_2}(E_2) \cup c_{d_1}(E_1) = c_{d_1+d_2}(E_1 \oplus E_2).$$

This proves the lemma and completes the proof of Theorem 3. □

§3. EXCESS FORMULA FOR GENERALIZED COHOMOLOGY THEORIES

In this section, we prove Theorem 5 containing an excess formula some specific cases of which are the excess formulas in [9, 10].

Let $\sigma : \widehat{X} \rightarrow X$ be the sigma-process with center Y , and let (\mathcal{A}, \cup, c) be a ring cohomology theory with Chern structure. We consider the diagram

$$\begin{array}{ccc} \mathbb{P}(N) & \xrightarrow{j} & \widehat{X} \\ \downarrow p & & \downarrow \sigma \\ Y & \xrightarrow{i} & X \end{array}$$

and the corresponding diagram of inverse images and Gysin homomorphisms

$$\begin{array}{ccc} \mathcal{A}(\mathbb{P}(N)) & \xrightarrow{j_{\text{gys}}} & \mathcal{A}(\widehat{X}) \\ \uparrow p^{\mathcal{A}} & & \uparrow \sigma^{\mathcal{A}} \\ \mathcal{A}(Y) & \xrightarrow{i_{\text{gys}}} & \mathcal{A}(X). \end{array}$$

Is this diagram commutative? It can easily be seen that the answer is no: for graded cohomology theories, the Gysin homomorphism often raises the degree of each element by the codimension of the subvariety; the upper homomorphism raises the degrees by 1, and the lower homomorphism raises the degrees by the codimension of Y in X (which is still denoted by n). In this connection, the following question arises: to what extent is this diagram noncommutative? The excess formula gives an answer to this question.

Since L_N is a subextension of $p^*(N)$, there exists a bundle E over $\mathbb{P}(N)$ and a short exact sequence of bundles

$$0 \longrightarrow L_N \longrightarrow p^*(N) \longrightarrow E \longrightarrow 0.$$

The bundle E will be called the *excess*.

Theorem 5 (Excess formula). *In the above notation, we have*

$$\sigma^{\mathcal{A}} \circ i_{\text{gys}}(x) = j_{\text{gys}}(p^{\mathcal{A}}(x) \cup c_{n-1}(E)).$$

3.1. Lemma on a diagram of a special form. Here, we prove a lemma to which the proof of the theorem will be reduced eventually.

Lemma 8. *Let Y be a subvariety of X such that there exists a retraction $r : X \longrightarrow Y$. We put $\mathcal{N} = r^*(N)$. Let X' be a variety such that $\mathbb{P}(N)$ is a subvariety in X' and X' itself is a subvariety in $\mathbb{P}(\mathcal{N})$ of codimension d . Consider the following commutative diagram of varieties:*

$$\begin{array}{ccccc} \mathbb{P}(N) & \xrightarrow{j} & X' & \xrightarrow{k} & \mathbb{P}(\mathcal{N}) \\ \downarrow p & & \downarrow \sigma & & \downarrow P \\ Y & \xrightarrow{i} & X & \xlongequal{\quad} & X, \end{array}$$

where j and k are the corresponding inclusions. Then there exists a bundle E of rank d over $\mathbb{P}(N)$ such that

$$\sigma^{\mathcal{A}} \circ i_{\text{gys}} = j_{\text{gys}} \circ \cup c_d(E) \circ p^{\mathcal{A}}.$$

Proof. We consider the equation

$$\sigma^{\mathcal{A}} \circ i_{\text{gys}} = k^{\mathcal{A}} \circ P^{\mathcal{A}} \circ i_{\text{gys}}$$

and the diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{N}) & \xrightarrow{P} & X \\ \uparrow k \circ j & & \uparrow i \\ \mathbb{P}(N) & \xrightarrow{p} & Y. \end{array}$$

Since this square is transversal, the property stated at the beginning of §2 implies the relation

$$k^{\mathcal{A}} \circ P^{\mathcal{A}} \circ i_{\text{gys}} = k^{\mathcal{A}} \circ (k \circ j)_{\text{gys}} \circ p^{\mathcal{A}}.$$

