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CAPACITY DIMENSION AND EMBEDDING OF HYPERBOLIC SPACES INTO A PRODUCT OF TREES

S. BUYALO

Abstract. It is proved that every visual Gromov hyperbolic space X whose boundary at infinity has finite capacity dimension, $\operatorname{cdim}(\partial_\infty X) < \infty$, admits a quasiisometric embedding into an n-fold product of metric trees with $n = \operatorname{cdim}(\partial_\infty X) + 1$.

§1. Introduction

Recall that a map $f : X \to Y$ between metric spaces is *quasiisometric* if

$$\frac{1}{\Lambda}|xx'| - \sigma \le |f(x)f(x')| \le \Lambda|xx'| + \sigma$$

for some constants $\Lambda \ge 1$, $\sigma \ge 0$ and all x, $x' \in X$. Our main result is the following embedding theorem.

Theorem 1.1. *Let X be a visual Gromov hyperbolic space whose boundary at infinity has finite capacity dimension*, $\operatorname{cdim}(\partial_\infty X) < \infty$. *Then there exists a quasiisometric embedding $f : X \to T_1 \times \cdots \times T_n$ of X into the n-fold product of metric trees $T_1, \dots, T_n$ with $n = \operatorname{cdim}(\partial_\infty X) + 1$.*

The property of a Gromov hyperbolic space X to be visual is a rough version of the property that every point $x \in X$ lies on a geodesic ray emanating from a fixed point $x_0 \in X$; see Subsection 4.2 for the precise definition.

The boundary at infinity $\partial_\infty X$ is taken with a visual metric, which is defined up to a quasisymmetry of $\partial_\infty X$. The notion of the capacity dimension of a metric space was introduced in [Bu], where it was proved that the capacity dimension is a quasisymmetry invariant (for a close notion of the Assouad–Nagata dimension this was proved earlier in [LS]). In particular, $\operatorname{cdim}(\partial_\infty X)$ is independent of the choice of a visual metric on $\partial_\infty X$.

It is well known that the asymptotic dimension of an n-fold product of metric trees is at most n for every $n \ge 1$ (see, e.g., [Ro, Chapter 9]). Thus, Theorem 1.1 generalizes the main result of [Bu] saying that $\operatorname{asdim} X \le \operatorname{cdim} \partial_\infty X + 1$ for every visual Gromov hyperbolic space X. In fact, Proposition 2.3 below, which is the main ingredient of the proof of Theorem 1.1, is a refined version of [Bu, Proposition 4.4].

In [BS1], for every $n \ge 2$ the real hyperbolic space H^n was embedded in a quasiisometric manner into an n-fold product of metric trees. It is known that $\operatorname{cdim} \partial_\infty \mathrm{H}^n = n - 1$ (see [Bu]), so that Theorem 1.1 also generalizes that result. Actually, our construction of the embedding is a version of the construction presented in [BS1].

The estimate of the number of tree-factors needed for a quasiisometric embedding given by Theorem 1.1 is sharp: by the results of [BS2], for every $n \ge 2$ the space H^n

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admits no quasiisometric embedding into the $(n-1)$-fold product of any metric trees stabilized by any Euclidean factor $\mathbb{R}^N$.

The class of Gromov hyperbolic spaces to which Theorem 1.1 applies contains all visual Gromov hyperbolic spaces with doubling boundary at infinity, or, which is the same, the spaces satisfying the bounded growth condition at some scale (see [BoS]). In particular, any Gromov hyperbolic group and any Hadamard manifold with pinched negative curvature are in this class. Indeed, the boundary at infinity $\partial_\infty X$ of any such space can be embedded quasisymmetrically in $\mathbb{R}^N$ for some $N \in \mathbb{N}$ by the Assouad embedding theorem (see [As], [He, Chapter 12]). Thus, $\operatorname{cdim} \partial_\infty X \le \operatorname{cdim} \mathbb{R}^N = N$ is finite. An open problem is to find conditions ensuring that the capacity dimension of $\partial_\infty X$ (more generally, of a metric space) coincides with the topological dimension.

For other similar results about embeddings into a product of trees, see [Dr, DZ, LS] and the references therein.

We briefly describe the layout of the paper. §2 is devoted to the capacity dimension. In Subsection 2.1 we fix notation and recall some notions related to coverings of a given metric space Z. Next, in Subsection 2.2, we recall one of many equivalent definitions of the capacity dimension, which is most suitable for our purposes.

The main feature of the capacity dimension is that, for the coverings involved in its definition, the Lebesgue number and the mesh are of the same order. This is the source of an astonishing flexibility in manipulating with coverings, which makes it possible to obtain many useful properties. The core of the paper is in Subsection 2.3, where we introduce the notion of a γ-separated characteristic sequence of coverings and prove the existence of such a sequence under the condition $\operatorname{cdim} Z < \infty$ (Proposition 2.3).

In §3 we use a γ-separated characteristic sequence of coverings of Z to construct a quasiisometric embedding of the hyperbolic cone $\operatorname{Co}(Z)$ over Z into the product of appropriate metric trees (Theorem 3.1). This construction is a version of that used in [BS1], adapted to the current situation.

In §4, we recall the basic notions of the theory of hyperbolic spaces and complete the proof of Theorem 1.1.

§2. CAPACITY DIMENSION

Let Z be a metric space. For U, $U' \subset Z$, we denote by $\operatorname{dist}(U, U')$ the distance between U and U', $\operatorname{dist}(U, U') = \inf\{|uu'| : u \in U,\ u' \in U'\}$, where $|uu'|$ is the distance between u and u'. For $r > 0$ we denote by $B_r(U)$ the open r-neighborhood of U, $B_r(U) = \{z \in Z : \operatorname{dist}(z, U) < r\}$, and by $\overline{B}_r(U)$ the closed r-neighborhood of U, $\overline{B}_r(U) = \{z \in Z : \operatorname{dist}(z, U) \le r\}$. We extend this notation to all real r by putting $B_r(U) = U$ for $r = 0$, and by defining $B_r(U)$ for $r < 0$ as the complement of the closed $|r|$-neighborhood of $Z \setminus U$, $B_r(U) = Z \setminus \overline{B}_{|r|}(Z \setminus U)$. It is straightforward to check the following statement.

Lemma 2.1. *Given $U \subset Z$, for every $0 < s < t$ we have*

$$B_{t-s}(U) \subset B_{-s}(B_t(U)). \qquad \square$$

A subset X of a metric space Z is a *net* in Z if there is a $\lambda > 0$ such that $\operatorname{dist}(z, X) \le \lambda$ for every $z \in Z$. In this case we say that X is a λ-net.

