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# ISOGENY CLASSES OF FORMAL GROUPS OVER COMPLETE DISCRETE VALUATION FIELDS WITH ARBITRARY RESIDUE FIELDS

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Abstract. An explicit construction is described for computing representatives in each isogeny class of one-dimensional formal groups over the ring of integers of a complete discrete valuation field of characteristic 0 with residue field of characteristic $p$. The logarithms of representatives are written out explicitly, and the number of nonisomorphic representatives of the form described in each isogeny class is computed. This result extends and generalizes the result obtained by Laffaile in the case of an algebraically closed residue field. The homomorphisms between the representatives constructed are described completely. The results obtained are applied to computation of the Newton polygon and the "fractional part" of the logarithm for an arbitrary one-dimensional formal group. Moreover, the valuations and the "residues" of the torsion elements of the formal module are calculated. A certain valuation of logarithms of formal groups is introduced and the equivalence of two definitions of the valuation is proved. One of these definitions is in terms of the valuations of the coefficients, and the other is in terms of the valuations of the roots of the logarithm (i.e., of the torsion elements of the formal module). This valuation only depends on the isomorphism class of a formal group, is nonpositive, and equals zero if and only if the formal group in question is isomorphic to one of the representatives considered.

The classification results of M. V. Bondarko and S. V. Vostokov on formal groups are employed, including invariant Cartier–Dieudonné modules and the fractional part invariant for the logarithm of a formal group.

## Introduction

In [6] it was proved that any isogeny class of one-dimensional formal groups over the ring of integers of a complete discrete valuation field of characteristic 0 with algebraically closed residue field of characteristic $p$ contains a formal group law the logarithm of which has a certain explicitly described form. This result was an important step towards the classification of one-dimensional formal group laws up to isogeny. Our aim in the present paper is to extend this result to arbitrary mixed characteristic complete discrete valuation fields. In contrast to [6], we describe explicitly the relationship between the logarithms of all formal groups in an isogeny class and the logarithm of the representative constructed. We also calculate the number of pairwise nonisomorphic representatives of the form described in each isogeny class. We completely calculate the homomorphisms between our representatives. As an application, we calculate the valuations and the "residues" of the torsion elements of the formal module. We also compute the "fractional part" of the logarithm for an arbitrary one-dimensional formal group (see the definition below and also [1, 2]). Recall that the fractional part of the logarithm is an invariant of formal group

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laws that classifies the formal groups up to an isogeny of a certain type, described in [2] explicitly. This invariant gives somewhat better information about the formal group than the functor constructed by Fontaine in [4]. Note that, for a nonperfect residue field, no classification results apart from those proved in [2] were known previously.

For the construction of canonical representatives and the verification of the main results, we apply the methods of the papers [1] and [2]. We recall the definitions and main properties of the invariant Cartier–Dieudonné modules. We also present the definition of the invariant for the fractional part of the logarithm and compute this fractional part for all one-dimensional formal groups. For simplicity, we formulate all definitions and results of [2] only for one-dimensional formal groups (though all of them can be extended naturally to commutative formal groups of arbitrary dimension).

We start the paper with citing the main definitions and results of [2]. At the beginning of §1, we recall the definition of a $p$-typical formal group. For simplicity, we mainly consider formal groups of that type, though our methods apply to arbitrary formal groups. One of the main differences of the case of an arbitrary residue field from the case of an algebraically closed (or merely perfect) residue field is that in the general case there is no canonical inertia subfield (and no Frobenius automorphism on it). Yet we recall an important structural theorem, proved in [2], which states that each absolutely unramified complete discrete valuation field admits (at least one) Frobenius operator $\sigma$. In [2] it was also proved that any complete discrete valuation field is a totally ramified extension of some $\sigma$-field. At the end of §1, we recall that the operator corresponding to the logarithm of a $p$-typical formal group over a totally ramified extension of a $\sigma$-field can be presented as a fraction of a certain sort.

In §2 we recall the definition of the invariant Cartier–Dieudonné module for a formal group. By using the properties of these modules, we construct a certain class of "nice" formal group laws explicitly.

In §3 we recall the properties of the "fractional part" for power series and formal groups. We define a certain valuation of formal group laws in terms of the coefficients of their logarithms. We describe homomorphisms between "nice" groups in terms of the fractional parts of their logarithms. Finally, we formulate the main theorem and prove that any formal group is isogeneous to a nice one.

In §4 we study the relationships among the Newton polygons, the valuations of the roots of the logarithms, and other invariants of formal groups. Using the results obtained, we calculate the fractional part of the logarithm for an arbitrary formal group. We prove that the valuation of the logarithm can be expressed in terms of valuations of its roots; hence, it depends only on the isomorphism class of formal groups. We verify that the valuation is 0 if and only if the formal group in question is isomorphic to a "nice" one. We also prove that the "niceness" of a formal group depends entirely on the fractional part of its logarithm. Next, we conclude the proof of the main Theorem 3.4.1. At the end of the section, we recall the definition and the properties of the invariant Cartier–Dieudonné modules for non-$p$-typical formal group laws. These properties allow us to decide whether or not an arbitrary (not necessarily $p$-typical) series is the logarithm of a formal group isomorphic to a given nice group.

In §5 we compare the fractional part of the logarithm functor with the functor constructed by Fontaine in [4]. Our results yield examples of formal groups whose Fontaine invariants coincide but whose fractional part invariants are nonequivalent (i.e., the consideration of the fractional parts shows that the formal groups are nonisomorphic).

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**Notation.** $K/N$ will denote a totally ramified mixed characteristic complete discrete valuation field extension, $e$ is the absolute ramification index of $K$, $\pi$ is a uniformizing

element of $K$, $\mathfrak{O}$ is the ring of integers of $N$, $\mathfrak{O}_K$ is the ring of integers of $K$, $\mathfrak{M}$ is the valuation ideal of $K$, and $k$ is the residue field of $K$. We write $\overline{a} \in k$ for the residue of an element $a \in \mathfrak{O}_K$; $v = v_K$ is the normalized valuation of $K$ extended to the completion $\mathbb{K}$ of the algebraic closure of $K$. We denote by $F$ a one-dimensional finite height formal group law over $\mathfrak{O}_K$; the reduction of $F$ modulo $\pi$ is denoted by $\overline{F}$.

All statements presented here without proof were proved in [2].

