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CONDITION NUMBERS OF LARGE MATRICES, AND ANALYTIC CAPACITIES

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Abstract. Given an operator $T : X \longrightarrow X$ on a Banach space X, we compare the condition number of T, $\mathrm{CN}(T) = \|T\| \cdot \|T^{-1}\|$, and the spectral condition number defined as $\mathrm{SCN}(T) = \|T\| \cdot r(T^{-1})$, where $r(\cdot)$ stands for the spectral radius. For a set Υ of operators, we put $\Phi(\Delta) = \sup\{\mathrm{CN}(T) : T \in \Upsilon, \mathrm{SCN}(T) \le \Delta\}$, $\Delta \in [1, \infty)$, and say that Υ is spectrally Φ-conditioned. As Υ we consider certain sets of $(n \times n)$-matrices or, more generally, algebraic operators with $\deg(T) \le n$ that admit a specific functional calculus. In particular, the following sets are included: Hilbert (Banach) space power bounded matrices (operators), polynomially bounded matrices, Kreiss type matrices, Tadmor–Ritt type matrices, and matrices (operators) admitting a Besov class $B^s_{p,q}$-functional calculus. The above function Φ is estimated in terms of the analytic capacity $\mathrm{cap}_A(\cdot)$ related to the corresponding function class A. In particular, for $A = B^s_{p,q}$, the quantity $\Phi(\Delta)$ is equivalent to $\Delta^n n^s$ as $\Delta \longrightarrow \infty$ (or as $n \longrightarrow \infty$) for $s > 0$, and is bounded by $\Delta^n (\log(n))^{1/q}$ for $s = 0$.

§1. Introduction

What this paper is about. How do we bound the resolvent of a matrix or an operator in terms of its "spectral data"? For instance, for $(n \times n)$-matrices acting on $\mathbb{C}^n$ the question is, given a set Υ_n of invertible $(n \times n)$-matrices, how do we find a function Φ_n such that

$$\|T^{-1}\| \le \Phi_n(\delta)$$

for every $T \in \Upsilon_n$, where δ stands for the minimum modulus of the eigenvalues of T, $\delta = \min |\lambda_i(T)|$? What does the best possible majorant Φ_n look like? How does it behave as $n \longrightarrow \infty$? Does it always exist?

It is well known that, to be meaningful, these questions require a kind of normalization (see below in this Introduction). In numerical analysis, the usual normalization is to replace $\|T^{-1}\|$ by the condition number $\mathrm{CN}(T) = \|T\| \cdot \|T^{-1}\|$ and then look for an estimate for $\mathrm{CN}(T)$ in terms of $\|T\|/\delta$. An equivalent approach is simply to include the normalization condition $\|T\| \le 1$ into the definition of Υ. See below for more comments and references in a more general Banach algebra setting.

In this paper, we use the second kind of normalization conditions and consider operators $T : X \longrightarrow X$ acting on a finite-dimensional Banach/Hilbert space X, $\dim X = N < \infty$ ("$(N \times N)$-matrices"). In order to have a more flexible classification of these operators than those given by the normalization $\|T\| \le 1$ mentioned above, we consider families Υ of operators obeying a functional calculus over a function space (algebra) A

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(see the definition in Subsection 3.1), i.e.,

$$\|f(T)\| \le C\|f\|_A$$

for every polynomial f. Let A_C be the set of all such operators. We shall see that, when using this functional calculus classification for $(N \times N)$-matrices, we should not view the dimension N as a true parameter for asymptotics of inverses or condition numbers; instead, we should take another integer n, namely, the degree of the minimal annihilating polynomial of T, $n = \deg(m_T) \le N$. That is why we can pass for free from $(N \times N)$-matrices to infinite-dimensional algebraic operators of degree not exceeding n on an arbitrary Banach/Hilbert space X. Recall that T is algebraic if there exists a polynomial $p \ne 0$ such that $p(T) = 0$; we write the minimal annihilating polynomial in a monic form,

$$m_T(z) = m_\sigma(z) = \prod_{1\le k\le n} (\lambda_k - z),$$

where $\sigma = \{\lambda_1, \dots, \lambda_n\}$ is the spectrum $\sigma(T)$ of T ($\sigma(T)$ consists of the eigenvalues of T, with possible multiplicities as they occur in the minimal polynomial).

Given a function space A on the unit disk $\mathbb{D} = \{z : |z| < 1\}$, a constant $C \ge 1$, and a family $\sigma = \{\lambda_1, \dots, \lambda_n\}$ in $\overline{\mathbb{D}}\backslash\{0\}$, we denote by

$$\Upsilon(m_\sigma, A_C)$$

the set of all algebraic operators T such that $T \in A_C$ and $m_\sigma(T) = 0$. In order to measure the size of inverses and condition numbers, we use the following quantities:

$$\begin{aligned}
\varphi(m_\sigma, A_C) &= \sup\big\{\|T^{-1}\| : T \in \Upsilon(m_\sigma, A_C)\big\},\\
\Phi_n(\Delta, A_C) &= \sup\big\{\varphi(m_\sigma, A_C) : \sigma \subset \{z : 1/\Delta \le |z| \le 1\}, \operatorname{card}(\sigma) \le n\big\}\\
&= \sup\big\{\|T^{-1}\| : T \in A_C, r(T^{-1}) \le \Delta, \deg(T) \le n\big\},
\end{aligned}$$

where $r(\cdot)$ stands for the spectral radius. If A is a Banach algebra, then the worst operator realizing the supremum in $\varphi(m_\sigma, A_1)$ is the quotient operator $S/m_\sigma A$ of the shift operator $S : A \longrightarrow A$, $Sf = zf$, or equivalently, its adjoint operator $S^*f = \frac{1}{z}(f - f(0))$ (the backward shift) acting on the subspace $K_{m_\sigma} \subset A^*$,

$$K_{m_\sigma} = \operatorname{span}\Big(\frac{1}{1-\overline{\lambda}z} : \lambda \in \sigma\Big),$$

with an obvious modification in the case of multiplicities in $\sigma = \{\lambda_1, \dots, \lambda_n\}$ (a point λ repeated m times gives the series $\frac{1}{(1-\overline{\lambda}z)^j}$, $1 \le j \le m$); see [N5, N3] for more information about such "model operators".

Notice also that the standard normalization $\|T\| \le 1$ is equivalent to the condition $T \in A_1$, where $\|\cdot\|_A$ is the uniform norm $\|\cdot\|_\infty$ for the case of a Hilbert space X, $\|f\|_\infty = \max_{|z|\le 1} |f(z)|$, and is the Wiener class norm $\|f\|_W = \sum_{k\ge 0} |\hat{f}(k)|$ for the case of an arbitrary Banach space X. Consequently, a specific A-functional calculus can be viewed as a qualified normalization for the above problem of bounding the inverses and condition numbers. This also makes the setting independent of the geometric nature of the space X, at least if we already know how to establish an A-functional calculus for a given operator $T : X \longrightarrow X$.

However, to begin with, in §2 we give a short proof for a slightly improved version of a well-known specifically Hilbert space result. Namely, we show that

$$\|T^{-1}\| \le (r(T^{-1}))^N$$

for every invertible contraction $T : \mathbb{C}^N \longrightarrow \mathbb{C}^N$, $\|T\| \le 1$, and describe all T for which $\|T^{-1}\| = (r(T^{-1}))^N$.

The case of N-dimensional operators (matrices) acting on a Banach space, $T : X \longrightarrow X$, $\dim X = N$, starts with a result of Schäffer [Sch] who, answering a question of B. L. van der Waerden, showed that $\|T^{-1}\| \cdot |\det(T)| \le k_N \|T\|^{N-1}$ for every invertible $(N \times N)$-matrix T, where $k_N \le \sqrt{eN}$ for an arbitrary X, and $k_N = 2$ for $X = l_N^1$ ($\mathbb{R}^N$ endowed with the l^1-norm). It follows that

$$\|T^{-1}\| \le \sqrt{eN}(r(T^{-1}))^N$$

for every contraction $\|T\| \le 1$ on an N-dimensional Banach space X. It was also conjectured in [Sch] that $k_N = 2$ for every space X. In [GMP], Gluskin, Meyer, and Pajor gave another proof to Schäffer's result and disproved Schäffer's conjecture, showing by a probabilistic method that $k_N \ge \frac{c_1}{\log\log(N)}\sqrt{\frac{N}{\log(N)}}$. The same paper contains a stronger counterexample by J. Bourgain giving $k_N \ge c_2\sqrt{\frac{N}{\log(N)}}$. Finally, Queffelec [Q] used a deterministic (number theory) approach to prove that Schäffer's inequality is sharp, i.e., $k_N \ge c_3\sqrt{N}$.

In this paper, considering an operator T in $\Upsilon(A_C, m_\sigma)$, we begin with the observation that, at least for $\sigma \subset \mathbb{D}\backslash\{0\}$, we have $\|f(T)\| \le C\|f\|_{A/m_\sigma A}$ for every polynomial f, where $A/m_\sigma A$ stands for the quotient algebra by the subspace of functions belonging to A and vanishing on σ. This reduces the problem of estimates of T^{-1} to an estimate of $\|f\|_{A/m_\sigma A}$, where f is a solution of the Bezout equation $zf + m_\sigma g = 1$, that is, to an estimate of $\|1/z\|_{A/m_\sigma A}$. In the case where A is a Banach algebra, this leads to a two-sided estimate

$$a \cdot \mathrm{cap}_A(\sigma) \le \varphi(m_\sigma, A_C) \le b \cdot \mathrm{cap}_A(\sigma),$$

where $a, b > 0$ are constants and cap_A stands for the A-zero capacity (at $z = 0$)

$$\mathrm{cap}_A(\sigma) = \inf\{\|f\|_A : f(0) = 1, f|\sigma = 0\}.$$

In complex analysis, the A-zero capacities are closely related to the problem of uniqueness sets for a function space A (the problem is to describe all σ such that $f \in A$, $f|\sigma = 0 \implies f = 0$). This means that they are of interest in the case of finite sets $\sigma \subset \mathbb{D}$ with $\max_{\lambda\in\sigma}|\lambda| \longrightarrow 1$. On the contrary, for bounding condition numbers and inverses, these capacities are of interest in the case of finite sets $\sigma \subset \mathbb{D}\backslash\{0\}$ with $\min_{\lambda\in\sigma}|\lambda| \longrightarrow 0$.

Therefore, in order to estimate $\varphi(m_\sigma, A_C)$, or to know an asymptotics for $\Phi_n(\Delta, A_C)$ as $\Delta \longrightarrow \infty$ and/or $n \longrightarrow \infty$, we need to bound $\mathrm{cap}_A(\sigma)$ or the following maximal capacities (an annular and a circular one):

$$\kappa_n(\Delta, A) = \sup\big\{\,\mathrm{cap}_A(\sigma) : \sigma \subset \{z : 1/\Delta \le |z| \le 1\},\ \mathrm{card}(\sigma) \le n\big\},$$
$$\underline{\kappa}_n(\Delta, A) = \sup\big\{\,\mathrm{cap}_A(\sigma) : \sigma \subset \{z : |z| = 1/\Delta\},\ \mathrm{card}(\sigma) \le n\big\}.$$

Applying this approach to the case of the Wiener algebra $A = W$ and using an unpublished estimate of Nazarov [Na], we obtain yet another short proof of the Schäffer upper estimate.

When applied to the analytic Besov spaces $A = B^s_{p,q}$ ($s \ge 0$, $1 \le p, q \le \infty$; see the definition in Subsection 3.1 below), the same idea gives

$$\mathrm{cap}_{B^s_{p,q}}(\sigma) \le c\frac{(\mathrm{card}(\sigma))^s}{|m_\sigma(0)|}, \quad \kappa_n(\Delta, B^s_{p,q}) \le c\Delta^n n^s$$

for $s > 0$, where $c = c(s, q)$, and

$$\mathrm{cap}_{B^s_{p,q}}(\sigma) \le c\frac{(\log(\mathrm{card}(\sigma)))^{1/q}}{|m_\sigma(0)|}, \quad \kappa_n(\Delta, B^s_{p,q}) \le c\Delta^n(\log(n))^{1/q}$$

for $s = 0$, where $c > 0$ is a numerical constant. It is also shown that for $s > 0$ these estimates are asymptotically sharp, namely

$$\frac{k}{2^{s+1}} n^s \leq \varliminf_{\Delta \longrightarrow \infty} \frac{\underline{\kappa}_n(\Delta, A)}{\Delta^n} \leq \varliminf_{\Delta \longrightarrow \infty} \frac{\kappa_n(\Delta, A)}{\Delta^n}.$$

The corresponding inequalities for the inverses and the condition numbers are immediate consequences of these estimates; see Corollary 3.35 in Subsection 3.6.

Recall that Besov classes functional calculi apply to the following operators.

(1) The set Υ of *Hilbert space* power bounded operators,

$$\sup_{k \geq 0} \|T^k\| = a.$$

Peller's result [Pe1] gives the bound $\|f(T)\| \leq k_G a^2 \|f\|_{B^0_{\infty 1}}$ for every polynomial f, where k_G is an absolute (Grothendieck) constant. This implies a logarithmic bound for the growth of inverses and condition numbers in terms of the degree $\deg(m_T) = n$ of the minimal polynomial of T:

$$\|T^{-1}\| \leq c \frac{k_G a^2 \log(n)}{|m_T(0)|}.$$

(2) The same bound holds for the *Banach space* Tadmor–Ritt operators $T : X \longrightarrow X$, that is, for the operators satisfying the resolvent estimate

$$\|R(\lambda, T)\| \leq C|\lambda - 1|^{-1}$$

for $|\lambda| > 1$. This result is based on Vitse's functional calculus for Tadmor–Ritt operators (see [Vi3], $\|f(T)\| \leq 300C^5 \|f\|_{B^0_{\infty 1}}$ for every polynomial f).

(3) The set of Banach space Kreiss operators T defined by the resolvent estimate

$$\|R(\lambda, T)\| \leq C(|\lambda| - 1)^{-1}$$

for $|\lambda| > 1$. In this case, we use yet another Vitse's functional calculus [Vi1], $\|f(T)\| \leq C\|f\|_{B^1_{1,1}}$ for every polynomial f (see also Peller [Pe2]). The corresponding asymptotics for the capacity $\kappa_n(\Delta, B^1_{1,1})$ and for inverses/condition numbers $\Phi_n(\Delta, (B^1_{1,1})_C)$ are $\Delta^n n$ (up to constants not depending on Δ and n), where as before $n = \deg(T)$, $\Delta > 1$. A higher growth of the resolvent $\|R(\lambda, T)\| \leq C(|\lambda| - 1)^{-s}$, $s > 1$, leads to a $B^s_{1,1}$ calculus (see [Pe2, Vi1]), and hence to the corresponding estimates for inverses and condition numbers in terms of the $B^s_{1,1}$ capacity (which is of the order of $\Delta^n n^s$).

(4) The set of *Hilbert space* operators T satisfying $C = \sup_{k \geq 1} \|T^k\|/k^\beta < \infty$, $\beta > 0$. By a theorem of Peller [Pe1], T obeys a $B^{2\beta}_{\infty,1}$-functional calculus. Therefore,

$$\|T^{-1}\| \leq c \frac{(\deg(T))^{2\beta}}{|m_T(0)|},$$

for every algebraic operator of this class. For $\beta < 1/2$, this is better than the estimate implied by the $l^1_A(k^\beta)$-functional calculus, which is valid for all Banach space operators satisfying $\sup_{n \geq 1} \|T^k\|/k^\beta < \infty$ (see the next paragraph). By the way, for $\beta > 1/2$, the obvious $l^1_A(k^\beta)$-functional calculus cannot be improved even for operators on a Hilbert space: there exists a Hilbert space operator satisfying $\sup_{k \geq 1} \|T^k\|/k^\beta < \infty$ and such that the norm $f \longmapsto \|f(T)\|$ is equivalent to $\|f\|_{l^1_A(k^\beta)}$; see Varopoulos [Va].

Another series of function spaces considered in Subsection 3.6 below is

$$l^q_A(w_k) = \Big\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{l^q_A(w_k)} = \Big(\sum_{k \geq 0} |\hat{f}(k)|^q w_k^q \Big)^{1/q} < \infty \Big\},$$

where $w_k > 0$ is such that $\lim_k w_k^{1/k} = 1$, $0 < \inf_k \frac{w_{k+1}}{w_k} \le \sup_k \frac{w_{k+1}}{w_k} < \infty$ and $1 \le q \le \infty$. It is shown that

$$\mathrm{cap}_{l^q_A(w_k)}(\sigma) \le \frac{\gamma_q(\mathrm{card}(\sigma))}{|m_\sigma(0)|},$$

where $\gamma_q(n)$ is defined in terms of the so-called Lagrange transform of the weight sequence $(w_k)_{k\ge 0}$; in particular, for $w_k = k^\beta$, $k \ge 1$ ($w_0 = 1$), we have

$$\gamma_q(n) \le n^\beta$$

for $q \ge 2$, and

$$\gamma_q(n) \le a n^{\beta + \frac{1}{q} - \frac{1}{2}} + b$$

for $1 \le q < 2$, where $a, b > 0$ are constants depending only on q and β. As a corollary, one can get the estimate

$$\|T^{-1}\| \le \frac{bCn^{\beta+(1/2)}}{|m_T(0)|}$$

(where $b > 0$ is a constant depending on (w_k)) for every Banach space operator T with $\deg(T) \le n$ and $C = \sup_{k\ge 1} \|T^k\|/k^\beta < \infty$.

Previous results can be viewed as bounding the ratio

$$\|1/z\|_{A/m_\sigma A} : \|1/z\|_{H^\infty/m_\sigma H^\infty},$$

since $\|1/z\|_{H^\infty/m_\sigma H^\infty} = 1/|m_\sigma(0)|$; see Subsection 3.3. In Subsection 3.4, we briefly consider the problem of comparison of the norms $a \longmapsto \|a\|_{A/m_\sigma A}$, where A is a function Banach algebra on $\mathbb{D}$, and $a \longmapsto \|a\|_{H^\infty/m_\sigma H^\infty}$ (H^∞ is the largest function Banach algebra on $\mathbb{D}$ having the weakest norm). This is a kind of Nevanlinna–Pick interpolation problem for a function algebra A. In fact, we consider three function algebras only, namely, $B^1_{1,1}$, W, and $B^0_{\infty,1}$, obtaining estimates for the quantity

$$k_n(A) = \sup\left\{k(\sigma, A) : \sigma \subset \mathbb{D},\ \mathrm{card}(\sigma) \le n\right\},$$

where

$$k(\sigma, A) = \sup\left\{\frac{\|a\|_{A/m_\sigma A}}{\|a\|_{H^\infty/m_\sigma H^\infty}} : a \in A, a \ne 0\right\}$$

(in Subsection 3.4 this notation is used for slightly different objects). For the case of $A = B^1_{1,1}$, the problem of estimation $k(\sigma, A)$ was raised in [Vi1], where it was related to the $B^1_{1,1}$-functional calculus for Kreiss operators (matrices). In Subsection 3.4 we show that $k_n \asymp n$ for all three algebras. Namely,

$$10^{-6} n \le k_n(B^0_{\infty,1}) \le k_n(W) \le \frac{3}{2} k_n(B^1_{1,1}) \le 9n$$

and $k_n(W) \le 2n$, despite the fact that these (strictly) embedded algebras $B^1_{1,1} \subset W \subset B^0_{\infty,1} \subset H^\infty \cap C(\overline{\mathbb{D}})$ are quite different. For instance, $\|z^n\|_{B^1_{1,1}}$ is equivalent to n as $n \longrightarrow \infty$, but for the other three algebras $\|z^n\|$ is bounded; the constants describing the asymptotics of the worst norm $\|T^{-1}\|$ as $T \in A_C$, $n = \deg(T) \longrightarrow \infty$, that is

$$k_n(1/z, A) = \sup\left\{\frac{\|1/z\|_{A/m_\sigma A}}{\|1/z\|_{H^\infty/m_\sigma H^\infty}} : \mathrm{card}(\sigma) \le n\right\},$$

also behave differently for all three algebras. Namely, from the preceding results it follows that

$$k_n(1/z, B^0_{\infty,1}) \le b \cdot \log(n+1),$$
$$a\sqrt{n} \le k_n(1/z, W) \le b\sqrt{n},$$
$$an \le k_n(1/z, B^1_{1,1}) \le bn$$

for the corresponding constants $a > 0$, $b > 0$. A slightly weaker estimate for $k_{n(}W)$, namely, $k_n(W) \le \pi n + 1$, has been known for a long time; see [KM]. For other similar results see [AFP, N2, VSh].

Subsection 3.7 contains a comparison of the above capacities $\mathrm{cap}_A(\cdot)$ and the Beurling–Carleson capacity $\mathrm{Carl}(\cdot)$ introduced in [GrN]. The latter is responsible for the uniqueness theorems for spaces of holomorphic functions in $\mathbb{D}$ having a "positive smoothness" up to the boundary,

$$\mathrm{Carl}(\sigma) = \frac{1}{|m_\sigma(0)|} \exp\Big(\int_{\mathbb{T}} \log \frac{2}{\mathrm{dist}(t,\sigma)}\, dm(t)\Big),$$

where $\sigma = \{\lambda_1, \dots, \lambda_n\}$ is a family in $\overline{\mathbb{D}}\backslash\{0\}$. We show that among all n-point families on the circle $\{z : |z| = 1/\Delta\}$, $\Delta > 1$, the worst $\mathrm{Carl}(\sigma)$ is attained at equidistributed sets σ_*, and

$$a\frac{n\Delta^n}{1+n(\Delta-1)} \le \mathrm{Carl}(\sigma_*) \le b\frac{n\Delta^n}{1+n(\Delta-1)}$$

for some numerical constants $a > 0$, $b > 0$. This contrasts with Queffelec's construction mentioned above for the worst $\mathrm{cap}_W(\sigma)$, since $\mathrm{cap}_W(\sigma_*) \le 2$ for an equidistributed σ_*. It is also shown that for a function space X such that $C_A^{(\alpha)}(\mathbb{D}) \subset X \subset C_A^{(\beta)}(\mathbb{D})$ for some $\alpha, \beta > 0$, where $C_A^{(\alpha)}(\mathbb{D})$ consists of holomorphic functions f on $\mathbb{D}$ having $(\alpha - [\alpha])$-Lipschitz derivative $f^{([\alpha])}$, there exist two increasing functions φ_X, ψ_X on $[1,\infty)$ such that $1 \le \varphi_\alpha(x) \le \lim_{x\longrightarrow\infty} \varphi_\alpha(x) = \infty$ (and similarly for ψ_α) and

$$\varphi_X\Big(a\frac{n\Delta^n}{1+n(\Delta-1)}\Big) \le \underline{\kappa}_n(\Delta, X) \le \psi_X\Big(b\frac{n\Delta^n}{1+n(\Delta-1)}\Big)$$

for every $\Delta > 1$. No estimates for φ_X, ψ_X are given.