2.1. Coverings.

Given a family $\mathcal{U}$ of subsets in a metric space Z, we define $\operatorname{mesh}(\mathcal{U}) = \sup\{\operatorname{diam} U : U \in \mathcal{U}\}$. The *multiplicity* $m(\mathcal{U})$ of $\mathcal{U}$ is the maximal number of members of $\mathcal{U}$ with nonempty intersection. For $r > 0$, the *r-multiplicity* $m_r(\mathcal{U})$ of $\mathcal{U}$ is the multiplicity of the family $\mathcal{U}_r$ obtained by taking the open r-neighborhoods of the members of $\mathcal{U}$. So

$m_r(\mathcal{U}) = m(\mathcal{U}_r)$. We say that a family $\mathcal{U}$ is *disjoint* if $m(\mathcal{U}) = 1$, and $\mathcal{U}$ is r-*disjoint* if $m_r(\mathcal{U}) = 1$.

A family $\mathcal{U}$ is called a *covering* of Z if $\bigcup\{U : U \in \mathcal{U}\} = Z$. A covering $\mathcal{U}$ is said to be *colored* if it is a union of $m \geq 1$ disjoint families, $\mathcal{U} = \bigcup_{a\in A} \mathcal{U}^a$, $|A| = m$. In this case we also say that $\mathcal{U}$ is m-colored. Clearly, the multiplicity of an m-colored covering is at most m.

Let $\mathcal{U}$ be an open covering of a metric space Z. Given $z \in Z$, we put $L'(\mathcal{U}, z) = \sup\{\operatorname{dist}(z, Z \setminus U) : U \in \mathcal{U}\}$. Let

$$L(\mathcal{U}, z) = \min\{L'(\mathcal{U}, z), \operatorname{mesh}(\mathcal{U})\}$$

be the Lebesgue number of $\mathcal{U}$ at z (the auxiliary number $L'(\mathcal{U}, z)$ can be larger than $\operatorname{mesh}(\mathcal{U})$, and even infinite, as, e.g., in the case where $Z = U$ for some $U \in \mathcal{U}$), and let $L(\mathcal{U}) = \inf_{z\in Z} L(\mathcal{U}, z)$ be the Lebesgue number of $\mathcal{U}$. In this definition of the Lebesgue number, open coverings must be used. For coverings that, for instance, are closed, the Lebesgue number is defined differently. We have $L(\mathcal{U}) \leq L(\mathcal{U}, z) \leq \operatorname{mesh}(\mathcal{U})$, and for every $z \in Z$ the open ball $B_r(z)$ of radius $r = L(\mathcal{U})$ centered at z is contained in some member of the covering $\mathcal{U}$.

We shall use the following obvious fact (see, e.g., [Bu]).

Lemma 2.2. *Let $\mathcal{U}$ be an open covering of Z with $L(\mathcal{U}) > 0$. Then for every $s \in (0, L(\mathcal{U}))$ the family $\mathcal{U}_{-s} = B_{-s}(\mathcal{U})$ is still an open covering of Z, and its s-multiplicity $m_s(\mathcal{U}_{-s})$ does not exceed $m(\mathcal{U})$.* □

2.2. Definition of the capacity dimension. There are several equivalent definitions of the capacity dimension (see [Bu]). In this paper we shall use the following one. Let $\mathcal{U}$ be an open covering of a metric space Z. We introduce the *capacity* of $\mathcal{U}$ by

$$\operatorname{cap}(\mathcal{U}) = \frac{L(\mathcal{U})}{\operatorname{mesh}(\mathcal{U})} \in [0, 1],$$

and if $\operatorname{mesh}(\mathcal{U}) = 0$ or $L(\mathcal{U}) = \operatorname{mesh}(\mathcal{U}) = \infty$, then we put $\operatorname{cap}(\mathcal{U}) = 1$ by definition. For $\tau > 0$, $\delta \in (0, 1)$, and an integer $m \geq 0$, we put

$$c_\tau(Z, m, \delta) = \sup_{\mathcal{U}} \operatorname{cap}(\mathcal{U}),$$

where the supremum is taken over all $(m + 1)$-colored open coverings $\mathcal{U}$ of Z with $\delta\tau \leq \operatorname{mesh}(\mathcal{U}) \leq \tau$.

Next, we take

$$c(Z, m, \delta) = \liminf_{\tau\to 0} c_\tau(Z, m, \delta).$$

The function $c(Z, m, \delta)$ is monotone in δ, $c(Z, m, \delta') \geq c(Z, m, \delta)$ for $\delta' < \delta$. Hence, the limit $c(Z, m) = \lim_{\delta\to 0} c(Z, m, \delta)$ exists. Now, the *capacity dimension* of Z is defined by

$$\operatorname{cdim}(Z) = \inf\{m : c(Z, m) > 0\}.$$

In other words, $\operatorname{cdim}(Z)$ is the smallest integer $m \geq 1$ with the following property: there is a constant $\delta > 0$ such that for every sufficiently small $\tau > 0$ there exists an $(m + 1)$-colored open covering $\mathcal{U}$ of Z with $\operatorname{mesh}(\mathcal{U}) \leq \tau$ and $L(\mathcal{U}) \geq \delta\tau$.

2.3. A characteristic sequence of coverings. Let Z be a metric space with finite capacity dimension, $n = \operatorname{cdim} Z < \infty$. We say that a sequence $\mathcal{U}_j$, $j \in \mathbb{N}$, of $(n + 1)$-colored (by a set A) open coverings of Z is *characteristic* with parameter $r \in (0, 1)$ if for some *characteristic* constants $\delta \in (0, 1)$ and $\lambda \geq 1$ the following conditions are fulfilled:

(1) $\operatorname{mesh} \mathcal{U}_j \leq r^j$ and $L(\mathcal{U}_j) \geq \delta r^j$ for every $j \in \mathbb{N}$;
(2) for every $a \in A$ and $j \in \mathbb{N}$, the family $\mathcal{U}_j^a$ is a λr^j-net in Z.

The existence of a sequence of coverings with property (1) follows directly from the definition of the capacity dimension; property (2) is auxiliary and is easy to achieve; see Lemma 2.4. A characteristic sequence of coverings $\{\mathcal{U}_j\}$ with parameter r is said to be *γ-separated*, $\gamma \in (0,1)$, if, in addition,

(3) for every $a \in A$ and for different members $U \in \mathcal{U}_j^a$, $U' \in \mathcal{U}_{j'}^a$ with $j' \le j$ we have either $B_s(U) \cap U' = \varnothing$, or $B_s(U) \subset U'$, and moreover, if $j' < j$, then there is $U'' \in \mathcal{U}_j^a$ with $B_s(U'') \subset U'$ for $s = \gamma r^j$.