## §1. Some results of [2]

**1.1. $p$-typical groups.** Throughout this paper we work with commutative formal group laws over characteristic zero rings; therefore, our formal group laws have logarithms. The following statement is well known (see [5, Theorem 16.4.14]).

**Proposition 1.1.1.** *Let $\mathfrak{A}$ be a commutative $\mathbb{Z}_p$-algebra. Then a formal group law with the logarithm $\lambda = x + \sum_{i>1} a_i x^i$, $a_i \in \mathbb{Q}_p\mathfrak{A}$, is strictly isomorphic over $\mathfrak{A}$ to a formal group law whose logarithm is $\lambda' = x + \sum_{i>0} a_{p^j} x^{p^j}$.*

Thus, we may assume that the logarithm $\lambda$ of the formal group $F$ is of the form $\Lambda(\Delta)(x)$, where $\Lambda \in K[[\Delta]]$. Here we define the action of $K[[\Delta]]$ on $x$, using the rule $a\Delta^s(x) = ax^{p^s}$ for any $s \geq 0$, $a \in K$. Such groups are said to be *$p$-typical*.

**1.2. $\sigma$-fields and their properties. The main structure theorem.** We recall a theorem that is important for the study of formal groups over rings with imperfect residue fields.

A pair $(N, \sigma)$ will be called a *$\sigma$-field* if $\sigma$ is an endomorphism of $N$ satisfying the congruence $\sigma(x) - x^p \in p\mathfrak{O}$ for any $x \in \mathfrak{O} = \mathfrak{O}_N$. Note that many rings admit more than one $\sigma$. For example, on the valuation ring (for the valuation of rank one) of the field $\mathbb{Q}_p\{\{t\}\}$ (the smallest two-dimensional local field of characteristics $(0, p, p)$) we can take $\sigma(t) = t^p$, so that $\sigma$ depends on the choice of $t$. Certainly, the absolute ramification index of any $\sigma$-field is 1.

We formulate the main structure theorem for complete discrete valuation fields. It was proved in [2] by using the method of [3, Chapter 2]. The author was not able to find any similar result in other sources.

**Theorem 1.2.1** (Main structure theorem)**.** *Any complete discrete valuation field $K$ is a totally ramified extension of some $\sigma$-field $N$.*

In particular, on any absolutely unramified field, a Frobenius operator $\sigma$ can be defined.

We introduce some notation to be used throughout this paper.

For any $\sigma$-field $N$ we consider a (noncommutative) ring $W$ that coincides with $\mathfrak{O}[[\Delta]]$ as a left $\mathfrak{O}$-module and satisfies the relation $\Delta a = \sigma(a)\Delta$ for any $a \in \mathfrak{O}$.

The $N$-module $N[[\Delta]]$ with the same multiplication will be denoted by $W'$; we also consider $NW = \bigcup_{i>0} p^{-i}W \subset W'$.

$w = (w_i)$ will denote some base of $\mathfrak{O}_K$ over $\mathfrak{O}$.

**1.3. Presentation of $\Lambda$ as a fraction.** From this point on, $N$ will be a $\sigma$-field. We have $e = [K : N]$.

The ring $K[[\Delta]]$ has a natural structure of a right $W$-module. Note that, in order to define this structure, we have no need to extend $\sigma$ to $K$ (since we do not consider products of the type $\Delta x$ for $x \in K \setminus N$). For $a_i \in K$, $b_i \in N$, $i \geq 0$, we define

$$\Big(\sum a_i\Delta^i\Big)\Big(\sum b_i\Delta^i\Big) = \sum_{i\geq 0} \sum_{j+l=i,\, j,l\geq 0} a_j\sigma^j(b_l)\Delta^i.$$

The following statement was proved in [2].

**Theorem 1.3.1.** 1. *Let $\Lambda \in K[[\Delta]]$ correspond to a formal group $F$ over $\mathfrak{O}_K$ (i.e., $\Lambda(\Delta)(x)$ it the logarithm of a $p$-typical group $F$). Then $\Lambda$ can be presented in the form $vu^{-1}$, where $vp^l\pi^{-p^l} \in \mathfrak{O}_K[[\Delta]]$, $l = [\log_p(pe/(p-1))]$, and $u \in p + W\Delta$.*

2. *Let $\Lambda = vu^{-1}$, $\Lambda' = v'u'^{-1}$, where $F$ and $F'$ are finite height formal groups. Then $u = u'\varepsilon$ for $\varepsilon \in W$ if and only if $\overline{F} = \overline{F}'$ (as series over $k$).*

3. *If $u = \sum u_i\Delta^i$, then the height of $F$ equals $\min\{i : p \nmid u_i\}$.*

We call $u$ a *special* element of $W$; the height of $u$ is defined as the height of $F$.

## §2. Invariant Cartier–Dieudonné modules; the construction of nice formal groups and their properties

**2.1. Invariant Cartier–Dieudonné modules.** For each $\alpha \in K$, we define an operator $\langle\alpha\rangle$ on $K[[\Delta]]$ by $\langle\alpha\rangle(\sum c_i\Delta^i) = \sum c_i\alpha^{p^i}\Delta^i$. Let $w$ be a base of $\mathfrak{O}_K$ over $\mathfrak{O}$.

**Definition 2.1.1.** For a formal group $F$ with logarithm $\lambda = \Lambda(x)$, $\Lambda \in K[[\Delta]]$, we define

$$D_F = \langle\langle w_i\rangle\Lambda\rangle W = \sum\langle w_i\rangle W \subset K[[\Delta]].$$

$D_F$ will be called the *invariant Cartier–Dieudonné module* of the formal group $F$.

**Proposition 2.1.2.** 1. *$D_F$ does not depend on the choice of the base $w$.*

2. *Let $a \in \mathfrak{O}_K$, and let $D_1$ and $D_2$ be the invariant Cartier–Dieudonné modules of $F_1$ and $F_2$, respectively. Then a homomorphism $f : F_1 :\to F_2$ such that $f(x) \equiv ax \bmod \deg 2$ exists if and only if $aD_1 \subset D_2$.*

3. *Formal groups $F_1$ and $F_2$ are strictly isomorphic if and only if $D_1 = D_2$.*

4.
$$D_F = \{f \in K[[\Delta]] : \exp_F(f(\Delta)(x)) \in \mathfrak{O}_K[[x]]\},$$
*where $\exp_F$ is the composition inverse to $\lambda_F$.*

5. *For any $\alpha \in \mathfrak{O}_K$, we have $\langle\alpha\rangle D_F \subset D_F$.*

Part 5 shows that $D_F \cap \mathfrak{O}_K$ is an ideal of $\mathfrak{O}_K$. We introduce the notation $D_F \cap \mathfrak{O}_K = \mathfrak{M}^{n(F)}$.