Subsection 3.8 contains a few comments on the limit case of the peripheral spectrum, that is, on the case of operators in $\Upsilon(m_\sigma, A_C)$ with $\sigma \subset \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. It was already observed in [GMP] that in this case $\|T^{-1}\| \le \overline{\lim}_{k\longrightarrow\infty} \|T^k\|$ for every Banach space algebraic operator T. We complete this observation by showing that $\mathrm{cap}_W(\sigma) \le 2$ for every finite $\sigma \subset \mathbb{T}$. The same phenomenon of a "$\sqrt{n}$ lost" with respect to σ's located inside the disk can be observed for estimates of an arbitrary function $f(T)$ (and not for $f = 1/z$ only). Namely, the well-known Helson and Rudin–Shapiro theorems (see [GrMcG]) imply that

$$\sqrt{\mathrm{card}(\sigma)/2} \le k(\sigma, A) \le \sqrt{\mathrm{card}(\sigma)}$$

for every finite set $\sigma \subset \mathbb{T}$.

Whenever possible, the author sought constants explicitly depending on all incoming parameters, without insisting, however, on their sharpness. I am sorry if this business would sometimes entail some boring computations.

A few general comments on condition numbers and efficient inversions. As is well known, condition numbers and the problem of efficient inversions (that is, inversions with numerical estimates of inverses) are ubiquitous subjects for many applied fields. For instance, they are related to the error analysis. Namely, recall that if we look for a numerical solution to a linear equation

$$Tx = y,$$

say, in a Banach space or simply in $\mathbb{C}^n$, then computational errors may occur, and in reality we deal with nearby data $y+\Delta y$ and the corresponding nearby (unknown) solution $x+\Delta x$. The starting problem is to bound the relative error $\frac{\|\Delta x\|}{\|x\|}$ of the solution in terms

of the relative error of the data $\frac{\|\Delta y\|}{\|y\|}$. Writing $T(x+\Delta x) = y + \Delta y$ and then $x + \Delta x = x + T^{-1}(\Delta y)$, one obtains

$$\|\Delta x\| \le \|T^{-1}\| \cdot \|\Delta y\| = \|T^{-1}\| \cdot \frac{\|\Delta y\|}{\|y\|} \cdot \|Tx\| \le \|T^{-1}\| \cdot \|T\| \cdot \frac{\|\Delta y\|}{\|y\|} \cdot \|x\|,$$

whence

$$\frac{\|\Delta x\|}{\|x\|} \le \|T^{-1}\| \cdot \|T\| \cdot \frac{\|\Delta y\|}{\|y\|}.$$

This simple estimate gives a reason (one of many) why the condition number $\mathrm{CN}(T) = \|T^{-1}\| \cdot \|T\|$ of a matrix T plays a crucial role in the numerical linear algebra, as well as in numerical realizations of many other equations (differential, convolution, etc.). From the applied analysis point of view, the question "Is T invertible?" is replaced now by "How large is $\|T^{-1}\| \cdot \|T\|$?". For more about that, we refer the reader, e.g., to [GVL, HRS].

Similar questions arise not only in various applications, suvh as signal processing or control theory, but also in many theoretical problems. For instance, in operator theory, when constructing a functional calculus

$$f \longmapsto f(T) = \frac{1}{2\pi i} \int_\gamma f(\lambda)(\lambda I - T)^{-1}\, d\lambda,$$

we need to bound the norm of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$ (usually, in terms of the distance $\mathrm{dist}(\lambda, \sigma(T)) = \frac{1}{r}((\lambda I - T)^{-1})$). Many other problems require similar estimates: constructions of "skew" resolutions of the identity for a nonselfadjoint operator, stability problems for semigroups of operators, completeness of eigenvectors and generalized eigenvectors, the similarity problem, the existence of nontrivial invariant subspaces, etc. etc.

In harmonic analysis, the action of a function on the Fourier transforms $\varphi \circ (\hat{f}) = \hat{g}$ on various classes of functions or distributions, the harmonic synthesis problem, and the so-called deconvolution formulas heavily depend on estimates of the functional calculus, and hence on the growth of the resolvent.

We refer to the survey [N7] and the books [N3, N5] for more explanations, examples, and references.

Returning to the initial question on the condition number of a matrix $T : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, we recall that for normal (Hermitian, unitary, ...) matrices the number $\mathrm{CN}(T)$ is simply the discrepancy of the set of eigenvalues, $\mathrm{CN}(T) = \frac{\max|\lambda_i|}{\min|\lambda_i|}$, and for the standard algebra $C(K)$ of all continuous functions, endowed with the uniform norm $\|f\| = \max_K |f|$, we also have $\mathrm{CN}(f) = \frac{\max|f|}{\min|f|}$. This may give an impression that the "wild" behaviour of the condition numbers can appear only for norms depending on derivatives, or — for matrix algebras — is due to the spectral multiplicity phenomenon, like Jordan blocks, etc. However, this is not the case. Even in "flat" commutative Banach algebras (without any smoothness or quasinilpotency), such as the Wiener algebra of absolutely convergent Fourier series or integrals, or $(n \times n)$-matrices with a rather sparse spectrum, the condition numbers may be arbitrarily larger than the corresponding "spectral condition numbers" defined geometrically. Their real behavior depends on some latent relations between the norm and the spectrum in a given algebra; these relations are not completely understood.

Now, we make a few comments on "efficient inversions" as they were treated in [N1, ENZ]. Namely, for a commutative Banach algebra A, one can look for an estimate of $\|a^{-1}\|$ in terms of $\delta = \min_{t \in \mathfrak{M}(A)} |\hat{a}(t)|$, where $\mathfrak{M}(A)$ stands for the space of maximal ideals of A and $\hat{a}$ for the Gelfand transform of an element $a \in A$. In fact, it often happens in applications that only a part of $\mathfrak{M}(A)$ is available, say $X \subset \mathfrak{M}(A)$, a "visible part" of

$\mathfrak{M}(A)$. In this case we need to bound $\|a^{-1}\|$ in terms of the "visible part" of the spectral data $\delta = \inf_{t\in X} |\hat{a}(t)|$.

It is known that, without any normalization, the above problem has a positive solution only for uniform algebras A, and this case is of limited interest for the problem. Namely, the following lemma (perhaps, known earlier) was proved in [N1]. For the reader's convenience we reproduce a short proof.

Lemma 1.1. *Let A be a unital commutative Banach algebra, and let $X \subset \mathfrak{M}(A)$. If for some positive δ and C the condition*

$$|\hat{a}(x)| \geq \delta (x \in X)$$

implies that $\|a^{-1}\| \leq C$, then A is a uniform algebra whose norm is equivalent to the sup norm on X, namely,

$$\|\hat{a}\|_X \leq \|a\| \leq (2C\delta + 1)\|\hat{a}\|_X$$

for all $a \in A$, where $\|\hat{a}\|_X = \sup_X |\hat{a}|$.

Proof. First, we observe that if $\inf_X |\hat{a}| = \epsilon > 0$, then $|\hat{a}(x)\delta/\epsilon| \geq \delta, x \in X$, and hence a is invertible and $\|a^{-1}\| \leq C\delta/\epsilon = C\delta\|\hat{a}^{-1}\|_X$. Now, given $b \in A$, we take $\lambda = \|\hat{b}\|_X + \epsilon$ and apply the previous remark to $a = (b - \lambda e)^{-1}$:

$$\begin{aligned}\|b\| &= \|b - \lambda e + \lambda e\| \leq C\delta\|\hat{b} - \lambda\|_X + \lambda \\ &\leq C\delta\|\hat{b}\|_X + (C\delta + 1)\lambda = (2C\delta + 1)\|\hat{b}\|_X + (C\delta + 1)\epsilon.\end{aligned}$$

Letting $\epsilon \longrightarrow 0$, we get $\|b\| \leq (2C\delta + 1)\|\hat{b}\|_X$ for all $b \in A$. □

As mentioned before, the problem can be "normalized" in two equivalent ways. First, we can simply restrict ourselves to elements $a \in A$ with $\|a\| \leq 1$. Second, following a custom of numerical analysis (see above), we can try to bound the classical condition number

$$\mathrm{CN}(a) = \|a\| \cdot \|a^{-1}\|$$

in terms of the corresponding *spectral condition number* defined by

$$\mathrm{SCN}(a) = \|a\| \cdot r(a^{-1}),$$

where $r(*)$ stands for the spectral radius of $*$. Recall that for a commutative algebra, $r(a^{-1}) = \frac{1}{\min_{t\in\mathfrak{M}(A)} |\hat{a}(t)|}$. In the case of an "incompletely visible" spectrum $X \subset \mathfrak{M}(A)$ we can set

$$\mathrm{SCN}(X, a) = \|a\| \cdot r_X(a^{-1}),$$

where $r_X(a^{-1}) = \frac{1}{\inf_{t\in X} |\hat{a}(t)|}$. Next, following [N1], we define the quantities

$$\begin{aligned}C_1(\Delta, B, X) &= \sup\big\{\, \mathrm{CN}(a) : a \in B, \mathrm{SCN}(X, a) \leq \Delta\big\}, \quad \Delta \geq 1, \\ C_1(\Delta, B) &= \sup\big\{\, \mathrm{CN}(a) : a \in B, \mathrm{SCN}(a) \leq \Delta\big\}, \quad \Delta \geq 1, \\ \Delta_1(B) &= \sup\big\{\Delta : C_1(\Delta, B) < \infty\big\}.\end{aligned}$$

Note that $\Delta_1(B)$ and $C_1(\Delta, B)$, where $B \subset A$, are well defined for an arbitrary Banach algebra A, and even for a family B containing elements of different Banach algebras, whereas their X-visible analogs refer to the maximal ideals, and so to a commutative Banach algebra.

Since the function $C_1(\cdot, B)$ increases,

$$I(B) = \{\Delta \geq 1 : C_1(\Delta, B) < \infty\} = [1, \Delta_1\rangle$$

is the largest interval where *the set B is well conditioned* in the following sense: there exists a function Φ such that

$$\mathrm{CN}(a) \leq \Phi(\mathrm{SCN}(a))$$

for every $a \in B$ with $\mathrm{SCN}(a) \in I$. On the complementary interval $[1,\infty)\backslash I$, the set B is *ill conditioned*. In the case where $\Delta_1 = \infty$ (i.e., $I = [1,\infty)$), we also say that elements of B are efficiently invertible.

As explained above, in this paper we deal with bounding functions Φ for sets of $(n \times n)$-matrices and algebraic operators satisfying a specific functional calculus. In a forthcoming paper, we plan to treat the same problem for $n \times n$ Toeplitz matrices, Toeplitz (Wiener–Hopf) operators on the Wiener algebra $W_A = \mathcal{F}l^1(\mathbb{Z}_+)$ and on the Hardy space H^2, as well as for l^p multipliers and Toeplitz operators acting on $\mathcal{F}l^p(\mathbb{Z}_+)$, $p \neq 1, 2, \infty$.

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§2. $C_1(\Delta)$ for the case of all matrices

We start with the algebra $\mathcal{A} = \mathcal{M}_n = L(\mathbb{C}^n)$ of all $(n \times n)$-matrices (equivalently, the algebra of operators on an n-dimensional Hilbert space) endowed with the usual operator norm $\|A\| = \|A : \mathbb{C}^n \longrightarrow \mathbb{C}^n\|$. The following theorem gives the (known) exact value of $C_1(\Delta, \mathcal{M}_n)$ and describes all matrices for which the supremum in the definition is attained. Note that, by von Neumann's inequality ($\|f(A)\| \leq \|f\|_{H^\infty}$ for every polynomial f and every contraction A ($\|A\| \leq 1$)), and a result of Subsection 3.3 below, we have $C_1(\Delta, \mathcal{M}_n) = \Phi_n(\Delta, H_1^\infty)$. But here we give an independent elementary proof to the following.

Theorem 2.1. *$C_1(\Delta, \mathcal{M}_n) = \Delta^n$ for every $\Delta \geq 1$, which means in particular that $\|T^{-1}\| \leq (r(T^{-1}))^n$ for every T with $\|T\| \leq 1$.*

Equality $\|T^{-1}\| = (r(T^{-1}))^n$ occurs for a matrix T with $\|T\| = 1$ if and only if

(1) either $r(T^{-1}) = 1$, and then T is an arbitrary unitary matrix;

(2) or $\Delta = r(T^{-1}) > 1$, and then T is unitarily equivalent to a matrix constructed by the following rule: take any family $\sigma = \{\lambda_0, \ldots, \lambda_{n-1}\}$ on the circle $|z| = 1/\Delta$ and find the solutions ζ_j, $j = 1, \ldots, n$, of the equation $B(\zeta) = -1$ sitting on $\mathbb{T}$, where B stands for the Blaschke product $\prod_{0 \leq j < n} b_{\lambda_j}$ (see the definition in the proof), then set $U = \mathrm{diag}(\zeta_j)$ and $x = (x_1, \ldots, x_n)$ with any vector having $|x_j|^2 = -\frac{a\overline{\zeta}_j}{B'(\zeta_j)}$, where $a = \frac{\Delta^n+1}{\Delta^n-1}$, and finally set

$$Tx = \frac{1}{\Delta^n} Ux, \quad \text{and} \quad T = U \text{ on the orthogonal complement of } x.$$

The spectrum of T is σ.

Proof. The upper estimate $C_1(\Delta, \mathcal{M}_n) \leq \Delta^n$ is an immediate consequence of the following identities:

$$|\lambda_0(T) \cdots \lambda_{n-1}(T)|^2 = |\det(T)|^2 = \det(T^*T) = s_0(T)^2 \cdots s_{n-1}(T)^2,$$

where the $\lambda_k(T)$ (respectively, $s_k(T) = \lambda_k((T^*T)^{1/2})$) are the eigenvalues (respectively, singular values) of T ordered so that the $|\lambda_j(T)|$ (respectively, the $s_j(T)$) decrease. (We refer to [G] for the properties of $s_k(T)$.) Indeed, the singular numbers $s_k(T)$ are precisely the diagonal elements of the positive factor in the polar decomposition of T, $T = UR$, where $R \geq 0$ and U is unitary, and hence $\|T^{-1}\| = 1/s_{n-1}(T)$, $\mathrm{CN}(T) = s_0(T)/s_{n-1}(T)$. By the above identities,

$$|\lambda_{n-1}(T)|^n \leq |\lambda_0(T) \cdots \lambda_{n-1}(T)| = s_0(T) \cdots s_{n-1}(T) \leq s_0(T)^{n-1} s_{n-1}(T),$$

which means

$$\mathrm{CN}(T) = \frac{s_0(T)}{s_{n-1}(T)} \le \Big(\frac{s_0(T)}{\lambda_{n-1}(T)}\Big)^n = (\mathrm{SCN}(T))^n.$$

This shows that $C_1(\Delta, \mathcal{M}_n) \le \Delta^n$.

To prove the reverse inequality $C_1(\Delta, \mathcal{M}_n) \ge \Delta^n$, one can use Horn's theorem (see [Hor] and also [MO, Chapter 9, Section E]), which gives a converse to the famous Weyl inequalities. But to describe all cases of equality $\|T^{-1}\| = r(T^{-1})^n$, $\|T\| = 1$, we need more, as is stated below in Lemma 2.2. Now, consider the matrices T turning the inequality

$$s_0(T) \cdots s_{n-1}(T) \le s_0(T)^{n-1} s_{n-1}(T)$$

into equality, that is, the matrices for which $s_0(T) = \cdots = s_{n-2}(T)$. Lemma 2.2 shows that if positive numbers s_0, s_{n-1} and complex numbers λ_k, $0 \le k < n$, satisfy $0 < s_{n-1} < s_0$ and $|\lambda_k| \le s_0$ for $0 \le k < n-1$, $|\lambda_0 \cdots \lambda_{n-1}| = s_0^{n-1} s_{n-1}$, then there exists a matrix $T = T(\lambda, s)$ such that $\lambda_k = \lambda_k(T)$ and $s_k = s_k(T)$ for $0 \le k < n$. Now, let $s_0 = 1$ and $|\lambda_k| = 1/\Delta < 1$ for every k, $0 \le k < n$. Then, the same lemma claims that every such matrix $T(\lambda, s)$ is of the form described in the statement of the theorem. Moreover, $s_{n-1}(T) = 1/\Delta^n$ and $\mathrm{CN}(T) = \Delta^n$, whereas $\mathrm{SCN}(T) = \Delta$. This shows that $C_1(\Delta, \mathcal{M}_n) \ge \Delta^n$ and gives the required description of all cases of equality $\|T^{-1}\| = r(T^{-1})^n$, $\|T\| = 1$. □

Lemma 2.2. *For any positive numbers $s_0 = s_1 = \cdots = s_{n-2} > s_{n-1} > 0$ and complex numbers λ_k, $0 \le k < n$, satisfying $|\lambda_k| \le s_0$ for $0 \le k < n$, and $|\lambda_0 \cdots \lambda_{n-1}| = s_0^{n-1} s_{n-1}$, there exists an $(n \times n)$-matrix T such that $\lambda_k = \lambda_k(T)$ and $s_k = s_k(T)$ for $0 \le k < n$.*

Moreover, for $s_0 = 1$ and $|\lambda_j| < 1$ $(0 \le j < n)$, all such matrices T are unitarily equivalent to the following matrix: let $\zeta_j, j = 1, \dots, n$, be all solutions of the equation $B(\zeta) = -1$ sitting on $\mathbb{T}$, where B stands for the Blaschke product $\prod_{0 \le j < n} b_{\lambda_j}$ (see the definition in the proof), then set $U = \mathrm{diag}(\zeta_j)$ and $x = (x_1, \dots, x_n)$ with any vector having $|x_j|^2 = \frac{-a\overline{\zeta}_j}{B'(\zeta_j)}$, where $a = \frac{1+s_{n-1}}{1-s_{n-1}}$, and finally define

$$Tx = s_{n-1} U x, \text{ and } T = U \text{ on the orthogonal complement of } x.$$

Proof. First, we consider the case where $|\lambda_k| < s_0$ for $0 \le k < n$. Let $T = UR$, where

$$R = \begin{pmatrix} s_0 & 0 & 0 & & & \\ 0 & s_0 & 0 & & & \\ 0 & 0 & & & & \\ & & & & 0 & 0 \\ & & & 0 & s_0 & 0 \\ & & & 0 & 0 & s_{n-1} \end{pmatrix}$$

in an orthonormal basis, and U is an unknown unitary matrix. This means that $T = U(s_0 I + (s_{n-1} - s_0)P)$, where $P = (\cdot, x)x$ is a rank one orthogonal projection. Without loss of generality, $s_0 = 1$. The equation for the eigenvalues of T is

$$0 = \det(zI - T) = \det(zI - U) \cdot \det(I - R(z, U)(s_{n-1} - 1)UP),$$

where $R(z, U) = (zI - U)^{-1}$ stands for the resolvent of U. Thus, for the eigenvalues z with $|z| < 1$ we have the equation

$$1 = (1 - s_{n-1}) \int_{\mathbb{T}} \zeta \frac{d\mu(\zeta)}{\zeta - z},$$

where $d\mu = (dEx, x)$ is a scalar spectral measure of U. Therefore, we look for a probability measure μ on the unit circle $\mathbb{T}$ such that $\mathrm{card}(\mathrm{supp}(\mu)) \le n$ and the roots of the

equation

$$\frac{1}{1-s_{n-1}} = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1-\overline{\zeta}z}$$

coincide with the given numbers λ_k, $0 \le k < n$. Writing the equation in the form

$$a = \frac{1+s_{n-1}}{1-s_{n-1}} = \int_{\mathbb{T}} \frac{1+\overline{\zeta}z}{1-\overline{\zeta}z}\, d\mu(\zeta),$$

denoting the integral on the right by f, and using the Herglotz theorem on holomorphic functions with positive real part (see, for instance, [N2, Theorem 3.9.2]), we can restate the problem as follows:

Find a rational function $f \in \mathrm{Hol}(\mathbb{D})$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, having its a-points $f(z) = a$ at the given places λ_k, $0 \le k < n$, and such that $\deg(f) \le n$, $\operatorname{Re} f(z) \ge 0$ in $\mathbb{D}$, $f(0) = 1$, and $\operatorname{Re} f(z) = 0$ on $\mathbb{T}$.

Setting $B = \frac{a-f}{a+f}$, we need to find a rational function B, $\deg(B) \le n$, with $|B(z)| = 1$ on $\mathbb{T}$, $B(0) = s_{n-1}$, and having the prescribed set of zeros λ_k, $0 \le k < n$. Clearly, such a function exists and is unique, namely, this is the Blaschke product $B = \prod_k b_{\lambda_k}$, where $b_\lambda(z) = \frac{\lambda - z}{1-\overline{\lambda}z} \cdot \frac{\overline{\lambda}}{|\lambda|}$. This completes the proof of the existence statement in the case where $|\lambda_k| < s_0$ for $0 \le k < n$.

To find $\mu = \sum_j p_j \delta_{\zeta_j}$, we observe that the ζ_j are points on $\mathbb{T}$ where $f = \infty$, that is $B = -1$, and the point masses at ζ_j are given by $p_j = \lim_{r \longrightarrow 1} \frac{1-r}{2} f(r\zeta_j)$. Once the measure μ is found, we can realize U as $\mathrm{diag}(\zeta_j)$ on the space $\mathbb{C}^n$. Then, $(dEx, x) = \sum_j |x_j|^2 \delta_{\zeta_j}$ and hence x is any vector having $p_j = |x_j|^2$. The matrix T is defined, therefore, by $Tx = s_{n-1}Ux$ and $T = U$ on the orthogonal complement of x (recall that we assume $s_0 = 1$).

