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ON THE SPECTRUM OF A LIMIT-PERIODIC SCHRÖDINGER OPERATOR

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In memory of O. A. Ladyzhenskaya

Abstract. The spectrum of the perturbed polyharmonic operator $H = (-\Delta)^l + V$ in $\mathsf{L}^2(\mathbb{R}^d)$ with a limit-periodic potential V is studied. It is shown that if V is periodic in one direction in $\mathbb{R}^d$ and $8l > d+3$, $d \neq 1 (\mathrm{mod}\, 4)$, then the spectrum of H contains a semiaxis. The proof is based on the properties of periodic operators.

§1. Introduction

Our aim in the present paper is to study the structure of the spectrum for the operators with limit-periodic potentials, which represent a subset of the almost-periodic potentials. Recall that a function V is said to be *limit-periodic* if it can be approximated by periodic functions $W_n, n = 1, 2, \dots$, whose period lattices form a sequence of sublattices of a given lattice Γ. We study the perturbed polyharmonic operator

$$H = H(V) = (-\Delta)^l + V, \tag{1.1}$$

in $\mathsf{L}^2(\mathbb{R}^d), d \geq 1, l > 0$, with a limit-periodic real-valued function V; see formula (2.2) for the precise definition. We are concerned with the structure of the spectrum $\sigma(H)$ as a set. The question we are asking is

(Q) Does the spectrum of H contain a half-line?

For smooth periodic potentials V the answer is always affirmative if $8l > d + 3$; see [10]. The following heuristic argument shows that the answer to the question (Q) might conceivably be negative for limit-periodic potentials. Consider a potential of the form $V = V_1 + \dots + V_d$, where $V_j = V_j(x_j), j = 1, \dots, d$, are limit-periodic functions of one variable, so that in the Schrödinger operator $H = -\Delta + V$ the variables separate. Then

$$\sigma(H) = \sum_{j=1}^{d} \sigma(H_j), \quad H_j = -\frac{d^2}{dx^2} + V_j, \;\; j = 1, 2, \dots, d.$$

In [2] it was shown that the spectrum of the one-dimensional Schrödinger operator with a limit-periodic potential is generically a Cantor set. Therefore, if the spectra of the H_j's are Cantor sets, then the above arithmetic sum of the spectra can be a Cantor set again (see, e.g., [4, 9] and the references therein). Thus, one may suspect that for a large set of limit-periodic potentials the spectrum of the multi-dimensional Schrödinger operator may be a Cantor set. We do not have a rigorous proof to justify the above argument, since it would be rather difficult to compare the properties of the Cantor sets arising

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in the context of the paper [2] with those considered in [4], [9]. However, this specific question is beyond the scope of this paper. At the same time, we note that in the recent communication [7] it was announced that for $d = 2$ and $l > 5$ the spectrum of H does contain a half-line.

In the present paper we prove that if the limit-periodic potential V is *periodic* in one direction in $\mathbb{R}^d$ and

$$8l > d + 3, \quad d \neq 1 (\operatorname{mod} 4),$$

then the spectrum of (1.1) does contain a half-line. The proof is based on the properties of periodic operators. The statement that the spectrum of H with a periodic potential W contains a half-line is referred to as the Bethe–Sommerfeld conjecture; see [3, 5] and [10]–[13]. This fact is based on the observation that, sufficiently far in the spectrum, the spectral bands overlap and hence have no gaps in between. We approximate the initial limit-periodic potential V with a sequence $W_n, n = 1, 2, \dots,$ of periodic functions for which the period lattices form a sequence of expanding sublattices of $\mathsf{\Gamma}$. In general, the spectral bands of the periodic operators $H(W_n)$ shrink in size as $n \to \infty$, which may lead to the creation of new gaps in the spectrum. However, as we observe in §7, the bands do not shrink and preserve their overlap if the approximating potentials W_n are all periodic in one fixed direction with a period independent of n. This allows us to conclude that the spectrum of the limiting operator H contains a half-line.

When studying the periodic operators $H(W_n)$, we closely follow paper [10], the methods of which seem to be most convenient for our purposes. The difference with [10] is that now we need to keep track of the dependence on the period lattices of the individual potentials W_n. Note that the condition $d \neq 1 (\operatorname{mod} 4)$ is dictated by the necessity to control this dependence (see Remark 7.2). As a by-product, in the course of the proof we obtain a new asymptotic estimate for the density of states of the limit-periodic operator (1.1); see (3.8).

§2. Main result

Let $\mathsf{\Gamma} \subset \mathbb{R}^d$ be a lattice in $\mathbb{R}^d$, $d \geq 2$. We denote by $\mathsf{\Gamma}^\dagger$ the dual lattice, and by $\mathcal{O}$ and $\mathcal{O}^\dagger$ the standard fundamental domains of $\mathsf{\Gamma}$ and $\mathsf{\Gamma}^\dagger$, respectively. If necessary, we indicate the dependence of these objects on the lattice and write $\mathcal{O}_{\mathsf{\Gamma}}$ and $\mathcal{O}^\dagger_{\mathsf{\Gamma}}$. The quantity $\mathrm{d}(\mathsf{\Gamma}) = \operatorname{vol} \mathcal{O}$ is called the determinant of the lattice $\mathsf{\Gamma}$. Note that $\mathrm{d}(\mathsf{\Gamma})\, \mathrm{d}(\mathsf{\Gamma}^\dagger) = (2\pi)^d$. Sometimes, the set $\mathcal{O}_{\mathsf{\Gamma}}$ is identified with the torus $\mathbb{T}^d = \mathbb{T}^d_{\mathsf{\Gamma}} = \mathbb{R}^d / \mathsf{\Gamma}$. For a function f periodic with respect to $\mathsf{\Gamma}$ we define its Fourier coefficients as follows:

$$\hat{f}(\boldsymbol{\theta}) = \frac{1}{\sqrt{\mathrm{d}(\mathsf{\Gamma})}} \int_{\mathcal{O}} e^{-i\boldsymbol{\theta}\mathbf{x}} f(\mathbf{x})\, d\mathbf{x}, \quad \boldsymbol{\theta} \in \mathsf{\Gamma}^\dagger.$$

Clearly, the value of the Fourier coefficient depends on the choice of the lattice with respect to which the function f is periodic. To avoid ambiguity, sometimes we reflect the dependence on the lattice by writing $\hat{f}(\boldsymbol{\theta}) = \hat{f}(\boldsymbol{\theta}; \mathsf{\Gamma})$. Denote

$$\| f \|_\nu = \frac{1}{\sqrt{\mathrm{d}(\mathsf{\Gamma})}} \sum_{\boldsymbol{\theta} \in \mathsf{\Gamma}^\dagger} (1 + |\boldsymbol{\theta}|^\nu) |\hat{f}(\boldsymbol{\theta}; \mathsf{\Gamma})|,$$

with some parameter $\nu \geq 0$ to be specified later. If $\mathsf{M} \subset \mathsf{\Gamma}$ is a sublattice and a function f is $\mathsf{\Gamma}$-periodic, then $\mathsf{\Gamma}^\dagger \subset \mathsf{M}^\dagger$ and

$$\frac{1}{\sqrt{\mathrm{d}(\mathsf{M})}} \hat{f}(\boldsymbol{\theta}; \mathsf{M}) = \begin{cases} \frac{1}{\sqrt{\mathrm{d}(\mathsf{\Gamma})}} \hat{f}(\boldsymbol{\theta}; \mathsf{\Gamma}) & \text{if } \boldsymbol{\theta} \in \mathsf{\Gamma}^\dagger, \\ 0 & \text{if } \boldsymbol{\theta} \in \mathsf{M}^\dagger \setminus \mathsf{\Gamma}^\dagger. \end{cases} \tag{2.1}$$

This shows that, for such a function, the quantity $\| f \|_\nu$ does not depend on the choice of the lattice.

In $\mathsf{L}^2(\mathbb{R}^d)$ we consider the operator

$$H = H_0 + V, \quad H_0 = (-\Delta)^l,$$

with a real-valued potential consisting of components periodic with respect to sublattices Λ of a fixed lattice $\Gamma \subset \mathbb{R}^d$. To describe the potential V more precisely, we introduce some notation. Let $\mathfrak{M}_p$ be the set of all sublattices of index at most p; that is, for each $\Lambda \in \mathfrak{M}_p$ we have $\mathrm{d}(\Lambda) = l\,\mathrm{d}(\Gamma)$ with some $l \le p$. The set $\mathfrak{N}_p = \mathfrak{M}_p \setminus \mathfrak{M}_{p-1}$ is the set of all sublattices of index p. We also need to single out a subset $\mathfrak{N}_p(\boldsymbol{\gamma}) \subset \mathfrak{N}_p$ of sublattices having a common vector $\boldsymbol{\gamma} \in \Gamma$. We are interested in the potentials of the form

$$V = \sum_{p=1}^{s} \sum_{\Lambda \in \mathfrak{N}_p} V_\Lambda, \quad s \le \infty, \tag{2.2}$$

where each V_Λ is a Ł-periodic function. For $s = \infty$ it is assumed that the series converges in the sup-norm, and then we say that the potential V is *limit-periodic*. In what follows we assume that $|\!|\!| V_\Lambda |\!|\!|_\nu$ is finite for the following values of the parameter ν:

$$\nu > \begin{cases} \frac{d-1}{2} & \text{if } d \ge 3; \\ \frac{2}{3}(l+1) & \text{if } d = 2. \end{cases} \tag{2.3}$$

Note that this condition is the same as in [6, §1]. Since the right-hand side of (2.2) comprises potentials with different, possibly growing periods, we need to introduce a new norm of V which attaches different weights to lattices of different indices. Let $m_p > 0, p = 1, 2, \dots,$ be an increasing sequence of real numbers such that

$$B_d < \infty, \quad B_d = \begin{cases} \sum_{p=1}^\infty p^d m_p^{-2}(|\ln m_p| + 1), & d \ge 3, \\ \sum_{p=1}^\infty p^d \big(m_p^{-\frac{3}{2}} + m_p^{-2}\big)(|\ln m_p| + 1), & d = 2. \end{cases} \tag{2.4}$$

Now we define

$$|\!|\!| V |\!|\!| = \sum_{p=1}^{s} m_p \sum_{\Lambda \in \mathfrak{N}_p} |\!|\!| V_\Lambda |\!|\!|_\nu. \tag{2.5}$$

Our objective is to prove the Bethe–Sommerfeld conjecture for H with a potential of the form (2.6), where the summation is restricted to the sublattices belonging to $\mathfrak{N}_p(\boldsymbol{\gamma})$, so that the resulting potential V is *$\boldsymbol{\gamma}$-periodic.*

Theorem 2.1. *Suppose $d \ge 2$ and $8l > d + 3$, $d \ne 1 (\mathrm{mod}\, 4)$. Let V be a real-valued function given by*

$$V = \sum_{p=1}^{s} \sum_{\Lambda \in \mathfrak{N}_p(\boldsymbol{\gamma})} V_\Lambda, \quad s \le \infty \tag{2.6}$$

with some vector $\boldsymbol{\gamma} \in \Gamma$, the series being absolutely convergent. Suppose that a number ν and a monotone increasing sequence $m_p > 0$ satisfy (2.3) and (2.4), respectively, and that $|\!|\!| V |\!|\!| < \infty$. Then the spectrum of H contains a semiaxis. More precisely, there exists a $l_0 = l_0(|\!|\!| V |\!|\!|, \Gamma) \in \mathbb{R}$, depending also on $d, l,$ and B_d, such that the entire semiaxis $[l_0, \infty)$ belongs to the spectrum of H.