Since $D_F$ is a $W$-module, we have $\mathfrak{M}^{n(F)}[[\Delta]] \subset D_F$.

For a formal group $F$ and $a \in \mathfrak{O}_K$, we denote by $F_a$ the formal group law $a^{-1}F(ax, ay)$. Obviously its coefficients are integral, and if $\lambda = \Lambda(x)$, then the operator $\Lambda_{F_a}$ corresponding to $F_a$ is equal to $a^{-1}\langle a\rangle\Lambda$.

**Proposition 2.1.3.** 1. *Let $\Lambda = \sum a_i\Delta^i$. Then $n(F) = \max(0, -[\inf_{i>0}\frac{v(a_i)}{p^i-1}])$.*

2. *$n(F_\pi) = \max(0, n(F) - 1)$.*

*Proof.* 1. If $\inf_{i>0}\frac{v(a_i)}{p^i-1} \geq 0$, then $\lambda \in \mathfrak{O}_K[[x]]$, and we are done.

Now, let $\inf_{i>0}\frac{v(a_i)}{p^i-1} = s < 0$. Since $\underline{\lim}_i\frac{v(a_i)}{p^i-1} = 0$, we see that $s \in \mathbb{Q}$ and there exists $j \in \mathbb{Z}$ such that $\frac{v(a_j)}{p^j-1} = s$. Let $\exp_F = \sum b_ix^i$. We consider the field $L = K[\pi^s]$ and introduce the variable $y = \pi^s x$. We have $\lambda(x) = \pi^{-s}f(y)$, $f \in y + \mathfrak{O}_L[[y]]y$. For the power series $g$ that is the composition inverse to $f$, we have $\exp_F(\pi^{-s}x) = \pi^{-s}g(x)$. We obtain $n(F) \geq -[\inf_{i>0}\frac{v(a_i)}{p^i-1}]$.

On the other hand, since the reduction of $f$ is a polynomial, the reduction of $g$ is not. Therefore, there exist arbitrarily large $j$ such that $v(b_j) = (j-1)s$. Hence, $\exp_F(ax) \notin \mathfrak{O}_K[[x]]$ whenever $v(a) < s$.

Part 2 follows from part 1 immediately. □

**2.2. The construction of formal groups.**

**Proposition 2.2.1.** *Let $s = -[e/(1-p)]$, let $a \in \mathfrak{O}_K$, and let $v_K(a) = l \le s$. Suppose that $\Lambda = vu^{-1}$ for some special $u \in W$, and that for a formal group law $F'$ over $\mathfrak{O}_K$ satisfying $a^{-1}\pi^s\mathfrak{O}_K W^m \subset D_{F'}$ we have $\langle a\rangle\Lambda \in aD_{F'}$ and $v \in \pi^s D_{F'}$. Then $\Lambda(X)$ is a logarithm of a formal group law $F$ over $\mathfrak{O}_K$, and $F'$ is strictly isomorphic to $F_a$.*

This statement was proved in [2].

**Proposition 2.2.2.** *Let the height of $u$ (see Subsection 1.3) be equal to $h$, and let $v = p + \sum_{i=1}^{\infty} v_i\Delta^i$. Suppose that the minimum of $\frac{e-v(v_i)}{p^i-1}$ is attained at $i = s$. We denote by $l = l(F)$ the quantity $\min_{0<s<h}(\frac{e-v(v_s)}{p^s-1}, \frac{e}{p^h-1})$. Let $v(v_i) \ge \max(l, e - l(p^i-1))$ for $i \ge 1$. Then for $\Lambda = vu^{-1}$ the series $\Lambda(x)$ is the logarithm of a formal group over $\mathfrak{O}_K$.*

*Proof.* Since the condition does not depend on $K$, we may assume that $p^s - 1 \mid e - v(v_s)$ for all $s < h$. As $F'$ we take the additive formal group law. We have $D_{F'} = \mathfrak{O}_K[[\Delta]]$. It is easily seen that for $\Lambda = \sum c_i\Delta^i$ we have $v(c_i) \ge v(v_{i \bmod h}) - e[i/h]$ (cf. Lemma 3.2.1 below). Hence, the conditions of Proposition 2.2.1 are fulfilled for $a = \pi^l$. □

We say that a formal group $F$ is *nice* if $v(v_i) \ge e\max(1 - \frac{i}{h}, \frac{1}{h})$. The logarithm of any nice group satisfies the assumptions of the assertion proved above.

Observe that the condition on $v$ is not affected by multiplication of $\Lambda$ by an element $\varepsilon \in 1 + W\Delta$. Therefore, the "niceness" of the logarithm does not depend on the choice of a representation of $\Lambda$ in the form $vu^{-1}$.

## §3. The properties of the fractional part; formulation of the main theorem

In this section we prove that any (one-dimensional) formal group is isogeneous to some nice group (see the definition in Subsection 2.2).

**3.1. The fractional part of the logarithm of a formal group.** In this subsection we define the fractional part invariant and recall that the fractional part of the logarithm classifies the formal groups up to isogeny of a certain type; this type can be described explicitly.

Consider the ring

$$R = \mathfrak{O}_K[[\Delta]] \otimes_{\mathfrak{O}_K} K = \mathfrak{O}_K[[\Delta]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \bigcup_{i\ge 0} p^i\mathfrak{O}_K[[\Delta]] \subset K[[\Delta]].$$

For a formal group $F$, we define $r(F)$ as the residue $\Lambda \bmod R$. We have $r(F) \in Ru^{-1}/R$, where $\Lambda = vu^{-1}$ is the presentation described in Theorem 1.3.1.

Note that $r(F)$ does not depend on the choice of the ground field.

We formulate the main results on fractional parts (for the proofs, see [2]).