If $|\lambda_k| = s_0$ for some k (notice that we do not need this case for the proof of Theorem 2.1), we can consider the modified eigenvalues $\lambda_k^* = r\lambda_k$, $0 \le k < n-1$, and $s_{n-1}^* = r^n s_{n-1}$, where $0 < r < 1$. Constructing a matrix T_r by the previous rule for these modified spectral data, we can then use compactness arguments and choose a sequence T_{r_j}, $r_j \longrightarrow 1$, tending to a desired matrix. □

Remark 2.3. *In order to prove merely the inequality $C_1(\Delta, \mathcal{M}_n) \ge \Delta^n$, in the lemma we only need* to exhibit an example of a matrix T with $|\lambda_k(T)| = \frac{1}{\Delta} = s_{n-1}^{1/n}$, $0 \le k < n$, and $s_0 = \cdots = s_{n-2}$. To this end, we can simply take $\mu = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{\zeta_k}$, where $\zeta_k = \zeta^k$ and $\zeta = e^{2\pi i/n}$ is the nth root of unity. Then

$$g(z) = \int \frac{d\mu(t)}{1-\overline{t}z} = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{j\ge 0}(\overline{\zeta}^k z)^j = \sum_{p \ge 0} z^{pn} = \frac{1}{1-z^n}$$

for every z, $|z| < 1$. Since the eigenvalues are the solutions of $(1-s_{n-1})g(z) = 1$, we get $\lambda_k = \zeta^k s_{n-1}^{1/n}$, $0 \le k < n$. The corresponding vector x is $x = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$.

§3. Matrices and operators obeying a functional calculus

Now, we consider the problem of bounding the inverses in a general framework of functional calculi. For a given operator, a bounded functional calculus over a function algebra or a function space is a kind of restriction of a more specialized form than a normalization. We shall see that such an approach is well adapted to operators acting both on a Hilbert space and on a Banach space. In this context, it is also natural to seek for bounds for general functions $a(T)$ of an operator T, not only for T^{-1}. We use the following language and notation.

3.1. Norms related to a functional calculus. A *function algebra* (*on* $\mathbb{D}$) is a unital Banach algebra, say A, such that

(i) A is continuously embedded into the algebra $\mathrm{Hol}(\mathbb{D})$ of all holomorphic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$;

(ii) A contains all polynomials and $\lim_n \|z^n\|_A^{1/n} = 1$; and

(iii)

$$(a \in A,\ \lambda \in \mathbb{D}, \text{ and } a(\lambda) = 0) \implies \frac{a}{z-\lambda} \in A.$$

An A-*functional calculus* for an operator $T : X \longrightarrow X$ on a Banach space X means a bounded homomorphism $A \longrightarrow L(X)$ of a function algebra A into the algebra $L(X)$ of bounded linear operators on X such that $z \longmapsto T$ and $1 \longmapsto I$. In particular,

$$\|a(T)\| \le C_T \|a\|_A$$

for every $a \in A$, where C_T stands for the norm of the homomorphism. Clearly, an operator T possessing a calculus over a function algebra on $\mathbb{D}$ has its spectrum in the closed unit disk, $\sigma(T) \subset \overline{\mathbb{D}}$.

In order to define and to use a calculus, we do not really need to require that A be an algebra. Below, we often work with merely *a function space in* $\mathbb{D}$, that is, a Banach space A satisfying (i)–(iii) and such that $pf \in A$ for every $f \in A$ and every polynomial p. The *calculus over such a space* A means a bounded mapping $f \longmapsto f(T)$ such that $(pf)(T) = p(T)f(T)$ for every $f \in A$ and every polynomial p. See also Remark 3.27 below.

Let Υ be a family of operators. We say that Υ obeys an A-calculus with a constant $C > 0$ if $C_T \le C$ for every $T \in \Upsilon$. This inequality plays a role of a normalization when we consider the problem of uniform estimates of inverses.

Finally, an operator T is *algebraic* (of degree $\deg(T) \le n$) if there exists a polynomial p such that $p \ne 0$, $p(T) = 0$, and $\deg(p) \le n$. By m_T we denote the *minimal annihilator* in the form of a monic polynomial

$$m_T(z) = \prod_{j=0}^{n-1} (\lambda_j - z).$$

It is clear that in this case the spectrum $\sigma(T) = \{\lambda_j : 0 \le j < n\}$ consists of eigenvalues, and T is invertible if and only if $m_T(0) \ne 0$.

We start with the following elementary lemma.

Lemma 3.1. *If an algebraic operator T admits a calculus over a function space/algebra A, then so it does over the quotient space/algebra A/mA for every polynomial m, $m \ne 0$, such that $m(T) = 0$ (in particular, for $m = m_T$), and*

$$\|a(T)\| \le C_T \|a\|_{A/mA}$$

for every $a \in A$, where $\|a\|_{A/mA} = \inf\{\|a + mg\|_A : g \in A\}$.

Proof. It is clear that $\|a(T)\| \le C_T \inf\{\|b\|_A : b = a + mg, g \in A\}$. The result follows. □

In this section, we consider some sets Υ of algebraic operators having the same annihilating polynomial m and satisfying an A-calculus (hence, an A/mA-calculus). Our aim is to compute or estimate the quantity

$$|a|_\Upsilon = \sup\{\|a(T)\| : T \in \Upsilon\},$$

where $a \in A$ (or $a \in A/mA$), in terms of m and the family $a(\sigma)$, where σ is the family of zeros of m. The case of the inverses corresponds to the choice $a = z^{-1} \in A/mA$.

It should be noted that for an arbitrary bounded set Υ of operators (or elements of a Banach algebra) the function $p \longmapsto |p|_\Upsilon$ is well defined on the algebra of complex polynomials and is a seminorm satisfying the normed algebra submultiplicativity conditions

$$|pq|_\Upsilon \le |p|_\Upsilon |q|_\Upsilon, \quad |1|_\Upsilon = 1.$$

In what follows, for the sake of simplicity and to avoid unessential notational complications, we often assume that *the minimal polynomial m_T of an algebraic operator T has simple zeros*. This does not result in any problem when arriving at final estimates of inverses or functions of operators because one can pass to simple zeros (and back) by the standard "small movings" of the spectrum. The key point is that the function α, $\alpha(T) = \sup\{\|f(T)\| : f \in A,\ \|f\|_A \le 1\}$, defined on the operators $T : X \longrightarrow X$ with $\dim X = N < \infty$ and having $\sigma(T) \subset \mathbb{D}$, is continuous.

Lemma 3.2. *Let T be an algebraic operator satisfying an A-calculus and having a minimal polynomial $m = m_\sigma$ with simple zeros, $\sigma = \{\lambda_0, \dots, \lambda_{n-1}\} \subset \mathbb{D}\backslash\{0\}$. Then*

$$\begin{aligned}\|T^{-1}\| &\le \frac{C_T}{|m(0)|}\Big\|\frac{m - m(0)}{z}\Big\|_{A/mA}\\ &= C_T \inf\Big\{\Big\|\frac{1-g}{z}\Big\|_A : g \in A, g(0) = 1, g(\lambda) = 0 \text{ for } \lambda \in \sigma\Big\}\\ &= C_T \inf\{\|h\|_A : h \in A, h(\lambda) = 1/\lambda \text{ for } \lambda \in \sigma\}.\end{aligned}$$

Proof. It is clear that $T^{-1} = a(T)$ for every $a \in A$ satisfying a Bezout equation

$$za + m_T g = 1,$$

where $g \in A$. Hence

$$\begin{aligned}\|T^{-1}\| &\le C_T \inf\{\|a\|_A : a(T) = T^{-1}\}\\ &= C_T \inf\{\|a + mh\|_A : h \in A\} = C_T\|a\|_{A/mA}.\end{aligned}$$

Since $a = \frac{m(0)-m}{zm(0)}$ is a solution of the Bezout equation, we get the inequality. Two other formulas also follow. □

Remark 3.3. Let $Sa = za$ and $S^*a = \frac{a-a(0)}{z}$ be the shift operator and the backward shift operator on A, respectively. Both S and S^* are bounded on A by the definition of a function space on $\mathbb{D}$. By using these operators, the quantities in Lemma 3.2 can be bounded as follows:

$$\frac{1}{\|S\|}\|1-g\|_A \le \Big\|\frac{1-g}{z}\Big\|_A \le \|S^*\| \cdot \|g\|_A.$$

We finish this subsection with a brief recalling on the (analytic) Besov spaces $B^s_{p,q}$ in the disk $\mathbb{D}$, which will be used afterwards. Namely, let $s \in \mathbb{R}$, $1 \le p$, $q \le \infty$, and let

$$B^s_{p,q} = \Big\{f = \sum_{n\ge 0} \hat{f}(n) z^n : \Big(\int_0^1 ((1-r)^{n-s-(1/q)}\|f_r^{(n)}\|_{L^p(\mathbb{T})})^q\, dr\Big)^{1/q} < \infty\Big\},$$

where $f_r^{(n)}(\zeta) = f^{(n)}(r\zeta)$, n being a nonnegative integer such that $n > s$. The space $B^s_{p,q}$ equipped with the norm

$$\|f\|_{B^s_{p,q}} = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \Big(\int_0^1 ((1-r)^{n-s-(1/q)}\|f_r^{(n)}\|_{L^p(\mathbb{T})})^q\, dr\Big)^{1/q}$$

is a function space on $\mathbb{D}$. We refer to [BL, P, T] for general properties of Besov spaces. In particular, the space $B^s_{p,q}$ allows yet another (harmonic analysis) definition (given first

by Lizorkin [L] and Peetre [P]) involving an equivalent norm:

$$B^s_{p,q} = \Big\{ f = \sum_{n\geq 0} \hat{f}(n) z^n : \|f\|_{s,p,q} = \Big(\sum_{n\geq 0} (2^{ns} \|f * W_n\|_{L^p(\mathbb{T})})^q \Big)^{1/q} < \infty \Big\},$$

$$c_* \|f\|_{B^s_{p,q}} \leq \|f\|_{s,p,q} \leq c^* \|f\|_{B^s_{p,q}},$$

where $c_* > 0$, $c^* > 0$ are constants depending on s, p, q, and the W_n are the modified de la Vallée–Poussin kernels determined by their Fourier coefficients as follows: $W_0 = 1 + z$, and for $n \geq 1$ we have $\hat{W}_n(k) = 1$ for $k = 2^n$, $\hat{W}_n(k) = 0$ for $k \notin (2^{n-1}, 2^{n+1})$, and $\hat{W}_n(k)$ is affine on $[2^{n-1}, 2^n]$ and $[2^n, 2^{n+1}]$; see [BL, P, T] for more details.

The duality of the spaces $B^s_{p,q}$ is given by the relation $(B^s_{p,q})^* = B^{-s}_{p',q'}$ and the Cauchy bilinear form $(f, g) = \sum_{k\geq 0} \hat{f}(k)\hat{g}(k)$ (for $q < \infty$), where $\frac{1}{p} + \frac{1}{p'} = 1$. By a duality constant we mean the smallest $k > 0$ such that

$$\frac{1}{k} \|f\|_{B^s_{p,q}} \leq \sup \big\{ |(f, g)| : \|g\|_{B^{-s}_{p',q'}} \leq 1 \big\} \leq k \|f\|_{B^s_{p,q}},$$

for all $f \in B^s_{p,q}$. In particular, for every operator $L : B^s_{p,q} \longrightarrow B^s_{p,q}$, we have $\|L\| \leq k^2 \|L^*\|$, where $L^* : B^{-s}_{p',q'} \longrightarrow B^{-s}_{p',q'}$ is defined by $(Lf, g) = (f, L^* g)$.

The space $B^s_{p,q}$ is an algebra with respect to pointwise multiplication if and only if $B^s_{p,q} \subset C_A(\mathbb{D}) = \mathrm{Hol}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$, and if and only if either $s > \frac{1}{p}$, or $s = \frac{1}{p}$ and $q = 1$; see [T, 2.6.2]. In this case, there exists a constant $K > 0$ such that

$$\|fg\| \leq K \|f\| \cdot \|g\|$$

for every $f, g \in B^s_{p,q}$. The corresponding multiplier norm,

$$\|f\|_* = \sup\{\|fg\| : \|g\| \leq 1\},$$

is equivalent to $\|\cdot\|$,

$$\frac{1}{\|1\|} \|f\| \leq \|f\|_* \leq K \|f\|,$$

and satisfies the usual Banach algebra properties $\|fg\|_* \leq \|f\|_* \cdot \|g\|_*$, $\|1\|_* = 1$.

From Peetre–Lizorkin's definition, it is obvious that $B^t_{p,q} \subset B^s_{p,q}$ for $t \geq s$, $B^s_{p,q} \subset B^s_{p,r}$ for $r \geq q$, and $B^s_{r,q} \subset B^s_{p,q}$ for $r \geq p$. It is also known (see, for instance, [BL, Theorem 6.5.1]) that $B^s_{p,q} \subset B^t_{P,Q}$ if $1 \leq p \leq P \leq \infty$, $1 \leq q \leq Q \leq \infty$, and $s - \frac{1}{p} = t - \frac{1}{P}$. Finally, it is known that $B^s_{p,q} \subset \mathrm{Lip}(\alpha)$ for $s - \frac{1}{p} > \alpha > 0$ (see [S.Ni, BL, T]).

3.2. The case of Banach space operators. Given a monic polynomial m and a constant $C \geq 1$, we denote by A_C the set of Banach space operators T admitting a calculus over a function algebra A with $C_T \leq C$, and set

$$\Upsilon(m, A_C) = \{T \in A_C : m(T) = 0\}.$$

Theorem 3.4. *Let m be a monic polynomial, A a function algebra, and $C \geq 1$. Then*

$$\|a\|_{A/mA} \leq |a|_{\Upsilon(m,A_C)} \leq C \|a\|_{A/mA}$$

for every $a \in A$.

Proof. The right-hand side inequality follows from Lemma 3.1. For the left inequality, we set $Sf = zf$ for every $f \in A/mA$. Since A/mA is a Banach algebra, it is clear that S is an operator on A/mA and $S \in \Upsilon(m, A_1)$. Moreover, since A is a unital algebra, $\|a(S)\| = \|a\|_{A/mA}$ for every $a \in A$. $\square$

Remark 3.5. *Note that, in fact, there exist many "universally worst" operators T in* $\Upsilon(m, A_C)$, in the sense that the norms $\|a(T)\|$ and $|a|_{\Upsilon(m,A_C)}$ are comparable (and even equal, as in the second example below) simultaneously for all $a \in A/mA$. Here are two of them.

In the previous proof, as a universal operator we chose the quotient S/mA of the shift operator S on A. Clearly, the adjoint of S/mA also satisfies the same property. If the duality between A and A^* is realized by the Cauchy bilinear form, $(f, g) = \sum_{k\geq 0} \hat{f}(k)\hat{g}(k)$ for $f \in A$, $g \in A^*$, then $(S/mA)^* = S^*|(mA)^\perp$, where S^* is the backward shift

$$S^*g = \frac{g - g(0)}{z} \quad \text{and} \quad (mA)^\perp = \operatorname{span}\Big(\frac{1}{1-\lambda_j z} : 0 \leq j < n\Big).$$

The latter n-dimensional space is endowed with the A^*-norm.

Yet another universally worst but less explicit construction is as follows. Consider the direct sum (product) of all operators in $\Upsilon = \Upsilon(m, A_C)$,

$$\mathcal{T} = \bigotimes_{T\in\Upsilon} T : \mathcal{Y} \longrightarrow \mathcal{Y},$$

defined coordinatewise on the direct sum $\mathcal{Y} = \bigoplus_{T\in\Upsilon} Y(T)$, $Y(T)$ being the space where T acts. (The direct sum of any l^p-type is convenient, $1 \leq p \leq \infty$.) Clearly, $\mathcal{T} \in \Upsilon(m, A_C)$ and $\|a(\mathcal{T})\| = |a|_{\Upsilon(m,A_C)}$ for every $a \in A/mA$.

By the way, if we everywhere restrict ourselves to Hilbert space operators only, and take an l^2-type direct sum above, we obtain a "universally worst" Hilbert space operator.

Now, we can apply the above approach to some usual classes of operators.

Corollary 3.6 (Banach space power bounded operators)**.** *Given a number $C \geq 1$ and a polynomial m, $m \neq 0$, let Υ be the set of all Banach space operators satisfying* $\sup_{n\geq 0} \|T^n\| \leq C$ *and* $m(T) = 0$. *Then*

$$\|a\|_{W/mW} \leq |a|_\Upsilon \leq C\|a\|_{W/mW}$$

for every $a \in W$, where

$$W = \Big\{ f = \sum_{n\geq 0} \hat{f}(n)z^n : \|f\|_W = \sum_{n\geq 0} |\hat{f}(n)| < \infty \Big\}$$

is the Wiener algebra on $\mathbb{D}$.

Indeed, Υ obviously obeys a W-calculus with the constant C. Applying Theorem 3.4, we get the result. □

Remark 3.7. For Hilbert space power bounded operators, a better estimate is known; see Remark 3.16 below.

Corollary 3.8 (Banach space contractions)**.** *Given a polynomial m, $m \neq 0$, let Υ be the set of all Banach space contractions satisfying $m(T) = 0$. Then*

$$|a|_\Upsilon = \|a\|_{W/mW}$$

for every $a \in A$.

Indeed, it suffices to take $C = 1$ in the preceding corollary. □

For $a = 1/z$ this corollary is known as the duality formula of Gluskin, Meyer, and Pajor [GMP] (the proof of these authors is different and is adapted specifically for $a = 1/z$). The next two corollaries are immediate consequences of the results of Vitse [Vi1, Vi3] and Peller [Pe2].

Corollary 3.9 (Banach space Kreiss operators)**.** *Given a number $C \geq 1$ and a polynomial m, $m \neq 0$, let Υ be the set of all Banach space operators satisfying $m(T) = 0$ and the Kreiss resolvent condition*

$$\|R(\lambda, T)\| \leq \frac{C}{|\lambda| - 1}, \quad |\lambda| > 1.$$

Then

$$c\|a\|_{B^1_{1,1}/mB^1_{1,1}} \leq |a|_\Upsilon \leq C\|a\|_{B^1_{1,1}/mB^1_{1,1}}$$

for every $a \in B^1_{1,1}$, where $c > 0$ is an absolute constant and $B^1_{1,1}$ stands for the (*analytic*) *Besov algebra* (*see Subsection* 3.1) *endowed with the equivalent norm*

$$\|f\|^*_{B^1_{1,1}} = \frac{2}{\pi} \int \int_{\mathbb{D}} |(z^2 f)''| \, dxdy.$$

Indeed, it was shown in [Vi1] (see also [Pe2]) that Υ obeys a $B^1_{1,1}$-functional calculus with the constant C. Moreover, since $B^1_{1,1}$ is a Banach algebra with respect to an equivalent norm and the multiplication operator $Tf = zf$ on $B^1_{1,1}$ satisfies the Kreiss condition (see [Pe2, Vi1]), again we can apply Theorem 3.4. The result follows (the constant $c > 0$ comes from the Banach algebra norming of $B^1_{1,1}$). □

Corollary 3.10 (Banach space Tadmor–Ritt operators)**.** *Given a number $C \geq 1$ and a monic polynomial $m = m_\sigma$, let Υ be the set of all Banach space operators satisfying $m(T) = 0$ and the Tadmor–Ritt resolvent condition*

$$\sup_{|\lambda|>1} \|R(\lambda, T)\| \cdot |\lambda - 1| = C < \infty;$$

here $\sigma \subset \Gamma_C = \mathrm{conv}(1, \{z \in \mathbb{D} : |z|^2 \leq 1 - \frac{1}{C^2}\})$, a Luzin type cone containing the spectrum $\sigma(T)$ (*see* [Vi2]). *Then*

$$|a|_\Upsilon \leq 300C^5 \|a\|_{B^0_{\infty,1}/mB^0_{\infty,1}}$$

for every $a \in B^0_{\infty,1}$, where $B^0_{\infty,1}$ stands for the (*analytic*) *Besov algebra endowed with the norm $\|\cdot\|_{0,\infty,1}$* (*see Subsection* 3.1 *for the definition*).

Indeed, it was shown in [Vi3] that $\|f(T)\| \leq 300C^5 \|f\|_{0,\infty,1}$ for every Banach space Tadmor–Ritt operator T. □

Remark 3.11. We do not know whether the norms $|\cdot|_\Upsilon$ and $\|\cdot\|_{B^0_{\infty,1}/mB^0_{\infty,1}}$, $m = m_\sigma$, are equivalent or not (uniformly with respect to $\sigma \subset \Gamma_C$). Using the methods of [Vi2, Vi3] we can only show that $|a|_\Upsilon \geq c\|a\|_{B^0_{\infty,1}(\Gamma_C)/mB^0_{\infty,1}(\Gamma_C)}$, where $B^0_{\infty,1}(\Gamma_C)$ is a Besov space over Γ_C.

In Subsection 3.6 below, we give some estimates of the inverses $\|T^{-1}\|$ for Kreiss and Tadmor–Ritt Banach space operators, as well as for Hilbert space power bounded operators (one more class obeying a $B^0_{\infty,1}$-calculus; see Subsection 3.4 below). But first, we consider operators obeying an H^∞-functional calculus (Subsection 3.3) and compare the norms $\|f(T)\|$ of arbitrary functions of an algebraic Kreiss and/or Tadmor–Ritt operator (Subsection 3.4).

3.3. The case of an H^∞-calculus. Let $\Upsilon(m, (C_A)_C)$ be the set of all operators $T : X \longrightarrow X$ obeying an H^∞-calculus of norm $C \geq 1$, that is, $\|a(T)\| \leq C\|a\|_{H^\infty}$ for every $a \in C_A(\mathbb{D})$, and such that $m(T) = 0$. Here, as before, m means a monic polynomial $m = m_\sigma$, or the corresponding Blaschke product $m = B_\sigma$. In particular, every algebraic Hilbert space contraction with $\sigma(T) \subset \sigma$ is in $\Upsilon(m_\sigma, (C_A)_1)$ (von Neumann's inequality). Moreover, as is well known, for a completely nonunitary (c.n.u., for short)

Hilbert space contraction T, $a(T)$ has a meaning for an arbitrary $a \in H^\infty$ (the Sz.-Nagy–Foiaş calculus); see [SzNF] or [N3, N5] for the details. So, given an inner function Θ, we may consider the set $\Upsilon(\Theta, (H^\infty)_C)$ of all Hilbert space operators in $(H^\infty)_C$ such that $\Theta(T) = 0$.