We illustrate Theorem 2.1. On $\mathbb{R}^2$, define the potential

$$V = \sum_{p=1}^{\infty} V^{(p)}, \quad V^{(p)}(x_1, x_2) = \sum_{n,m \in \mathbb{Z}} v_{nm}^{(p)} e^{inx_1} e^{imp^{-1}x_2},$$

with a sequence of complex numbers $v_{nm}^{(p)}$ such that $v_{nm}^{(p)} = \overline{v_{-n,-m}^{(p)}}$. Clearly, if the series for $V^{(p)}$ converges, it defines a real-valued function, which is 2π-periodic in x_1 and $2\pi p$-periodic in x_2. We take $m_p = p^\alpha$ with some $\alpha > 2$, so that (2.4) (with $d = 2$) is satisfied. Assume also that ν satisfies (2.3) (with $d = 2$). Then under the condition

$$\sum_{p=1}^{\infty} p^\alpha \sum_{m,n\in\mathbb{Z}} \bigl(1 + n^\nu + (mp^{-1})^\nu\bigr)|v_{nm}^{(p)}| < \infty$$

the norm (2.5) of the function V is finite, so that the conditions of Theorem 2.1 are fulfilled for $d = 2$. Consequently, for all $l > 5/8$ the spectrum of the operator H contains a half-line.

The proof of Theorem 2.1 is based on the spectral properties of periodic operators. The next section contains the necessary information.

§3. Periodic operators

3.1. Floquet decomposition. We identify the space $\mathcal{H} = \mathsf{L}^2(\mathbb{R}^d)$ with the direct integral

$$\mathfrak{G} = \int_{\mathcal{O}^\dagger} \mathfrak{H}\, d\mathbf{k}, \quad \mathfrak{H} = \mathsf{L}^2(\mathcal{O}).$$

This identification is implemented by the Gelfand transformation

$$(Uu)(\mathbf{x},\mathbf{k}) = \frac{1}{\sqrt{\mathrm{d}(\Gamma^\dagger)}} e^{-i\mathbf{k}\mathbf{x}} \sum_{\boldsymbol{\gamma}\in\Gamma} e^{-i\mathbf{k}\boldsymbol{\gamma}} u(\mathbf{x}+\boldsymbol{\gamma}), \quad \mathbf{k}\in\mathcal{O}^\dagger, \tag{3.1}$$

which is initially defined on $u \in \mathsf{S}(\mathbb{R}^d)$ and extends by continuity to a unitary mapping from $\mathcal{H}$ onto $\mathfrak{G}$. It is readily seen that

$$(UH_0U^{-1}u)(\,\cdot\,,\mathbf{k}) = H_0(\mathbf{k})u(\,\cdot\,,\mathbf{k}), \;\; H_0(\mathbf{k}) = \bigl(\mathbf{D}+\mathbf{k}\bigr)^{2l}, \quad \mathbf{k}\in\mathbb{R}^d,$$

with the domain $D(H_0(\mathbf{k})) = \mathsf{H}^{2l}(\mathbb{T}^d)$. The family $H(\mathbf{k}) = H_0(\mathbf{k}) + V$ realizes the expansion of H in the direct integral:

$$UHU^{-1} = \int_{\mathcal{O}^\dagger} H(\mathbf{k})\, d\mathbf{k}.$$

The spectrum of each $H(\mathbf{k})$ consists of discrete eigenvalues $l_j(\mathbf{k}) = l_j(H(\mathbf{k})), j = 1, 2, \dots,$ which we arrange in nondecreasing order counting multiplicity. It is clear that the $l_j(\,\cdot\,)$ are continuous functions of $\mathbf{k}$. In general, for any bounded selfadjoint continuous operator-valued function $B(\mathbf{k})$, the spectrum of $\hat{H}(\mathbf{k}) = H_0(\mathbf{k}) + B(\mathbf{k})$ is discrete, and the eigenvalues $l_j(\hat{H}(\mathbf{k}))$ are continuous functions of $\mathbf{k}\in\mathcal{O}^\dagger$. We introduce the counting function

$$N(l;\hat{H}(\mathbf{k})) = N_\Gamma\bigl(l;\hat{H}(\mathbf{k})\bigr) = \sharp\{n : l_n(\hat{H}(\mathbf{k})) \le l\}, \quad \mathbf{k}\in\mathcal{O}^\dagger.$$

When $B(\mathbf{k}) = V$, we use the notation $N(l;\mathbf{k})$. The images

$$\ell_j = \bigcup_{\mathbf{k}\in\overline{\mathcal{O}^\dagger}} l_j(\mathbf{k})$$

of the functions l_j are called *spectral bands*. The spectrum of the initial operator H is the union of the bands:

$$\sigma(H) = \bigcup_j \ell_j.$$

Our aim is to show that the bands with distinct numbers overlap if j is sufficiently large. To this end, we make the following simple observation: a point l belongs to a band ℓ_j if and only if the number $N(l;\mathbf{k})$ is not constant as a function of $\mathbf{k}$, or, in other words, if the deviation of $N(l;\mathbf{k})$ from its average value is not zero. To characterize this deviation, we define

$$S(l) = S(l;H) = \int_{\mathcal{O}^\dagger} \bigl|N(l;\mathbf{k}) - \overline{N}(l)\bigr|\, d\mathbf{k}, \quad \overline{N}(l) = \frac{1}{\mathrm{d}(\Gamma^\dagger)} \int_{\mathcal{O}^\dagger} N(l;\mathbf{k})\, d\mathbf{k}. \tag{3.2}$$

If $H = H_0$, then we denote $N_0(l;\mathbf{k}) = N(l;\mathbf{k})$, $\overline{N}_0(l) = \overline{N}(l)$, and $S_0(l) = S(l)$.

Another quantity characterizing the band structure, the overlap length, was introduced in [11]. This quantity is defined as follows:

$$z(l) = \sup\Bigl\{t > 0 : \max_{\mathbf{k}} N(l-t;\mathbf{k}) > \min_{\mathbf{k}} N(l+t;\mathbf{k})\Bigr\}. \tag{3.3}$$

It is not hard to check (see [11]) that the function $z(l)$ is given by the formula

$$z(l) = \max_j \max\{t : [l-t, l+t] \subset \ell_j\},$$

or, in words, $z(l)$ is half the length of the maximal interval centered at l that fits in at least one spectral band. Clearly, the fact that $z(l) \ge z_0 > 0$ for all $l \ge l_0$ means that

a) the length of each band with center to the right of l_0 is at least $2z_0$, and

b) every point $l \ge l_0$ is inside some band at a distance of at least z_0 from its ends.

The overlap length $z(l)$ can be estimated by using the function $S(l)$.

Lemma 3.1. *Let $S(l)$ and $z(l)$ be as defined above. Then*

$$z(l) \ge \sup\biggl\{t > 0 : \overline{N}(l-t) + \frac{1}{2\,\mathrm{d}(\Gamma^\dagger)} S(l-t) > \overline{N}(l+t) - \frac{1}{2\,\mathrm{d}(\Gamma^\dagger)} S(l+t)\biggr\}.$$

Proof. By the definition of $\overline{N}$, we have

$$\int_{\mathcal{O}^\dagger} \bigl(N(l;\mathbf{k}) - \overline{N}(l)\bigr)\, d\mathbf{k} = 0.$$

Consequently,

$$\mathrm{d}(\Gamma^\dagger) \max_{\mathbf{k}} \bigl(N(l;\mathbf{k}) - \overline{N}(l)\bigr)_\pm \ge \int_{\mathcal{O}^\dagger} \bigl(N(l;\mathbf{k}) - \overline{N}(l)\bigr)_\pm\, d\mathbf{k} = \frac{1}{2} S(l).$$

Here $f_\pm = (|f| \pm f)/2$. The above inequality implies that

$$\max_{\mathbf{k}} N(l;\mathbf{k}) \ge \overline{N}(l) + \frac{1}{2\,\mathrm{d}(\Gamma^\dagger)} S(l),$$
$$\min_{\mathbf{k}} N(l;\mathbf{k}) \le \overline{N}(l) - \frac{1}{2\,\mathrm{d}(\Gamma^\dagger)} S(l).$$

Now the claim follows from (3.3). □

We do not use the function $z(l)$ in our proofs; it is introduced purely for illustration purposes.

3.2. Estimates for periodic operators. A key ingredient in the proof of Theorem 2.1 is an estimate for the counting function $N(l;\mathbf{k})$ of the operator $H_0(\mathbf{k}) + V$ with a periodic potential V represented by the sum (2.2) with finite s, where each V_Λ is Λ-periodic, $\|V_\Lambda\|_\nu$ is finite, and ν satisfies (2.3). Since $s < \infty$, the potential V is periodic with respect to the lattice $\mathsf{M} = s\Gamma$ of index s^d, obtained by stretching the lattice Γ by a factor of s in all directions. Below we use the notation $\mathcal{O} = \mathcal{O}_{\mathsf{M}}$ and $\mathcal{O}^\dagger = \mathcal{O}^\dagger_{\mathsf{M}}$. Throughout the paper we denote by C and c (with or without indices) various positive constants whose precise value is of no importance. The estimate stated in the next theorem is uniform with respect to the number s occurring in (2.2).