**Theorem 3.1.1.** I. 1. *A homomorphism $f$ from $F$ to $F'$, $f(x) \equiv ap^s x \bmod \deg 2$ for some $s \in \mathbb{Z}$, exists if and only if $ar = r'\varepsilon$ for some $\varepsilon \in NW$. Here $r = r(F)$ and $r' = r(F')$.*

2. *Let $F$ and $F'$ satisfy the conditions of part 1, let $F$ be a finite height formal group, and let $\Lambda = vu^{-1}$. A homomorphism $f$ from $F$ to $F'$, $f(x) \equiv ap^s x \bmod \deg 2$ for fixed $s \in \mathbb{Z}$, exists if and only if $ap^s v \in D_{F'}$.*

II. *Let $F$ and $F'$ be finite height formal groups, and let $a \in K$, $b \in \mathfrak{O}$, $m \in \mathbb{Z}$. Present $u$ as $p - \sum u_i\Delta^i$, $u_i \in \mathfrak{O}$. Then $u_h \in \mathfrak{O}^*$, $u_i \in p\mathfrak{O}$ for $i < h$, where $h$ is the height of $F$.*

*If there exists an isogeny $f = \sum a_i x^i \equiv ax \bmod \deg 2$ from $F$ to $F'$ such that the height*

*of $f$ is equal to $m$ and $a_{p^m} \equiv b \bmod \pi$, then*

$$ar(F) = r(F')b\varepsilon\Delta^m \tag{1}$$

*for some $\varepsilon \in 1 + W\Delta$.*

*If the residue field $k$ is perfect, then relation* (1) *characterizes all possible $r(F')$ completely.*

**3.2. Valuation of the fractional part of the logarithm.** Suppose the height of $F$ equals $h$, $\Lambda = vu^{-1}$. It is easily seen that the residue of $v$ modulo $u$ can be found, i.e., for some $w = \sum_{i=0}^{h-1} w_i\Delta^i$ and $y \in R$ we have $v = w + yu$. The proof of this is almost the same as for the standard statement about division with remainder for polynomials. Hence, $r(\Lambda) = r(wu^{-1})$.

We denote by $b_i = b_i(F)$ the numbers $j$, $0 \le j < h$, for which the minimum of $v(w_i) + \frac{ie}{h}$ is attained, $1 \le i \le s$.

We denote $s = \#b_i$ by $s(F)$.

We introduce a valuation $v_h$ on $K[[\Delta]] \setminus \{0\}$ by the formula

$$v_h\Big(\sum_{i\ge 0} a_i\Delta^i\Big) = \inf\Big(v(a_i) + \frac{ie}{h}\Big)$$

(in general, $v_h(f)$ can be equal to $-\infty$).

Let $v(F)$ denote $v_h(w) - e$.

**Lemma 3.2.1.** *Let $\Lambda = \sum_{i\ge 0} a_i\Delta^i$. Then for all $i$ exceeding some $I > 0$ we have*

1) *$v(F) = v(a_i) + \frac{ie}{h}$ if and only if $v(w_{i \bmod h}) = v(F) + e(1 - \{\frac{i}{h}\})$;*
2) *$v(a_i) + \frac{ie}{h} > v_h(F)$ for all other $i > I$.*

*Proof.* Consider the valuation subring $W_h \subset W'$ with respect to the valuation $v_h$, and let $\mathfrak{M}_h$ be its valuation ideal. We expand $u$ as $p - \sum u_i\Delta^i$, $u_i \in \mathfrak{O}$. In $W_h$ we have $u \equiv p - u_h\Delta^h \bmod p\mathfrak{M}_h$. Therefore,

$$u^{-1} \equiv (p - u_h\Delta^h)^{-1} \equiv \sum_{j\ge 0} p^{j-1}u_h\sigma^h(u_h)\cdots\sigma^{jh-h}(u_h)\Delta^{jh} \mod p^{-1}\mathfrak{M}_h. \tag{2}$$

Moreover, $wu^{-1} = vu^{-1} + f(\Delta)$, $f(\Delta) \in R$. For any real number $y$ we denote by $\mathfrak{M}_{h,y}$ the set $\{f = \sum a_ix^i \in K[[\Delta]] : v(a_i) \ge y - ei/h\}$. We have

$$\Lambda \equiv \sum_{j\ge 0} p^{j-1}wu_h\sigma^h(u_h)\cdots\sigma^{jh-h}(u_h)\Delta^{jh} \mod(\mathfrak{M}_{h,v(F)+1/h}, R), \tag{3}$$

which implies the claim. □

**3.3. The properties of nice groups.** We use the properties of the fractional part of the logarithm for the study of nice formal groups.

**Proposition 3.3.1.** 1. *A formal group $F'$ with logarithm $\lambda' = \Lambda'(x)$ is isomorphic to a nice group $F$ with logarithm $\lambda = \Lambda(x)$ if and only if $n(F') \le -[\frac{e}{h-ph}]$ and $\Lambda'\varepsilon \equiv a\Lambda \bmod R$ for some $\varepsilon \in W^*$, $a \in \mathfrak{O}_K^*$.*

*Moreover, the series giving an isomorphism of $F$ and $F'$ is congruent to $ax$ modulo degree* 2.

2. *Let $F$ and $F'$ be nice groups, and let $r$ and $r'$ be the fractional parts of their logarithms. Then the following statement is true for any $a \in \mathfrak{O}_K$:*

*There exists a homomorphism from $F$ into $F'$, and the series giving the homomorphism is congruent to $ax \bmod \deg 2$ if and only if for some $\varepsilon \in W$ we have $a\Lambda \equiv \Lambda'\varepsilon \bmod R$.*

*Proof.* If $F$ is isomorphic to $F'$, then $D_{F'} = aD_F$, whence $n(F) = n(F')$. Now, by Proposition 2.1.3, we obtain $n(F) \leq -[\frac{e}{h-ph}]$. Next, by part II of Theorem 3.1.1, we have $a\Lambda \equiv \Lambda'\varepsilon$ for some $\varepsilon \in W^*$, $a \in \mathfrak{O}_K^*$, so that the series giving the isomorphism is congruent to $ax \bmod \deg 2$.

We prove the converse implication. We must check the conditions of part I.2 of Theorem 3.1.1 for the groups $F$, $F'$. Let $\Lambda = vu^{-1}$. Since $v \in \mathfrak{M}^{-[-e/h]}[[\Delta]]$, the assumption on $n(F)$ implies that $av \in D_{F'}$. Hence, there exists a series giving an isomorphism of $F$ and $F'$ that is also congruent to $ax$ modulo degree 2.