Theorem 3.12. *Let Θ be an inner function. Then*

$$|a|_{\Upsilon(\Theta,(H^\infty)_1)} = \|a\|_{H^\infty/\Theta H^\infty} = \operatorname{dist}_{L^\infty}(a\overline{\Theta}, H^\infty)$$

for every $a \in H^\infty$ (or $a \in H^\infty/\Theta H^\infty$), and the worst operator is $T = M_\Theta$, i.e., $|a|_{\Upsilon(\Theta,(C_A)_1)} = \|a(M_\Theta)\|$, where $M_\Theta f = P_{K_\Theta} zf$, $f \in K_\Theta = H^2 \ominus \Theta H^2$, stands for the so-called model operator with the characteristic function Θ (here P_{K_Θ} is the orthogonal projection onto K_Θ; see [N3, N5] *for the details). Moreover,*

$$\Big|\frac{1}{z}\Big|_{\Upsilon(\Theta,(H^\infty)_1)} = \frac{1}{|\Theta(0)|},$$

where $1/z$ in $H^\infty/\Theta H^\infty$ means any solution a of the Bezout equation $za + \Theta g = 1$ $(a, g \in H^\infty)$. In particular, in the case of $\Theta = m_\sigma/\prod(1-\overline{\lambda}_j z)$ we have

$$\sup\big\{\|T^{-1}\| : T \in \Upsilon(m_\sigma, (C_A)_1)\big\} = \frac{1}{\prod_{\lambda\in\sigma}|\lambda|}.$$

For an arbitrary value of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$, we have

$$\frac{|\Theta(\lambda)|^{-1} - |\lambda|}{1-|\lambda|^2} \le \Big|\frac{1}{\lambda - z}\Big|_{\Upsilon(\Theta,(H^\infty)_1)} \le \frac{|\Theta(\lambda)|^{-1}}{1-|\lambda|}$$

for every $\lambda \in \mathbb{D}$, $\Theta(\lambda) \neq 0$.

Proof. By Theorem 3.4, we only need to calculate $\|a\|_{H^\infty/\Theta H^\infty}$ for the corresponding functions a. From Sarason's lifting theorem we know that $\|a\|_{H^\infty/\Theta H^\infty} = \|a(M_\Theta)\|$ for every $a \in H^\infty$. Now, for the inverse M_Θ^{-1}, we use the fact that the operator M_Θ is an isometry on the orthogonal complement $D^\perp$ of the defect subspace $D = \operatorname{range}(I - M_\Theta^* M_\Theta)$ and the modulus of the restriction $M_\Theta|D$ is equivalent to multiplication by $|\Theta(0)|$; see [SzNF, N3, N5] for the details. Hence, $\|M_\Theta^{-1}\| = \frac{1}{|\Theta(0)|}$.

For the resolvent bounds we use again $|\frac{1}{\lambda-z}|_{\Upsilon(\Theta,(H^\infty)_1)} = \|\frac{1}{\lambda-z}\|_{H^\infty/\Theta H^\infty}$ and refer to an inequality in [N3, p. 122]. □

Remark 3.13. For the sets $\Upsilon(\Theta, (H^\infty)_C)$ or $\Upsilon(\Theta, (C_A)_C)$ with $C > 1$ we must replace all identities by inequalities:

$$\|a\|_{H^\infty/\Theta H^\infty} \le |a|_{\Upsilon(\Theta,(H^\infty)_1)} \le C\|a\|_{H^\infty/\Theta H^\infty}.$$

3.4. The norms $|\cdot|_\Upsilon$ and the Nevanlinna–Pick problem in Besov algebras. Now, after reducing (in some cases) the seminorm $|a|_\Upsilon$ to $\|a\|_{A/mA}$ (see Subsection 3.2), we specify our problem as follows. As above, we denote by Υ a family of Banach space operators, by A a function space on $\mathbb{D}$, and by $m = m_\sigma$ a monic polynomial with simple zeros $\sigma \subset \mathbb{D}$. In the case where $\Upsilon = \Upsilon(m, A_C)$, we want to compare the norms $|a|_\Upsilon$ and $\|a\|_{A/mA}$ with the "size" of the local data $a(\sigma)$. As a "size" of $a(\sigma)$ we choose the weakest function algebra norm, i.e., $\|a\|_{H^\infty/mH^\infty}$. (Note that H^∞ is the largest Banach algebra of holomorphic functions in $\mathbb{D}$.) Therefore, we consider the problem to bound or calculate the following quantities.

(1) For $m = m_\sigma$, a function $a \neq 0$ on a set $\sigma \subset \mathbb{D}$, and a constant $C \ge 1$, let

$$k(a, m, A_C) = \frac{|a|_{\Upsilon(m,A_C)}}{\|a\|_{H^\infty/mH^\infty}}.$$

The case of inverses corresponds to $m(0) \neq 0$ and $a = 1/z$. Notice also that, by Theorem 3.4 above, in the case of a function algebra A we have $k(a, m, A_1) = \frac{\|a\|_{A/mA}}{\|a\|_{H^\infty/mH^\infty}}$.

(2) For a function a defined on a set $\Sigma \subset \mathbb{D}$ and numbers $n \geq 1$, $C \geq 1$, let

$$k_n(a, A_C) = \sup\big\{k(a, m_\sigma, A_C) : \sigma \subset \Sigma,\ \mathrm{card}(\sigma) \leq n\big\}.$$

The case of inverses corresponds to $\Sigma = \mathbb{D}\backslash\{0\}$ and $a = 1/z$. Corollary 3.6 and the definition of the function $C_1(\Delta, \Upsilon)$ (see Introduction) imply

$$C_1(\Delta, \mathcal{B}_n) \leq \Delta^n k_n(1/z, W_1),$$

where $\mathcal{B}_n$ stands for the set of all linear operators on Banach spaces of dimension n. We return to this quantity in Subsection 3.6 below.

(3) For a given finite set $\sigma \subset \mathbb{D}$ and $m = m_\sigma$, let

$$k(m, A_C) = \sup\big\{k(a, m, A_C) : a \in A\big\}.$$

(4) Finally,

$$k_n(A_C) = \sup\big\{k(m, A_C) : \deg(m) \leq n\big\} = \sup\big\{k_n(a, A_C) : a \in A\big\}.$$

Below we compare these quantities for the algebras $B^1_{1,1}, W$, $B^0_{\infty,1}$ that arise for particularly interesting functional calculi. It is known that the following continuous (strict) embeddings hold:

$$B^1_{1,1} \subset W \subset B^0_{\infty,1} \subset H^\infty$$

(see, e.g., [BL, P] or [N2, Section B.8.7]). In fact, $B^0_{\infty,1} \subset C_A(\mathbb{D})$, where $C_A(\mathbb{D}) = C(\overline{\mathbb{D}}) \cap \mathrm{Hol}(\mathbb{D})$ is the disk algebra.

Theorem 3.14. *For every $n \geq 1$ and $C \geq 1$, we have*

$$k_n((B^0_{\infty,1})_C) \leq k_n(W_C) \leq k_n((B^1_{1,1})_{3C/2}) \leq 9Cn,$$

where $B^1_{1,1}$ and $B^0_{\infty,1}$ are equipped with the Peetre–Lizorkin norms. Moreover, $k_n(W_C) \leq 2Cn$.

Conversely,

$$\frac{n}{8} \leq k_n((B^0_{\infty,1})_{10^5}).$$

Proof. We have

$$\begin{aligned}\|a\|_{0,\infty,1} &= \sum_{n\geq 0} \|a * W_n\|_\infty \\ &\leq \sum_{n\geq 0} \|a * W_n\|_W = \sum_{n\geq 0}\sum_{k\geq 0} |\hat{a}(k)|\hat{W}_n(k) = \|a\|_W,\end{aligned}$$

which means that $(B^0_{\infty,1})_C \subset W_C$, and the first inequality follows.

For the second inequality, we simply use the fact that $|\hat{a}(k)|\hat{W}_n(k) \leq \|a * W_n\|_{L^1(\mathbb{T})}$ for every $k, n \in \mathbb{Z}_+$, whence

$$\begin{aligned}\|a\|_W &= \sum_{k\geq 0} |\hat{a}(k)| = \sum_{n\geq 0}\sum_{k\geq 0} |\hat{a}(k)|\hat{W}_n(k) \\ &\leq \sum_{n\geq 0} 3 \cdot 2^{n-1}\|a * W_n\|_{L^1(\mathbb{T})} = \frac{3}{2}\|a\|_{1,1,1}.\end{aligned}$$

Therefore, $W_C \subset (B^1_{1,1})_{3C/2}$.

For the last inequality, it suffices to estimate $\|a\|_{B^1_{1,1}/mB^1_{1,1}}$ for every $m = m_\sigma$ with $\mathrm{card}(\sigma) \leq n$ and for every $a \in B^1_{1,1}$ such that $\|a\|_{H^\infty} \leq 1$. By the Pick theorem (see, for example, [N5, p. 204]), there exists a Blaschke product $B = \prod_{j=0}^{n-1} \frac{w_j - z}{1 - \overline{w}_j z}$ and

$\lambda \in \mathbb{C}$, $|\lambda| \le 1$, such that $a = \lambda B$ on σ, and hence $a = \lambda B$ in $B^1_{1,1}/mB^1_{1,1}$. But from Peller's theorem on the trace-class Hankel operators $\Gamma_B \in \mathfrak{S}_1$ we know that $\|B\|_{1,1,1} = \sum_{n\ge 0} 2^n \|B * W_n\|_{L^1(\mathbb{T})} \le 6\|\Gamma_B\|_{\mathfrak{S}_1} \le 6n$ (see the proof in [N2, pp. 357–359]). Therefore, $\|a\|_{B^1_{1,1}/mB^1_{1,1}} \le 6n\|a\|_{H^\infty/mH^\infty}$.

To get a better bound for $k_n(W_C)$ claimed in the statement, it suffices to show that $\|a\|_{W/mW} \le 2n\|a\|_{H^\infty/mH^\infty}$ for every $a \in W$ and every m, $\deg(m) \le n$, which turns into $\|a\|_{W/mW} \le 2n$ for every $a \in W$, $\|a\|_{H^\infty} \le 1$. We employ a similar argument: by the Pick theorem there exists a Blaschke product $B = \prod_{j=0}^{n-1} \frac{\lambda_j - z}{1-\overline{\lambda}_j z}$ and $\lambda \in \mathbb{C}$, $|\lambda| \le 1$, such that $a = \lambda B$ in W/mW. Therefore, by a Vinogradov–Peller inequality (see [N2, pp. 368, 375]), $\|a\|_{W/mW} \le \|B\|_W \le 2n$.

In order to prove a lower estimate of $k_n((B^0_{\infty,1})_C)$, say $k_n(A_C) \ge kn$ with a constant $k > 0$, we use Corollary 3.10. Then, for every $\epsilon > 0$, it suffices to find an operator $T \in \Upsilon(m, A_C)$, $\deg(m) \le n$, and $a \in A$, $a \ne 0$, such that $\|a(T)\| \ge (k-\epsilon)n\|a\|_{H^\infty/mH^\infty}$. Let $Te_j = \lambda_j e_j$, where $(e_j)_{j=0}^{n-1}$ stands for the standard basis in the space $bv_n = (\mathbb{C}^n, \|\cdot\|_{bv})$ endowed with the variation norm

$$\|x\|_{bv} = |x_0| + \sum_{1\le j<n} |x_j - x_{j-1}|,$$

and let $0 < \lambda_0 < \cdots < \lambda_{n-1} < 1$, $\sigma = (\lambda_j)$. It was shown in [Vi2] that T is a so-called Tadmor–Ritt operator, that is, $\|R(\lambda, T)\| \le \frac{c}{|\lambda-1|}$ for every $|\lambda| > 1$, where $c \le \frac{\pi}{2} + 1$. From [Vi3] we know also that $T \in (B^0_{\infty,1})_C$ with $C \le 300c^5 < 10^5$, where $B^0_{\infty,1}$ is endowed with the Peetre–Lizorkin norm.

Next, choosing $\lambda_j = 1-q^j$ with $0 < q < 1$, we obtain a set σ with the Carleson constant $\delta(\sigma) = \inf_j (1-|\lambda_j|^2)|B'_\sigma(\lambda_j)|$ satisfying $\delta(\sigma) \ge (\prod_{m\ge 1} \frac{1-q^m}{1+q^m})^2$ (see, for example, [N5, p. 160]). By the Jones–Vinogradov theorem ([N5, p. 189] or [N3, p. 179]), there exists a function a (in fact, due to Pick's theorem, a can be chosen to be a finite Blaschke product times a constant) such that $a(\lambda_j) = (-1)^j$, $0 \le j < n$, and $\|a\|_\infty \le 8/\delta(\sigma)^2$. The operator $a(T)$ is a projection in bv_n satisfying $\|a(T)\| \ge n$ (see [Vi2]). Making q sufficiently small, we get $\|a(T)\| \ge (8-\epsilon)n\|a\|_{H^\infty/mH^\infty}$. The result follows. □

Remark 3.15. Here we state explicitly the related estimates for the usual classes of operators. Sometimes, such an estimate follows directly from Theorem 3.14, sometimes the constants are better if we use other sources. Let $C \ge 1$. For the set Υ_1 of operators satisfying the Kreiss condition $\|R(\lambda, T)\| \le \frac{C}{|\lambda|-1}$, $|\lambda| > 1$, we have (see [Vi1])

$$\|a(T)\| \le \frac{16Cn}{\pi}\|a\|_{H^\infty/mH^\infty}$$

for every $T \in \Upsilon_1$ satisfying $m(T) = 0$, $m = m_\sigma$, $\mathrm{card}(\sigma) \le n$, and for every function a on σ.

Let Υ_2 be the set of all Banach space operators with $\sup_{k\ge 0}\|T^k\| \le C$. Then

$$\|a(T)\| \le 2Cn\|a\|_{H^\infty/mH^\infty}$$

for every $T \in \Upsilon_2$ satisfying $m(T) = 0$, $m = m_\sigma$, $\mathrm{card}(\sigma) \le n$, and for every function a on σ. It should be mentioned that a slightly worse inequality with the constant $\pi n+1$ instead of $2n$ was already known to Katsnelson and Matsaev [KM]; see also [N3, C.2.5.4(d)].

For the set Υ_3 of operators satisfying $m(T) = 0$, $\deg(m) \le n$, and the Tadmor–Ritt condition $\sup_{|\lambda|>1}\|R(\lambda, T)\| \cdot |\lambda - 1| = C < \infty$, we get $\sup_{k\ge 0}\|T^k\| \le C^2$ (see [ER]) and hence

$$\|a(T)\| \le 2C^2 n\|a\|_{H^\infty/mH^\infty}$$

for every $T \in \Upsilon_3$ satisfying $m(T) = 0$, $m = m_\sigma$, $\mathrm{card}(\sigma) \le n$, and for every function a on σ.

Moreover, from the proof of Theorem 3.14 it is clear that for every $n \ge 1$ there exists T satisfying $m(T) = 0$, $\deg(m) \le n$ and the Tadmor–Ritt condition with the constant 10^5, and there exists a rational function a of degree $\deg(a) \le n$ such that $\|a(T)\| \ge \frac{n}{8}\|a\|_{H^\infty/mH^\infty}$.

Remark 3.16. Operators defined on a specific Banach space (in particular, on a Hilbert space) and satisfying formally the same resolvent or power conditions may have better functional calculi than operators on an arbitrary Banach space, and hence may admit better estimates for inverses and for other functions of the operator. For instance, Peller [Pe1] proved that

$$\|f(T)\| \le C_T^2 K \|f\|_{0,\infty,1}$$

for every *Hilbert space* power bounded operator T such that $\sup_{n\ge 0}\|T^n\| = C_T < \infty$, where K is an absolute constant and $\|\cdot\|_{0,\infty,1}$ stands for the $B^0_{\infty,1}$-norm. (Compare with an arbitrary Banach space operator of the same class, where we have only $\|f(T)\| \le C\|f\|_W$.) Therefore, for the set $\Upsilon(m, C)$ of Hilbert space power bounded operators satisfying $C_T \le C$ and $m(T) = 0$, Corollary 3.6 can be improved up to the estimate from Corollary 3.10, namely,

$$|a|_{\Upsilon(m,C)} \le C^2 K \|a\|_{B^0_{\infty,1}/mB^0_{\infty,1}}$$

for every $a \in B^0_{\infty,1}$ and $T \in \Upsilon(m, C)$. Just as in the latter corollary, the question of the (uniform) sharpness of this bound can be raised, but here it has a *negative answer* (in contrast to the case of all Banach space operators satisfying a $B^0_{\infty,1}$-calculus; see Theorem 3.4). Indeed, in [Pe1] it was shown that $B^0_{\infty,1}$ is not an operator algebra, that is, there exists no Hilbert space operator T such that $c_1\|f\|_{0,\infty,1} \le \|f(T)\| \le c_2\|f\|_{0,\infty,1}$ for every polynomial f and some constants c_1, $c_2 > 0$ independent of f. Now, if we suppose that there exists $k > 0$ such that

$$|a|_{\Upsilon(m,C)} \ge k\|a\|_{B^0_{\infty,1}/mB^0_{\infty,1}}$$

for every $a \in B^0_{\infty,1}$ and m, then, given a, we can find a polynomial $m = m_{\sigma(a)}$ such that $\|a\|_{B^0_{\infty,1}} \le 2\|a\|_{B^0_{\infty,1}/mB^0_{\infty,1}}$ (the partial sums of the Taylor series of a indexed by 2^n converge to a; hence a polynomial m of degree 2^n with zeros $\sigma(a)$ close to 0 is convenient). Then, we take an operator $T_a \in \Upsilon(m, C)$ such that $\|a(T_a)\| \ge 2^{-1}|a|_{\Upsilon(m,C)}$, and finally, define $T = \bigoplus_a T_a$ acting coordinatewise on the l^2-sum $\bigoplus H(T_a)$ of spaces where the T_a are defined. Then, obviously, T is power bounded and $\|a(T)\| \ge (k/4)\|a\|_{B^0_{\infty,1}}$ for every $a \in B^0_{\infty,1}$, which contradicts Peller's result. □

For the classes of *Hilbert space* Kreiss and Tadmor–Ritt operators, estimates better than those mentioned in Subsection 3.2 are also, probably, possible.

Remark 3.17. For $C = 1$, Theorem 3.14 gives a comparison of the norms $\|a\|_{A/mA}$ and $\|a\|_{H^\infty/mH^\infty}$ for $A = B^1_{1,1}$, W, and $B^0_{\infty,1}$. The maximum of the ratio of these norms, bounded in Theorem 3.14, can be regarded as a distortion coefficient for the Nevanlinna–Pick type interpolation problem for these algebras. In fact, for $A = B^1_{1,1}$, the result is essentially contained in [AFP] (without evaluating the constants). In [VSh], among other results, the authors found two-sided estimates for a similar quantity $\sup\{\|B_\sigma\|_A : \mathrm{card}(\sigma) \le n\}$, where $B_\sigma = \prod_{\lambda\in\sigma} b_\lambda$, $b_\lambda = \frac{\lambda - z}{1-\overline{\lambda}z}$, stands for a finite Blaschke product and $A = B^{1/p}_{p,q}$, $1 \le p, q \le \infty$ (in particular, for $A = B^0_{\infty,1}$). The proofs in those papers are different from ours.

3.5. Condition numbers and analytic capacities. Here we return to the problem of bounding the condition numbers of large matrices, and more generally, the quantities $|\frac{1}{z}|_{\Upsilon(m_\sigma,A_C)}$, $\|\frac{1}{z}\|_{A/m_\sigma A}$, and $k(\frac{1}{z}, m_\sigma, A_C)$ introduced in Subsections 3.1 and 3.4, respectively. More precisely, we study the behavior of these quantities as functions of the spectrum σ and of $\mathrm{card}(\sigma)$. In this subsection, we start with power bounded Banach space operators, $\sup_{n\geq 0}\|T^n\| < \infty$, and the corresponding algebra W (the analytic Wiener algebra). The inverses and the resolvents of such operators are estimated. This will lead to a general definition of the A-capacity cap_A for a given function space A on the disk $\mathbb{D}$. In forthcoming sections we study cap_A for the Besov classes $B^s_{p,q}$ and the weighted Beurling–Sobolev spaces $\mathcal{F}l^q(\mathbb{Z}_+, w_n)$.

Inverses, capacities, and resolvents. The above program forces us to generalize our previous approach to bounding $\mathrm{CN}(T)$ in terms of $\mathrm{SCN}(T)$, as follows. Recall that in order to bound condition numbers in terms of the spectral condition numbers, we introduced the quantities

$$\begin{aligned} C_1(\Delta, \Upsilon) &= \sup\big\{\,\mathrm{CN}(T) : T \in \Upsilon,\ \mathrm{SCN}(T) \leq \Delta\big\} \\ &= \sup\big\{\|T\|\cdot\|T^{-1}\| : T \in \Upsilon,\ \|T\| r(T^{-1}) \leq \Delta\big\}. \end{aligned}$$

It is obvious that for a positively homogeneous set Υ we have

$$C_1(\Delta, \Upsilon) = \sup\big\{\|T^{-1}\| : T \in \Upsilon, \|T\| \leq 1,\ r(T^{-1}) \leq \Delta\big\},$$

and, on the other hand, the contractivity condition $\|T\| \leq 1$ can be written as a calculus condition $T \in W_1$. Since an operator may have a better functional calculus, it seems of interest to study the following functions generalizing $C_1(\Delta, W_1)$ and already mentioned in the Introduction:

$$\begin{aligned} \varphi(m_\sigma, A_C) &= \Big|\frac{1}{z}\Big|_{\Upsilon(m_\sigma,A_C)} = \sup\{\|T^{-1}\| : T \in A_C, m_\sigma(T) = 0\}, \\ \Phi_n(\Delta, A_C) &= \sup\big\{\varphi(m_\sigma, A_C) : \sigma \subset \sigma_{1/\Delta},\ \mathrm{card}(\sigma) \leq n\big\} \\ &= \sup\big\{\|T^{-1}\| : T \in A_C, r(T^{-1}) \leq \Delta,\ \deg(T) \leq n\big\}, \end{aligned}$$

and

$$\underline{\Phi}_n(\Delta, A_C) = \sup\big\{\varphi(m_\sigma, A_C) : \sigma \subset \{z : |z| = 1/\Delta\},\ \mathrm{card}(\sigma) \leq n\big\},$$

where $\sigma_\delta = \{z \in \mathbb{C} : \delta \leq |z| \leq 1\}$, A is a function space in $\mathbb{D}$, and as before, $\Upsilon = \Upsilon(m, A_C)$ stands for the set of Banach space operators satisfying $m(T) = 0$ and $\|f(T)\| \leq C\|f\|_A$ for all polynomials f. In particular,

$$\varphi(m_\sigma, A_C) = \Big\|\frac{1}{z}\Big\|_{H^\infty/m_\sigma H^\infty} k(1/z, m_\sigma, A_C) = \frac{1}{\prod_{j=0}^{n-1}|\lambda_j|} k(1/z, m_\sigma, A_C).$$

Note that the functions φ and Φ_n contain information on the norms of inverses and on condition numbers for families of operators with *separated constraints* on their norms ($T \in A_C$) and spectral data ($r(T^{-1}) \leq \Delta$), whereas $C_1(\Delta, \Upsilon)$ is a homogeneous quantity depending on the spectral condition numbers $\|T\| r(T^{-1}) \leq \Delta$, $T \in \Upsilon$, which combine norm and spectral conditions. Moreover, for the quantity $C_1(\Delta, \Upsilon)$, the functional calculus type restrictions (such as $\Upsilon \subset A_C$) play almost no role, as the following lemma shows.