Theorem 3.2. *Suppose $d \geq 2$ and $2l \geq 1$. Let V be represented by* (2.2) *with some finite s, where each V_Λ is Λ-periodic with $\| V_\Lambda \|_\nu < \infty$, and let $\mathsf{M} = s\Gamma$. Suppose that a number ν and a positive monotone nondecreasing sequence m_p satisfy* (2.3) *and* (2.4)*, respectively. Let*

$$V_0 = \sum_{p=1}^{s} \sum_{\Lambda \in \mathfrak{N}_p} \frac{1}{\sqrt{\mathrm{d}(\Lambda)}} \hat{V}_\Lambda(\mathbf{0}; \Lambda) \tag{3.4}$$

be the mean value of the function V. Then there is a constant $\rho_0 = \rho_0(V, \Gamma) > 0$ such that

$$\int_{\mathcal{O}^\dagger_{\mathsf{M}}} |N(\rho^{2l}; H(\mathbf{k})) - N\big(\rho^{2l} - V_0; H_0(\mathbf{k})\big)| \, d\mathbf{k} \leq C B_d A(V) \rho^{d+1-4l} (|\ln \rho| + 1) \tag{3.5}$$

for all $\rho \geq \rho_0$, where

$$A(V) = \begin{cases} \big(\| V \|^2 + \| V \|^{\frac{3}{2}}\big)\big(1 + |\ln \| V \||\big) & \text{if } d = 2, \\ \| V \|^2 \big(1 + |\ln \| V \||\big) & \text{if } d \geq 3, \end{cases} \tag{3.6}$$

with a constant C depending only on the numbers l, d, ν, and the lattice Γ.

An estimate of the form (3.5) was proved in [6] with a constant depending on the lattice of periods. Although in our proof of Theorem 3.2 we follow the idea of [6], a number of technical details require revision.

Let $V^{(s)}$ be the potential given by formula (2.6) with a finite s, and let $V_0^{(s)}$ be its mean value given by (3.4). Theorem 3.2 immediately gives an estimate for the density of states (DOS) of the periodic operator $H^{(s)} = H_0 + V^{(s)}$; the latter is defined by the formula

$$D(l; H^{(s)}) = \frac{1}{(2\pi)^d} \int_{\mathcal{O}^\dagger_{\mathsf{M}}} N\big(l; H^{(s)}(\mathbf{k})\big) \, d\mathbf{k}.$$

More precisely, recall that for the unperturbed operator $H_0 = (-\Delta)^l$, the DOS is given by

$$D(l; H_0) = \frac{\mathrm{w}_d}{(2\pi)^d} l^{\frac{d}{2l}}, \quad l \geq 0,$$

where w_d is the volume of the unit ball in $\mathbb{R}^d$. Thus, Theorem 3.2 implies that

$$\begin{aligned} D(\rho^{2l}; H^{(s)}) &= D\big(\rho^{2l} - V_0^{(s)}; H_0\big) + O(\rho^{d+1-4l} \ln \rho) \\ &= \frac{\mathrm{w}_d}{(2\pi)^d} \rho^d - \frac{d\,\mathrm{w}_d}{2l} V_0^{(s)} \rho^{d-2l} + O(\rho^{d+1-4l} \ln \rho) \end{aligned} \tag{3.7}$$

for sufficiently large ρ *uniformly in s*. This uniformity shows that the DOS $D(\rho^{2l}; H)$ for the full operator H with a potential given by the infinite sum (2.2) satisfies the same formula. Indeed, an elementary perturbation-theoretic argument yields

$$D(l - \epsilon_s; H^{(s)}) \leq D(l; H) \leq D(l + \epsilon_s; H^{(s)}), \quad \epsilon_s = \|V^{(s)} - V\|,$$

for all s. Since $\epsilon_s \to 0$ as $s \to \infty$, relation (3.7) implies that

$$D(\rho^{2l}; H) = \frac{\mathrm{w}_d}{(2\pi)^d} \rho^d - \frac{d\,\mathrm{w}_d}{2l} V_0 \rho^{d-2l} + O(\rho^{d+1-4l} \ln \rho) \tag{3.8}$$

for sufficiently large ρ. This estimate refines the well-known asymptotic estimate

$$D(\rho^{2l}; H) = \frac{\mathrm{w}_d}{(2\pi)^d} \rho^d + O(\rho^{d-2})$$

of the DOS for almost periodic operators (see [14]).

§4. Counting function

4.1. In this and the next section we prepare for the proof of Theorem 3.2. Observing that

$$N(l; H(\mathbf{k})) = N\big(l - V_0; H(\mathbf{k}) - V_0\big),$$

we may assume without loss of generality that $V_0 = 0$. Furthermore, until the end of the proof it suffices to assume, instead of the special form (2.2), that V is simply a potential periodic with respect to some lattice M. We denote by $\Theta \subset \mathsf{M}^\dagger \setminus \{\mathbf{0}\}$ the set of all lattice points $\boldsymbol{\theta}$ for which $\hat{V}(\boldsymbol{\theta}) = \hat{V}(\boldsymbol{\theta}; \mathsf{M}) \neq 0$. The representation

$$V = \sum_{\boldsymbol{\theta} \in \Theta} V_{\boldsymbol{\theta}}, \quad V_{\boldsymbol{\theta}}(\mathbf{x}) = \hat{V}(\boldsymbol{\theta}) \mathbf{e}_{\boldsymbol{\theta}}, \quad \mathbf{e}_{\boldsymbol{\theta}}(\mathbf{x}) = \frac{1}{\sqrt{\mathrm{d}(\mathsf{M})}} e^{i\boldsymbol{\theta}\mathbf{x}}$$

will be useful. Instead of the finiteness of the norm (2.5), we assume that the weighted norm

$$\|V\|_w = \frac{1}{\sqrt{\mathrm{d}(\mathsf{M})}} \sum_{\boldsymbol{\theta} \in \Theta} |\hat{V}(\boldsymbol{\theta})| w(\boldsymbol{\theta})$$

is finite for some positive function w. The precise definition of the function w will be needed only at the end of the proof.

4.2. Pseudodifferential operators. At the first step of the proof it is convenient to use pseudo-differential operators (PDO's). Let $b(\mathbf{x}, \boldsymbol{\phi}), \mathbf{x} \in \mathbb{T}^d = \mathbb{T}^d_{\mathsf{M}}$, $\boldsymbol{\phi} \in \mathsf{M}^\dagger$, be a function such that $\sup_x |b(\mathbf{x}, \boldsymbol{\phi})| \leq C(|\boldsymbol{\phi}|^m + 1)$ with some $m \in \mathbb{R}$. Then the PDO $B = \mathrm{Op}(b)$ on the torus $\mathbb{T}^d_{\mathsf{M}}$, with the symbol b is defined to be

$$(Bf)(\mathbf{x}) = \sum \mathbf{e}_{\boldsymbol{\phi}}(\mathbf{x}) b(\mathbf{x}, \boldsymbol{\phi}) \hat{f}(\boldsymbol{\phi}),$$

for any $f \in \mathsf{C}^\infty(\mathbb{T}^d)$. If the Fourier coefficients $\hat{b}(\boldsymbol{\theta}, \boldsymbol{\phi})$ satisfy the condition

$$\langle b \rangle = \frac{1}{\sqrt{\mathrm{d}(\mathsf{M})}} \sum_{\boldsymbol{\theta} \in \mathsf{M}^\dagger} \sup_{\boldsymbol{\phi} \in \mathsf{M}^\dagger} |\hat{b}(\boldsymbol{\theta}, \boldsymbol{\phi})| < \infty,$$

then the operator $\mathrm{Op}(b)$ is easily shown to be bounded on $\mathsf{L}^2(\mathbb{T}^d)$, and

$$\|\mathrm{Op}(b)\| \leq \langle b \rangle. \tag{4.1}$$

The symbol of the operator $H_0(\mathbf{k})$ is $|\boldsymbol{\phi} + \mathbf{k}|^{2l}$.

4.3. Reduced operator. Let

$$p(\boldsymbol{\xi}; \rho) = |\boldsymbol{\xi}|^{2l} - \rho^{2l}.$$

For every $\boldsymbol{\theta} \in \Theta$ and $\rho > 0$, we introduce the set

$$\Omega_{\boldsymbol{\theta}}(\rho; \mathbf{k}) = \Big\{\boldsymbol{\xi} \in \mathbb{R}^d : \big|p(\boldsymbol{\xi} + \mathbf{k}; \rho)\big|\big|p(\boldsymbol{\xi} + \boldsymbol{\theta} + \mathbf{k}; \rho)\big| \leq 16\|V\|_w^2 w(\boldsymbol{\theta})^{-2}\Big\} \tag{4.2}$$

and set $\Omega_{\boldsymbol{\theta}}(\rho) = \Omega_{\boldsymbol{\theta}}(\rho; \mathbf{0})$. Let $P_{\boldsymbol{\theta}}(\mathbf{k}) = P_{\boldsymbol{\theta}}(\rho; \mathbf{k})$, $\mathbf{k} \in \mathcal{O}^\dagger$, denote the projection in $\mathsf{L}^2(\mathbb{T}^d)$ to the exponentials "living" in the set $\Omega_{\boldsymbol{\theta}}(\rho; \mathbf{k})$, i.e.,

$$P_{\boldsymbol{\theta}} e_{\boldsymbol{\phi}} = \begin{cases} e_{\boldsymbol{\phi}} & \text{if } \boldsymbol{\phi} \in \Omega_{\boldsymbol{\theta}}(\rho; \mathbf{k}) \cap \mathsf{M}^\dagger, \\ 0 & \text{if } \boldsymbol{\phi} \notin \Omega_{\boldsymbol{\theta}}(\rho; \mathbf{k}) \cap \mathsf{M}^\dagger. \end{cases}$$

In other words, $P_{\boldsymbol{\theta}}(\mathbf{k})$ is a pseudodifferential operator with the symbol $\chi(\boldsymbol{\phi} + \mathbf{k}; \Omega_{\boldsymbol{\theta}})$, $\boldsymbol{\phi} \in \mathsf{M}^\dagger$, where $\chi(\,\cdot\,; \mathcal{C})$ is the characteristic function of the set $\mathcal{C} \subset \mathbb{R}^d$. Also, we denote

$Q_{\boldsymbol\theta}(\mathbf{k}) = I - P_{\boldsymbol\theta}(\mathbf{k})$. Along with the original operator $H(\mathbf{k}) = H_0(\mathbf{k}) + V$ in $\mathcal{H} = \mathsf{L}^2(\mathbb{T}^d)$, we introduce the auxiliary PDO

$$\hat H(\rho;\mathbf{k},g) = H_0(\mathbf{k}) + gW(\rho;\mathbf{k}), \quad W(\rho;\mathbf{k}) = \sum_{\boldsymbol\theta} Q_{\boldsymbol\theta}(\mathbf{k})V_{\boldsymbol\theta}Q_{\boldsymbol\theta}(\mathbf{k}), \tag{4.3}$$

with a coupling constant $g \ge 0$. If $g = 1$, we drop g in the notation and write simply $\hat H(\rho;\mathbf{k})$.