Sequences of coverings satisfying some versions of (1)–(3) have been used in a number of papers (see, e.g., [Bu, BS1, Dr, LS]), for various purposes, basically for construction of embeddings with specific properties.

Proposition 2.3. *Suppose that Z is a metric space with finite capacity dimension,* $\operatorname{cdim} Z \le n$. *Then there are constants δ, $\gamma \in (0,1)$, and $\lambda \ge 1$ such that for every sufficiently small $r > 0$ there exists a γ-separated characteristic sequence of coverings $\mathcal{U}_j$, $j \in \mathbb{N}$, of Z with the parameter r and the characteristic constants δ, λ.*

The proof of this proposition follows the lines of the proof of [Bu, Proposition 4.4]; cf. also [LS, Proposition 4.1]. We first construct a characteristic sequence of coverings of Z and then modify it to obtain property (3).

Lemma 2.4. *Under the conditions of Proposition* 2.3, *there are constants $\delta \in (0,1)$ and $\lambda \ge 1$ such that for every sufficiently small $r > 0$ there exists a characteristic sequence of coverings $\widehat{\mathcal{U}}_j$, $j \in \mathbb{N}$, of Z with the parameter r and the characteristic constants δ, λ. Moreover, for every $j \in \mathbb{N}$, each $U \in \widehat{\mathcal{U}}_j$ contains a ball of radius δr^j, and for every $a \in A$, the family $\widehat{\mathcal{U}}_j^a$ is δr^j-disjoint.*

Proof. We have $c = c(Z,n)/8 > 0$ by the definition of $\operatorname{cdim} Z$, and $c(Z,n,\delta') \ge 4c$ for all sufficiently small $\delta' > 0$. We fix such a number δ'; then $c_\tau(Z,n,\delta') \ge 2c$ for all τ, $0 < \tau \le \tau_0$. This means that for every $\tau \in (0,\tau_0]$ there is an $(n+1)$-colored open covering $\mathcal{U}_\tau$ of Z with $\delta'\tau \le \operatorname{mesh}(\mathcal{U}_\tau) \le \tau$ and with capacity arbitrarily close to $c_\tau(Z,n,\delta')$; in particular, $L(\mathcal{U}_\tau) \ge c \operatorname{mesh}(\mathcal{U}_\tau) \ge c\delta'\tau$.

We take a positive $r < r_0 = \min\{c\delta'/2, \tau_0\}$ and for every $j \in \mathbb{N}$ consider the covering $\mathcal{U}_j = \mathcal{U}_{\tau_j}$, where $\tau_j = r^j$. Then the sequence $\mathcal{U}_j$, $j \in \mathbb{N}$, of $(n+1)$-colored coverings of Z satisfies condition (1) (with characteristic constant $c\delta'$). Fixing $j \in \mathbb{N}$, for $s = c\delta' r^j/2$ we consider the family $\widehat{\mathcal{U}}_j = B_{-s}(\mathcal{U}_j)$. By Lemma 2.2, $\widehat{\mathcal{U}}_j$ is an open covering of Z, and we have $\operatorname{mesh}(\widehat{\mathcal{U}}_j) \le \operatorname{mesh}(\mathcal{U}_j) \le r^j$ and $L(\widehat{\mathcal{U}}_j) \ge \frac{1}{2} c\delta' r^j$; thus, (1) is satisfied for $\widehat{\mathcal{U}}_j$ with $\delta = c\delta'/2$.

We can assume additionally that every $U \in \widehat{\mathcal{U}}_j$ contains a ball of radius $\delta r^j \le L(\widehat{\mathcal{U}}_j)$, because otherwise U is covered by other members of $\widehat{\mathcal{U}}_j$ and thus U can be deleted from $\widehat{\mathcal{U}}_j$ without violation of property (1).

Furthermore, the family $\widehat{\mathcal{U}}_j^a$ is δr^j-disjoint for every color $a \in A$. Now, starting with a given family $\widehat{\mathcal{U}}_j^a$, $a \in A$, and adding copies of members of other colors in $\widehat{\mathcal{U}}_j$ that are δr^j-disjoint with $\widehat{\mathcal{U}}_j^a$, we can produce a λr^j-net with $\lambda = 1 + 2\delta$. This does not change the mesh and does not decrease the Lebesgue number of $\widehat{\mathcal{U}}_j$. Thus, (2) is satisfied for every $\widehat{\mathcal{U}}_j$, $j \in \mathbb{N}$. □

We shall use the following modification of a construction employed in [Bu]. Let $\mathcal{U}$, $\mathcal{U}'$ be families of sets in Z, $s > 0$. We denote by $\mathcal{U} *_s \mathcal{U}'$ the family obtained by taking, for every $U \in \mathcal{U}$, the s-neighborhood of the union V of U and all members $U' \in \mathcal{U}'$ with nonempty intersection $B_s(U) \cap B_s(U')$, $\mathcal{U} *_s \mathcal{U}' = \{B_s(V) : U \in \mathcal{U}\}$.

Lemma 2.5. *Assume that a family* $\widehat{\mathcal{U}}$ *of sets is* δs*-disjoint for* $\delta \in (0, 2/3]$ *and* $s > 0$, *and that* $\operatorname{mesh}(\widehat{\mathcal{U}}) \leq 2s$. *Then the operation*

$$U \mapsto U^* = B_{-4s}(U) *_{\delta s} \widehat{\mathcal{U}}$$

does not increase any set $U \subset Z$ *(i.e.,* $U^* \subset U$*) and for every* $\widehat{U} \in \widehat{\mathcal{U}}$ *either* $B_{\delta s}(\widehat{U}) \cap U^* = \varnothing$, *or* $B_{\delta s}(\widehat{U}) \subset U^*$.

Proof. For every $\widehat{U} \in \widehat{\mathcal{U}}$, we have $\operatorname{diam} B_{\delta s}(\widehat{U}) \leq 2s + 2\delta s$, so that $\delta s + \operatorname{diam} B_{\delta s}(\widehat{U}) \leq 2s(1 + 3\delta/2) \leq 4s$. Hence, if $B_{\delta s}(\widehat{U})$ intersects $B_{\delta s}(B_{-4s}(U))$, then $B_{\delta s}(B_{-4s}(U) \cup \widehat{U}) \subset U$, and $U^* \subset U$; in particular, in this case we have $B_{\delta s}(\widehat{U}) \subset U^*$. Otherwise, $B_{\delta s}(\widehat{U})$ misses the δs-neighborhood of $B_{-4s}(U)$ as well as such a neighborhood of every other member of $\widehat{\mathcal{U}}$, whence $B_{\delta s}(\widehat{U}) \cap U^* = \varnothing$. □

Lemma 2.6. *Under the conditions of Lemma* 2.5, *assume that for some sets* U_1, $U_2 \subset Z$ *we have* $B_t(U_1) \subset U_2$, $t > 4s$. *Then* $B_{t'}(U_1^*) \subset U_2^*$ *for* $t' = t - 4s$.