By the functoriality of the Gysin homomorphism, we have

$$k^{\mathcal{A}} \circ (k \circ j)_{\text{gys}} \circ p^{\mathcal{A}} = k^{\mathcal{A}} \circ k_{\text{gys}} \circ j_{\text{gys}} \circ p^{\mathcal{A}}.$$

By Theorem 2,

$$k^{\mathcal{A}} \circ k_{\text{gys}} \circ j_{\text{gys}} \circ p^{\mathcal{A}} = j_{\text{gys}} \circ \cup c_d(j^*(N_{\mathbb{P}(N)/X'})) \circ p^{\mathcal{A}}.$$

Let $E = j^*(N_{\mathbb{P}(N)/X'})$. Collecting the above relations, we obtain

$$\sigma^{\mathcal{A}} \circ i_{\text{gys}} = j_{\text{gys}} \circ \cup c_d(E) \circ p^{\mathcal{A}}. \quad \square$$

3.2. Reduction to the normal bundle. We must prove that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A}(Y) & \xrightarrow{p^{\mathcal{A}}} & \mathcal{A}(\mathbb{P}(N)) & \xrightarrow{\cup c_{n-1}(E)} & \mathcal{A}(\mathbb{P}(N)) \\ \downarrow i_{\text{gys}} & & & & \downarrow j_{\text{gys}} \\ \mathcal{A}(X) & \xlongequal{\quad} & \mathcal{A}(X) & \xrightarrow{\sigma^{\mathcal{A}}} & \mathcal{A}(\widehat{X}). \end{array}$$

We want to apply Lemma 8, but the varieties X and \widehat{X} are not suitable for this: we need a retraction $r : X \rightarrow Y$. We can use the normal bundle N instead of X and the natural projection as a retraction. We prove that this replacement preserves the form of the excess formula.

We consider the following diagram, in which the extreme paths decode the mappings that occur in the preceding diagram:

$$\begin{array}{ccccccc} \mathcal{A}(Y) & \xrightarrow{p^{\mathcal{A}}} & \mathcal{A}(\mathbb{P}(N)) & \xrightarrow{\cup c_{n-1}(E)} & \mathcal{A}(\mathbb{P}(N)) & & \\ \downarrow p^{\mathcal{A}} & & & & \downarrow p^{\mathcal{A}} & & \\ \mathcal{A}(N) & & & & \mathcal{A}(L_N) & & \\ \downarrow \cup \text{th}(N) & & & & \downarrow \cup \text{th}(L_N) & & \\ \mathcal{A}(N, N \setminus Y) & \xlongequal{\quad} & \mathcal{A}(N, N \setminus Y) & \xrightarrow{\tilde{\sigma}^{\mathcal{A}}} & \mathcal{A}(L_N, L_N \setminus \mathbb{P}(N)) & & \\ \simeq \uparrow i_0^{\mathcal{A}} & & & & \simeq \uparrow I_0^{\mathcal{A}} & & \\ \mathcal{A}(X_t, X_t \setminus \mathbb{A}^1 \times Y) & \xlongequal{\quad} & \mathcal{A}(X_t, X_t \setminus \mathbb{A}^1 \times Y) & \xrightarrow{\sigma_t^{\mathcal{A}}} & \mathcal{A}(\widehat{X}_t, \widehat{X}_t \setminus \mathbb{A}^1 \times \mathbb{P}(N)) & & \\ \simeq \downarrow i_1^{\mathcal{A}} & & & & \simeq \downarrow I_1^{\mathcal{A}} & & \\ \mathcal{A}(X, X \setminus Y) & \xlongequal{\quad} & \mathcal{A}(X, X \setminus Y) & \xrightarrow{\sigma^{\mathcal{A}}} & \mathcal{A}(\widehat{X}, \widehat{X} \setminus \mathbb{P}(N)) & & \\ \downarrow i^{\mathcal{A}} & & & & \downarrow i^{\mathcal{A}} & & \\ \mathcal{A}(X) & \xlongequal{\quad} & \mathcal{A}(X) & \xrightarrow{\sigma^{\mathcal{A}}} & \mathcal{A}(\widehat{X}). & & \end{array}$$