2. The existence of a homomorphism implies the condition on the fractional parts, by part II of Theorem 3.1.1. The converse implication is obtained from part I.2 of Theorem 3.1.1 in the same way as above. □

*Remark* 3.3.2. We have proved that the homomorphism group for a pair of nice formal groups can be recovered if we know the fractional parts of their logarithms. This is similar to the corresponding result for homomorphisms of formal groups over fields with low ramification (i.e., $e < p$; see [1], [4], and [2]).

Below we shall see that the condition on $n(F)$ follows from the condition on the fractional parts automatically.

Now we study the endomorphism ring of a nice group.

**Proposition 3.3.3.** *Let $A \subset \mathfrak{O}_K$ be the ring of $\mathfrak{O}_K$-endomorphisms of a nice group $F$. Then $A$ is the ring of integers of some local field (i.e., a finite extension of $\mathbb{Q}_p$) $L \subset K$.*

*Proof.* Let $L$ be the fraction field of $A$. Since the height of $F$ is finite, $L$ is a local field (see [5]). We must prove that $\alpha \in L \cap \mathfrak{O}_K$ implies $\alpha \in A$. For some $s \in \mathbb{Z}$ we have $p^s\alpha \in A$. By part I.2 of Theorem 3.1.1, it suffices to verify that $\alpha v \in D_F$. This fact follows from $v \in \mathfrak{M}^{-[-e/h]}\mathfrak{O}_K[[\Delta]] \subset D_F$. □

**3.4. The main theorem (statement and the beginning of the proof).** For $u = p + \sum a_i\Delta^i$ with $a_i \in \mathfrak{O}$ and $r > 0$, we denote by $u(r)$ the power series $p + \sum \sigma^r(a_i)\Delta^i \in W$. We have $u(r)\Delta^r = \Delta^r u$.

**Theorem 3.4.1.** *Let $\Lambda = vu^{-1}$, let $w$ be the residue of $v$ modulo $u$, and let $wu^{-1} = \sum s_i\Delta^i$. We denote by $d$ the absolute ramification index of the endomorphism ring of $F$ (i.e., of its field of fractions).*

*Recall that the $b_i = b_i(F)$, $1 \leq i \leq s$, denote the numbers $j$, $0 \leq j < h$, at which the minimum of $v(w_j) + \frac{je}{h}$ is attained.*

1. *A nice group is isogeneous to $F$ if and only if it is isomorphic to $F_i$ for some $1 \leq i \leq s$. Here $F_i$ is a formal group with the logarithm $\Lambda_i(x)$, where $\Lambda_i = s_{b_i}^{-1}\sum_{j\geq 0} s_{j+b_i}\Delta^j$.*

2. *$F_i$ is isomorphic to $F_j$ if and only if $d \mid (i-j)$.*

3. *The number of pairwise nonisomorphic nice formal groups in the isogeny class of $F$ is equal to $s(F)/d$.*

*Proof.* First, we check that $F$ is isogeneous to all $F_i$.

Denote $s_{b_i}$ by $c$. We have $c\Lambda_i\Delta^{b_i} \equiv \Lambda \bmod R$. Therefore, it suffices to verify that $\Lambda_i$ actually gives some nice formal group. Indeed, by Theorem 3.1.1, this group would be isogeneous to $F$.

We denote $b_i$ by $r$. In the field $K((\Delta))$ we have

$$\Lambda_i = c^{-1}\Big(wu^{-1} - \sum_{j=0}^{r-1} cs_j\Delta^j\Big)\Delta^{-r} = c^{-1}\Big(w - \sum_{j=0}^{r-1} cs_j\Delta^j u\Big)\Delta^{-r}u(r)^{-1}.$$

By definition, $\Lambda_i \equiv 1 \bmod \Delta$. Next, $v(c) = v(F) - \frac{er}{h}$. So, $c^{-1}(w - \sum_{j=0}^{r-1} cs_j \Delta^j u)\Delta^{-r}$ satisfies the condition on $v$ in the definition of a nice group. Taking $u(r)$ for the role of $u$ in that definition, we see that $F_i$ is a nice group.

The proof will be concluded in the next section. □

## §4. Newton polygons and the valuations of roots

**4.1. General results.** For an arbitrary power series $f = \sum_{i>0} b_i x^i$, $b_i \in K$, its Newton polygon is defined as the convex hull of the points $(i, v(b_i))$.

Let $v(b_1) = 0$ and suppose that the derivative $f'$ of the series $f$ belongs to $\mathfrak{O}_K[[x]]$. We shall say that the line $z = cy+b$ contains a side of a Newton polygon for $f$ if $v(b_i) \geq ci+a$ for all $i > 0$ and equality occurs for at least two different $i$. The pairs $(i, v(b_i))$ for which equality is attained are called the *vertices* of the Newton polygon.

We fix a uniformizing element $\pi$ of $K$. Let $\mathbb{K}$ denote the completion of the algebraic closure of $K$. In $\mathbb{K}$ we choose a compatible system of $\pi^{1/s}$, $s > 1$, and extend the valuation $v$ to it. Denoting by $\mathfrak{M}_{\mathbb{K}}$ the valuation ideal of $\mathbb{K}$, we introduce the function $r : \mathbb{K}^* \to k^{\mathrm{alg}}$, $r(x) = x\pi^{-v(x)} \bmod \mathfrak{M}_{\mathbb{K}}$. Obviously, $r(xy) = r(x)r(y)$.

**Theorem 4.1.1.** *Suppose $f$ satisfies the conditions described above.*

1. *For $v(x) > 0$, $vx \in \mathbb{K}$, the series $f(x)$ converges.*
2. *$f$ has no multiple roots in $\mathfrak{M}_{\mathbb{K}}$.*
3. *Let $c < 0$, $c \in \mathbb{Q}$, let the line $z = cy + a$ contain a side of the Newton polygon for $f$, and let $v(b_{n_j}) = cn_j + a$, $1 \leq j \leq s$, $n_1 < n_2 < \ldots n_s$. Suppose that for $j \neq n_i$ we have $v(b_i) > ci + a$. Then $f$ has $n_s - n_1$ roots $r_i \in \mathbb{K}$ with valuation $-c$, and the values $r(r_i)$ coincide (with regard to multiplicities) with the set of roots of $h(x) = \sum_{1 \leq j \leq s} \overline{b_{n_j} \pi^{-cn_j - a}} x^{n_j - n_1}$.*
4. *$f$ has no other roots (with positive valuations).*
5. *We arrange the torsion elements of $f$ in the order of decreasing valuations and denote them by $t_i$, $i > 0$ (so that $t_1 = 0$). The relation*

$$\sum_{i=2}^{l} v(t_i) = -v(b_l) \tag{4}$$

*is fulfilled if and only if $(l, v(b_l))$ is a vertex of the Newton polygon; otherwise $\sum_{i=2}^{l} v(t_i) > -v(b_l)$.*

*Proof.* 1. We have $v(b_i x^i) = iv(x) + v(b_i) \to +\infty$.