Proof. By Lemma 2.5, we obtain $B_{t'}(U_1^*) \subset B_{t'}(U_1)$. By Lemma 2.1, $B_{t-4s}(U_1) \subset B_{-4s}(B_t(U_1))$. Finally, $B_{-4s}(B_t(U_1)) \subset B_{-4s}(U_2) \subset U_2^*$. □

Proof of Proposition 2.3. Using Lemma 2.4, we find a characteristic sequence of coverings $\widehat{\mathcal{U}}_j$, $j \in \mathbb{N}$, with characteristic constants $\delta \in (0,1)$ and $\lambda \geq 1$ and with an arbitrarily small parameter r such that, for every $j \in \mathbb{N}$, every $U \in \widehat{\mathcal{U}}_j$ contains a ball of radius δr^j, and for every color $a \in A$ the family $\widehat{\mathcal{U}}_j^a$ is δr^j-disjoint. In what follows, we assume that $\frac{2r}{1-r} \leq \delta/4 \leq 1/6$ and $(\lambda + 1)r < \delta/2$.

We fix a color $a \in A$ and define $\mathcal{V}_1^a = \mathcal{U}_{1,1}^a := \widehat{\mathcal{U}}_1^a$. Then the family $\mathcal{U}_{1,1}^a$ is δr-disjoint and $\operatorname{mesh}(\mathcal{U}_{1,1}^a) \leq r$.

Assume that for $k \geq 1$ the family $\mathcal{V}_k^a$ is already defined and has the following properties:

(i) $\mathcal{V}_k^a = \bigcup_{j=1}^{k} \mathcal{U}_{j,k}^a$;

(ii) for every $1 \leq j \leq k$, the family $\mathcal{U}_{j,k}^a$ is δr^j-disjoint and $\operatorname{mesh}(\mathcal{U}_{j,k}^a) \leq r^j$;

(iii) given $1 \leq j' < j \leq k$, for every $U' \in \mathcal{U}_{j',k}^a$, $U \in \mathcal{U}_{j,k}^a$ we have either $B_t(U) \cap U' = \varnothing$ with $t = \delta r^j/2$ or $B_t(U) \subset U'$ with $t = \gamma_{k,j} r^j$, where $\gamma_{k,j}$ is defined recursively by $\gamma_{j,j} = \delta/2$ and $\gamma_{k,j} = \gamma_{k-1,j} - 2r^{k-j}$ for $k > j$.

We define

$$\mathcal{V}_{k+1}^a := B_{-4s}(\mathcal{V}_k^a) *_{\delta s} \widehat{\mathcal{U}}_{k+1}^a \cup \widehat{\mathcal{U}}_{k+1}^a$$

with $s = r^{k+1}/2$. Then $\mathcal{V}_{k+1}^a = \bigcup_{j=1}^{k+1} \mathcal{U}_{j,k+1}^a$, where

$$\mathcal{U}_{j,k+1}^a = B_{-4s}(\mathcal{U}_{j,k}^a) *_{\delta s} \widehat{\mathcal{U}}_{k+1}^a$$

for $1 \leq j \leq k$ and $\mathcal{U}_{k+1,k+1}^a = \widehat{\mathcal{U}}_{k+1}^a$. Since the family $\widehat{\mathcal{U}}_{k+1}^a$ is $2\delta s$-disjoint, we can apply Lemma 2.5 to see that every $U^* \in \mathcal{U}_{j,k+1}^a$ with $1 \leq j \leq k$ is contained in an appropriate $U \in \mathcal{U}_{j,k}^a$; in particular, the family $\mathcal{U}_{j,k+1}^a$ is δr^j-disjoint and $\operatorname{mesh}(\mathcal{U}_{j,k+1}^a) \leq \operatorname{mesh}(\mathcal{U}_{j,k}^a) \leq r^j$. Furthermore, for every $\widehat{U} \in \mathcal{U}_{k+1,k+1}^a$ we have either $B_{\delta s}(\widehat{U}) \cap U^* = \varnothing$, or $B_{\delta s}(\widehat{U}) \subset U^*$.

Now, if $U' \in \mathcal{U}_{j',k}^a$ with $j' < j$ and $B_t(U) \cap U' = \varnothing$ for $t = \delta r^j/2$, then $B_t(U^*) \cap U'^* = \varnothing$ because $U^* \subset U$ and $U'^* \subset U'$. In the case where $B_t(U) \subset U'$ with $t = \gamma_{j,k} r^j$, by Lemma 2.6 we have $B_{t-2r^{k+1}}(U^*) \subset U'^*$ and $t - 2r^{k+1} = \gamma_{k+1,j} r^j$ with $\gamma_{k+1,j} = \gamma_{k,j} - 2r^{k+1-j}$. Note that $\lim_{k\to\infty} \gamma_{k,j} = \delta/2 - \frac{2r}{1-r} \geq \delta/4$ for every $j > 1$.

Therefore, for every color $a \in A$, we have a sequence $\mathcal{V}_k^a$, $k \in \mathbb{N}$, of families of sets in Z with properties (i)–(iii). The definition of the $*$-operation implies that every $U^* \in \mathcal{V}_{k+1}^a$ is contained in its well-defined predecessor $U \in \mathcal{V}_k^a$, and moreover, $U^* \in \mathcal{U}_{j,k+1}^a$ if and only if $U \in \mathcal{U}_{j,k}^a$. In this sense, the sequence $\mathcal{V}_k^a$ is monotone, $\mathcal{V}_k^a \supset \mathcal{V}_{k+1}^a$, and we define $\mathcal{U}_j^a = \operatorname{Int}(\bigcap_{k\geq j} \mathcal{U}_{j,k}^a)$, $\mathcal{U}_j = \bigcup_{a\in A} \mathcal{U}_j^a$ for every $j \in \mathbb{N}$.