The following theorem describes the counting function of $\hat H$. With some minor modifications, this theorem is borrowed from [6], where it was given for a slightly different construction of the reduced operator $\hat H$.

Theorem 4.1. *Suppose $\hat V(\mathbf{0}) = 0$ and $\|V\|_w < \infty$ for some weight w. Then*

$$N(\rho^{2l}; \hat H(\rho;\mathbf{k})) = N(\rho^{2l}; H_0(\mathbf{k})) \tag{4.4}$$

for all $\mathbf{k} \in \mathcal{O}^\dagger$ and $\rho > 0$.

Proof. It suffices to prove the formula under the assumption that ρ^{2l} is not in the spectrum of $H_0(\mathbf{k})$, i.e., $p(\boldsymbol\phi + \mathbf{k}; \rho) \neq 0$ for all $\boldsymbol\phi \in \mathsf{M}^\dagger$. For the remaining ρ the required relation will follow by the upper semicontinuity of the counting function.

Consider the resolvent

$$R = R(\rho;\mathbf{k},g) = \big(\hat H(\rho;\mathbf{k},g) - \rho^{2l}\big)^{-1}, \quad R_0 = R_0(\rho;\mathbf{k}) = R(\rho;\mathbf{k},0).$$

We shall show that the operator $R(\rho;\mathbf{k},g)$ is norm-continuous with respect to the coupling parameter $g \in [0,1]$. For this, we write the resolvent identity, using the representation $R_0 = |R_0|U$ with a unitary U:

$$|R_0|^{-1/2}R = |R_0|^{1/2}U - g|R_0|^{1/2}UW|R_0|^{1/2}|R_0|^{-1/2}R.$$

The operators $|R_0|, U$ are easy to compute: their symbols are

$$|p(\boldsymbol\phi + \mathbf{k};\rho)|^{-1}, \quad \frac{p(\boldsymbol\phi + \mathbf{k};\rho)}{|p(\boldsymbol\phi + \mathbf{k};\rho)|},$$

respectively. Since, clearly, U commutes with $|R_0|^{1/2}$, we have

$$|R_0|^{-1/2}R = |R_0|^{1/2}U - gUA|R_0|^{-1/2}R, \quad A = |R_0|^{1/2}W|R_0|^{1/2}. \tag{4.5}$$

We prove that the norm of the operator

$$A = \sum_{\boldsymbol\theta} A_{\boldsymbol\theta}, \quad A_{\boldsymbol\theta} = |R_0|^{1/2}Q_{\boldsymbol\theta}V_{\boldsymbol\theta}Q_{\boldsymbol\theta}|R_0|^{1/2},$$

occurring in (4.5), does not exceed 1/4. It is straightforward to find the symbol of $A_{\boldsymbol\theta}$:

$$\frac{1}{\sqrt{\mathrm{d}(\mathsf{M})}} \frac{\hat V(\boldsymbol\theta)e^{i\boldsymbol\theta\mathbf{x}}(1 - \chi(\boldsymbol\phi + \mathbf{k};\Omega_{\boldsymbol\theta}))(1 - \chi(\boldsymbol\phi + \boldsymbol\theta + \mathbf{k};\Omega_{\boldsymbol\theta}))}{\sqrt{p(\boldsymbol\phi + \mathbf{k};\rho)p(\boldsymbol\phi + \mathbf{k} + \boldsymbol\theta;\rho)}}.$$

Obviously, this symbol has only one nontrivial Fourier coefficient, which corresponds to the exponential $\mathbf{e}_{\boldsymbol\theta}$. By the definition (4.2), this coefficient does not exceed

$$|\hat V(\boldsymbol\theta)| \frac{w(\boldsymbol\theta)}{4\|V\|_w},$$

whence, by (4.1),

$$\|A_{\boldsymbol\theta}\| \le \frac{1}{\sqrt{\mathrm{d}(\mathsf{M})}} |\hat V(\boldsymbol\theta)| \frac{w(\boldsymbol\theta)}{4\|V\|_w},$$

so that

$$\|A\| \le \sum_{\boldsymbol\theta} \|A_{\boldsymbol\theta}\| \le \frac{1}{4},$$

by the definition of $\|V\|_w$. This means that for all $g \in [0,1]$, equation (4.5) can be solved by using a simple von Neumann decomposition:

$$R = |R_0|^{1/2}(I + gUA)^{-1}|R_0|^{1/2}U = R_0 + \sum_{j=1}(-1)^j g^j |R_0|^{1/2}(UA)^j |R_0|^{1/2}U. \tag{4.6}$$

This relation has the following two implications. First, $l = \rho^{2l}$ is not an eigenvalue of the operator $\hat{H}(\rho;\mathbf{k},g)$ for all $g \in [0,1]$, because the operator $R(\rho;\mathbf{k},g)$ is bounded together with $R_0(\rho;\mathbf{k})$. Second, the operator $R(\rho;\mathbf{k},g)$ is obviously norm-continuous in $g \in [0,1]$. Consequently, the eigenvalues of $R(\rho;\mathbf{k},g)$ are continuous functions of g. In combination with the boundedness of the perturbed resolvent for all g, this implies that the eigenvalues of the operator $\hat{H}(\rho;\mathbf{k},g)$ do not cross the point ρ^{2l} as the parameter g varies from 0 to 1, and (4.4) follows. □

§5. Counting lattice points

As in §4, we assume that V is M-periodic and $\hat{H}$ is given by (4.3). To estimate the difference between $N(\rho^{2l}; H(\mathbf{k}))$ and $N(\rho^{2l}; \hat{H}(\rho;\mathbf{k}))$, we need to estimate the rank of the perturbation

$$T(\rho;\mathbf{k}) = \sum_{\boldsymbol{\theta}} (P_{\boldsymbol{\theta}} V_{\boldsymbol{\theta}} Q_{\boldsymbol{\theta}} + Q_{\boldsymbol{\theta}} V_{\boldsymbol{\theta}} P_{\boldsymbol{\theta}} + P_{\boldsymbol{\theta}} V_{\boldsymbol{\theta}} P_{\boldsymbol{\theta}}). \tag{5.1}$$

The dimension of this operator does not exceed $\sum_{\boldsymbol{\theta}\in\Theta} 2\mathcal{N}[\Omega_{\boldsymbol{\theta}}(\rho;\mathbf{k})]$, where $\mathcal{N}[\mathcal{C}]$ stands for the number of lattice points $\boldsymbol{\phi} \in \Gamma^\dagger$ in the set $\mathcal{C}$.

We are interested in the number $\mathcal{N}$ averaged in $\mathbf{k}$, i.e., we want to estimate the L^1-norm

$$\|\mathcal{N}[\Omega_{\boldsymbol{\theta}}(\rho;\,\cdot\,)]\|_1 = \int_{\mathcal{O}^\dagger} \mathcal{N}[\Omega_{\boldsymbol{\theta}}(\rho;\mathbf{k})]\, d\mathbf{k}.$$

Rewriting the counting function in the form

$$\mathcal{N}[\Omega_{\boldsymbol{\theta}}(\rho;\mathbf{k})] = \sum_{\boldsymbol{\phi}\in\mathsf{M}^\dagger} \chi\big(\boldsymbol{\phi}+\mathbf{k}; \Omega_{\boldsymbol{\theta}}(\rho)\big), \quad \Omega_{\boldsymbol{\theta}}(\rho) = \Omega_{\boldsymbol{\theta}}(\rho;\mathbf{0}),$$

we immediately conclude that

$$\|\mathcal{N}[\Omega_{\boldsymbol{\theta}}(\rho;\,\cdot\,)]\|_1 = \operatorname{vol}\Omega_{\boldsymbol{\theta}}(\rho).$$

This volume will be estimated individually for each $\boldsymbol{\theta}$.

Lemma 5.1. *Suppose that* $2l \ge 1$ *and* $8\|V\|_w w(\pm\boldsymbol{\theta})^{-1} \le \rho^{2l}$. *Then*

$$\begin{aligned} \operatorname{vol}\Omega_{\boldsymbol{\theta}}(\rho) &\le C\Big[\|V\|_w^2 w(\boldsymbol{\theta})^{-2} + \|V\|_w^{3/2} w(\boldsymbol{\theta})^{-3/2}|\boldsymbol{\theta}|^l\Big] \\ &\quad\times \rho^{d+1-4l}|\boldsymbol{\theta}|^{-1}\Big[|\ln\rho| + |\ln\|V\|_w| + |\ln w(\boldsymbol{\theta})| + |\ln|\boldsymbol{\theta}|| + 1\Big], \quad d=2, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \operatorname{vol}\Omega_{\boldsymbol{\theta}}(\rho) &\le C\|V\|_w^2 w(\boldsymbol{\theta})^{-2}\rho^{d+1-4l}|\boldsymbol{\theta}|^{-1} \\ &\quad\times\Big[|\ln\rho| + |\ln\|V\|_w| + |\ln w(\boldsymbol{\theta})| + |\ln|\boldsymbol{\theta}|| + 1\Big], \quad d\ge 3. \end{aligned} \tag{5.3}$$

We start with some preparatory constructions. We split $\Omega_{\boldsymbol{\theta}}(\rho;\mathbf{k})$ in two pieces:

$$\begin{aligned} \Omega_{\boldsymbol{\theta}}(\rho;\mathbf{k}) &= \overline{\Omega^{>}_{\boldsymbol{\theta}}(\rho;\mathbf{k}) \cup \Omega^{<}_{\boldsymbol{\theta}}(\rho;\mathbf{k})}, \\ \Omega^{>}_{\boldsymbol{\theta}}(\rho;\mathbf{k}) &= \Big\{\boldsymbol{\xi}\in\Omega_{\boldsymbol{\theta}}(\rho;\mathbf{k}) : |p(\boldsymbol{\xi}+\boldsymbol{\theta}+\mathbf{k};\rho)| > |p(\boldsymbol{\xi}+\mathbf{k};\rho)|\Big\}, \\ \Omega^{<}_{\boldsymbol{\theta}}(\rho;\mathbf{k}) &= \Big\{\boldsymbol{\xi}\in\Omega_{\boldsymbol{\theta}}(\rho;\mathbf{k}) : |p(\boldsymbol{\xi}+\boldsymbol{\theta}+\mathbf{k};\rho)| < |p(\boldsymbol{\xi}+\mathbf{k};\rho)|\Big\}. \end{aligned} \tag{5.4}$$

Observe that $\Omega_{\boldsymbol\theta}(\rho;\mathbf{k}) = \Omega_{-\boldsymbol\theta}(\rho;\mathbf{k}+\boldsymbol\theta)$ and $\Omega^{<}_{\boldsymbol\theta}(\rho;\mathbf{k}) = \Omega^{>}_{-\boldsymbol\theta}(\rho;\mathbf{k}+\boldsymbol\theta)$. Consequently, $\operatorname{vol}\Omega^{<}_{\boldsymbol\theta}(\rho) = \operatorname{vol}\Omega^{>}_{-\boldsymbol\theta}(\rho)$. Thus, it suffices to estimate the volume of the set $\Omega^{>}_{\boldsymbol\theta}(\rho)$ from above for each individual $\boldsymbol\theta$.