Lemma 3.18. *Let A be a function space on the disk $\mathbb{D}$ and $C > 0$ a constant such that $C > \sup\{|f(0)| : \|f\|_A \leq 1\}$. Let*

$$\begin{aligned} \Upsilon^1_n &= \big\{T \in L(X) : X \text{ a Banach space},\ \deg(T) \leq n\big\}, \\ \Upsilon^2_n &= \big\{T \in L(X) : X \text{ a Banach space},\ \dim(X) \leq n\big\}. \end{aligned}$$

Then

$$C_1(\Delta, \Upsilon_n^1) = C_1(\Delta, \Upsilon_n^1 \cap A_C) \geq C_1(\Delta, \Upsilon_n^2) = C_1(\Delta, \Upsilon_n^2 \cap A_C)$$

for every $\Delta \geq 1$.

Proof. Clearly,

$$\begin{aligned} C_1(\Delta, \Upsilon_n^1) &\geq C_1(\Delta, \Upsilon_n^2), \\ C_1(\Delta, \Upsilon_n^1) &\geq C_1(\Delta, \Upsilon_n^1 \cap A_C), \\ C_1(\Delta, \Upsilon_n^2) &\geq C_1(\Delta, \Upsilon_n^2 \cap A_C). \end{aligned}$$

To prove that equality occurs in the last two inequalities, observe that, by the hypothesis, there exists $r > 0$ such that $\|f\|_{D(r)} \leq C\|f\|_A$ for every polynomial $f = \sum_{k\geq 0} \hat{f}(k) z^k$, where $\|f\|_{D(r)} = \max_{|z|\leq r} |f(z)|$. Furthermore, by a theorem of Bohr [Bo] (see also [N3, p. 120]),

$$\sum_{k\geq 0} \frac{|\hat{f}(k)| r^k}{3^k} \leq \|f\|_{D(r)}.$$

Therefore, for every operator $T : X \longrightarrow X$ there exists $\epsilon > 0$ such that $\epsilon T \in A_C$. Indeed, if $\|\epsilon T\| \leq \frac{r}{3}$ and f is a polynomial, then

$$\|f(\epsilon T)\| \leq \sum_{k\geq 0} \frac{|\hat{f}(n)| r^k}{3^k} \leq \|f\|_{D(r)} \leq C\|f\|_A.$$

Since $\mathrm{CN}(T) = \mathrm{CN}(\epsilon T)$ and $\mathrm{SCN}(T) = \mathrm{SCN}(\epsilon T)$, we get

$$C_1(\Delta, \Upsilon_n^1) = C_1(\Delta, \Upsilon_n^1 \cap A_C), \quad C_1(\Delta, \Upsilon_n^2) = C_1(\Delta, \Upsilon_n^2 \cap A_C). \qquad \square$$

Now, we link the functions φ and Φ_n with condition numbers.

Lemma 3.19. *In the previous notation, we have*

(1) $\varphi(m_\sigma, A_C) \leq C\|\frac{1}{z}\|_{A/m_\sigma A}$;

(2) $\sup\{\mathrm{CN}(T) : T \in \Upsilon(m_\sigma, A_C)\} \leq C\|z\|_{A/m_\sigma A}\varphi(m_\sigma, A_C)$;

(3) $\sup_{\Delta\geq 1} \frac{\Phi_n(\Delta, A_C)}{\Delta^n} \leq k_n(\frac{1}{z}, A_C)$ (*the last constant was defined at the beginning of Subsection* 3.4).

In the case of a function algebra A, *we have in addition*

(4) $\|\frac{1}{z}\|_{A/m_\sigma A} \leq \varphi(m_\sigma, A_C) \leq C\|\frac{1}{z}\|_{A/m_\sigma A}$;

(5) $\frac{1}{C}\|z\|_{A/m_\sigma A}\varphi(m_\sigma, A_C) \leq \sup\{\mathrm{CN}(T) : T \in \Upsilon(m_\sigma, A_C)\} \leq C\|z\|_{A/m_\sigma A}\varphi(m_\sigma, A_C)$;

(6)

$$\begin{aligned} \frac{1}{C}\Phi_n(\Delta, A_C) &\leq \sup\{\mathrm{CN}(T) : T \in A_C, r(T^{-1}) \leq \Delta, \deg(T) \leq n\} \\ &\leq C\|z\|_A \Phi_n(\Delta, A_C). \end{aligned}$$

Proof. Items (1) and (2) are straightforward consequences of the definitions. Item (3) follows from

$$\begin{aligned} k_n\Big(\frac{1}{z}, A_C\Big) &= \sup\{|m_\sigma(0)|\varphi(m_\sigma, A_C) : \mathrm{card}(\sigma) \leq n\} \\ &= \sup_{\Delta\geq 1} \sup\{|m_\sigma(0)|\varphi(m_\sigma, A_C) : \sigma \subset \sigma_{1/\Delta},\ \mathrm{card}(\sigma) \leq n\} \\ &\geq \sup_{\Delta\geq 1} \frac{\Phi_n(\Delta, A_C)}{\Delta^n}. \end{aligned}$$

For the left-hand side inequalities in (4) and (5), we use (1) and the quotient operator $T = z/m_\sigma A$ on $A/m_\sigma A$.

For the left-hand side inequality in (6), note that every function Banach algebra is contractively embedded in H^∞, whence $1 = \|z\|_{H^\infty/mH^\infty} \le \|z\|_{A/mA} \le \|z\|_A$ for every m with $m(0) \neq 0$. Therefore, (6) follows from (5). □

Below we bound the functions Φ_n and φ in terms of a certain capacity related to the space A. The capacities themselves are studied in the next section. But first we consider a special case already mentioned. The facts equivalent to the following theorem are contained in [GMP] and [VSh], but we give a new short proof using an idea of Nazarov [Na]. Notice that the same trick of smoothing an interpolating function $f_{\{\lambda_k\}}(z)$ taking given values at $z = \lambda_k$ by passing to $f_{\{r\lambda_k\}}(rz)$ was also used in [Vi2], for different purposes.

Theorem 3.20. *Given a number $C \ge 1$ and a monic polynomial $m = m_\sigma$, $\sigma = \{\lambda_0, \dots, \lambda_{n-1}\}$, $m(0) \neq 0$, let $\Upsilon = \Upsilon(m, W_C)$ be the set of all Banach space operators satisfying $\sup_{n\ge 0} \|T^n\| \le C$ and $m(T) = 0$. Then*

$$\sup_{T\in\Upsilon} \|T^{-1}\| = \Big|\frac{1}{z}\Big|_\Upsilon \le C\Big\|\frac{1}{z}\Big\|_{W/mW} \le \frac{C\sqrt{e(n+1)}}{\prod_{j=0}^{n-1}|\lambda_j|},$$

or equivalently, $\Phi_n(\Delta, W_C) \le \Delta^n C\sqrt{e(n+1)}$, $k_n(\frac{1}{z}, W_C) \le C\sqrt{e(n+1)}$. In particular,

$$\mathrm{CN}(T) \le C^2\sqrt{e(n+1)}(r(T^{-1}))^n$$

for every $T \in \Upsilon(m, W_C)$.

Proof. Given a function $f \in H^2$ and a number $0 < r < 1$, we denote $f_r = f(rz) = \sum_{k\ge 0} \hat{f}(k) r^k z^k$ and observe that

$$\|f_r\|_W \le \Big(\sum_{k\ge 0} |\hat{f}(k)|^2\Big)^{1/2}\Big(\frac{1}{1-r^2}\Big)^{1/2} \le \|f\|_{H^\infty}\Big(\frac{1}{1-r^2}\Big)^{1/2}.$$

Now, let $B = \prod_{j=0}^{n-1} \frac{\lambda_j r - z}{1-\overline{\lambda}_j rz}$ be a Blaschke product and $f = B/B(0) = \frac{1}{r^n m(0)}B$. Then $f_r \in W$, $f_r(\lambda_j) = 0$ for $0 \le j < n$, and $f_r(0) = 1$. Consequently (see also Lemma 3.2),

$$\Big\|\frac{1}{z}\Big\|_{W/mW} \le \Big\|\frac{1-f_r}{z}\Big\|_W = \|f_r\|_W - 1 \le \frac{1}{r^n|m(0)|}\cdot\Big(\frac{1}{1-r^2}\Big)^{1/2} - 1.$$

Taking $r^2 = 1 - \frac{1}{n+1}$ and using $(\frac{n+1}{n})^n \le e$, we get the first of the inequalities claimed.

The second inequality follows from the first. □

Remark 3.21. The norms $\|\frac{1}{z}\|_{A/m_\sigma A}$, where A is a function space (and, in particular, the norm $\|\frac{1}{z}\|_{W/m_\sigma W}$), can be viewed as a kind of "A-capacity" of a set $\sigma \subset \mathbb{D}\backslash\{0\}$.

Namely, let

$$\mathrm{cap}_A^*(\sigma) = \inf\{\|g\|_A : g|\sigma = 1, g(0) = 0\},$$

and let $\mathrm{cap}_A(\sigma)$ be the *zero A-capacity*, that is,

$$\mathrm{cap}_A(\sigma) = \inf\{\|g\|_A : g|\sigma = 0, g(0) = 1\}.$$

If A is included in the space of continuous functions on $\overline{\mathbb{D}}$, $A \subset C(\overline{\mathbb{D}})$, then the above capacity $\mathrm{cap}_A(\sigma)$ is still well defined for sets in the closed unit disk $\sigma \subset \overline{\mathbb{D}}\backslash\{0\}$.

Next, let

$$\kappa_n(\Delta, A) = \sup\big\{\,\mathrm{cap}_A(\sigma) : \sigma \subset \sigma_{1/\Delta},\ \mathrm{card}(\sigma) \le n\big\},$$
$$\underline{\kappa}_n(\Delta, A) = \sup\big\{\,\mathrm{cap}_A(\sigma) : \sigma \subset \{z : |z| = 1/\Delta\},\ \mathrm{card}(\sigma) \le n\big\},$$

where, as before, $\Delta > 1$, $\sigma_\delta = \{z \in \mathbb{C} : \delta \le |z| \le 1\}$, and finally

$$\kappa_n^*(A) = \sup\big\{|m_\sigma(0)|\,\mathrm{cap}_A(\sigma) : \mathrm{card}(\sigma) \le n\big\}.$$

The following lemma shows how these capacities are related to the functions φ, Φ_n, k_n introduced previously. The first assertion of the lemma is contained in Remark 3.3 above, and the second through fourth easily follow from the definitions and the same arguments as in Lemma 3.19 above. All these properties are stated for sets σ in $\mathbb{D}\backslash\{0\}$, but if $A \subset C(\overline{\mathbb{D}})$, then they extend easily to sets σ in $\overline{\mathbb{D}}\backslash\{0\}$.

Lemma 3.22. (1) *For every function space on $\mathbb{D}$ and every set $\sigma \subset \mathbb{D}\backslash\{0\}$, we have*

$$\frac{1}{\|S\|}\operatorname{cap}_A^*(\sigma) \le \Big\|\frac{1}{z}\Big\|_{A/mA} \le \|S^*\|\operatorname{cap}_A(\sigma),$$

and if the shift S is an isometry on A, then

$$\Big\|\frac{1}{z}\Big\|_{A/mA} = \operatorname{cap}_A^*(\sigma).$$

Moreover,

$$|\operatorname{cap}_A(\sigma) - \operatorname{cap}_A^*(\sigma)| \le \|1\|_A;\ \ \operatorname{cap}_W(\sigma) = 1 + \operatorname{cap}_W^*(\sigma) \quad \text{for } A = W.$$

(2) $\varphi(m_\sigma, A_C) \le C\|S^*\|\operatorname{cap}_A(\sigma)$ *and* $\Phi_n(\Delta, A_C) \le C\|S^*\|\kappa_n(\Delta, A)$, *and in the case of a Banach algebra A we have*

$$\frac{1}{\|S\|}\operatorname{cap}_A^*(\sigma) \le \varphi(m_\sigma, A_C) \le C\|S^*\|\operatorname{cap}_A(\sigma),$$

$$\frac{1}{\|S\|}(\kappa_n(\Delta, A) - 1) \le \Phi_n(\Delta, A_C) \le C\|S^*\|\kappa_n(\Delta, A).$$

(3) *Similarly,*

$$k\Big(\frac{1}{z}, m_\sigma, A_C\Big) = |m_\sigma(0)|\varphi(m_\sigma, A_C) \le C\|S^*\| \cdot |m_\sigma(0)|\operatorname{cap}_A(\sigma)$$

and

$$k_n\Big(\frac{1}{z}, A_C\Big) \le C\|S^*\|\kappa_n^*(A),$$

and in the case of a Banach algebra A, we have

$$\frac{|m_\sigma(0)|}{\|S\|}\operatorname{cap}_A^*(\sigma) \le k\Big(\frac{1}{z}, m_\sigma, A_C\Big) \le C\|S^*\| \cdot |m_\sigma(0)|\operatorname{cap}_A(\sigma),$$

$$\frac{1}{\|S\|}(\kappa_n^*(A) - 1) \le k_n\Big(\frac{1}{z}, A_C\Big) \le C\|S^*\|\kappa_n^*(A).$$

(4) $\underline{\kappa}_n(\Delta, A) \le \kappa_n(\Delta, A) \le \Delta^n\kappa_n^*(A)$ *for every* $\Delta > 1$. □

Notice that the capacity cap_A^* is defined in the spirit of classical potential theory (see, e.g., Wermer [W]), whereas the capacity cap_A is closely related to the problem of uniqueness sets for A. Namely, assume that a function space A satisfies the following Fatou property: if $f_n \in A$, $\sup_n \|f_n\|_A < \infty$ and $\lim_n f_n(z) = f(z)$ for $z \in \mathbb{D}$, then $f \in A$. Then an infinite subset $\sigma \subset \mathbb{D}\backslash\{0\}$ is a uniqueness set for A (i.e., $f \in A$, $f|\sigma = 0 \implies f = 0$) if and only if

$$\sup\{\operatorname{cap}_A(\sigma') : \sigma' \text{ finite}, \sigma' \subset \sigma\} = \infty.$$

In particular, this criterion applies to $A = W$, where the problem of uniqueness sets is open. Finally, we explicitly indicate the bounds for cap_W obtained in Theorem 3.20 (see the proof) and Theorem 3.12.

Corollary 3.23. *For every $\sigma = \{\lambda_0, \dots, \lambda_{n-1}\} \subset \mathbb{D}\backslash\{0\}$, we have*

$$\frac{1}{\prod_{j=0}^{n-1}|\lambda_j|} \le \operatorname{cap}_W(\sigma) \le \frac{\sqrt{e(n+1)}}{\prod_{j=0}^{n-1}|\lambda_j|}.$$

Both bounds are attained (up to a constant) for subsets $\sigma = \{\lambda_0, \dots, \lambda_{n-1}\}$ of a circle $C_\delta = \{|\lambda| = \delta\}$. Namely, the left-hand side inequality is sharp on C_δ for every $0 < \delta < 1$, and the right-hand side inequality is sharp on C_δ (at least) for $\delta = 1 - \frac{c}{n}$, $c > 0$.

Indeed, for the sharpness of the left-hand side inequality, we simply set $g = 1 - \frac{z^n}{\delta^n}$ and $\sigma = \{\delta e^{2\pi i k/n} : 0 \le k < n\}$. Then

$$\mathrm{cap}_W(\sigma) \le \|g\|_W = 1 + \frac{1}{\delta^n} = 1 + \frac{1}{\prod_{j=0}^{n-1} |\lambda_j|}.$$

The sharpness statement for the right-hand side inequality is a striking result of Queffelec [Q]. □

Clearly, the left-hand side determinant of the last corollary, $\mathrm{cap}_{H^\infty}(\sigma) = \prod_{j=0}^{n-1} \frac{1}{|\lambda_j|}$, corresponds to the Blaschke uniqueness condition. One more point to notice about the relationships between condition numbers and uniqueness theorems is a kind of discrepancy between them. Namely, for the needs of the uniqueness theorems, the above capacities are interesting in the case where $\lim_n(\max_{0\le k<n} |\lambda_k|) = 1$, and in the framework of the theory of condition numbers, in the case where $\lim_n(\min_{0\le k<n} |\lambda_k|) = 0$. In particular, for a function algebra A, the asymptotics of $\kappa_n(\Delta, A)$ as $\Delta \longrightarrow \infty$ describes the worst case behavior of the norms of inverses and the condition numbers for operators from A_C as the lower bound of the spectrum tends to zero. Unfortunately, the last corollary contains only partial information about such behavior for $A = W$, giving only $\Delta^n \le \kappa_n(\Delta, W) \le \sqrt{e(n+1)}\Delta^n$. For Besov spaces $A = B^s_{p,q}$, we shall obtain some better estimates.

We finish this section with an estimate of the resolvent of an algebraic power bounded operator.

Theorem 3.24. *In the notation of Theorem 3.20, suppose $T \in \Upsilon = \Upsilon(m, W_C)$, $\lambda \in \mathbb{D}\backslash\sigma$, and $R(\lambda, T) = (\lambda I - T)^{-1}$ is the resolvent of T. Then*

$$\|R(\lambda, T)\| \le \frac{2Ce^{3/2}\sqrt{n}}{(1-|\lambda|)^{3/2}} \cdot \frac{1}{|B(\lambda)|},$$

where $B(\lambda) = \prod_{k=0}^{n-1} b_{\lambda_k}(\lambda)$, $b_\mu(z) = \frac{\mu - z}{1-\bar{\mu}z} \cdot \frac{|\mu|}{\mu}$, is the corresponding Blaschke product. For $|\lambda| > 1$, we obviously have $\|R(\lambda, T)\| \le \frac{C}{|\lambda|-1}$.

Proof. We follow the same method as in Theorem 3.20. Namely, $\|R(\lambda, T)\| \le |\frac{1}{\lambda - z}|_\Upsilon \le C\|\frac{1}{\lambda - z}\|_{W/mW}$ and

$$\begin{aligned}\Big\|\frac{1}{\lambda - z}\Big\|_{W/mW} &= \inf\Big\{\|f\|_W : f(\lambda_j) = \frac{1}{\lambda - \lambda_j},\ 0 \le j < n\Big\} \\ &= \inf\Big\{\Big\|\frac{g}{\lambda - z}\Big\|_W : g(\lambda_j) = 1, 0 \le j < n,\ g(\lambda) = 0\Big\}.\end{aligned}$$

It is known and easy to check that $\frac{g}{\lambda - z} = S^*(I - \lambda S^*)^{-1} g$, where S^* is the backward shift operator on W (see above). Since $\|(I - \lambda S^*)^{-1}\| = (1 - |\lambda|)^{-1}$, we obtain

$$\Big\|\frac{1}{\lambda - z}\Big\|_{W/mW} \le \frac{1}{1-|\lambda|} \inf\{\|g\|_W : g(\lambda_j) = 1, 0 \le j < n, g(\lambda) = 0\}.$$

Denoting

$$\mathrm{cap}_{W,\lambda}(\sigma) = \inf\big\{\|g\|_W : g(\lambda_j) = 0, 0 \le j < n, g(\lambda) = 1\big\},$$

we get

$$\Big\|\frac{1}{\lambda - z}\Big\|_{W/mW} \le \frac{1}{1-|\lambda|}(1 + \mathrm{cap}_{W,\lambda}(\sigma)).$$

Let $B = \prod_{j=0}^{n-1} \frac{\lambda_j r - z}{1-\overline{\lambda}_j r z}$ be the same Blaschke product as in Theorem 3.20, and let $h(z) = B(rz)$. Then $h \in W$, $h(\lambda_j) = 0$ for $0 \le j < n$, and

$$\|h\|_W \le \frac{1}{(1-r^2)^{1/2}}.$$

Setting $g = h/B(r\lambda)$, we obtain

$$\mathrm{cap}_{W,\lambda}(\sigma) \le \|g\|_W \le \frac{1}{(1-r^2)^{1/2}|B(r\lambda)|}.$$

Next,

$$\frac{1}{B(r\lambda)} = r^{-n} \prod_{j=0}^{n-1} \frac{1-\overline{\lambda}_j r^2 \lambda}{\lambda_j - \lambda} = \frac{1}{r^n B(\lambda)} \prod\nolimits_1,$$

$$\mathrm{cap}_{W,\lambda}(\sigma) \le \frac{|\prod_1|}{(1-r^2)^{1/2} r^n |B(\lambda)|},$$

where

$$\prod\nolimits_1 = \prod_{j=0}^{n-1} \tfrac{1-\overline{\lambda}_j r^2 \lambda}{1-\overline{\lambda}_j \lambda} = \prod_{j=0}^{n-1} \Big(1 + \tfrac{\overline{\lambda}_j \lambda (1-r^2)}{1-\overline{\lambda}_j \lambda}\Big).$$

Since $|\prod_1| \le (1 + \frac{1-r^2}{1-|\lambda|})^n$, we can set $1 - r^2 = (1-|\lambda|)/n$ and get

$$\mathrm{cap}_{W,\lambda}(\sigma) \le \frac{e}{((1-|\lambda|)/n)^{1/2} r^n |B(\lambda)|},$$

where $r^n = (1 - \frac{1-|\lambda|}{n})^{n/2} \ge \exp(-(1-|\lambda|)/2) \ge e^{-1/2}$. Finally,

$$\mathrm{cap}_{W,\lambda}(\sigma) \le \Big(\frac{e^3 n}{1-|\lambda|}\Big)^{1/2} \frac{1}{|B(\lambda)|},$$

$$\|R(\lambda, T)\| \le \frac{C}{1-|\lambda|}\Big(1 + \Big(\frac{e^3 n}{1-|\lambda|}\Big)^{1/2} \frac{1}{|B(\lambda)|}\Big)$$

$$\le \frac{2C}{1-|\lambda|}\Big(\frac{e^3 n}{1-|\lambda|}\Big)^{1/2} \frac{1}{|B(\lambda)|}. \qquad \square$$

Remark 3.25. Recall that for an algebraic polynomially bounded operator T such that $\|f(T)\| \le C\|f\|_\infty$ for every polynomial f, there exists a better estimate $\|R(\lambda, T)\| \le \frac{C}{(1-|\lambda|)|B(\lambda)|}$, see Theorem 3.12. Concerning the estimate for $\mathrm{cap}_{W,\lambda}$ in the last theorem, it should be mentioned that it cannot be an automatic "conformal transfer" of an estimate for $\mathrm{cap}_W = \mathrm{cap}_{W,0}$, since the composition map $f \longrightarrow f \circ b_\lambda$ is bounded on W only for $\lambda = 0$, see [K, GrMcG].