We put $\widehat{s}_j = \sum_{k\geq j} 2r^{k+1} = \frac{2r}{1-r} r^j$. Then $\widehat{s}_j \leq \delta r^j/4 < L(\widehat{\mathcal{U}}_j)$, and $B_{-\widehat{s}_j}(\widehat{\mathcal{U}}_j^a) \subset \mathcal{U}_j^a$ for every $a \in A$, $j \in \mathbb{N}$. By Lemma 2.2, the family $\mathcal{U}_j$ is still an $(n+1)$-colored open covering of Z with $L(\mathcal{U}_j) \geq \delta r^j - \widehat{s}_j \geq \delta r^j/2$. Property (ii) shows that $\operatorname{mesh}(\mathcal{U}_j) \leq r^j$, and the family $\mathcal{U}_j^a$ is δr^j-disjoint for every $a \in A$, $j \in \mathbb{N}$. By property (iii), given $1 \leq j' < j$, for every $U' \in \mathcal{U}_{j'}^a$, $U \in \mathcal{U}_j^a$ we have either $B_t(U) \cap U' = \varnothing$, or $B_t(U) \subset U'$ for $t = \gamma r^j$, $\gamma \geq \delta/4$.

Finally, recall that, for every $j \in \mathbb{N}$ and $a \in A$, the family $\widehat{\mathcal{U}}_j^a$ is a λr^j-net with $\lambda \geq 1$ independent of j, and every $\widehat{U} \in \widehat{\mathcal{U}}_j$ includes a ball of radius δr^j. Since every $U \in \mathcal{U}_j$ includes some $\widehat{U} \in B_{-\widehat{s}_j}(\widehat{\mathcal{U}}_j)$, it also includes a ball $B \subset \widehat{U}$ of radius $\delta r^j - \widehat{s}_j \geq \delta r^j/2$. Therefore, for every color $a \in A$ and every $z \in Z$ there is $U \in \mathcal{U}_j^a$ with $\operatorname{dist}(z, U) \leq (\lambda+1)r^j$, which means that the family $\mathcal{U}_j^a$ is a $(\lambda+1)r^j$-net. Since $(\lambda+1)r < \delta/2$, this also shows that for every $U' \in \mathcal{U}_{j'}^a$ with $j' < j$ there is $U'' \in \mathcal{U}_j^a$ with $U'' \cap U' \neq \varnothing$, whence $B_{\gamma r^j}(U'') \subset U'$. This completes the proof of Proposition 2.3. □

§3. THE EMBEDDING CONSTRUCTION

3.1. The hyperbolic cone. Let Z be a bounded metric space. Assuming that $\operatorname{diam} Z > 0$, we put $\mu = \pi/\operatorname{diam} Z$ and note that $\mu|zz'| \in [0, \pi]$ for every z, $z' \in Z$. Recall that the hyperbolic cone $\operatorname{Co}(Z)$ over Z is the space $Z \times [0,\infty)/Z \times \{0\}$ equipped with the metric defined as follows. Given $x = (z,t)$, $x' = (z',t') \in \operatorname{Co}(Z)$, we consider a triangle $\overline{o}\,\overline{x}\,\overline{x}' \subset \mathrm{H}^2$ such that $|\overline{o}\,\overline{x}| = t$, $|\overline{o}\,\overline{x}'| = t'$, and the angle $\angle_{\overline{o}}(\overline{x}, \overline{x}')$ is equal to $\mu|zz'|$. Now, we put $|xx'| := |\overline{x}\,\overline{x}'|$. In the degenerate case where $Z = \{\mathrm{pt}\}$, we define $\operatorname{Co}(Z) = \{\mathrm{pt}\} \times [0,\infty)$ as a metric product. The point $o = Z \times \{0\} \in \operatorname{Co}(Z)$ is called the *vertex* of $\operatorname{Co}(Z)$.

Theorem 3.1. *Let Z be a bounded metric space with finite capacity dimension*, $\operatorname{cdim} Z = n < \infty$. *Then there exists a quasiisometric embedding $f : \operatorname{Co}(Z) \to \prod_{a\in A} T_a$ of $\operatorname{Co}(Z)$ into the $(n+1)$-fold product of metric trees T_a, $|A| = n+1$.*

To obtain the embedding $f : \operatorname{Co}(Z) \to \prod_{a\in A} T_a$, we use a construction similar to that employed in [BS1].

3.2. Construction of the trees T_a. For every $a \in A$, the tree T_a is a rooted simplicial tree with root $v_a \in T_a$, and every edge of T_a has length 1. Using Proposition 2.3, we find constants δ, $\gamma \in (0,1)$, $\lambda \geq 1$, and, for a sufficiently small $r > 0$, a γ-separated characteristic sequence of $(n+1)$-colored open coverings $\mathcal{U}_j = \bigcup_{a\in A} \mathcal{U}_j^a$, $j \in \mathbb{N}$, of Z with the parameter r and characteristic constants δ, λ. We assume that $\lambda r < \delta$ and $r < \operatorname{diam} Z$.

For every $a \in A$, we define a graph T_a as follows. Its vertex set V^a is the disjoint union $\bigcup_{j\geq 0} V_j^a$, where the vertex set V_j^a of level $j \geq 0$ is identified with the set $\mathcal{U}_j^a$, $\mathcal{U}_0^a = \{Z\}$, i.e., the root v_a corresponds to the set Z. The edges only connect vertices of distinct levels, and vertices $v \in V_j^a$ and $v' \in V_{j'}^a$, $j' < j$, are connected by a (unique) edge if and only if for the corresponding members $U \in \mathcal{U}_j^a$ and $U' \in \mathcal{U}_{j'}^a$ we have $U \subset U'$ and j' is maximal with this property.

By this definition, every vertex $v \in V_j^a$, $j \in \mathbb{N}$, is connected with a lower level vertex v' by an edge, and every vertex in V_1^a is connected with v_a. Thus, the graph T_a is connected. By the properties of the sequence $\{\mathcal{U}_j\}$, for every vertex $v \in V_j^a$, $j \in \mathbb{N}$, there is at most one edge leading to a lower level vertex; therefore, T_a is a tree.

3.3. Construction of the embedding $f : \mathrm{Co}(Z) \to \prod_{a \in A} T_a$**.** We denote by Z_t the metric sphere of radius $t > 0$ around o in $\mathrm{Co}(Z)$. There are natural polar coordinates $x = (z, t)$, $z \in Z$, $t \geq 0$, in $\mathrm{Co}(Z)$. Then $Z_t = \{(z, t) : z \in Z\}$ is the copy of Z at the level t. For $t > 0$ we denote by $\pi_t : Z_t \to Z$ the canonical homeomorphism, $\pi_t(z, t) = z$.