The next lemma provides the required estimates for suitable "model" sets.

Lemma 5.2. *Suppose $d \ge 2$ and $2l \ge 1$. For numbers $\theta > 0$ and $a > 0$, define the following sets in the space $\mathbb{R}^d$. If $d = 2$, then*

(5.5)
$$\text{Ł}^{(2)}_\theta(\rho,a) = \Big\{\boldsymbol\xi \in \mathbb{R}^2 : \xi_1 \ge 0, \big|p(\boldsymbol\xi;\rho)\big|\big|p(\xi_1,\xi_2+\theta;\rho)\big| \le a^2, |p(\boldsymbol\xi;\rho)| < |p(\xi_1,\xi_2+\theta;\rho)|\Big\}.$$

If $d \ge 3$, then

$$\text{Ł}^{(d)}_\theta(\rho,a) = \Big\{\boldsymbol\xi \in \mathbb{R}^d : \big|p(\boldsymbol\xi;\rho)\big|\big|p(\hat{\boldsymbol\xi},\xi_d+\theta;\rho)\big| \le a^2, |p(\boldsymbol\xi;\rho)| < |p(\hat{\boldsymbol\xi},\xi_d+\theta;\rho)|\Big\}, \tag{5.6}$$

$\hat{\boldsymbol\xi} = (\xi_1,\xi_2,\dots,\xi_{d-1})$. *Assume that $2a \le \rho^{2l}$. Then*

(5.7)
$$\operatorname{vol}\text{Ł}^{(d)}_\theta(\rho,a) \le \begin{cases} C\big(a^2 + a^{3/2}\theta^l\big)\rho^{3-4l}\theta^{-1}\big[|\ln(\rho\theta^{-1})| + |\ln(a\theta^{-2l})| + 1\big] & \text{if } d = 2,\\ Ca^2\rho^{d+1-4l}\theta^{-1}\big[|\ln(\rho\theta^{-1})| + |\ln(a\theta^{-2l})| + 1\big] & \text{if } d \ge 3,\end{cases}$$

with a constant C depending only on the parameter l and the dimension d.

Proof of Lemma 5.2. Note that

$$\operatorname{vol}\text{Ł}^{(d)}_\theta(\rho,a) = \theta^d \operatorname{vol}\text{Ł}^{(d)}_1(\rho\theta^{-1}, a\theta^{-2l}).$$

Therefore, it suffices to prove estimates (5.7) for $\theta = 1$ under the condition $2a \le \rho^{2l}$; that is,

$$\operatorname{vol}\text{Ł}^{(d)}_1(\rho,a) \le \begin{cases} C\big(a^2 + a^{3/2}\big)\rho^{3-4l}\big[|\ln\rho| + |\ln a| + 1\big] & \text{if } d = 2,\\ Ca^2\rho^{d+1-4l}\big[|\ln\rho| + |\ln a| + 1\big] & \text{if } d \ge 3.\end{cases} \tag{5.8}$$

From now on we assume that $\theta = 1$.

Our first step is to show that the problem for $d \ge 3$ can be reduced to $d = 2$. Indeed, assume that $d \ge 3$. Since the set $\text{Ł}^{(d)}_1(\rho,a)$ is axially symmetric, it is convenient to introduce the cylindrical coordinates:

$$\boldsymbol\xi = \{t,\eta,\boldsymbol\omega\}, \quad t \ge 0,\ \ \eta \in \mathbb{R},\ \ \boldsymbol\omega \in \mathbb{S}^{d-2},$$

$$\eta = \xi_d, \quad t = \sqrt{|\boldsymbol\xi|^2 - \eta^2}, \quad \boldsymbol\omega = \frac{\hat{\boldsymbol\xi}}{|\hat{\boldsymbol\xi}|}, \quad \hat{\boldsymbol\xi} = \{\xi_1,\xi_2,\dots,\xi_{d-1}\}.$$

In these coordinates,

$$p(\boldsymbol\xi;\rho) = (t^2+\eta^2)^l - \rho^{2l}, \quad p(\hat{\boldsymbol\xi},\xi_d+1;\rho) = \big(t^2 + (\eta+1)^2\big)^l - \rho^{2l},$$

whence

$$\operatorname{vol}\text{Ł}^{(d)}_1(\rho,a) = \int_{\boldsymbol\omega\in\mathbb{S}^{d-2}}\int_{(t,\eta)\in\text{Ł}^{(2)}_1(\rho,a)} t^{d-2}\,dtd\eta d\boldsymbol\omega \le C\rho^{d-3}\int_{(t,\eta)\in\text{Ł}^{(2)}_1(\rho,a)} t\,dtd\eta.$$

Recalling the formula

$$\operatorname{vol}\text{Ł}^{(2)}_1(\rho,a) = \int_{(t,\eta)\in\text{Ł}^{(2)}_1(\rho,a)} dtd\eta,$$

we see that, in order to get estimates (5.8), we need to show that the integral

$$V_\sigma\big(\text{Ł}^{(2)}_1(\rho,a)\big), \ \ V_\sigma\big(\text{Ł}\big) = \int_{(t,\eta)\in\text{Ł}} |t|^\sigma\,dtd\eta, \quad \sigma = 0,1, \tag{5.9}$$

admits the following bounds:

$$V_\sigma\big(Ł_1^{(2)}(\rho,a)\big) \le \begin{cases} C\big(a^2+a^{3/2}\big)\rho^{3-4l}\big[|\ln\rho|+|\ln a|+1\big] & \text{if } s=0;\\ Ca^2\rho^{4-4l}\big[|\ln\rho|+|\ln a|+1\big] & \text{if } \sigma=1.\end{cases} \tag{5.10}$$

As the issue is now reduced to the two-dimensional case, we omit "(2)" from the notation and write simply $Ł_1(\rho,a)$. From now on $\boldsymbol\xi\in\mathbb{R}^2$.

By the definition of $Ł_1(\rho,a)$, we have

$$|p(\boldsymbol\xi;\rho)| = \big||\boldsymbol\xi|^{2l}-\rho^{2l}\big| \le a, \quad \boldsymbol\xi\in Ł_1(\rho,a).$$

Since $2a\le\rho^{2l}$ and $2l\ge 1$, we have

$$\frac12\rho \le |\boldsymbol\xi| \le \frac32\rho, \quad \boldsymbol\xi\in Ł_1(\rho,a). \tag{5.11}$$

Moreover,

$$\begin{cases} \big||\boldsymbol\xi|^2-\rho^2\big| \le C_1|p(\boldsymbol\xi;\rho)|\rho^{2-2l} \le C_1 a\rho^{2-2l},\\ \quad \big||\boldsymbol\xi|-\rho\big| \le \tilde C_1|p(\boldsymbol\xi;\rho)|\rho^{1-2l} \le \tilde C_1 a\rho^{1-2l},\end{cases} \quad \boldsymbol\xi\in Ł_1(\rho,a), \tag{5.12}$$

with constants C_1 and $\tilde C_1$ depending only on l. With the notation $s=(\xi_2+1)^2-\xi_2^2$, from the first inequality in (5.12) we obtain

$$\big|(|\boldsymbol\xi|^2+s)^l-\rho^{2l}\big| \ge C_1^{-1}\rho^{2l-2}\big||\boldsymbol\xi|^2+s-\rho^2\big| \ge C_1^{-1}\big||s|\rho^{2l-2}-C_1a\big|. \tag{5.13}$$

We split $Ł_1(\rho,a)$ into two disjoint subsets:

$$Ł^{(+)}(\rho,a) = \{\boldsymbol\xi\in Ł_1(\rho,a) : |s|\rho^{2l-2}\ge 2C_1a\},$$
$$Ł^{(-)}(\rho,a) = \{\boldsymbol\xi\in Ł_1(\rho,a) : |s|\rho^{2l-2} < 2C_1a\}.$$

By (5.13) and the definition of $Ł_1(\rho,a)$, for all $\boldsymbol\xi\in Ł^{(+)}(\rho,a)$ we have

$$\big||\boldsymbol\xi|^{2l}-\rho^{2l}\big| \le \frac{a^2}{\big|(|\boldsymbol\xi|^2+s)^l-\rho^{2l}\big|} \le 2C_1a^2\rho^{2-2l}|s|^{-1},$$

which immediately implies the estimate

$$\big||\boldsymbol\xi|-\rho\big| \le \tilde C_1a^2\rho^{3-4l}|s|^{-1}, \quad \boldsymbol\xi\in Ł^{(+)}(\rho,a). \tag{5.14}$$

From this point on, we divide the proof into several steps. The cases where $\sigma=0$ and $\sigma=1$ are treated simultaneously. Note that the case of $\sigma=1$ is covered entirely by Case 1 below.

Case 1: Proof of (5.10) for $\rho>0$ and $\sigma=1$, or $\rho\ge 2$ and $\sigma=0$. We split the domain $Ł^{(+)}(\rho,a)$ into two disjoint subdomains:

$$\Phi_1=\Phi_1(\rho,a)=\Big\{\boldsymbol\xi\in Ł^{(+)}(\rho,a) : \big||\boldsymbol\xi|-\rho\big|\le 8\tilde C_1a^2\rho^{2-4l}\Big\},$$
$$\Phi_2=\Phi_2(\rho,a)=\Big\{\boldsymbol\xi\in Ł^{(+)}(\rho,a) : \big||\boldsymbol\xi|-\rho\big| > 8\tilde C_1a^2\rho^{2-4l}\Big\},$$

so that

$$Ł_1(\rho,a) = Ł^{(-)}(\rho,a)\cup\Phi_1(\rho,a)\cup\Phi_2(\rho,a). \tag{5.15}$$

The integral (5.9) over Φ_1 does not exceed $Ca^2\rho^{\sigma+3-4l}$ with a suitable constant C, which gives (5.10). It remains to estimate the integral over Φ_2. By (5.14), on the set Φ_2 we have

$$|s| \le \frac{\rho}{8}. \tag{5.16}$$

Using the coordinates ξ_2 and $r = |\boldsymbol{\xi}|$, we have

$$V_\sigma(\Phi_2) = \int_{\Phi_2} \xi_1^\sigma \, d\xi_1 d\xi_2 = \int_{\Phi_2} r(r^2 - \xi_2^2)^{\frac{\sigma-1}{2}} \, d\xi_2 dr.$$

We prove that

$$r(r^2 - \xi_2^2)^{\frac{\sigma-1}{2}} \le C\rho^\sigma, \quad \boldsymbol{\xi} \in \Phi_2, \tag{5.17}$$

if $\sigma = 1$ and $\rho > 0$, or $\sigma = 0$ and $\rho \ge 2$. For $\sigma = 1$ the estimate follows immediately from the inequality $r \le 3\rho/2$ (see (5.11)).