3.6. $B^s_{p,q}$-capacities and operators obeying a $B^s_{p,q}$-functional calculus. Here we are looking for estimates of inverses and condition numbers for operators obeying a calculus over a Besov space $B^s_{p,q}$ with $s \ge 0$, $1 \le p$, $q \le \infty$. To this end, we establish upper and lower bounds for capacities and functions κ_n corresponding to the spaces $B^s_{p,q}$. Having in mind the behavior of condition numbers as the spectral minimum tends to 0 and/or the size of the matrices tends to infinity, we pay special attention to $\kappa_n(\Delta, B^s_{p,q})$ as $\Delta \longrightarrow \infty$ and/or $n \longrightarrow \infty$.

A space $B^s_{p,q}$ is not always an algebra with respect to pointwise multiplication; see the references in Subsection 3.1 above. However, whatever s, p, q may be, if an operator T satisfies $\|f(T)\| \le C\|f\|_{B^s_{p,q}}$ for all polynomials f, we still speak about a $B^s_{p,q}$-calculus for T. See also Remark 3.27 and Lemma 3.28 below for a discussion of generalized calculi over spaces that are not function algebras.

The next theorem contains some upper bounds for the $B^s_{p,q}$ capacities. For the case where $s > 0$, we show estimates for $\Delta^{-n}\kappa_n(\Delta, B^s_{p,q})$ and $|m_\sigma(0)|\,\mathrm{cap}_{B^s_{p,q}}(\sigma)$, and hence for

$\Delta^{-n}\Phi_n(\Delta, B^s_{p,q})$ and $k_n(\frac{1}{z}, (B^s_{p,q})_C)$, which depend mostly on s and asymptotically are $\mathcal{O}(n^s)$. For $s = 0$, the same estimates depend on q and asymptotically are $\mathcal{O}((\log n)^{1/q})$. The case of $B^0_{\infty,1}$ is especially interesting in view of the functional calculi (discussed above) for Hilbert space power bounded operators and Banach space Tadmor–Ritt operators. It also follows that for $s = 0$ and $q = \infty$ the quantities $k_n(\frac{1}{z}, (B^s_{p,q})_C)$ are uniformly bounded for $n \geq 1$, which is not surprising because $H^\infty \subset B^0_{p,\infty}$ for every p, $1 \leq p \leq \infty$; see also Lemma 3.28 below. A statement equivalent to the case of $s > 0$ in this theorem (without evaluating the constants) can be found in [VSh], but our proof is shorter. We state the result for subsets $\sigma \subset \mathbb{D}\backslash\{0\}$, but if $B^s_{p,q} \subset C_A(\mathbb{D})$ (see Subsection 3.1 above), then the capacity is well defined for $\sigma \subset \overline{\mathbb{D}}\backslash\{0\}$ and the theorem is still true in this case.

Theorem 3.26. *Let $A = (B^s_{p,q}, \|\cdot\|_{s,p,q})$, where $s \geq 0$, $1 \leq p, q \leq \infty$, and $\sigma = \{\lambda_0, \dots, \lambda_{n-1}\} \subset \mathbb{D}\backslash\{0\}$. Then*

$$\operatorname{cap}_A(\sigma) \leq \gamma \frac{(n+1)^s}{|m_\sigma(0)|} = \gamma \frac{(n+1)^s}{\prod_{j=0}^{n-1} |\lambda_j|},$$

where

$$\gamma \leq \frac{3e2^s}{\log 2} \cdot \Big\{ \min\Big((\log n)^{1/q}, \frac{1}{(sq)^{1/q}} \Big) + c\Big(\frac{s}{e}\Big)^s + 2 \Big\}$$

and $c > 0$ is an absolute constant. In particular,

$$\gamma \leq \gamma(s,q) = \frac{3e2^s}{\log 2} \cdot \Big\{ \frac{1}{(sq)^{1/q}} + c\Big(\frac{s}{e}\Big)^s + 2 \Big\}$$

for $s > 0$, and

$$\gamma \leq \frac{3e}{\log 2} \cdot \{(\log n)^{1/q} + 2\} \leq \frac{9e}{\log 2} \cdot (\log(ne))^{1/q}$$

for $s = 0$.

Proof. We use the same test function as in the proof of Theorem 3.20. Namely, let $B = \prod_{j=0}^{n-1} \frac{\lambda_j r - z}{1 - \overline{\lambda}_j r z}$, $0 < r < 1$, be a Blaschke product, and let $f = B/B(0) = \frac{1}{r^n m(0)} B$. Then $f_r \in B^s_{p,q}$, $f_r(0) = 1$, and $f_r(\lambda_j) = 0$ for $0 \leq j < n$. Hence, $\operatorname{cap}_A(\sigma) \leq \|f_r\|_{s,p,q}$. Now, for $r = 1 - \frac{1}{n+1}$,

$$\begin{aligned}
\|f_r\|^q_{s,p,q} &= \sum_{2^{k-1} \leq n} 2^{ksq} \|f_r * W_k\|^q_{L^p(\mathbb{T})} + \sum_{2^{k-1} > n} 2^{ksq} \|f_r * W_k\|^q_{L^p(\mathbb{T})} \\
&\leq \sum_{2^{k-1} \leq n} 2^{ksq} \Big(\frac{3}{2r^n |m(0)|} \Big)^q \\
&\quad + \sum_{2^{k-1} > n} 2^{ksq} \|r^{2^{k-1}} z^{2^{k-1}} (z^{-2^{k-1}} f)_r * (z^{-2^{k-1}} W_k)\|^q_{L^p(\mathbb{T})} \\
&\leq 2^{sq} \frac{n^{sq} 2^{sq} - 1}{2^{sq} - 1} \cdot \Big(\frac{3e}{2|m(0)|} \Big)^q \\
&\quad + 2^{sq} \sum_{2^{k-1} \leq n} 2^{(k-1)sq} \Big(1 - \frac{1}{n+1} \Big)^{2^{k-1} q} \Big(\frac{3e}{2|m(0)|} \Big)^q \\
&\leq \Big(\frac{3e2^s}{2|m(0)|} \Big)^q \{(\alpha + 2) n^{sq} + S\},
\end{aligned}$$

where

$$\alpha \leq \frac{1}{\log 2} \cdot \min\Big(\log n, \frac{2}{sq} \Big)$$

and $S = \sum_{2^j > n} 2^{jsq} e^{-\frac{2^j q}{n+1}}$. Since the function $x \longmapsto xs - \frac{2^x}{n+1}$ decreases for $2^x \geq n+1$ if $s \leq 1$ and has precisely one extremum (maximum) at $2^x = s(n+1)$ if $s > 1$, we obtain

$$S \leq \int_{2^x \geq n+1} 2^{xsq} e^{-\frac{2^x q}{n+1}}\, dx + (s(n+1))^{sq} e^{-sq} = I + \Big(\frac{s(n+1)}{e}\Big)^{sq}$$

for $s > 1$, and $S \leq I$ for $s \leq 1$. Moreover, setting $y = 2^x/(n+1)$, we get

$$I = \frac{(n+1)^{sq}}{\log 2} \int_1^\infty y^{sq-1} e^{-yq}\, dy.$$

Therefore, if $sq \leq 1$, we have

$$S \leq I \leq \frac{(n+1)^{sq}}{q \cdot \log 2} = \beta_1 (n+1)^{sq},$$

and finally

$$\|f_r\|_{s,p,q} \leq \gamma_1 \frac{(n+1)^s}{|m(0)|},$$

where

$$\begin{aligned}
\gamma_1 &\leq 3e2^{s-1}\{\alpha + \beta_1 + 2\}^{1/q} \leq 3e2^{s-1}\{\alpha^{1/q} + \beta_1^{1/q} + 2^{1/q}\} \\
&\leq 3e2^{s-1}\Big\{\frac{1}{(\log 2)^{1/q}} \cdot \min\Big((\log n)^{1/q}, \frac{2^{1/q}}{(sq)^{1/q}}\Big) + \frac{1}{(q \cdot \log 2)^{1/q}} + 2^{1/q}\Big\} \\
&\leq \frac{3e2^s}{\log 2} \cdot \Big\{\min\Big((\log n)^{1/q}, \frac{1}{(sq)^{1/q}}\Big) + 2\Big\}.
\end{aligned}$$

If $sq > 1$, we have

$$\begin{aligned}
I &\leq \frac{(n+1)^{sq}}{\log 2} \int_0^\infty y^{sq-1} e^{-yq}\, dy = \frac{(n+1)^{sq}}{q^{sq} \log 2} \int_0^\infty t^{sq-1} e^{-t}\, dt \\
&= \frac{(n+1)^{sq} \Gamma(sq)}{q^{sq} \log 2},
\end{aligned}$$

where Γ stands for the Euler Γ-function,

$$S \leq \frac{(n+1)^{sq}\Gamma(sq)}{q^{sq} \log 2} + \Big(\frac{s(n+1)}{e}\Big)^{sq} = \beta_2 (n+1)^{sq},$$

and finally

$$\|f_r\|_{s,p,q} \leq \gamma_2 \frac{(n+1)^s}{|m(0)|},$$

where

$$\begin{aligned}
\gamma_2 &\leq 3e2^{s-1}\{\alpha + \beta_2 + 2\}^{1/q} \\
&\leq 3e2^{s-1}\Big\{\frac{1}{(\log 2)^{1/q}} \cdot \min\Big((\log n)^{1/q}, \frac{2^{1/q}}{(sq)^{1/q}}\Big) \\
&\qquad + \frac{\Gamma(sq)^{1/q}}{q^s (\log 2)^{1/q}} + \Big(\frac{s}{e}\Big)^s + 2^{1/q}\Big\}.
\end{aligned}$$

Using the Stirling formula, we observe that

$$\Gamma(sq)^{1/q} \leq \Big(a\Big(\frac{sq}{e}\Big)^{sq} \sqrt{2\pi sq}\Big)^{1/q} \leq bq^s (s/e)^s$$

for all $sq \geq 1$, where $a > 0$, $b > 0$ are absolute constants. Thus,

$$\gamma_2 \leq \frac{3e2^s}{\log 2} \cdot \Big\{\min\Big((\log n)^{1/q}, \frac{1}{(sq)^{1/q}}\Big) + c\Big(\frac{s}{e}\Big)^s + 2\Big\},$$

where $c > 0$. Taking $\gamma = \max(\gamma_1, \gamma_2) = \gamma_2$, we complete the proof. $\square$

Remark 3.27. In the case where the space $B^s_{p,q}$ is not an algebra with respect to pointwise multiplication, quite a special situation occurs for an operator obeying a $B^s_{p,q}$-calculus. Namely, if this is the case (see conditions on s, p, q above), then one can check that $\lim_{|\lambda|\longrightarrow+1}(|\lambda|-1)\|\frac{1}{\lambda-z}\|_{B^s_{p,q}} = 0$ and hence $\lim_{|\lambda|\longrightarrow+1}(|\lambda|-1)\|\frac{1}{\lambda-z}\|_{s,p,q} = 0$. Now, if $T \in (B^s_{p,q})_C$, then, using the inequalities

$$\frac{1}{\operatorname{dist}(\lambda, \sigma(T))} \le \|R(\lambda, T)\| \le C\Big\|\frac{1}{\lambda - z}\Big\|_{s,p,q},$$

we obtain the following numerical bound for the spectral radius $r(T)$:

$$r(T) \le \rho = 1 - \sup\Big\{\frac{1}{C\|\frac{1}{\lambda-z}\|_{s,p,q}} - (|\lambda|-1)\Big\} < 1.$$

This means that in the nonalgebra case, a $B^s_{p,q}$-calculus imposes some implicit constraints on the spectral data, namely $|\lambda_j| \le \rho$, $0 \le j < n$.

The above situation can be generalized as follows. Let X be a function space on the disk $\mathbb{D}$ (see Subsection 3.1), let $C > 0$ be a constant, and let $\Upsilon(X_C)$ be the set of all operators T obeying an X-calculus with constant C, i.e., such that $\|f(T)\|_X \le C\|f\|_X$ for every polynomial f. Let $0 \le r < 1$ and

$$M_X(r) = \sup\{|f(z)| : |z| \le r,\ \|f\|_X \le 1\}.$$

The following lemma shows that the operators in $\Upsilon(X_C)$ also obey a functional calculus defined on a Banach algebra which often is much larger than X.

Lemma 3.28. *Let X be a Banach function space on $\mathbb{D}$ not included in H^∞, and assume that X is rotation invariant: $f \in X \implies f_\zeta \in X$ and $\|f_\zeta\|_X = \|f\|_X$, where $\zeta \in \mathbb{T}$ and $f_\zeta(z) = f(z\zeta)$. Then M_X is a continuous strictly increasing function on $[0, 1)$ and $\lim_{r\longrightarrow 1} M_X(r) = \infty$.*

Moreover, if $C > M_X(0)$ and $r = M_X^{-1}(C)$, then $0 < r < 1$, and the seminorm $|\cdot|_{\Upsilon(X_C)}$ (see Subsection 3.1) is a Banach algebra norm defined on polynomials and such that

$$\|f(T)\| \le |f|_{\Upsilon(X_C)} \le C\|f\|_X$$

for every $T \in \Upsilon(X_C)$ and every polynomial f, and

$$\|f\|_{H^\infty(D_r)} \le |f|_{\Upsilon(X_C)} \le c_\epsilon\|f\|_{H^\infty(D_{r+\epsilon})}$$

for every polynomial f and every $\epsilon > 0$, where $D_r = \{z \in \mathbb{C} : |z| \le r\}$ and c_ϵ is a constant.

Proof. The claimed properties of M_X follow from the maximum modulus principle and the definition of a function space on $\mathbb{D}$.

The estimate for $\|f(T)\|$ is obvious. Furthermore, if $C > M_X(0)$, then $D_r \subset \{\lambda \in \mathbb{C} : |f(\lambda)| \le C\|f\|_X \quad \text{for every polynomial } f\}$. The reverse inclusion follows from the rotational symmetry of X: if $|f(\lambda)| \le C\|f\|_X$ for every polynomial f, then $\max_{|z|=|\lambda|} |f(z)| \le C\|f\|_X$ and hence $\lambda \in D_r$. Therefore,

$$D_r = \{\lambda \in \mathbb{C} : |f(\lambda)| \le C\|f\|_X \text{ for every polynomial } f\}.$$

Consequently, $T = \lambda I \in \Upsilon(X_C)$ for every $\lambda \in D_r$. This implies the left-hand side inequality for $|f|_{\Upsilon(X_C)}$.

In order to prove the right-hand side inequality, observe first that by the spectral mapping theorem $|f(\lambda)| \le \|f(T)\| \le C\|f\|_X$ for every $T \in \Upsilon(X_C)$, every $\lambda \in \sigma(T)$, and every polynomial f. This and the previous formula for D_r mean that $\sigma(T) \subset D_r$. Next, let $\mathcal{T} = \bigoplus_{T\in\Upsilon(X_C)} T$ be the direct sum of all operators in $\Upsilon(X_C)$ acting coordinatewise on the l^∞-type direct sum $\bigoplus_{l^\infty} Y$, where $Y = Y(T)$ stands for a Banach space on which the operator $T \in \Upsilon(X_C)$ is defined. Obviously, $\mathcal{T} \in \Upsilon(X_C)$ and therefore $\sigma(\mathcal{T}) \subset D_r$.

Now, the standard Riesz–Dunford calculus implies that for every $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ such that

$$|f|_{\Upsilon(X_C)} = \|f(\mathcal{T})\| \le c_\epsilon \|f\|_{H^\infty(D_{r+\epsilon})}$$

for every polynomial f. □

Example 3.29. Applying Lemma 3.28 to the Hardy space $X = H^p$, we get $M_X(r) = \frac{1}{(1-r^2)^{1/p}}$, whence $r = (1 - \frac{1}{C^p})^{1/2}$; see also [N4] for other similar estimates of spectral radii.

Upper estimates for capacities similar to those of Theorem 3.26 exist for some other spaces, e.g., for the Beurling–Sobolev spaces $l^q_A(w_k) = \mathcal{F}l^q(\mathbb{Z}, w_k)$,

$$l^q_A(w_k) = \Big\{ f = \sum_{k\ge 0} \hat{f}(k) z^k : \|f\|_{l^q_A(w_k)} = \Big(\sum_{k\ge 0} |\hat{f}(k)|^q w_k^q \Big)^{1/q} \Big\},$$

where $w_k > 0$. We assume that X is a function space in $\mathbb{D}$, in particular, $\lim_k w_k^{1/k} = 1$. For conditions for $l^q_A(w_k)$ to be an algebra with respect to pointwise multiplication, see [N6] or [ENZ]. For instance, if the function $k \longmapsto w_k$ is sufficiently "regular" (example: $k \longmapsto \log(k^s w_k)$ is concave or convex for every $s \in \mathbb{R}$), then $l^q_A(w_k)$ is an algebra if and only if $l^q_A(w_k) \subset H^\infty(\mathbb{D})$, and if and only if $(1/w_k) \in l^{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$, see [ENZ]. The $l^q_A(w_k)$-capacities are bounded in terms of the so-called Lagrange transforms of the sequence $w = (w_k)$:

$$L_\alpha(w, r) = \Big(\sum_{k\ge 0} r^{k\alpha} w_k^\alpha \Big)^{1/\alpha}, \quad 0 < r < 1.$$

Theorem 3.30. *Let $X = l^q_A(w_k)$, where $w_k > 0$, $1 \le q \le \infty$, and let $\sigma = \{\lambda_0, \dots, \lambda_{n-1}\} \subset \mathbb{D}\backslash\{0\}$. Then*

$$\mathrm{cap}_X(\sigma) \le \frac{\gamma_q(n)}{|m_\sigma(0)|} = \frac{\gamma_q(n)}{\prod_{j=0}^{n-1} |\lambda_j|},$$

where

$$\gamma_q(n) = \inf_{0<r<1} r^{-n} L_\alpha(r),$$

and $\alpha = \infty$ for $q \ge 2$, $\alpha = \frac{2q}{2-q}$ for $1 \le q < 2$. In particular, for $w_k = k^\beta$, $k \ge 1$ ($w_0 = 1$), we have $\gamma_q(n) \le n^\beta$ for $q \ge 2$ and $\gamma_q(n) \le a n^{\beta + \frac{1}{q} - \frac{1}{2}} + b$ for $1 \le q < 2$, where $a, b > 0$ are constants depending on q and β.

Proof. We use the same test function as in Theorem 3.20. Namely, let $B = \prod_{j=0}^{n-1} \frac{\lambda_j r - z}{1 - \overline{\lambda}_j r z}$ be a Blaschke product, and let $f = B/B(0) = \frac{1}{r^n m(0)} B$. Then $f_r \in X$, $f_r(\lambda_j) = 0$ for $0 \le j < n$, and $f_r(0) = 1$. Hence,

$$\mathrm{cap}_A(\sigma) \le \|f_r\|_X.$$

Now, for $q \ge 2$ we have

$$\begin{aligned} \|f_r\|_X &\le \Big(\sum_{k\ge 0} |\hat{f}(k)|^2 r^{2k} w_k^2 \Big)^{1/2} \le \sup_{k\ge 0} (r^k w_k) \|f\|_2 \\ &= \frac{1}{r^n |m_\sigma(0)|} L_\infty(w, r). \end{aligned}$$

For $1 \le q < 2$, by Hölder,

$$\|f_r\|_X = \Big(\sum_{k\ge 0} |\hat{f}(k)|^q r^{kq} w_k^q\Big)^{1/q}$$
$$\le \|f\|_2 \Big(\sum_{k\ge 0} r^{k\alpha} w_k^\alpha\Big)^{1/\alpha} = \frac{1}{r^n |m_\sigma(0)|} L_\alpha(w,r),$$

where $\alpha = \frac{2q}{2-q}$. The result follows.

For $w_k = k^\beta$ and $q \ge 2$, we have

$$L_\infty(r) = \sup_{k\ge 0} r^k w_k \le \max\Big(1, \sup_{t>0} e^{-t\log(1/r)} t^\beta\Big)$$

and

$$\sup_{t>0} e^{-t\log(1/r)} t^\beta = e^{-\beta}\Big(\beta/\log\frac{1}{r}\Big)^\beta$$

(and the right-hand side is equivalent to $L_\infty(r)$ as $r \longrightarrow 1$). Therefore, $\gamma_q(n) \le n^\beta$ (and $\lim_{n\longrightarrow\infty} n^{-\beta}\gamma_q(n) = 1$).