We denote $R = \ln \frac{1}{r}$ and, for $j \in \mathbb{N}$, put $Z_j = Z_{jR}$, $\pi_j = \pi_{jR}$. Let $Z_0 = o$ be the vertex of $\mathrm{Co}(Z)$; we consider the set $X = \bigcup_{j \geq 0} Z_j \subset \mathrm{Co}(Z)$. Given $a \in A$, we define $f_a : X \to T_a$ as follows. We put $f_a(o) = v_a$ and then, for $x \in Z_j$, $j \in \mathbb{N}$, take a member $U \in \mathcal{U}_j^a$ closest to $\pi_j(x) \in Z$ and let $f_a(x) = v \in V_j^a$ be the vertex corresponding to U.

We shall need to pass from distances in Z to distances in any sphere Z_j, $j \in \mathbb{N}$. This passage is described in the important technical lemma below (Lemma 3.2). We say that $A \asymp B$ up to a multiplicative error not exceeding C if

$$\frac{1}{C} \leq \frac{A}{B} \leq C.$$

Lemma 3.2. *Given* z, $z' \in Z$, $j \in \mathbb{N}$, *we put* $\mu|zz'| = \tau$, $\tau_j = |z_j z_j'|$, *where* $z_j = \pi_j^{-1}(z)$, $z_j' = \pi_j^{-1}(z') \in Z_j$. *Then*

$$\sinh(\tau_j/2) \asymp \tau/r^j$$

up to a universally bounded multiplicative error.

Proof. Using the hyperbolic cosine law $\cosh a(t, \alpha) = \cosh^2(t) - \sinh^2(t) \cos \alpha$ for the base $a(t, \alpha)$ of an isosceles triangle in H^2 with sides t and angle α between them, we see that

$$\begin{aligned} \cosh \tau_j &= \cosh^2(jR) - \sinh^2(jR) \cos \tau \\ &= 1 + \sinh^2(jR) 2 \sin^2\left(\frac{\tau}{2}\right) \end{aligned}$$

by the definition of the cone metric. The claim follows. □

3.3.1. *The large scale Lipschitz property of* f_a*.* The proof of the large scale Lipschitz property of f_a follows the lines of [BS1, Proposition 2.7]. We need the following elementary fact (see [BS1, Lemma 2.6]).

Lemma 3.3. *Let* p, q, t *be the side lengths of a triangle in a metric space such that* $t \geq p$. *Then* $p + q \leq 3t$. □

Proposition 3.4. *For every* $a \in A$, *the map* $f_a : X \to T_a$ *is roughly Lipschitz, i.e., there are constants* $\Lambda > 0$ *and* $\sigma \geq 0$ *such that*

$$|f_a(x) f_a(x')| \leq \Lambda |xx'| + \sigma$$

for any x, $x' \in X$.

Proof. We fix x, $x' \in X$ and consider $v = f_a(x)$, $v' = f_a(x')$. There is no loss of generality in assuming that $x \in Z_j$, $x' \in Z_{j'}$ with $j \geq j' \geq 0$. Then $v \in V_j^a$ and $v' \in V_{j'}^a$, and we also assume that $v \neq v'$.

First, we consider the case where $j' = j$. By the properties of the edges, no shortest path in T_a has an interior vertex with locally maximal level. Thus, the shortest path in T_a between v and v' has a unique vertex v_0 of the lowest level j_0, $j_0 < j$. On the geodesic segments $v_0 v$, $v_0 v' \subset T_a$, we take the vertices $v_1 \in v_0 v$, $v_1' \in v_0 v'$ adjacent to v_0. We can assume that $v_1 \in V_{j_1}^a$, $v_1' \in V_{j_1'}^a$, where $j_0 < j_1' \leq j_1 \leq j$.

For the covering members $U_1 \in \mathcal{U}_{j_1}^a$ and $U_1' \in \mathcal{U}_{j_1'}^a$ corresponding to v_1 and v_1', respectively, we have either $B_s(U_1) \cap U_1' = \varnothing$, or $B_s(U_1) \subset U_1'$ for $s = \gamma r^{j_1}$, by the separation property (property (3)) of the sequence $\{\mathcal{U}_j\}$. The latter possibility is excluded, because otherwise there is a path in T_a connecting v and v', and missing v_0, and hence the initial path $vv_0 \cup v_0v'$ is not shortest, a contradiction. Therefore, $\operatorname{dist}(U_1, U_1') \ge \gamma r^{j_1}$ in Z, and by Lemma 3.2 we have

$$\operatorname{dist}(\widehat{U}_1, \widehat{U}_1') \ge 2R(j - j_1) - \sigma'$$

for some constant σ' depending only on $\mu\gamma$, where $\widehat{U}_1 = \pi_j^{-1}(U_1)$ and $\widehat{U}_1' = \pi_j^{-1}(U_1') \subset Z_j$. We have $\operatorname{dist}(\pi_j(x), U_1) \le \operatorname{dist}(\pi_j(x), U) \le \lambda r^j$, where $U \in \mathcal{U}_j^a$ corresponds to the vertex $f_a(x) \in T_a$, and similarly, $\operatorname{dist}(\pi_j(x'), U_1') \le \lambda r^j$. Thus, using Lemma 3.2, we obtain $|xx'| \ge \operatorname{dist}(\widehat{U}_1, \widehat{U}_1') - 2\lambda'$ with $\lambda' > 0$ depending only on $\lambda\mu$.

Let j_2' be the level of the vertex $v_2' \in v_1'v'$ adjacent to v_1', $v_2' \in V_{j_2'}^a$. There are two possibilities, which are treated differently.

(1) $j_2' \ge j_1$. Then for the distances in the tree T_a, we have $|v_0v| = |v_1v| + 1 \le j - j_1 + 1$ and $|v_0v'| = |v_2'v'| + 2 \le j - j_2' + 2 \le j - j_1 + 2$. Therefore,

$$|vv'| \le 2(j - j_1) + 3 \le \Lambda|xx'| + \sigma$$

for $\Lambda = 1/R$ and some $\sigma \ge 0$ independent of x, x'.

(2) $j_2' < j_1$. Let $U_2' \in \mathcal{U}_{j_2'}^a$ be the member corresponding to the vertex v_2'. By the separation property, $B_s(U_2') \subset U_1'$ for $s = \gamma r^{j_2'}$. Since $U_1 \cap U_1' = \varnothing$, we have $\operatorname{dist}(U_1, U_2') \ge \gamma r^{j_2'}$. Applying Lemma 3.2, we obtain $\operatorname{dist}(\widehat{U}_1, \widehat{U}_2') \ge 2R(j - j_2') - \sigma'$, where $\widehat{U}_2' = \pi_j^{-1}(U_2')$. As in case (1), this yields $|vv'| \le \Lambda|xx'| + \sigma$.