For $\sigma = 0$ and $\rho \ge 2$, we recall that $|s| = |2\xi_2 + 1| \le \rho/8$ (see (5.16)), so that, by (5.11),

$$|\xi_2| \le \frac{\rho}{16} + \frac{1}{2} \le \frac{5\rho}{16} \le \frac{5}{8} r$$

for all $\boldsymbol{\xi} \in \Phi_2$. Consequently, $r^2 - \xi_2^2 \ge r^2/2$, and (5.17) with $\sigma = 0$ is satisfied. From (5.17) and (5.14) it follows that, both for $\sigma = 0$ and for $\sigma = 1$,

$$\begin{aligned} V_\sigma(\Phi_2) &\le C\rho^\sigma \int_{8\tilde{C}_1 a^2 \rho^{2-4l} \le |r-\rho| \le \tilde{C}_1 a \rho^{1-2l}} \int_{|2\xi_2+1| < Ca^2\rho^{3-4l}\frac{1}{|r-\rho|}} d\xi_2 dr \\ &\le \tilde{C} a^2 \rho^{\sigma+3-4l} \int_{8\tilde{C}_1 a^2 \rho^{2-4l} \le |r-\rho| \le \tilde{C}_1 a \rho^{1-2l}} \frac{1}{|r-\rho|} dr \\ &\le \tilde{C}' a^2 \rho^{\sigma+3-4l} \big(|\ln \rho| + |\ln a| + 1 \big). \end{aligned} \tag{5.18}$$

Similarly,

$$\begin{aligned} V_\sigma(\text{Ł}^{(-)}(\rho, a)) &\le C\rho^\sigma \int_{|r-\rho| \le \tilde{C}_1 a \rho^{1-2l}} \int_{|2\xi_2+1| < 2C_1 a \rho^{2-2l}} d\xi_2 dr \\ &\le \tilde{C} a^2 \rho^{\sigma+3-4l}. \end{aligned}$$

Collecting the estimates for Φ_1, Φ_2, and $\text{Ł}^{(-)}(\rho, a)$, we arrive at (5.10).

Since the case of $\sigma = 1$ is already covered, we concentrate on $\sigma = 0$ only.

Case 2: $\rho \le C_3 = \min\big(6^{-1}, (2C_1)^{-1/2}\big)$, $\sigma = 0$. Since $|\xi_2| \le |\boldsymbol{\xi}| \le 3/2C_3 \le 1/4$ (see (5.11)), we have $|s| \ge 1 - 2|\xi_2| \ge 1/2$. Moreover, since $2a \le \rho^{2l}$ and $\rho \le (2C_1)^{-1/2}$, the set $\text{Ł}^{(-)}(\rho, a)$ is empty. As a consequence, the estimate (5.14) ensures that

$$V_0(\text{Ł}_1(\rho, a)) = \operatorname{vol} \text{Ł}^{(+)}(\rho, a) \le Ca^2\rho^{4-4l} \le CC_3 a^2 \rho^{3-4l},$$

which yields (5.10).

Case 3: $C_3 \le \rho \le 2$, $\sigma = 0$. Again, we use the partition (5.15). The integral over Φ_1 is estimated as at Step 1. In its turn, the domain Φ_2 splits into two disjoint subdomains: Φ_{21} with $|\xi_2| > r - a$ and Φ_{22} with $|\xi_2| \le r - a$. We use the estimate

$$\frac{r}{\sqrt{r^2 - \xi_2^2}} \le C\sqrt{\frac{r}{a}} \quad \text{if } |\xi_2| \le r - a \tag{5.19}$$

and repeat estimate (5.18) replacing ρ^σ with $\sqrt{\rho a^{-1}} \le \sqrt{2a^{-1}}$:

$$V_0(\Phi_{22}) \le Ca^{3/2}\rho^{3-4l}(|\ln \rho| + |\ln a| + 1).$$

In order to handle Φ_{21}, we estimate

$$\begin{aligned} V_0(\Phi_{21}) &\le \int_{|r-\rho| \le \tilde{C}_1 a \rho^{1-2l}} \int_{r-a \le |\xi_2| \le r} \frac{r}{\sqrt{r^2 - \xi_2^2}} d\xi_2 dr \\ &\le \tilde{C}\sqrt{a} \int_{|r-\rho| \le \tilde{C}_1 a \rho^{1-2l}} \sqrt{r} \, dr \\ &\le \tilde{C}' a^{3/2} \rho^{\frac{3}{2}-2l} \le \tilde{C}'' a^{3/2} \rho^{3-4l}. \end{aligned} \tag{5.20}$$

For the set $Ł^{(-)}(\rho, a)$ we write

$$V_0\big(Ł^{(-)}(\rho,a)\big) \le \int_{|r-\rho|\le \tilde C_1 a\rho^{1-2l}} \int_{|\xi_2|\le r-a, |2\xi_2+1|\le 2C_1 a\rho^{2-2l}} \frac{r}{\sqrt{r^2-\xi_2^2}}\, d\xi_2 dr + \int_{|r-\rho|\le \tilde C_1 a\rho^{1-2l}} \int_{r-a\le|\xi_2|\le r} \frac{r}{\sqrt{r^2-\xi_2^2}}\, d\xi_2 dr.$$

The second term is treated as in (5.20). By using (5.19), the first term can be estimated from above by

$$\frac{C}{\sqrt{a}} \int_{|r-\rho|\le \tilde C_1 a\rho^{1-2l}} \int_{|2\xi_2+1|\le 2C_1 a\rho^{2-2l}} \sqrt{r}\, d\xi_2 dr \le \tilde C a^{3/2}\rho^{3-4l}.$$

Combining the upper bounds for Φ_1, Φ_{21}, Φ_{22}, and $Ł^{(-)}$, we get the required estimate (5.10).

The proof of estimate (5.10) is complete. As was explained earlier, (5.10) implies (5.8) and, with it, (5.7), as required. □

Now we are prepared to prove Lemma 5.1.

Proof of Lemma 5.1. The remark before Lemma 5.2 shows that it suffices to estimate the volume of the set $\Omega^{>}_{\boldsymbol\theta}(\rho)$. We use Lemma 5.2 with $a = 4\|V\|_w w(\boldsymbol\theta)^{-1}$. Note that the condition $2a \le \rho^{2l}$ is satisfied.

By rotating $\Omega^{>}_{\boldsymbol\theta}(\rho)$, we can align the basis vector $(0, 0, \dots, 1)$ with the direction of $\boldsymbol\theta$,

$$p(\boldsymbol\xi+\boldsymbol\theta;\rho) = p(\hat{\boldsymbol\xi}, \xi_d+\theta;\rho), \quad \theta = |\boldsymbol\theta|.$$

By the definitions (4.2), (5.4), and (5.6), the set $\Omega^{>}_{\boldsymbol\theta}(\rho)$ coincides with $Ł^{(d)}(\rho, a)$ if $d \ge 3$. If $d = 2$, then, observing that $\Omega^{>}_{\boldsymbol\theta}(\rho)$ is symmetric with respect to the reflection $\xi_1 \to -\xi_1$, we conclude that $\operatorname{vol}\Omega^{>}_{\boldsymbol\theta}(\rho) = 2\operatorname{vol}Ł^{(2)}_\theta(\rho, a)$. Now the required estimate follows directly from Lemma 5.2. □

§6. Proof of Theorem 3.2

Now we can use Lemma 5.1 to estimate the rank of the perturbation (5.1).

Theorem 6.1. *Let V be an M-periodic potential with $\hat V(\mathbf{0}) = 0$, and let $2l \ge 1$. Suppose that $\|V\|_w < \infty$ for some weight function w such that w^{-1} is bounded and*

$$Y(w) = \sum_{\boldsymbol\theta\in\Theta} w(\boldsymbol\theta)^{-2}|\boldsymbol\theta|^{-1}\big(|\ln w(\boldsymbol\theta)| + |\ln|\boldsymbol\theta|| + 1\big) < \infty.$$

Then, under the condition

$$\rho^{2l} \ge 8\|V\|_w \sup_{\boldsymbol\theta\in\mathsf{M}^\dagger} w(\boldsymbol\theta)^{-1},$$

for $d \ge 3$ we have

$$\int_{\mathcal{O}^\dagger} \dim T(\rho;\mathbf{k})\, d\mathbf{k} \le CY(w)\|V\|_w^2(1+|\ln\|V\|_w|)\rho^{d+1-4l}\big(|\ln\rho|+1\big) \tag{6.1}$$

with a constant C depending only on l and the dimension d. If, moreover,

$$Z(w) = \sum_{\boldsymbol\theta\in\Theta} w(\boldsymbol\theta)^{-\frac32}|\boldsymbol\theta|^{l-1}\big(|\ln w(\boldsymbol\theta)| + |\ln|\boldsymbol\theta|| + 1\big) < \infty,$$

then for $d = 2$ we have

$$\int_{\mathcal{O}^\dagger} \dim T(\rho;\mathbf{k})\, d\mathbf{k} \le C\big(Y(w)+Z(w)\big)\big(\|V\|_w^2 + \|V\|_w^{3/2}\big)(1+|\ln\|V\|_w|)\rho^{3-4l}\big(|\ln\rho|+1\big) \tag{6.2}$$

with a constant C depending only on l.

Proof. By the definition (5.1),

$$\int_{\mathcal{O}^\dagger} \dim T(\rho; \mathbf{k}) \le 2 \sum_{\boldsymbol{\theta}\in\Theta} \operatorname{vol}\Omega_{\boldsymbol{\theta}}(\rho).$$

It remains to apply Lemma 5.1. □

Before proving Theorem 3.2, we establish an elementary lemma.