For $1 \le q < 2$, we have $L_\alpha(r) \le \max(1, I(r)^{1/\alpha})$, where

$$I(r) = \int_0^\infty e^{-\alpha t\log(1/r)} t^{\alpha\beta}\,dt = \frac{1}{\big(\alpha\log\frac{1}{r}\big)^{\alpha\beta+1}} \int_0^\infty e^{-t} t^{\alpha\beta}\,dt$$
$$= \frac{\Gamma(\alpha\beta+1)}{\big(\alpha\log\frac{1}{r}\big)^{\alpha\beta+1}} = \frac{c^\alpha}{\big(\log\frac{1}{r}\big)^{\alpha\beta+1}},$$

with $c > 0$ defined by the last equation. Hence,

$$\gamma_q(n) \le \inf_{0<r<1} \frac{cr^{-n}}{\big(\log\frac{1}{r}\big)^{\beta+(1/\alpha)}}$$
$$= ce^{\beta+(1/\alpha)}\Big(\frac{n}{\beta+\frac{1}{\alpha}}\Big)^{\beta+\frac{1}{\alpha}}. \qquad \square$$

Now, we show a lower bound for the $B^s_{p,q}$- capacities ($s > 0$), having the same order as the upper bound in Theorem 3.26 but only for $\sigma = (\lambda_j)$ with sufficiently small $|\lambda_j|$. Next, we apply these estimates for bounding condition numbers and the norms of inverses.

Theorem 3.31. *Let $\sigma = \{\lambda_0, \dots, \lambda_{n-1}\} \subset \mathbb{D}\backslash\{0\}$, $s > 0$, $1 \le p, q \le \infty$ and $A = (B^s_{p,q}, \|\cdot\|_{s,p,q})$. Then*

$$\mathrm{cap}_A(\sigma) \ge c\frac{(n+1)^s}{\prod_{0\le j<n}|\lambda_j|}\Big(1 + \frac{c_*}{c^* k^2} - \prod_{0\le j<n}(1+|\lambda_j|)\Big),$$

where $c = \frac{c^* k^2}{2^{s+1} c_*}$, the constants c_*, c^* come from the equivalence of the norms $\|\cdot\|_{s,p,q}$ and $\|\cdot\|_{B^s_{p,q}}$, and $k > 0$ is the duality constant for $(B^s_{p,q}, \|\cdot\|_{B^s_{p,q}})^* = B^{-s}_{p',q'}$ (or $(B^{-s}_{p',q'})^* = B^s_{p,q}$ if $q = \infty$); see Subsection 3.1.

Proof. Let $g \in B^s_{p,q}$, $g(0) = 1$, and let $f = g\prod_{0\le j<n}(z - \lambda_j)$. Then, denoting $\|f\| = \|f\|_{s,p,q}$, we obtain

$$\|f\| = \Big\|z^n g + \sum_{i=0}^{n-1} z^i g \sum_{j_l \ne j_m} (-1)^{n-i}\lambda_{j_1}\cdots\lambda_{j_{n-i}}\Big\|.$$

First, observe that

$$\|z^i g\| \le c^*\|z^i g\|_{B^s_{p,q}} = c^*\|S^{*(n-i)}(z^n g)\|_{B^s_{p,q}}.$$

Then, using the duality $(B^s_{p,q})^* = B^{-s}_{p',q'}$ (or $(B^{-s}_{p',q'})^* = B^s_{p,q}$, if $q=\infty$) and the duality constant $k>0$ (see Subsection 3.1), we arrive at the inequality

$$\|S^{*(n-i)}(z^n g)\|_{B^s_{p,q}} \le \|S^{*(n-i)}\| \cdot \|z^n g\|_{B^s_{p,q}} \le k^2 \|S^{n-i}\| \cdot \|z^n g\|_{B^s_{p,q}},$$

where $\|S^{n-i}\|$ stands for the norm of S^{n-i} on the space $(B^{-s}_{p',q'}, \|\cdot\|_{B^{-s}_{p',q'}})$. With $s>0$, we have

$$\|f\|_{B^{-s}_{p',q'}} = \Big(\int_0^1 ((1-r)^{s-\frac{1}{q'}} \|f_r\|_{L^{p'}(\mathbb{T})})^{q'}\, dr\Big)^{1/q'},$$

whence $\|S^{n-i}\| \le 1$. Therefore,

$$\begin{aligned}
\|z^i g\| &\le \frac{c^* k^2}{c_*} \|z^n g\|,\\
\|f\| &\ge \|z^n g\| - \sum_{i=0}^{n-1} \frac{c^* k^2}{c_*} \|z^n g\| \sum_{j_l \ne j_m} |\lambda_{j_1} \cdots \lambda_{j_{n-i}}|\\
&= \|z^n g\| \Big(1 + \frac{c^* k^2}{c_*} - \frac{c^* k^2}{c_*} \prod_{0\le j<n} (1+|\lambda_j|)\Big).
\end{aligned}$$

Moreover,

$$\|z^n g\| \ge \|z^n g\|_{s,1,\infty} = \sup_{k\ge 0} 2^{ks} \|W_k * (z^n g)\|_{L^1}.$$

If $2^k < n+1 \le 2^{k+1}$, then $\hat{W}_k(n) + \hat{W}_{k+1}(n) = 1$, and $\max(\hat{W}_k(n), \hat{W}_{k+1}(n)) \ge 1/2$; let $\hat{W}_k(n) \ge 1/2$. Then

$$\|z^n g\| \ge 2^{ks} \|W_k * (z^n g)\|_{L^1} \ge 2^{ks} \hat{W}_k(n) \hat{g}(0) \ge 2^{ks-1} \ge \frac{(n+1)^s}{2^{s+1}}.$$

The result follows. □

The following capacity bounds are straightforward consequences of Theorems 3.26 and 3.31.

Corollary 3.32. *Let $A = (B^s_{p,q}, \|\cdot\|_{s,p,q})$. For $s>0$, we have*

$$\begin{aligned}
\frac{k}{2^{s+1}}(n+1)^s &\le \varliminf_{\Delta\longrightarrow\infty} \frac{\underline{\kappa}_n(\Delta, A)}{\Delta^n} \le \varliminf_{\Delta\longrightarrow\infty} \frac{\kappa_n(\Delta, A)}{\Delta^n}\\
&\le \sup_{\Delta>1} \frac{\kappa_n(\Delta, A)}{\Delta^n} \le \kappa^*_n(A) \le \gamma \cdot (n+1)^s,
\end{aligned}$$

where $\underline{\kappa}_n(\Delta, A)$ and $\kappa^*_n(A)$ are the quantities defined immediately before Lemma 3.22, $k>0$ is the duality constant for $(B^s_{p,q}, \|\cdot\|_{B^s_{p,q}})^* = B^{-s}_{p',q'}$ (or $(B^{-s}_{p',q'})^* = B^s_{p,q}$ if $q=\infty$; see Subsection 3.1),

$$\gamma \le \gamma(s,q) = \frac{3e2^s}{\log 2} \cdot \Big\{\frac{1}{(sq)^{1/q}} + c\Big(\frac{s}{e}\Big)^s + 2\Big\},$$

and $c>0$ is an absolute constant.

For $s = 0$ we have

$$\sup_{\Delta>1} \frac{\kappa_n(\Delta, A)}{\Delta^n} \le \kappa_n^*(A) \le \frac{9e}{\log 2} \cdot (\log(ne))^{1/q}. \qquad \square$$

Remark 3.33. Here are two comments on the lower estimate of Theorem 3.26.

First, the estimate obtained starts to matter for circular sets $\sigma \subset \{z : |z| = 1/\Delta\}$ when Δ and n are related, say, by

$$\frac{n}{\Delta} \le \log\Big(1 + \frac{c_*}{2c^* k^2}\Big),$$

where c_*, c^*, k depend on s, p, q. A lower estimate similar to those of Theorem 3.26 was also claimed in [VSh, Theorem 3.2]. However, in [VSh], the limit case was only considered, where σ is replaced by a degenerate family $\sigma = \{\lambda_j\}$ with $\lambda_1 = \cdots = \lambda_n = 0$ (i.e., the case of a zero of multiplicity n at the origin). (By the way, the proof in [VSh] contains a snag, making use of an inclusion $B^s_{p,q} \subset B^s_{1,1}$, which is not true for $q > 1$. However, as the authors of [VSh] mentioned (private communication), the use of this inclusion is not necessary and the proof goes well if we simply forget it. The fact itself claimed in [VSh] is true and easy to check; namely, if $g(0) = 1$ and $2^k < n + 1 \le 2^{k+1}$, then, as in the proof of Theorem 3.26 above, $\|z^n g\|_{s,p,q} \ge 2^{ks}\|(z^n g) * W_k\|_p \ge 2^{ks-1} \ge (n+1)^s/2^{s+1}$.)

Second, we comment on the case of $s = 0$. In this case, when trying to show that the upper bound in Theorem 3.26/Corollary3.32 is sharp, one must seek a more sophisticated lower estimate for $\mathrm{cap}_{B^s_{p,q}}$ than those of Theorem 3.31. For example, it is easily seen that for uniformly distributed sets $\sigma = \{\lambda_0, \dots, \lambda_{n-1}\}$, $\lambda_j = \delta e^{2\pi i j/n}$, $0 < \delta < 1$, the $B^0_{p,q}$-capacity behaves as the H^∞-capacity. Indeed, setting $F = 1 - \frac{z^n}{\delta^n}$, we get $F(0) = 1$ and $\|F\|_{0,p,q} = (1 + \delta^{-nq}\|z^n\|^q_{0,p,q})^{1/q}$, which gives

$$\mathrm{cap}_{B^0_{p,q}}(\sigma) \le 1 + \frac{1}{\delta^n} = 1 + \frac{1}{\prod_{0\le j<n} |\lambda_j|}.$$

It is quite clear that this (counter)example works for every function space X such that $\sup_{n\ge0} \|z^n\|_X < \infty$ (in particular, for $X = W$). In this case, in order to get, nevertheless, a nontrivial lower estimate of $\underline{\kappa}_n(\Delta, X)$ (for $X = B^0_{p,q}$ the optimal lower bound would be $\underline{\kappa}_n(\Delta, X) \ge c(\log(n))^{1/q}\Delta^n$), we need specifically distributed λ_j, $|\lambda_j| = 1/\Delta$. For $X = W$, one of such distributions was exhibited by Queffelec [Q] when he showed that

$$\sup_{\Delta>1} \frac{\underline{\kappa}_n(\Delta, W)}{\Delta^n} \ge \sqrt{e^{-1}n};$$

see also the next remark. One may expect that Queffelec type distributions of λ_j, $|\lambda_j| = \delta$, could give the required lower bound also for $\mathrm{cap}_{B^0_{p,q}}(\sigma)$ and $\mathrm{cap}_X(\sigma)$ for other spaces X satisfying $\sup_{n\ge0} \|z^n\|_X < \infty$.

Remark 3.34. Returning to the case of the Wiener algebra W and power bounded operators $T \in W_C$, we can restate Theorem 3.20 and Corollary 3.23 (see also Lemma 3.22) as follows:

$$\sqrt{e^{-1}n} \le \sup_{\Delta>1} \frac{\underline{\kappa}_n(\Delta, W)}{\Delta^n} \le \sup_{\Delta>1} \frac{\kappa_n(\Delta, W)}{\Delta^n} \le \kappa_n^*(W) \le \sqrt{e(n+1)},$$

$$\sqrt{e^{-1}n} - 1 \le \sup_{\Delta>1} \Delta^{-n}\underline{\Phi}_n(\Delta, W_C) \le \sup_{\Delta>1} \Delta^{-n}\Phi_n(\Delta, W_C) \le C\sqrt{e(n+1)}.$$

It was also shown in [Q] (in an equivalent form) that

$$\varliminf_{n\longrightarrow\infty} \Big(\frac{1}{\sqrt{n}} \sup_{\Delta>1} \frac{\underline{\kappa}_n(\Delta, W)}{\Delta^n}\Big) \ge 1.$$

On the contrary, asymptotics for $\frac{\underline{\kappa}_n(\Delta,W)}{\Delta^n}$ or $\frac{\kappa_n(\Delta,W)}{\Delta^n}$ as $\Delta \longrightarrow \infty$, or as $n \longrightarrow \infty$ (for a fixed $\Delta > 1$) are not studied.

Corollary 3.32 and Lemmas 3.22 and 3.19 yield the following estimates of inverses and condition numbers for operators obeying a $B^s_{p,q}$-functional calculus.

Corollary 3.35. *Let $A = (B^s_{p,q}, \|\cdot\|_{s,p,q})$, where $s \geq 0$, $1 \leq p, q \leq \infty$, and $C \geq 1$.*

I. *Let $s > 0$. Then*

(1) $\Phi_n(\Delta, A_C) = \sup\{\|T^{-1}\| : T \in A_C, r(T^{-1}) \leq \Delta, \deg(T) \leq n\} \leq C\gamma\|S^*\|\Delta^n(n+1)^s$ *and*

$$\sup\Big\{\operatorname{CN}(T) : T \in A_C, r(T^{-1}) \leq \Delta, \deg(T) \leq n\Big\} \leq C^2\gamma\|z\|_A\|S^*\|\Delta^n(n+1)^s$$

for every $\Delta > 1$ *and* $n \geq 1$, *where γ is a constant occurring in Corollary* 3.32.

(2) *If A is an algebra with respect to pointwise multiplication* (*see Subsection* 3.1), *and hence $\|fg\|_{s,p,q} \leq K\|f\|_{s,p,q}\|g\|_{s,p,q}$ for every $f, g \in A$, and if $C \geq K$, then, moreover,*

$$\frac{k}{\|S\|2^{s+1}}(n+1)^s \leq \varliminf_{\Delta\longrightarrow\infty} \frac{\underline{\Phi}_n(\Delta, A)}{\Delta^n} \leq \varliminf_{\Delta\longrightarrow\infty} \frac{\Phi_n(\Delta, A)}{\Delta^n},$$

where $k > 0$ has the same meaning as in Theorem 3.31, *and*

$$\begin{aligned}\frac{k}{C\|S\|2^{s+1}}&(n+1)^s\\ &\leq \varliminf_{\Delta\longrightarrow\infty} \Delta^{-n} \sup\big\{\operatorname{CN}(T) : T \in A_C, \sigma(T) \subset \{z : |z| = 1/\Delta\}, \deg(T) \leq n\big\}.\end{aligned}$$

(3) *Similarly, for*

$$k_n\Big(\frac{1}{z}, A_C\Big) = \sup\{|m_\sigma(0)| \cdot \|T^{-1}\| : T \in A_C,\ m_\sigma(T) = 0,\ \operatorname{card}(\sigma) \leq n\},$$

we have

$$k_n\Big(\frac{1}{z}, A_C\Big) \leq C\gamma\|S^*\|(n+1)^s,$$

and if A is an algebra, a lower bound is

$$\frac{k}{\|S\|2^{s+1}}(n+1)^s \leq k_n\Big(\frac{1}{z}, A_C\Big) + \frac{1}{\|S\|}.$$

II. *Let $s = 0$. Then*

$$\begin{aligned}\Phi_n(\Delta, A_C) &\leq C\|S^*\|\frac{9e}{\log 2} \cdot \Delta^n(\log(ne))^{1/q},\\ \sup\Big\{\operatorname{CN}(T) &: T \in A_C, r(T^{-1}) \leq \Delta, \deg(T) \leq n\Big\}\\ &\leq C^2\gamma\|z\|_A\|S^*\|\Delta^n(\log(ne))^{1/q},\\ k_n\Big(\frac{1}{z}, A_C\Big) &\leq C\gamma\|S^*\|\frac{9e}{\log 2} \cdot (\log(ne))^{1/q}.\end{aligned}$$

Indeed, all upper estimates are immediate consequences of Lemma 3.19 and Theorems 3.26 and 3.31. For the lower estimates, assume that A is an algebra with $\|fg\|_{s,p,q} \leq K\|f\|_{s,p,q}\|g\|_{s,p,q}$ for every $f, g \in A$, and let $\|\cdot\|_*$ be an equivalent Banach algebra norm on A (see Subsection 3.1 above). Next, let $\sigma \subset \mathbb{D}\backslash\{0\}$, $\operatorname{card}(\sigma) \leq n$, and let $T : A_*/m_\sigma A_* \longrightarrow A_*/m_\sigma A_*$ be the quotient multiplication operator $T(f + m_\sigma A_*) = zf + m_\sigma A_*$, where $A_* = (A, \|\cdot\|_*)$. Then

$$\|\varphi(T)\| = \|\varphi\|_{A_*/m_\sigma A_*} \leq \|\varphi\|_* \leq K\|\varphi\|$$

for every polynomial φ, whence $T \in A_C$ (if $C \geq K$). Hence, by Lemma 3.22,

$$\varphi_n(\sigma, A_C) \geq \Big\|\frac{1}{z}\Big\|_{A_*/m_\sigma A_*} \geq \frac{1}{\|S\|_{A_*\longrightarrow A_*}}(\operatorname{cap}_{A_*}(\sigma) - 1).$$

Moreover, by the definition of cap and the inequality $\|\cdot\|_* \geq \|\cdot\|_A/\|1\|_A$, we have $\mathrm{cap}_{A_*}(\sigma) \geq \frac{1}{\|1\|_A}\,\mathrm{cap}_A(\sigma)$. Since $\|1\|_{s,p,q} = 1$ for every s, p, q, and $\|S\|_{A_* \longrightarrow A_*} = \|z\|_{A_*} = \|S\|$, we obtain

$$\varphi_n(\sigma, A_C) \geq \frac{1}{\|S\|}(\mathrm{cap}_A(\sigma) - 1).$$

Passing to the supremum over $\sigma \subset \sigma_{1/\Delta}$, or, respectively, over $\sigma \subset \{z : |z| = 1/\Delta\}$, we get

$$\Phi_n(\Delta, A_C) \geq \frac{1}{\|S\|}(\kappa_n(\Delta, A) - 1)$$

and

$$\underline{\Phi}_n(\Delta, A_C) \geq \frac{1}{\|S\|}(\underline{\kappa}_n(\Delta, A) - 1).$$

Now, the required lower estimates follow from Corollary 3.32, and similarly for the lower bound for k_n. □

3.7. The Beurling–Carleson capacity. In [C], Carleson described the uniqueness sets for the spaces $C_A^{(\alpha)}(\mathbb{D})$,

$$C_A^{(\alpha)}(\mathbb{D}) = \{f \in \mathrm{Hol}(\mathbb{D}) : |f^{([\alpha])}(z) - f^{([\alpha])}(z')| \leq C_f|z - z'|^{\alpha - [\alpha]}\},$$

where $\alpha > 0$ and $[\alpha]$ stands for the integral part of α. Namely, a set $\sigma \subset \overline{\mathbb{D}}$ is NOT a uniqueness set for $C_A^{(\alpha)}(\mathbb{D})$ if and only if $\sigma \cap \mathbb{D}$ satisfies the Blaschke condition $\sum_{\lambda\in\sigma}(1 - |\lambda|) < \infty$ and $\int_{\mathbb{T}} \log \frac{1}{\mathrm{dist}(t,\sigma)}\, dm(t) < \infty$, where m is the normalized Lebesgue measure on $\mathbb{T}$. Moreover, in [GrN] the following quantity $\mathrm{Carl}(\sigma)$ was introduced (in an equivalent form):

$$\mathrm{Carl}(\sigma) = \Big(\prod_{\lambda\in\sigma} \frac{1}{|\lambda|}\Big) \exp\Big(\int_{\mathbb{T}} \log \frac{2}{\mathrm{dist}(t,\sigma)}\, dm(t)\Big),$$

and it was shown that there exist functions φ_α and ψ_α on $[1, \infty)$ such that $1 \leq \varphi_\alpha(x) \leq \lim_{x\longrightarrow\infty} \varphi_\alpha(x) = \infty$ (and similarly for ψ_α) and

$$\varphi_\alpha(\mathrm{Carl}(\sigma)) \leq \mathrm{cap}_{C_A^{(\alpha)}}(\sigma) \leq \psi_\alpha(\mathrm{Carl}(\sigma))$$

for every $\sigma \subset \{z : 1/2 \leq |z| \leq 1\}$. (General capacities $\mathrm{cap}_X(\sigma)$ were also introduced in [GrN].) Presumably, the functions $\varphi_\alpha(x)$ and $\psi_\alpha(x)$ behave at ∞ as $\mathrm{const} \cdot x^\alpha$, but this seems to be not completely clear (in [GrN] it was shown that $\psi_\alpha(x) \leq \mathrm{const} \cdot x^\alpha$). The same Beurling–Carleson condition characterizes the nonuniqueness sets for many other spaces. For instance, for the Besov spaces $B^s_{p,q}$, $s > \frac{1}{p}$, this follows from the well-known embedding theorems (see Subsection 3.1). For the spaces $B^{1/p}_{p,1}$, $1 \leq p < \infty$ (which are not included in any $C_A^{(\alpha)}(\mathbb{D})$), this was proved by Shirokov [Sh].

Although the case where $\sigma \subset \{z : 1/2 \leq |z| \leq 1\}$ is not a priority setting for the theory of condition numbers, it is of interest to know the asymptotics of the circular capacity $\underline{\kappa}_n(\Delta, X)$ as $n \longrightarrow \infty$ and/or $\Delta \longrightarrow 1$. The following theorem and corollaries contain some information on such asymptotics for general spaces of smooth functions as well as for the spaces $B^s_{p,q}$ and $l^q_A(k^\beta)$.

Theorem 3.36. *Let X be a function space in $\mathbb{D}$ such that $C_A^{(\alpha)}(\mathbb{D}) \subset X \subset C_A^{(\beta)}(\mathbb{D})$ for some $\alpha, \beta > 0$.*

(1) *There exist two increasing functions φ_X, ψ_X on $[1, \infty)$ such that $1 \leq \varphi_\alpha(x) \leq \lim_{x\longrightarrow\infty} \varphi_\alpha(x) = \infty$ (and similarly for ψ_α) and*

$$\varphi_X(\mathrm{Carl}(\sigma)) \leq \mathrm{cap}_X(\sigma) \leq \psi_X(\mathrm{Carl}(\sigma))$$

for every $\sigma \subset \{z : 1/2 \leq |z| \leq 1\}$.