Finally, we consider the case where $j' < j$. Let $U' \in \mathcal{U}_{j'}^a$ be the member corresponding to the vertex v', and let $\widehat{U}' = \pi_{j'}^{-1}(U') \subset Z_{j'}$. Since $\operatorname{dist}(\pi_{j'}(x'), U') \le \lambda r^{j'}$ and $\operatorname{diam} U' \le r^{j'}$, we have $|x'y| \le \lambda''$ for every $y \in \widehat{U}'$ by Lemma 3.2, and the constant λ'' depends only on λ, μ. Thus, without loss of generality, as x' we can take any point in $\widehat{U}'$. Recall that, by property (3) of the sequence $\{\mathcal{U}_j\}$, there is $U'' \in \mathcal{U}_j^a$ with $B_s(U'') \subset U'$ for $s = \gamma r^j$. We take $x' \in \widehat{U}'$ so that $\pi_{j'}(x') \in U''$.

Now, consider $x'' \in Z_j$ sitting over x', i.e., $\pi_j(x'') = \pi_{j'}(x')$. Then $x'' \in \widehat{U}'' = \pi_j^{-1}(U'')$ and thus $f_a(x'') = v'' \in V_j^a$ corresponds to U''. It follows that v' is the lowest level vertex of the segment $v'v'' \subset T_a$, and $|v'v''| \le j - j' = \Lambda|x'x''|$. By what we have already proved, $|vv''| \le \Lambda|xx''| + \sigma$. On the other hand, obviously, $|x'x''| \le |x'x|$, whence $|xx''| + |x''x'| \le 3|xx'|$ by Lemma 3.3. Therefore,

$$|vv'| \le |vv''| + |v''v'| \le \Lambda(|xx''| + |x''x'|) + \sigma \le 3\Lambda|xx'| + \sigma,$$

which completes the proof of the proposition. □

3.3.2. *The large scale bi-Lipschitz property of f.* The map $f : X \to \prod_{a \in A} T_a$ defined by its coordinate maps $f_a : X \to T_a$ is roughly Lipschitz by Proposition 3.4. To prove that f is roughly bi-Lipschitz, we begin with the following lemma, which is the main ingredient of the proof.

Lemma 3.5. *Given $x = (z, Rj) \in Z_j$, $j \in \mathbb{N}$, for every nonnegative integer $i \le j$ there is a color $a \in A$ such that $\operatorname{dist}(f_a(x), V_i^a) \ge M$ with $M + 1 \ge (j - i + 1)/|A|$. Furthermore, if for $k \le i$ a vertex $v \in V_k^a$ is the lowest level vertex of the segment $f_a(x)v \subset T_a$, then $|f_a(x)v| \ge M$.*

Proof. Recall that the Lebesgue number $L(\mathcal{U}_j)$ is at least δr^j for every integer $j \ge 0$, because the sequence $\{\mathcal{U}_j\}$ of coverings is characteristic. Therefore, for every j there is $U_j \in \mathcal{U}_j$ with $B_{\delta r^j}(z) \subset U_j$. There exists a color $a \in A$ such that the set $\{U_i, \dots, U_j\}$ contains $M + 1 \ge (j - i + 1)/|A|$ members of color a, i.e., every such U_k belongs to $\mathcal{U}_k^a$. Let $U \in \mathcal{U}_j^a$ be the member corresponding to the vertex $f_a(x) \in T_a$. Since $\lambda r < \delta$, we have $\operatorname{dist}(z, U) \le \lambda r^j < \delta r^k$ for every $k < j$. Thus, $B_{\delta r^k}(z) \cap U \ne \varnothing$ and, by the separation property, $U \subset U_k$ for every $k < j$ with U_k of color a.

Using the separation property once again, we see that any path in T_a between $f_a(x)$ and the set V_i^a must contain at least $M+1$ vertices, whence $\operatorname{dist}(f_a(x), V_i^a) \geq M$.

Finally, let $V \in \mathcal{U}_k^a$ be the set corresponding to the vertex v. By the assumption on v, the set V includes U and every set among $\{U_i, \dots, U_j\}$ that is of color a. Hence, $|f_a(x)v| \geq M$. $\square$

Proposition 3.6. *There are constants* $\Lambda > 0$ *and* $\sigma \geq 0$ *such that*

$$|xx'| \leq \Lambda |f(x)f(x')| + \sigma$$

for all x, $x' \in X$.

Proof. We have $x = (z, Rj)$, $x' = (z', Rj')$ in the polar coordinates in $\operatorname{Co}(Z)$. We may assume that $j \geq j'$. Furthermore, if $j' = 0$, then x' coincides with o, the vertex of $\operatorname{Co}(Z)$, and we assume that $z' = z$ in this case.

First, suppose that $|zz'| \leq \lambda r^j + (\lambda+1) r^{j'}$. For $x'' = (z, Rj')$ we have $|x''x'| \leq \sigma'$ by Lemma 3.2, where σ' depends only on λ and μ. Then $|xx'| \leq |xx''| + |x''x'| \leq (j-j')R + \sigma'$.

On the other hand, $f_a(x') \in V_{j'}^a$ for all $a \in A$, and Lemma 3.5 implies

$$|f_a(x)f_a(x')| \geq \operatorname{dist}(f_a(x), V_{j'}^a) \geq M$$

with $M+1 \geq (j-j'+1)/|A|$ for some color $a \in A$. Therefore, $|xx'| \leq \Lambda |f(x)f(x')| + \sigma$ with $\Lambda = |A|R$ and σ depending only on $|A|$, R, and σ'.

Second, consider the case where $|zz'| > \lambda r^j + (\lambda+1) r^{j'}$. There is an integer $l \geq 0$ with

$$\lambda r^j + r^{l+1} + \lambda r^{j'} < |zz'| \leq \lambda r^j + r^l + \lambda r^{j'},$$

where we can take $r^0 := \operatorname{diam} Z$ because $r < \operatorname{diam} Z$. Then $r^{l+1} \geq r^{j'}$, so that $l+1 \leq j'$. Now, we show that, for every color $a \in A$, any path in T_a between $f_a(x)$ and $f_a(x')$ passes through a vertex of some level $k \leq l$. Indeed, let $v \in T_a$ be a lowest level vertex of the segment $f_a(x)f_a(x') \subset T_a$, $v \in V_k^a$. Then the set $V \in \mathcal{U}_k^a$ corresponding to v contains the members $U \in \mathcal{U}_j^a$ and $U' \in \mathcal{U}_{j'}^a$ corresponding to $f_a(x)$ and $f_a(x')$, respectively, and we have

$$|zz'| \leq \operatorname{dist}(z, U) + \operatorname{diam} V + \operatorname{dist}(z', U') \leq \lambda r^j + r^k + \lambda r^{j'}.$$

It follows that $r^{l+1} < r^k$, whence $k \leq l$.