2. Since $v(f'(x)) = v(b_1) = 0$, there are no multiple roots.

3. Consider $g = \sum \pi^{-a-ci} b_i x^i$. The power series $g$ is integral and is defined over a finite extension of $K$. We have $\overline{g(y)} = h(y)y^{n_1}$. We expand $g$ in accordance with the Weierstrass Preparation Theorem. We have $g(y) = cj(y)k(y)$, where $c \in \mathfrak{O}_{\mathbb{K}}$, $j$ is a unitary polynomial, and $k(y) \in \mathfrak{O}_{\mathbb{K}}[[y]]^*$. Therefore, the roots of $g$ with nonnegative valuations are the roots of $j(y)$. Let $j(y) = \prod(y - c_i)$ be the expansion of $j$ as a product of linear terms. We have $\overline{j(y)} = y^{n_1}\overline{h}(y)$, whence the $\overline{c_i}$ with 0 excluded are exactly the roots of $h$. Since $h(x) = \pi^a g(x\pi^c)$, we obtain the claim.

4. Suppose that $f$ has a root $t$ with valuation $u > 0$. Then the minimum of $v(b_i t^i)$ must be attained at more than one $i$. Let this minimum be equal to $b$. Then the line $z = b - uy$ contains some side of the Newton polygon of $f$. Therefore, $t$ is one of the roots described above.

5. We prove (4) for the vertices of the polygon by induction on $l > 0$. For $l = 1$ the relation is obvious. Let $(l_1, b_{l_1})$ and $(l_2, b_{l_2})$ be two neighboring vertices. Then, by

part 3, $v(t_i) = \frac{v(b_{l_1})-v(b_{l_2})}{l_2-l_1}$ for $l_1 < i \le l_2$. Thus, if (4) is fulfilled for $(l_1, b_{l_1})$, then it is fulfilled for $(l_2, b_{l_2})$ also.

Now, suppose that $(l, v(b_l))$ is not a vertex of the Newton polygon. We choose the largest $j < l$ such that $(j, v(b_j))$ is a vertex of the Newton polygon and let $(w, v(b_w))$ be the next vertex. Since $(l, v(b_l))$ is not a vertex, we obtain

$$v(b_l) > v(b_j) + \frac{(l-j)(v(b_w) - v(b_j))}{(w-j)}.$$

On the other hand, for $j < m \le w$ we have $v(t_m) = \frac{v(b_j)-v(b_w)}{w-j}$. Therefore,

$$v(t_m) > \frac{v(b_l) - v(b_j)}{j-l} \quad \text{for } j < m \le l. \tag{5}$$

Summing all inequalities (5) and adding relation (4) for $(j, v(b_j))$, we obtain the claim. □

**4.2. Logarithms and isogenies of formal groups.** The preceding statement allows us to deduce a formula that expresses $n(F)$ in terms of the roots of the logarithm of $F$.

**Proposition 4.2.1.** $n(F) = -[\min(-v(t_i))]$, *where the* $t_i$ *are the nonzero roots of* $\lambda$, *i.e., the torsion elements of* $F(\mathfrak{M}_{\mathbb{K}})$.

*Proof.* By Proposition 2.1.3, we have $n(F) = \max(0, -[\inf_{i>0} \frac{v(a_i)}{p^i-1}])$. On the other hand, applying Theorem 4.1.1 to $\lambda_F$, we see that if $\lambda_F$ has roots with positive valuation, then $-\inf_{i>0} \frac{v(a_i)}{p^i-1}$ is equal to the largest valuation of the nonzero roots. □

**Proposition 4.2.2.** *Let* $F$ *and* $G$ *be finite height groups, let* $f = \sum_{i>0} a_i x^i$ *be an isogeny from* $F$ *to* $G$, *let* $T$ *be the kernel of* $f$, *and let* $s = \#T$. *Then the following hold.*

1. $r(a_1) = r(a_s) \prod_{t \in T \setminus \{0\}} r(t)$.
2. *Suppose* $x \in \mathfrak{M}_{\mathbb{K}}$, $v(x) = l$, *and for each* $t \in T$ *we have* $v(x +_F t) \le l$. *Then*

$$v(f(x)) = \sum_{t \in T, v(t) < l} v(t) + l\#\{t \in T, v(t) \ge l\}, \tag{6}$$

*and*

$$r(f(x)) = r(a_s) \prod_{t \in T, v(t) < l} r(t) \cdot \prod_{t \in T, v(t) = l} (r(t) + r(x)) \cdot r(x)^{\#\{t \in T, v(t) > l\}}. \tag{7}$$

*Proof.* We denote by $P_T$ the power series $\prod_{t \in T}(x +_F t)$. By the result of Lubin (see [5, §35.2]), $f$ can be represented as a composition of $P_T$ with an isomorphism. Therefore,

$$f = cx \circ g(x) \circ P_T(x), \tag{8}$$

where $c \in \mathfrak{O}_K^*$ and $g(x) \equiv x \bmod \deg 2$. Since the reduction of $P_T(x)$ equals $x^s$, composing it with $cx \circ g(x)$ does not affect the validity of the assertion, i.e., we may assume that $f = P_T$. For any $t \in T$ we have $t +_F x \equiv t + x \bmod xt\mathfrak{O}_{\mathbb{K}}[[x]]$, which yields part 1 immediately. Part 2 follows directly from the following fact: for $y, z \in \mathfrak{M}_{\mathbb{K}}$ with $v(y) \ge v(z)$ we have $r(x +_F y) = r(z)$ if $v(y) > v(z)$ and $r(x +_F y) = r(x) + r(y)$ if $v(x) = v(y) = v(x+y)$. □

*Remark* 4.2.3. For $x \in \mathfrak{M}_{\mathbb{K}}$, suppose that the maximum of $v(x +_F t)$ at $t \in T$ is attained for $t = t_0$. Then $f(x) = f(x +_F t_0)$, and $x +_F t_0$ satisfies the conditions of part 2. Thus, we can compute $v(f(x))$ and $r(f(x))$ for any $x$. In particular, we can use the resulting statement for $x$ belonging to the torsion of $F(\mathfrak{M}_{\mathbb{K}})$; in this case $x +_F t_0$ will also belong to the torsion.