The three lower squares are commutative, because they are induced by the following commutative diagram of varieties:

$$\begin{array}{ccc} (N, N \setminus Y) & \xleftarrow{\tilde{\sigma}} & (L_N, L_N \setminus \mathbb{P}(N)) \\ \downarrow i_0 & & \downarrow I_0 \\ (X_t, X_t \setminus \mathbb{A}^1 \times Y) & \xleftarrow{\sigma_t} & (\widehat{X}_t, \widehat{X}_t \setminus \mathbb{A}^1 \times \mathbb{P}(N)) \\ \uparrow i_1 & & \uparrow I_1 \\ (X, X \setminus Y) & \xleftarrow{\sigma} & (\widehat{X}, \widehat{X} \setminus \mathbb{P}(N)) \\ \uparrow i & & \uparrow i \\ X & \xleftarrow{\sigma} & \widehat{X}. \end{array}$$

The upper square contain only normal bundles. We describe the mapping σ_t . For this, consider the mapping $\sigma \times id : \widehat{X} \times \mathbb{A}^1 \rightarrow X \times \mathbb{A}^1$. Then, by [4, Chapter 2, Proposition 7.15], there exists a mapping of the blown-up varieties $\widehat{X} \times \widehat{\mathbb{A}^1}_{\mathbb{P}(N) \times 0} \rightarrow X \times \widehat{\mathbb{A}^1}_{Y \times 0}$, which will provide the required mapping after excision of the proper inverse transforms.

It remains to observe that if $i : Y \rightarrow N$ is an inclusion, then i_{gys} coincides with the composition (see [7])

$$\mathcal{A}(Y) \xrightarrow{p^A} \mathcal{A}(N) \xrightarrow{\cup \text{th}(N)} \mathcal{A}(N, N \setminus Y) \xrightarrow{i^A} \mathcal{A}(N).$$

Thus, reduction to the case of a normal bundle is performed.

3.3. Completion of the proof of the excess formula. Now, it remains to check the conditions of Lemma 8 and to prove that the excess arising in this lemma coincides with the bundle E constructed at the beginning of this section.

We denote the projection $N \rightarrow Y$ by r and consider the bundle $\mathcal{N} = r^*(N)$, as in Lemma 8. Then the required diagram must have the following form:

$$\begin{array}{ccccc} \mathbb{P}(N) & \xrightarrow{j} & L_N & \xrightarrow{k} & \mathbb{P}(\mathcal{N}) \\ \downarrow p & & \downarrow \tilde{\sigma} & & \downarrow P \\ Y & \xrightarrow{i} & N & \xlongequal{\quad} & N. \end{array}$$

To apply the lemma, we need to describe the inclusion k . Observe that

$$\mathcal{N} = \{(x, y) \in N \times N \mid p(x) = p(y)\} = \mathbb{P}(N) \times_Y N.$$

Projecting $\mathbb{P}(\mathcal{N})$ onto $\mathbb{P}(N)$, we can identify $\mathbb{P}(\mathcal{N})$ with $p^*(N)$. As a result, we obtain

$$\mathbb{P}(\mathcal{N}) = p^*(N) \supset L_N.$$

Thus, applying Lemma 8, we see that it remains to prove that $E = j^*(N_{\mathbb{P}(\mathcal{N})/L_N})$. For this, we consider the following diagram in which the rows are short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N_{L_N/\mathbb{P}(N)} & \longrightarrow & N_{\mathbb{P}(\mathcal{N})/\mathbb{P}(N)} & \longrightarrow & j^*N_{\mathbb{P}(\mathcal{N})/L_N} \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & L_N & \longrightarrow & p^*(N) & \longrightarrow & E \longrightarrow 0. \end{array}$$

Since the upper row is a short exact sequence and the first two equality signs are obvious, we see that $E = j^*(N_{\mathbb{P}(\mathcal{N})/L_N})$. Theorem 5 is proved completely.

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Received 18/SEP/2004

Translated by B. BEKKER