By Lemma 3.5, there is a color $a \in A$ such that $\operatorname{dist}(f_a(x), V_l^a) \geq M$ with $M+1 \geq (j-l+1)/|A|$. Let $v \in V_k^a$ be the lowest level vertex of the segment $f_a(x)f_a(x')$. Then $|f_a(x)f_a(x')| \geq |f_a(x)v|$, and by Lemma 3.5 we have $|f_a(x)v| \geq M$ since $k \leq l$.

Consider the points $x_l = (z, Rl)$, $x_l' = (z', Rl) \in Z_l$. Using the estimate

$$|zz'| \leq \lambda r^j + r^l + \lambda r^{j'} \leq (2\lambda+1) r^l$$

and Lemma 3.2, we obtain $|x_l x_l'| \leq \sigma'$ with σ' depending only on λ and μ. Therefore,

$$|xx'| \leq |xx_l| + |x_l x_l'| + |x_l' x'| \leq 2(j-l)R + \sigma' \leq \Lambda |f(x)f(x')| + \sigma,$$

where $\Lambda = 2|A|R$, and σ depends only on $|A|$, R, and σ'. $\square$

3.4. Proof of Theorem 3.1. The map $f : X \to \prod_{a \in A} T_a$ is a quasiisometric embedding by Propositions 3.4 and 3.6. Being the union of equidistant spheres Z_j, $j \geq 0$, the set $X \subset \operatorname{Co}(Z)$ is obviously an $R/2$-net in $\operatorname{Co}(Z)$, $R = \ln \frac{1}{r}$. Thus, to define a quasiisometric embedding of $\operatorname{Co}(Z)$ into something, it suffices to define this embedding on X. Consequently, the map f is a well-defined quasiisometric embedding $\operatorname{Co}(Z) \to \prod_{a \in A} T_a$. $\square$

§4. PROOF OF THEOREM 1.1

4.1. Basics on hyperbolic spaces. We briefly recall the necessary facts about hyperbolic spaces. For more details the reader may consult, e.g., [BoS].

Let X be a metric space. We fix a base point $o \in X$ and put $(x|x')_o = \frac{1}{2}(|xo| + |x'o| - |xx'|)$ for x, $x' \in X$. The number $(x|x')_o$, which is nonnegative by the triangle inequality, is called the *Gromov product* of x, x' with respect to o.

A metric space X is (*Gromov*) *hyperbolic* if the *δ-inequality*

$$(x|x'')_o \geq \min\{(x|x')_o, (x'|x'')_o\} - \delta$$

is fulfilled for some $\delta \geq 0$, some base point $o \in X$, and all x, x', $x'' \in X$.

Let X be a hyperbolic space, and let $o \in X$ be a base point. A sequence of points $\{x_i\} \subset X$ *converges to infinity* if

$$\lim_{i,j\to\infty} (x_i|x_j)_o = \infty.$$

Two sequences $\{x_i\}$, $\{x_i'\}$ that converge to infinity are *equivalent* if

$$\lim_{i\to\infty} (x_i|x_i')_o = \infty.$$

The *boundary at infinity* $\partial_\infty X$ of X is defined as the set of equivalence classes of sequences converging to infinity. The Gromov product extends to $X \cup \partial_\infty X$ as follows. For points ξ, $\xi' \in \partial_\infty X$, the Gromov product is defined by

$$(\xi|\xi')_o = \inf \liminf_{i\to\infty} (x_i|x_i')_o,$$

where the infimum is taken over all sequences $\{x_i\} \in \xi$, $\{x_i'\} \in \xi'$. Note that $(\xi|\xi')_o$ takes values in $[0, \infty]$, and that $(\xi|\xi')_o = \infty$ if and only if $\xi = \xi'$.

Similarly, the Gromov product

$$(x|\xi)_o = \inf \liminf_{i\to\infty} (x|x_i)_o$$

is defined for any $x \in X$ and $\xi \in \partial_\infty X$, where the infimum is taken over all sequences $\{x_i\} \in \xi$.

A metric d on the boundary at infinity $\partial_\infty X$ of X is said to be *visual* if there is a point $o \in X$, a number $a > 1$, and positive constants c_1, c_2 such that

$$c_1 a^{-(\xi|\xi')_o} \leq d(\xi, \xi') \leq c_2 a^{-(\xi|\xi')_o}$$

for all ξ, $\xi' \in \partial_\infty X$. In this case, we say that d is a visual metric with respect to the base point o and the parameter a. The boundary at infinity is bounded and complete with respect to any visual metric, and if $a > 1$ is sufficiently close to 1, then a visual metric with respect to a does exist.

4.2. Visual hyperbolic spaces. A hyperbolic space X is said to be *visual* if for some base point $x_0 \in X$ there is a positive constant D such that for every $x \in X$ there exists $\xi \in \partial_\infty X$ with $|xx_0| \leq (x|\xi)_{x_0} + D$ (it is easily seen that this property is independent of the choice of x_0).

A map $g : X \to Y$ to a metric space Y is said to be *roughly homothetic* if

$$|g(x)g(x')| \doteq \Lambda |xx'|$$

up to a uniformly bounded additive error for some constant $\Lambda > 0$ and all x, $x' \in X$. We say that the space X is *roughly similar* to the image $g(X)$.

The proof of the following proposition can be found in [Bu, Proposition 6.2] and can also be extracted from [BoS].

Proposition 4.1. *Every visual hyperbolic space X is roughly similar to a subspace of the hyperbolic cone* $\mathrm{Co}(\partial_\infty X)$ *over the boundary at infinity, where $\partial_\infty X$ is taken with a visual metric.* ☐

Combining Proposition 4.1 and Theorem 3.1, we obtain a quasiisometric embedding $X \to \prod_{a\in A} T_a$ with $|A| = \operatorname{cdim} \partial_\infty X + 1$, which completes the proof of Theorem 1.1. ☐

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ST. PETERSBURG BRANCH, STEKLOV MATHEMATICAL INSTITUTE, RUSSIAN ACADEMY OF SCIENCES, FONTANKA 27, ST. PETERSBURG 191023, RUSSIA

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