(2) *Let σ be an n-point set sitting on the circle $|\lambda| = 1/\Delta$, and $\sigma_* = \sigma_*(\Delta, n)$ an n-point equidistributed set, $\sigma_* = \{e^{2\pi i j/n}/\Delta : 0 \le j < n\}$, where $1 < \Delta \le 2$. Then*

$$\mathrm{Carl}(\sigma) \le \mathrm{Carl}(\sigma_*),$$

$$2\frac{\Delta^n}{((1-1/\Delta)^2 + (\pi/n)^2)^{1/2}} \le \mathrm{Carl}(\sigma_*) \le \frac{\pi^2 e}{4}\frac{\Delta^n}{((1-1/\Delta)^2 + (\pi/n)^2)^{1/2}},$$

$$a\frac{n\Delta^n}{1+n(\Delta-1)} \le \mathrm{Carl}(\sigma_*) \le b\frac{n\Delta^n}{1+n(\Delta-1)},$$

where $n \ge 2$ and $a, b > 0$ are numerical constants; therefore

$$\frac{2n}{\pi} \le \sup_{1<\Delta\le 2} \frac{\mathrm{Carl}(\sigma_*)}{\Delta^n} = \lim_{\Delta\longrightarrow 1} \frac{\mathrm{Carl}(\sigma_*)}{\Delta^n} \le \frac{\pi e n}{4}.$$

(3) *Consequently,*

$$\varphi_X\Big(\frac{2n}{\pi}\Big) \le \varliminf_{\Delta\longrightarrow 1} \underline{\kappa}_n(\Delta, X) \le \varlimsup_{\Delta\longrightarrow 1} \underline{\kappa}_n(\Delta, X) \le \psi_X\Big(\frac{\pi e n}{4}\Big),$$

$$\varphi_X\Big(a\frac{n\Delta^n}{1+n(\Delta-1)}\Big) \le \kappa_n(\Delta, X) \le \psi_X\Big(b\frac{n\Delta^n}{1+n(\Delta-1)}\Big)$$

for every $\Delta, 1 < \Delta \le 2$,

$$\varphi_X\Big(\frac{a}{(1+A)(1+B)}e^A n\Big) \le \underline{\kappa}_n(\Delta, X) \le \psi_X(be^B n)$$

for $1 + \frac{A}{n} \le \Delta \le 1 + \frac{B}{n}$, where $A < B$, and

$$\varphi_X\Big(\frac{a\Delta^n}{(1+c)\epsilon}\Big) \le \underline{\kappa}_n(\Delta, X) \le \psi_X\Big(\frac{b\Delta^n}{\epsilon}\Big)$$

for $\frac{1}{n} \le \epsilon \le \Delta - 1 \le c\epsilon$, where $0 < \epsilon < 1$.

Proof. (1) Since the embeddings $C_A^{(\alpha)}(\mathbb{D}) \subset X \subset C_A^{(\beta)}(\mathbb{D})$ are continuous, we can take $\varphi_X = c_1\varphi_\beta$ and $\psi_X = c_2\psi_\alpha$ with constants coming from these embeddings, where φ_α, ψ_α are the functions defined immediately before the theorem.

(2) Let $\sigma = \{\lambda_j : 0 \le j < n\}$ be an n-point subset of a circle $|z| = 1/\Delta$, $1 < \Delta \le 2$. Then

$$\mathrm{Carl}(\sigma) = \Delta^n \exp\Big(\sum_{0\le j<n} \int_{I(j)} \log\frac{2}{|t-\lambda_j|}\, dm(t)\Big),$$

where the $I(j)$ are pairwise disjoint arcs of the unit circle $\mathbb{T}$ such that $\bigcup_j I(j) = \mathbb{T}$ and $\mathrm{dist}(t, \sigma) = |t - \lambda_j|$ for $t \in I(j)$. Clearly, there is a strictly decreasing function f on $[0, \pi]$ that does not depend on σ and is such that $0 \le f(\theta) \le \log\frac{2\Delta}{\Delta-1}$ and

$$\int_{I(j)} \log\frac{2}{|t-\lambda_j|}\, dm(t) = \frac{1}{\pi}\int_0^{\pi m(I(j))} f(\theta)\, d\theta$$

for every j. The case of the equidistributed set σ_* corresponds to arcs $I(j)$ of the same length, $m(I(j)) = 1/n$ for every j. Setting

$$\Sigma(\sigma) = \sum_{0\le j<n} \int_0^{\pi m(I(j))} f(\theta)\, d\theta$$

and assuming that there exist two adjacent arcs with $a = \pi m(I(j)) < b = \pi m(I(j+1))$, we get

$$\int_0^a f(\theta)\, d\theta + \int_0^b f(\theta)\, d\theta < \int_0^{(a+b)/2} f(\theta)\, d\theta + \int_0^{(a+b)/2} f(\theta)\, d\theta.$$

This shows that the only sets σ where the maximum $\max_\sigma \Sigma(\sigma)$ (over all n-point sets σ) is attained are $\sigma = \sigma_*$. Therefore, $\mathrm{Carl}(\sigma) \le \mathrm{Carl}(\sigma_*)$.

Now, we prove the second inequality of (2). We have

$$\mathrm{Carl}(\sigma_*) = \Delta^n \exp\Big(\frac{n}{\pi}\int_0^{\pi/n} \log\frac{2}{|e^{i\theta}-(1/\Delta)|}\,d\theta\Big)$$

and

$$\int_0^{\pi/n} \log\frac{2}{|e^{i\theta}-(1/\Delta)|}\,d\theta = \frac{1}{2}\int_0^{\pi/n} \log\frac{4}{\delta^2+4\sin^2(\theta/2)}\,d\theta,$$

where $\delta = 1-\frac{1}{\Delta}$. Assuming $n \ge 2$ (and hence $\frac{\sqrt{2}}{\pi}\theta \le \sin(\theta/2) \le \theta/2$ for $0 \le \theta \le \pi/n$), we obtain $\frac{8}{\pi^2}(\delta^2+\theta^2) \le \delta^2 + 4\sin^2(\theta/2) \le \delta^2+\theta^2$, and then

$$\Delta^n D(\Delta,n) \le \mathrm{Carl}(\sigma_*) \le \frac{\pi^2}{8}\Delta^n D(\Delta,n),$$

where

$$\begin{aligned} D(\Delta,n) &= \exp\Big(\frac{n}{2\pi}\int_0^{\pi/n}\log\frac{4}{\delta^2+\theta^2}\,d\theta\Big)\\ &= 2\exp\Big(\frac{n}{2\pi}\Big\{\frac{\pi}{n}\log\frac{1}{\delta^2+(\pi/n)^2}+\frac{2\pi}{n}-2\delta\arctan\frac{\pi}{n\delta}\Big\}\Big). \end{aligned}$$

Since $0 \le 1-\frac{\delta n}{\pi}\arctan\frac{\pi}{\delta n} \le 1$, we get $\frac{2}{(\delta^2+(\pi/n)^2)^{1/2}} \le D(\Delta,n) \le \frac{2e}{(\delta^2+(\pi/n)^2)^{1/2}}$, whence

$$2\frac{\Delta^n}{(\delta^2+(\pi/n)^2)^{1/2}} \le \mathrm{Carl}(\sigma_*) \le \frac{\pi^2 e}{4}\frac{\Delta^n}{(\delta^2+(\pi/n)^2)^{1/2}}.$$

In order to check the third inequality of (2), we simply replace the l^2-norm in the denominator by the l^1-norm and use the inequalities $1 < \Delta \le 2$.

The fourth estimate of (2) follows from the second and the obvious fact that

$$\Delta \longmapsto \Delta^{-n}\,\mathrm{Carl}(\sigma_*(\Delta,n))$$

is a decreasing function.

(3) The first estimate of (3) follows from the second estimate of (2) and the fact that $\Delta \longmapsto \Delta^{-n}\,\mathrm{Carl}(\sigma_*(\Delta,n))$ is a decreasing function.

The second estimate of (3) follows from the third one of (2).

For the third estimate of (3), we have

$$\varphi_X(\mathrm{Carl}(\sigma_*)) \le \mathrm{cap}_X(\sigma_*) \le \underline{\kappa}_n(\Delta,X) \le \psi_X(\mathrm{Carl}(\sigma_*)) \le \psi_X(be^B n),$$

since $\mathrm{Carl}(\sigma_*) \le b\frac{\Delta^n n}{1+n(\Delta-1)} \le b\frac{e^B n}{1+A} \le be^B n$. Similarly,

$$\mathrm{Carl}(\sigma_*) \ge a\frac{\Delta^n n}{1+n(\Delta-1)} \ge a\frac{e^A n}{(1+B)(1+A/n)},$$

and the inequality follows.

The last estimate of (3) is similar to the previous one if we use the inequalities

$$\frac{\Delta^n}{(1+c)\epsilon} \le \frac{\Delta^n n}{n\epsilon+n(\Delta-1)} \le \frac{\Delta^n n}{1+n(\Delta-1)} \le \frac{\Delta^n n}{n\epsilon}. \qquad \square$$

Corollary 3.37. *The previous theorem applies to*

(1) $X = B^s_{p,q}$ *with* $s > \frac{1}{p}$ *and* $\varphi_X = c_1\varphi_\alpha$, *where* $s-\frac{1}{p} \ge \alpha > 0$; *and* $\psi_X = c_2\psi_\gamma$, *where* $\gamma > s$, *and*

(2) $X = l^q_A(k^\beta)$ *with* $\beta > 1-\frac{1}{q}$ *and* $\varphi_X = c_1\varphi_\alpha$, $0 < \alpha < \beta-1+\frac{1}{q}$ *and* $\psi_X = c_2\psi_\gamma$, $\gamma > \beta+\frac{1}{q}$.

(*The functions* φ_α *and* ψ_α *are defined immediately before Theorem* 3.36.)

Indeed, under the hypotheses of (1), we have $C_A^{(\gamma)}(\mathbb{D}) \subset B^s_{p,q} \subset C_A^{(\alpha)}(\mathbb{D})$ (see Subsection 3.1), and under the hypotheses of (2), $C_A^{(\gamma)}(\mathbb{D}) \subset l^q_A(k^\beta) \subset l^1_A(k^\alpha) \subset C_A^{(\alpha)}(\mathbb{D})$ (by Hölder's inequality and the fact that $f \in C_A^{(\gamma)}(\mathbb{D}) \implies \hat{f}(n) = O(|n|^{-\gamma})$). □

3.8. Boundary spectrum and Helson's constant. Here we briefly consider the situation where $T \in X_C$, $m_\sigma(T) = 0$, and $\sigma \subset \mathbb{T}$. In this case the linear manifold $m_\sigma X$ is no longer closed in X for most of the usual function spaces X in $\mathbb{D}$ (for all such spaces?). Assuming that $X \subset C_A(\mathbb{D})$, one can easily show that $\operatorname{clos}_X(m_\sigma X) = \{f \in X : f(\lambda) = 0 \text{ for } \lambda \in \sigma\}$. Therefore, the quotient space $X/m_\sigma X$ should be replaced by the restriction space

$$X(\sigma) = X|\sigma = X/\operatorname{clos}_X(m_\sigma X)$$

endowed with the usual quotient norm: a function $a \in C(\sigma)$ is in $X(\sigma)$ if and only if $a = f|\sigma$ for a function $f \in X$, and $\|a\|_{X(\sigma)} = \inf \|f\|_X$, where the infimum is taken over all such extensions of a. For certain function spaces X these restriction spaces are studied quite well; for example, see [K, GrMcG] for $X = W$, and [D, BD] for the Hölder classes $X = C_A^{(\alpha)}(\mathbb{D})$ and for some other spaces. By the way, an X-capacity of subsets $\sigma \subset \mathbb{T}$ can be defined exactly in the same way as for $\sigma \subset \mathbb{D}$:

$$\operatorname{cap}_X(\sigma) = \inf\{\|f\|_X : f(0) = 1,\ f|\sigma = 0\}.$$

In the following theorem we use an observation from [GMP] in order to estimate the inverses T^{-1} and the capacities $\operatorname{cap}_W(\sigma)$ for sets $\sigma \subset \mathbb{T}$, $\operatorname{card}(\sigma) < \infty$ (here σ is precisely a subset, not a family of points).

Theorem 3.38. (1) *Suppose $T : Y \longrightarrow Y$ is an algebraic Banach space operator, and $m_\sigma(T) = 0$, where $\sigma \subset \mathbb{T}$, $\operatorname{card}(\sigma) < \infty$. Then T is power bounded, $\sup_{n\ge 0} \|T^n\| = C < \infty$, invertible, and $\|T^{-1}\| \le C$.*

(2) $\operatorname{cap}_W(\sigma) \le 2$ *for every finite subset $\sigma \subset \mathbb{T}$.*

Proof. (1) Since the family $(\operatorname{Ker}(T - \lambda I) : \lambda \in \sigma)$ is a basis of Y (use the Lagrange interpolating polynomials applied to T), the operator T is power bounded. Next, the classical Kronecker "solenoid theorem" (see, for example, [GrMcG]) implies that for every finite independent set $E \subset \mathbb{T}$ (i.e., a set such that the conditions $\lambda_j \in E$ and $\lambda_1^{n_1} \cdots \lambda_k^{n_k} = 1$, $n_j \in \mathbb{Z}$, imply $n_j = 0$ for all j) there exists a sequence $N_k \in \mathbb{Z}_+$ tending to infinity and such that $\lim_k \lambda^{N_k} = \lambda^{-1}$ for every $\lambda \in E$. Since for every finite set $\sigma \subset \mathbb{T}$ there exists an independent basis $E = \{\lambda_1, \dots, \lambda_k\}$ (i.e., such that $\sigma \subset \{\lambda_1^{n_1} \cdots \lambda_k^{n_k} : n_j \in \mathbb{Z}\}$), it follows that $\lim_k \lambda^{N_k} = \lambda^{-1}$ for every $\lambda \in \sigma$. This entails that $\lim_k T^{N_k} x = \lambda^{-1} x$ for every eigenvector $Tx = \lambda x$, $\lambda \in \sigma$. Therefore, there exists a limit $Ax = \lim_k T^{N_k} x$ for every $x \in Y$, which obviously determines the inverse operator $ATx = TAx = x$. Clearly, $\|T^{-1}\| \le \underline{\lim}_k \|T^{N_k}\| \le C$.

(2) Let $T(f + m_\sigma W) = zf + m_\sigma W$ be the quotient multiplication operator on $W(\sigma)$. Then, clearly, $\|T\| = 1$ and $m_\sigma(T) = 0$. By (1), $\|T^{-1}\| = 1$. This means that $z + m_\sigma W$ is invertible in the Banach algebra $W(\sigma)$ and $\|(z + m_\sigma W)^{-1}\|_{W(\sigma)} = 1$; hence, for every $\epsilon > 0$ there exists $\varphi \in W$ such that $z\varphi = 1$ on σ and $\|\varphi\|_W < 1 + \epsilon$. Setting $f = 1 - z\varphi$, we obtain $f|\sigma = 0$, $f(0) = 1$, and $\|f\|_W \le 2 + \epsilon$. Therefore, $\operatorname{cap}_W(\sigma) \le 2$. □

Remark 3.39. Here we comment on several curious points related to the last theorem.

(1) Comparing Theorem 3.20 (see also Remark 3.34) and Theorem 3.38, we can see that the circular capacity function $\Delta \longmapsto \underline{\kappa}_n(\Delta, W)$ is not continuous at $\Delta = 1$. Moreover, Theorem 3.38 and Queffelec's theorem [Q] imply the following stronger result: for any integer $n \ge 1$, there exists a subset $\sigma \subset \mathbb{T}$, $\operatorname{card}(\sigma) = n < \infty$, such that

$$\underline{\lim}_{\Delta \longrightarrow 1+} \operatorname{cap}_W\left(\frac{1}{\Delta}\sigma\right) \ge \sqrt{n},$$

but $\mathrm{cap}_W(\sigma) \le 2$. It seems plausible that for the spaces $B^s_{p,q}$ embedded in a space $C^{(\alpha)}(\mathbb{D})$, $\alpha > 0$ (i.e., for $s > 1/p$, see Subsection 3.1 above) such a phenomenon is not possible because the function $\Delta \longmapsto \mathrm{Carl}(\frac{1}{\Delta}\sigma)$ is continuous on $(0,1]$ for every finite set $\sigma \subset \mathbb{T}$.

(2) A similar discrepancy in $\sqrt{n}$ times can be observed between the behavior of the quantity $k(m_\sigma, W_1)$ in the disk $\sigma \subset \mathbb{D}$ and on the circle $\sigma \subset \mathbb{T}$, $\mathrm{card}(\sigma) = n$. To see this, recall that for a finite set $\sigma \subset \mathbb{D}$ we have

$$k(m_\sigma, W_1) = \sup\Big\{ \frac{\|a\|_{W/m_\sigma W}}{\|a\|_{H^\infty/m_\sigma H^\infty}} : a \in C(\sigma)\Big\}$$

(see Subsection 3.4). In order to consider this function on the closed unit disk, $\sigma \subset \overline{\mathbb{D}}$, we should rewrite this as

$$k(m_\sigma, W_1) = \sup\Big\{ \frac{\|a\|_{W(\sigma)}}{\|a\|_{C_A(\sigma)}} : a \in C(\sigma)\Big\},$$

where $W(\sigma)$ is defined above and $C_A(\sigma)$ stands for the similar restriction space $C_A(\mathbb{D})|\sigma$ with the quotient norm $\|a\|_{C_A(\sigma)} = \inf\{\|f\|_{C_A(\mathbb{D})} : f|\sigma = a\}$. We also introduce the following function $k_n(\Delta, W)$ defined for $\Delta \in [1,\infty)$ (a norm analog of $\kappa_n(\Delta, A)$ appeared in §3):

$$k_n(\Delta, W) = \sup\big\{k(m_\sigma, W_1) : \sigma \subset \{z : 1/\Delta \le |z| \le 1\},\ \mathrm{card}(\sigma) \le n\big\}.$$

Now, Theorem 3.14 implies that $k_n(\Delta, W) \ge an$ for every $\Delta > 1$, where $a > 0$ is a numerical constant.

For $\Delta = 1$, from known facts it follows that $k_n(1, W) \le \sqrt{n}$. Indeed, on the one hand, the Rudin–Carleson theorem (see, e.g., [N2, p. 156]) tells us that $\|a\|_{C_A(\sigma)} = \|a\|_{C(\sigma)}$ for every $a \in C(\sigma)$ and every finite set $\sigma \subset \mathbb{T}$, and hence $k_n(1, W) = H_n$, where

$$H_n = \sup\Big\{ \frac{\|a\|_{W(\sigma)}}{\|a\|_{C(\sigma)}} : a \in C(\sigma),\ \sigma \subset \mathbb{T},\ \mathrm{card}(\sigma) \le n\Big\}.$$

On the other hand, the latter quantity, known as "Helson's constant", has the following bounds (see [GrMcG, p. 34]).

Fact.

$$\sqrt{n/2} \le H_n \le \sqrt{n}.$$

A short proof (*for the reader's convenience*). *The right-hand side inequality.* By a duality argument, the quantity H_n is the smallest constant k such that $\mathrm{Var}(\mu) \le k\|\hat{\mu}\|_{l^\infty(\mathbb{Z})}$ for every measure μ of the form $\mu = \sum_{k=1}^n a_k \delta_{\zeta_k}$, where $\zeta_k \in \mathbb{T}$ and $\hat{\mu}(k)$, $k \in \mathbb{Z}$, stand for the Fourier coefficients. Since

$$\mathrm{Var}(\mu) = \sum_{k=1}^n |a_k| \le \sqrt{n}\Big(\sum_{k=1}^n |a_k|^2\Big)^{1/2}$$

and

$$\Big(\sum_{k=1}^n |a_k|^2\Big)^{1/2} = \lim_n \Big(\frac{1}{2n+1}\sum_{k=-n}^n |\hat{\mu}(k)|^2\Big)^{1/2} \le \|\hat{\mu}\|_{l^\infty(\mathbb{Z})}$$

by the Parseval identity for almost periodic functions (Wiener's theorem; see, for example, [GrMcG]), we get $H_n \le \sqrt{n}$.

The left-hand side inequality. We invoke the so-called Rudin–Shapiro measures. Let $\{\lambda_1, \dots, \lambda_n\}$ be an independent set on $\mathbb{T}$. Setting $\mu_0 = \nu_0 = \delta_1$ (the Dirac measure at 1), we define by induction $\mu_{j+1} = \mu_j + \nu_j * \delta_{\lambda_{j+1}}$ and $\nu_{j+1} = \mu_j - \nu_j * \delta_{\lambda_{j+1}}$. The measures

μ_n and ν_n are of the form $\sum_\lambda \pm\delta_\lambda$, where λ runs over all points $\lambda = \lambda_1^{\alpha_1}\cdots\lambda_n^{\alpha_n}$ with $\alpha_j = 0$ or 1. The parallelogram identity and $|\hat{\delta}_{\lambda_{j+1}}(n)| = |\overline{\lambda}_{j+1}^n| = 1$ imply that

$$|\hat{\mu}_{j+1}(k)|^2 + |\hat{\nu}_{j+1}(k)|^2 = 2(|\hat{\mu}_j(k)|^2 + |\hat{\nu}_j(k)|^2) = 2^{j+2}$$

for every $k \in \mathbb{Z}$ and every j. Consequently, at least one of the measures μ_n, ν_n satisfies $\|\hat{\mu}\|_{l^\infty(\mathbb{Z})} \le \|\hat{\mu}\|_{l^2(\mathbb{Z})} \le 2^{n/2}$ and $\mathrm{Var}(\mu) = 2^n$. For an arbitrary $m \ge 1$, we choose n such that $2^n < m \le 2^{n+1}$ and get $H_m \ge \mathrm{Var}(\mu)/\|\hat{\mu}\|_{l^\infty(\mathbb{Z})} \ge \sqrt{m/2}$. □

We do not know whether discontinuities of this kind occur for function spaces on $\mathbb{D}$ different from W.

(3) The invertibility of T in Theorem 3.38 is a consequence of power-boundedness. Moreover, for a given log-concave sequence $(w_n)_{n\ge0}$, $\lim_n w_n^{1/n} = 1$, the following statements are equivalent; see [N6, 3.3].

(a) Every Banach space operator $T : X \longrightarrow X$ such that $\|T^n\| \le Cw_n$, $n \ge 0$, and $\mathrm{span}(\mathrm{Ker}(T - \lambda I) : \lambda \in \sigma) = X$, where $\sigma \subset \mathbb{T}$, is invertible.

(b) Every Hilbert space operator $T : H \longrightarrow H$ such that $\|T^n\| \le Cw_n, n \ge 0$, and $\mathrm{span}(\mathrm{Ker}(T - \lambda I) : \lambda \in \sigma) = H$, where $\sigma \subset \mathbb{T}$, is invertible.

(c) $\sup_{n\ge0} w_n < \infty$.

In fact, $\sigma(T) = \mathrm{clos}(\sigma)$ if (c) is fulfilled.

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