Lemma 6.2. *Let $f \in \mathsf{C}^1(\mathbb{R}^d \setminus \{0\}) \cap \mathsf{L}^1(\mathbb{R}^d)$ be a real-valued function with positive values and such that*

$$|\nabla f(\boldsymbol{\xi})| \le C|\boldsymbol{\xi}|^{-1}|f(\boldsymbol{\xi})|, \quad \boldsymbol{\xi} \ne 0. \tag{6.3}$$

Then for any lattice Γ and any integer $p \ge 1$ we have

$$\sum_{\mathbf{0}\ne\boldsymbol{\theta}\in(p\Gamma)^\dagger} f(\boldsymbol{\theta}) \le \tilde{C} p^d \int_{\mathbb{R}^d} f(\boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{6.4}$$

with a constant $\tilde{C}$ depending only on Γ and the constant C in (6.3).

Proof. Let $p = 1$. Let $K_{\mathbf{0}} = \mathcal{O}^\dagger_{\Gamma'}$ be the cell of the lattice $\Gamma' = 2\Gamma$, obtained from the cell of Γ by dividing it into 2^d equal parallelepipeds. Denote $K_{\boldsymbol{\theta}} = K_{\mathbf{0}} + \boldsymbol{\theta}$. By (6.3), for all $\boldsymbol{\xi} \in K_{\boldsymbol{\theta}}$ we have

$$|\ln f(\boldsymbol{\xi}) - \ln f(\boldsymbol{\theta})| \le \max_{\boldsymbol{\eta}\in K_{\boldsymbol{\theta}}} \frac{|\nabla f(\boldsymbol{\eta})|}{f(\boldsymbol{\eta})} |\boldsymbol{\xi} - \boldsymbol{\theta}| \le C \max_{\boldsymbol{\eta}\in K_{\boldsymbol{\theta}}} \frac{1}{|\boldsymbol{\eta}|} |\boldsymbol{\xi} - \boldsymbol{\theta}|.$$

Since

$$\sup_{\boldsymbol{\theta}\in\Gamma^\dagger} \max_{\boldsymbol{\xi}\in K_{\boldsymbol{\theta}}} |\boldsymbol{\xi} - \boldsymbol{\theta}| = C_\Gamma < \infty, \quad \sup_{\boldsymbol{\theta}\in\Gamma^\dagger} \max_{\boldsymbol{\xi}\in K_{\boldsymbol{\theta}}} |\boldsymbol{\xi}|^{-1} = C'_\Gamma < \infty,$$

it follows that

$$c \le \frac{f(\boldsymbol{\xi})}{f(\boldsymbol{\theta})} \le C, \quad \boldsymbol{\xi} \in K_{\boldsymbol{\theta}},$$

uniformly in $\boldsymbol{\theta}$, whence

$$\operatorname{vol} K_{\boldsymbol{\theta}} f(\boldsymbol{\theta}) \le c^{-1} \int_{K_{\boldsymbol{\theta}}} f(\boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

This leads to (6.4) with $p = 1$.

For arbitrary $p \ge 1$, note that

$$\sum_{\boldsymbol{\theta}\in(p\Gamma)^\dagger} f(\boldsymbol{\theta}) = \sum_{\boldsymbol{\theta}\in\Gamma^\dagger} g(\boldsymbol{\theta}), \quad g(\boldsymbol{\theta}) = f(p^{-1}\boldsymbol{\theta}),$$

and the function g satisfies (6.3) together with f. This observation reduces the problem to the case where $p = 1$; hence, the first part of the proof yields the required upper bound (6.4). □

Proof of Theorem 3.2. Recall that the function V given by (2.2) is periodic with respect to the lattice $\mathsf{M} = s\Gamma$ of index s^d. Without loss of generality, we assume that $V_0 = 0$.

In order to use the estimates established in Theorem 6.1, we need to introduce an appropriate weight function w. Since V is given by (2.2), we have

$$\Theta \subset \mathsf{N}_s^\dagger = \bigcup_{\Lambda\in\mathfrak{M}_s} \Lambda^\dagger.$$

We decompose the set $\mathsf{N}_s^\dagger$ into a union of disjoint sets of the form

$$\mathsf{K}_p^\dagger = \mathsf{N}_p^\dagger \setminus \mathsf{N}_{p-1}^\dagger, \ \ p \ge 2; \quad \mathsf{K}_1^\dagger = \Gamma^\dagger.$$

For all $\boldsymbol\theta \in \mathsf{N}_s^\dagger$, define

$$w(\boldsymbol\theta) = m_p(1 + |\boldsymbol\theta|^\nu), \quad \boldsymbol\theta \in \mathsf{K}_p^\dagger, \ \ 1 \le p \le s, \tag{6.5}$$

so that

$$w(\boldsymbol\theta) \le m_p(1 + |\boldsymbol\theta|^\nu), \quad \boldsymbol\theta \in \mathsf{N}_p^\dagger, \ \ 1 \le p \le s,$$

because the sequence m_p is monotone. Since for any $\Lambda \in \mathfrak{N}_p$ we have $\Lambda^\dagger \subset \mathsf{N}_p^\dagger$, it follows that if $\Lambda \in \mathfrak{N}_p$, then

$$w(\boldsymbol\theta) \le m_p(1 + |\boldsymbol\theta|^\nu), \quad \boldsymbol\theta \in \Lambda^\dagger.$$

Consequently, for any potential V_Λ that is Λ-periodic with $\Lambda \in \mathfrak{N}_p$ we have

$$\|V_\Lambda\|_w = \frac{1}{\sqrt{\mathrm{d}(\Lambda)}} \sum_{\boldsymbol\theta \in \Lambda^\dagger} |\hat V(\boldsymbol\theta; \Lambda)| w(\boldsymbol\theta) \le m_p \,\mathbf{|} V_\Lambda \mathbf{|}_\nu.$$

Thus, a potential of the form (2.2) satisfies

$$\|V\|_w \le \sum_{p=1}^s m_p \sum_{\Lambda \in \mathfrak{N}_p} \mathbf{|} V_\Lambda \mathbf{|}_\nu = \mathbf{|} V \mathbf{|};$$

see the definition (2.5). Now we estimate the sums $Y(w)$ and $Z(w)$, keeping in mind that $\Theta \subset \mathsf{N}_s^\dagger$:

$$\begin{aligned} Y(w) &\le \sum_{p=1}^s \sum_{0 \ne \boldsymbol\theta \in \mathsf{K}_p^\dagger} w(\boldsymbol\theta)^{-2} |\boldsymbol\theta|^{-1} \big(|\ln w(\boldsymbol\theta)| + |\ln|\boldsymbol\theta|| + 1 \big) \\ &\le C \sum_{p=1}^s m_p^{-2} (|\ln m_p| + 1) \sum_{0 \ne \boldsymbol\theta \in \mathsf{K}_p^\dagger} (1 + |\boldsymbol\theta|^\nu)^{-2} |\boldsymbol\theta|^{-1} \big(|\ln|\boldsymbol\theta|| + 1 \big). \end{aligned}$$

To find the sum over $\mathsf{K}_p^\dagger$, observe that it does not exceed the sum over $\Lambda^\dagger$, where $\Lambda = p\Gamma$ is the lattice of index p^d. Since ν satisfies (2.3), the function

$$f(\boldsymbol\xi) = (1 + |\boldsymbol\xi|^\nu)^{-2} |\boldsymbol\xi|^{-1} (|\ln|\boldsymbol\xi|| + 1)$$

is integrable. Since f satisfies (6.3), we can use Lemma 6.2 to get the estimate

$$Y(w) \le C_\Gamma \sum_{p=1}^s p^d m_p^{-2} (|\ln m_p| + 1).$$

Similarly,

$$Z(w) \le C'_\Gamma \sum_{p=1}^s p^d m_p^{-\frac{3}{2}} (|\ln m_p| + 1).$$

Recalling the definitions (2.4) and (3.6), now we can rewrite (6.1) and (6.2) as one formula:

$$\int_{\mathcal{O}^\dagger} \dim T(\rho; \mathbf{k})\, d\mathbf{k} \le C B_d A(V) \rho^{d+1-4l} \big(|\ln \rho| + 1 \big) \tag{6.6}$$

with a constant C depending only on the lattice Γ and the numbers ν, l, and d.

By the definitions (4.3) and (5.1), we have $H(\mathbf{k}) = \hat H(\mathbf{k}) + T(\rho; \mathbf{k})$. By Theorem 4.1, a straightforward perturbation-theoretic argument shows that

$$|N(\rho; H(\mathbf{k})) - N(\rho; H_0(\mathbf{k}))| = |N(\rho; H(\mathbf{k})) - N(\rho; \hat H(\mathbf{k}))| \le \dim T(\rho; \mathbf{k}).$$

A reference to (6.6) completes the proof of Theorem 3.2. □

§7. Proof of the main theorem

7.1. Lattice points in large balls. A core of the proof of Theorem 2.1 is the fact that the counting function $N(l; H_0(\mathbf{k}))$ is directly related to the counting function for lattice points. More precisely, let $\mathcal{N}(\rho; \mathbf{k})$ be the number of lattice points $\boldsymbol{\beta} \in \mathsf{M}^\dagger$ in the ball of radius $\rho \geq 0$ centered at $-\mathbf{k}$, i.e.,

$$\mathcal{N}(\rho; \mathbf{k}) = \sharp\{\boldsymbol{\beta} \in \mathsf{M}^\dagger : |\boldsymbol{\beta}| \leq \rho\}.$$

Then, clearly,

$$N_0(\rho^{2l}; \mathbf{k}) = \mathcal{N}(\rho; \mathbf{k}), \quad \rho \geq 0. \tag{7.1}$$

We introduce the Fourier transform and the mean value of $\mathcal{N}$:

$$\hat{\mathcal{N}}(\rho; \boldsymbol{\mu}) = \int_{\mathcal{O}^\dagger} \mathcal{N}(\rho; \mathbf{k}) e^{i\mathbf{k}\boldsymbol{\mu}}\, d\mathbf{k}, \quad \boldsymbol{\mu} \in \mathsf{M},$$

$$\overline{\mathcal{N}}(\rho) = \frac{1}{\mathrm{d}(\mathsf{M}^\dagger)} \hat{\mathcal{N}}(\rho; \mathbf{0}).$$

From the elementary formula

$$\mathcal{N}(\rho; \mathbf{k}) = \sum_{\boldsymbol{\beta} \in \mathsf{M}^\dagger} \chi_\rho(\boldsymbol{\beta} + \mathbf{k}),$$

where χ_ρ is the characteristic function of the closed ball of radius ρ centered at the origin, we deduce that

$$\hat{\mathcal{N}}(\rho; \boldsymbol{\mu}) = \int_{|\mathbf{k}| \leq \rho} e^{i\boldsymbol{\mu}\mathbf{k}}\, d\mathbf{k}, \quad \overline{\mathcal{N}}(\rho) = \frac{1}{\mathrm{d}(\mathsf{M}^\dagger)}\, \mathrm{w}_d\, \rho^d. \tag{7.2}$$

Here w_d denotes the volume of the unit ball in $\mathbb{R}^d$.