The piecewise-linear function occurring in formula (6) is the Herbrand function for the numbers $v(t)$, $t \in T$.

**Proposition 4.2.4.** *Let $F$, $G$, $T$, and $s = p^l$ be as in Proposition 4.2.2. Then for any formal group $G'$ strictly isomorphic to $G$ we have*

$$r(G')\varepsilon\Delta^s = a_1 r(F)$$

*for some $\varepsilon \in W$, $\varepsilon \equiv r(a_1)\prod_{t\in T\setminus\{0\}} r(t)^{-1} \bmod (p, \Delta)$. For a perfect field, this relation characterizes all possible $r(G')$.*

*Proof.* This follows immediately from Theorem 3.1.1. □

So, for an arbitrary formal group the fractional part of the logarithm can easily be expressed in terms of the fractional part of the logarithm for a nice group isogeneous to the initial formal group. Indeed, the valuations and the residues of the torsion elements of the formal module for a nice group are known. This can also be done in the case of a nonperfect residue field, by using the decomposition (8).

Moreover, the valuations and the "residues" of $t_i$ (i.e., $r(t_i)$) can be computed with the help of Theorem 4.1.1 and the decomposition (8).

**4.3. The properties of $v(F)$.** We put the torsion elements of $F(\mathfrak{M}_{\mathbb{K}})$ in the order of decreasing valuations and denote them by $t_i$, $i > 0$ ($t_1 = 0$). We may assume (permuting the $t_i$ with equal valuations if needed) that for each $j > 0$ the set of $t_i$, $1 \le i \le p^j$, is a group with respect to the operation $F$. Obviously, for each $s \ge 0$ we have $v(t_{p^s+1}) = v(t_{p^s+2}) = \dots = v(t_{p^{s+1}})$. It is well known that the torsion of $F(\mathfrak{M}_{\mathbb{K}})$ coincides with the roots of $\lambda = \Lambda(x)$. Moreover, $\lambda$ satisfies the conditions of Theorem 4.1.1.

**Proposition 4.3.1.** 1. *We have*

$$v(F) = \min_{j>0}\Big(\frac{je}{h} - \sum_{i=2}^{p^j} v(t_i)\Big) = \underline{\lim}_j \Big(\frac{je}{h} - \sum_{i=2}^{p^j} v(t_i)\Big).$$

2. *For any formal group $F$ we have $v(F) \le 0$.*
3. *$v(F) = 0$ if and only if $F$ is isomorphic to a nice group.*

*Proof.* 1. By Lemma 3.2.1, for sufficiently large $i$ the valuations of the coefficients of $\lambda = \sum a_{p^i} x^{p^i}$ are at least $v(F) - \frac{ei}{h}$, with equality occurring for $i = b_l + hj$, $j \in \mathbb{Z}$, $1 \le l < s$. Since the function $f(x) = v(F) - \frac{e\log_p(x)}{h}$ is convex, for all $i$ that exceed some $I$ all $(p^{b_l+hj}, v(F) - \frac{b_l e}{h} - je)$ are vertices of the Newton polygon of $f$. All other vertices (to the right from $x = I$) are of the form $(p^m, y(m) - \frac{me}{h})$ for some $m \in \mathbb{Z}$, whence $y(m) > v(F)$. Part 5 of Theorem 4.1.1 implies that $v(F) = \underline{\lim}_j \frac{je}{h} - \sum_{i=2}^{p^j} v(t_i)$.

Also, we see that the minimum of $\frac{je}{h} - \sum_{i=2}^{p^j} v(t_i)$ is attained for some $j = j_0$. Now we check that this minimum repeats periodically. We consider $g = [p]_F$ and decompose $g_i = g - t_i$ by using the Weierstrass Preparation Theorem. This decomposition shows that, since the height of $g$ is equal to $h$ (i.e., the reduction of $g$ is equal to $cs^{p^h}$ for some $c \ne 0$), the series $g_i$ has exactly $p^h$ roots. Since $g_i \equiv -t_i \bmod x$, the sum of valuations of the roots of $g_i$ equals $v(t_i)$. Hence, the sum of valuations of all $t_s$ such that $s > 1$ and $[p]_F(t_s) = t_{s'}$ for $s' \le p^{j_0}$ is equal to

$$\sum_{i=2}^{p^j} v(t_i) + \sum_{[p]_f(x)=0, x\ne 0} v(x),$$

and the number of them is equal to $p^{j_0+h} - 1$. Now we decompose the series $g$ in accordance with the Weierstrass Preparation Theorem. Since $g \equiv px \bmod \deg 2$, the sum of valuations of all roots of $g$ (except 0) equals $e$. Therefore, the sum of some set

of $p^{h+j_0}-1$ numbers $v(t_s)$ for pairwise distinct $t_s>1$ equals $\sum_{i=2}^{p^{j_0}} v(t_i)+e$. Thus, $\sum_{i=2}^{p^{h+j_0}} v(t_i)$ is not less than $\sum_{i=2}^{p^{j_0}} v(t_i)+e$, and we see that the minimum is equal to the lower limit.

2. For an arbitrary $j>0$, we decompose the isogeny $[p^j]_F$ by the Weierstrass Preparation Theorem. Since $g\equiv p^j x \bmod \deg 2$, the sum of valuations of the roots of $g$ (with 0 excluded) is $ej$. Therefore, the sum of some set of $p^{jh}-1$ numbers $v(t_i)$ for pairwise distinct $t_i>1$ equals $ej$. Thus, $\sum_{i=2}^{p^{jh}} v(t_i)$ is not less than $ej$. The expression of $v(F)$ as a minimum yields the claim.

3. Obviously, if $F$ is a nice group, then $v(F)=0$. Part 1 implies that $v(F)$ is the same for isomorphic groups. Hence, $v(F)=0$ for all groups isomorphic to a nice group.

Now suppose that $v(F)=0$. We check that $n(F)\le -[\frac{e}{h-ph}]$. By Proposition 4.2.1, it suffices to check that $v(t_2)=v(t_3)=\cdots=v(t_p)\le -[\frac{e}{h-ph}]$. This follows immediately from the expression of $v(F)$ as a minimum. Indeed, we obtain $v(F)\ge \frac{e}{h}-(p-1)v(t_2)$.