In accordance with our strategy described in Subsection 3.1, in order to show that the spectral bands overlap we prove that the quantity $S(l)$ defined in (3.2) is positive. We start with the unperturbed case, i.e., with a lower bound for

$$\mathcal{S}(\rho) = \int_{\mathcal{O}^\dagger} \big|\mathcal{N}(\rho; \mathbf{k}) - \overline{\mathcal{N}}(\rho)\big|\, d\mathbf{k}. \tag{7.3}$$

The next theorem states a well-known lower bound for $\mathcal{S}(\rho)$; see [3, 5, 10]. Nevertheless, we provide a complete proof since we are interested in the dependence of the constants on the lattice.

Theorem 7.1. *Let $d \geq 2$, $d \neq 1 (\mathrm{mod}\, 4)$, and let $\mu = \min_{\boldsymbol{\mu} \in \mathsf{M}} |\boldsymbol{\mu}|$. Then for all sufficiently large $\rho\mu$ we have the estimate*

$$\mathcal{S}(\rho) \geq c_d \mu^{-\frac{d+1}{2}} \rho^{\frac{d-1}{2}} \tag{7.4}$$

with a constant c_d depending only on the dimension d.

Proof. Observe that

$$\mathcal{S}(\rho) = \int_{\mathcal{O}^\dagger} |\mathcal{N}(\rho, \mathbf{k}) - \overline{\mathcal{N}}(\rho)|\, d\mathbf{k} \geq |\hat{\mathcal{N}}(\rho; \boldsymbol{\mu})|, \quad \boldsymbol{\mu} \in \mathsf{M} \setminus \{\mathbf{0}\}. \tag{7.5}$$

Computing the Fourier coefficient (7.2), we see that

$$\hat{\mathcal{N}}(\rho; \boldsymbol{\mu}) = (2\pi)^{d/2} \mu^{-d/2} \rho^{d/2} J_{d/2}(\rho\mu), \quad \mu = |\boldsymbol{\mu}| \neq 0. \tag{7.6}$$

We need the following elementary property of the Bessel functions:

$$|J_\nu(z)| + |J_\nu(2z)| \geq c_\nu z^{-1/2}, \quad 2\nu \neq 1 (\mathrm{mod}\, 4), \tag{7.7}$$

for all sufficiently large $z > 0$. Indeed, the asymptotics of the Bessel function looks like this (see [1]):

$$J_\nu(z) = -\sqrt{\frac{2}{\pi z}}g(z) + O(z^{-3/2}), \quad z \to \infty,$$

with

$$g(z) = \sin(z + a\pi), \quad a = -\frac{2\nu - 1}{4}.$$

The roots of $g(z)$ and $g(2z)$ are $-a\pi + \pi n$ and $-a\pi/2 + \pi m/2$, $m, n \in \mathbb{Z}$, respectively. Since a is not an integer, these roots never coincide. This proves (7.7).

By (7.5) and (7.6), we have

$$2\mathcal{S}(\rho) \ge |\hat{\mathcal{N}}(\rho;\boldsymbol{\mu})| + |\hat{\mathcal{N}}(\rho;2\boldsymbol{\mu})| \ge c'\mu^{-\frac{d}{2}}\rho^{\frac{d}{2}}\big(|J_{d/2}(\rho\mu)| + |J_{d/2}(2\rho\mu)|\big).$$

Now, (7.7) immediately yields the required lower bound (7.4). □

Remark 7.2. It is worth pointing out that if $d = 1 (\mathrm{mod}\, 4)$, then a lower bound slightly different from (7.4) can be written for $\mathcal{S}(\rho)$ (see [8]). The proof of that bound is more sophisticated, and it is not clear how to control the dependence on the lattice. This is the main reason why we have imposed the condition $d \ne 1 (\mathrm{mod}\, 4)$ in Theorem 2.1.

7.2. Proof of Theorem 2.1. The proof is divided into two steps. First, we establish an estimate similar to (7.4) for $S(l)$ (see (3.2) for the definition) and a lower bound for $z(l)$ (see (3.3) for the definition):

Lemma 7.3. *Suppose that $d \ge 2$, $d \ne 1 (\mathrm{mod}\, 4)$, $8l \ge d + 3$. Let V be a potential satisfying the conditions of Theorem 2.1 with a finite s and some vector $\boldsymbol{\gamma} \in \Gamma$. Suppose also that $\lVert V \rVert \le v$ with some constant v. Then there exists a constant $\rho_0 > 0$ depending only on v and the numbers $\gamma = |\boldsymbol{\gamma}|$, l, d, such that for $\rho \ge \rho_0$ we have*

$$S(\rho^{2l}) \ge c\gamma^{-\frac{d+1}{2}}\rho^{\frac{d-1}{2}}, \tag{7.8}$$

$$z(\rho^{2l}) \ge c\gamma^{-\frac{d+1}{2}}\rho^{\frac{4l-d-1}{2}}, \tag{7.9}$$

with a constant c depending only on l and the dimension d.

Estimate (7.9) shows that the spectral bands of the operator H do not shrink and preserve their overlap as $s \to \infty$.

Proof. Since all the sublattices $\Lambda \in \mathfrak{N}_p(\boldsymbol{\gamma})$ in (2.6) include the vector $\boldsymbol{\gamma}$, the potential V is periodic with respect to a sublattice $\mathsf{M} \subset s\Gamma$ that contains $\boldsymbol{\gamma}$. For any function $f \in \mathsf{L}^1(\mathcal{O}_{\mathsf{M}}^{\dagger})$, its norm is denoted by $\|f\|_1$. We put $l = \rho^{2l}, l_1 = \rho_1^{2l} = l - V_0$ (see (3.4) for the definition of V_0) and use (7.1) and the definition (3.2) to write

$$\begin{aligned} S(l) &= \|N(l;\,\cdot\,) - \overline{N}(l)\|_1 \\ &\ge \|N_0(l_1;\,\cdot\,) - \overline{N}_0(l_1)\|_1 - \|\overline{N}(l) - \overline{N}_0(l_1)\|_1 - \|N(l;\,\cdot\,) - N_0(l_1;\,\cdot\,)\|_1 \\ &\ge \mathcal{S}(\rho_1) - 2\|N(l;\,\cdot\,) - N_0(l_1;\,\cdot\,)\|_1. \end{aligned}$$

By Theorem 3.2, the L^1-norm on the right-hand side does not exceed

$$CB_dA(V)\rho^{d+1-4l}(|\ln\rho| + 1).$$

On the other hand, since $\min_{\boldsymbol{\mu}\in\mathsf{M}} |\boldsymbol{\mu}| \le \gamma$, we have

$$\mathcal{S}(\rho_1) \ge c\gamma^{-\frac{d+1}{2}}\rho^{\frac{d-1}{2}}$$

by (7.4). Under the condition $8l > d + 3$, we have $d + 1 - 4l < (d - 1)/2$; hence, there exists a number $\rho_0 = \rho_0(\gamma, v)$ such that

$$8\|N(l;\,\cdot\,) - N_0(l_1;\,\cdot\,)\|_1 \le \mathcal{S}(\rho_1), \quad \rho \ge \rho_0,$$

whence $S(l) \geq \mathcal{S}(\rho_1)/2$. This leads to the required inequality (7.8).

To prove (7.9), we use Lemma 3.1. We write

$$\overline{N}(l) + \frac{1}{2\,\mathrm{d}(\mathsf{M}^\dagger)} S(l) \geq \overline{N_0}(l_1) - \frac{1}{\mathrm{d}(\mathsf{M}^\dagger)} \|N(l;\,\cdot\,) - N_0(l_1;\,\cdot\,)\|_1 + \frac{1}{4\,\mathrm{d}(\mathsf{M}^\dagger)} \mathcal{S}(\rho_1)$$
$$\geq \overline{N_0}(l_1) + \frac{1}{8\,\mathrm{d}(\mathsf{M}^\dagger)} \mathcal{S}(\rho_1).$$

Consequently, by (7.2) and (7.8), if $0 \leq t \leq c\rho^2$ with a suitable small c, then

$$\begin{aligned} \mathrm{d}(\mathsf{M}^\dagger)\overline{N}(l-t) + \frac{1}{2} S(l-t) &\geq \mathrm{w}_d\, \rho_1^d - \frac{d}{2l}\, \mathrm{w}_d\, t\rho_1^{d-2l} + c\gamma^{-\frac{d+1}{2}} \rho_1^{\frac{d-1}{2}} + O(t^2 \rho_1^{d-4l}) \\ &\geq \mathrm{w}_d\, \rho_1^d - \frac{d}{4l}\, \mathrm{w}_d\, t\rho_1^{d-2l} + c\gamma^{-\frac{d+1}{2}} \rho_1^{\frac{d-1}{2}}. \end{aligned} \tag{7.10}$$

Similarly,

$$\mathrm{d}(\mathsf{M}^\dagger)\overline{N}(l+t) - \frac{1}{2} S(l+t) \leq \mathrm{w}_d\, \rho_1^d + \frac{d}{4l}\, \mathrm{w}_d\, t\rho_1^{d-2l} - c\gamma^{-\frac{d+1}{2}} \rho_1^{\frac{d-1}{2}}. \tag{7.11}$$

The right-hand side in (7.10) is greater than that of (7.11) provided that

$$t \leq C\gamma^{-\frac{d+1}{2}} \rho^{\frac{4l-d-1}{2}},$$

with a suitable constant C depending only on d and l. Now, a reference to Lemma 3.1 results in (7.9). □

Proof of Theorem 2.1. We approximate the potential V given by (2.6) by sums with finite s, i.e., for any $\epsilon > 0$ we find a partial sum V' of the form (2.6) such that $\|V - V'\|_{\mathsf{L}^\infty} < \epsilon$ and $\lVert V' \rVert \leq 2 \lVert V \rVert$. Thus, the number $\rho_0 = \rho_0(2\lVert V\rVert, \gamma)$ in Lemma 7.3 is one and the same for all such approximations. Suppose that the operator H has a gap of width δ lying above the point ρ_0^{2l}. Lemma 7.3 shows that the spectrum of $H_0 + V'$ has no gaps above ρ_0^{2l}. By the elementary perturbation theory, any perturbation with norm not exceeding ϵ can only open a gap of maximal size 2ϵ. Choosing $\epsilon < \delta/2$, we arrive at a contradiction, which proves the theorem. □

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