In the proof of part 2 it was shown that $\sum_{i=2}^{p^{jh}} v(t_i)=ej$ for any $j>0$. Therefore, $b_1=0$. Let $\Lambda=vu^{-1}$, and let $w$ be the residue of $v$ modulo $u$. We see that $w=ap+\sum_{i=1}^{h-1} w_i\Delta^i$, where $a\in\mathfrak{O}_K^*$. Then, by Proposition 3.3.1, $F$ is isomorphic to the nice group that corresponds to the operator $a^{-1}wu^{-1}$. □

**4.4. Logarithms and isogenies of nice groups.** We lift the condition on $n(F)$ occurring in Proposition 3.3.1.

**Proposition 4.4.1.** *The group $F'$ that corresponds to $\lambda'=\Lambda'(x)$ is isomorphic to a nice group $F$ corresponding to $\Lambda$ if and only if $\Lambda'\varepsilon\equiv a\Lambda \bmod R$ for some $\varepsilon\in W^*$, $a\in\mathfrak{O}_K^*$.*

*Proof.* It suffices to verify that the condition on the fractional parts implies that $n(F')\le [\frac{e}{h-ph}]$. Since the fractional part determines $v(F')$, we easily obtain $v(F')=v(F)=0$. It remains to apply Proposition 4.3.1. □

Now we describe the kernels of the isogenies from arbitrary formal groups into nice groups.

**Proposition 4.4.2.** *Let $f:F\to G$ be an isogeny of formal groups. Then $G$ is isomorphic to a nice group if and only if for some $s\ge 0$ the kernel of $f$ equals $\{t_i: 1\le i\le p^s\}$ and $\frac{se}{h}-\sum_{i=2}^{p^s} v(t_i)=v(F)$.*

*Proof.* Let $T$ denote the kernel of $f$, and let $\#T=p^s$. Let $F,F'$ be formal groups satisfying $ar(F)=r(F')\varepsilon\Delta^m$, where $\varepsilon\in W^*$. Lemma 3.2.1 shows that $v(F')=v(F)+v(a)-\frac{me}{h}$. Therefore, $v(G)=v(F)-\frac{se}{h}+\sum_{t\in T} v(t)$. Hence, $v(G)=0$ if and only if $\sum_{t\in T} v(t)=\frac{se}{h}-v(F)$. We verify that $T$ has the form $\{t_i: 1\le i\le p^s\}$. Indeed, the sum of valuations of the elements of $T\setminus\{0\}$ is equal to $\sum_{2\le i\le p^s} v(t_i)$. It remains to check that no other set of $t_i$ admits the same sum of valuations. For this, it suffices to verify that $v(t_{p^s+1})<v(t_{p^s})$.

Indeed, otherwise

$$v(t_{p^s+1})=v(t_{p^s+2})=\cdots=v(t_{p^{s+1}})=v(t_{p^s}).$$

Since $\sum_{2\le i\le p^{s-1}} v(t_i)\le \frac{se-e}{h}-v(F)$, we have

$$v(t_{p^s})=\frac{\sum_{p^{s-1}+1\le i\le p^s} v(t_i)}{(p-1)p^{s-1}}\ge \frac{e}{h(p-1)p^{s-1}},$$

whence

$$\sum_{2\le i\le p^{s+1}} v(t_i)=\sum_{2\le i\le p^s} v(t_i)+(p-1)p^s v(t_{p^s})\ge \frac{(s+p)e}{j}-v(F),$$

which contradicts part 1 of Proposition 4.3.1. □

**4.5. The conclusion of the proof of the main theorem.** Immediately from the definition (see Theorem 3.4.1), it follows that the group $F_i$ constructed for $F$ coincides with the $F_i$ constructed for $F_1$. Observe also that $d$ and $s$ in the main theorem are constant within any isogeny class. Therefore, it suffices to finish the proof of Theorem 3.4.1 in the case where $F$ is a nice formal group.

We prove statement 1 of the theorem. We use the following well-known fact: if $f_1$ and $f_2$ are isogenies from $F$ into formal groups $G_1$ and $G_2$, respectively, and the kernels of $f_1$ and $f_2$ coincide, then $G_1 \cong G_2$.

Hence, by Proposition 4.4.2, if $G$ is a nice group isogeneous to $F$, then the isomorphism class of $G$ can be recovered uniquely by the height of the isogeny.

If $f$ is an isogeny of height $m$ from $F$ into $G$, then $h = [p]_G \circ f$ is also an isogeny from $F$ into $G$, and the height of $h$ equals $m + h$. Thus, if $G$ is a nice group isogeneous to $F$, then the isomorphism class of $G$ is completely determined by the residue of the height of the isogeny modulo $h$. By Proposition 4.4.2, the heights of the isogenies from $F$ into nice groups that exceed some $I$ are precisely the numbers of the form $b_i + lh$, $1 \le i \le s = s(F)$. On the other hand, for each $1 \le i \le s(F)$, by Theorem 3.1.1 there exists an isogeny from $F$ into $F_i$ of height $b_i + nh$ for some $n \in \mathbb{Z}$. We arrive at statement 1 of the theorem.

We verify statement 2. Let $f \in \mathrm{End}(F_i)$ be an endomorphism of $F_i$ that is a uniformizing element of $\mathrm{End}(F_i)$. It is easily seen that the heights of isomorphisms give a valuation on the endomorphism ring of $F$, and the height of $[p]_F$ is equal to $h$. Therefore, the height of $f$ is $h/d$. Let $i + s/d \le s$, and let $b_{i'} = b_i + h/d$. As above, we see that $f$ gives an isomorphism of $F_i$ and some group isomorphic to $F_{i'}$. Hence, if $h/d \mid b_i - b_j$, then $F_i$ is isomorphic to $F_j$. It is easily seen that this condition is fulfilled if and only if $s/d \mid i - j$.

Now, suppose that $s/d \nmid i - j$. By Theorem 3.1.1, there exists a homomorphism $f$ from $F_i$ into $F_j$, and $h/d$ does not divide the height of $f$. Hence, the height of $f$ does not coincide with the height of any element of $\mathrm{End}(F_i)$, whence $F_i$ is not isomorphic to $F_j$.

Statement 3 follows immediately from statements 1 and 2.

**4.6. $D_F$ for non-$p$-typical formal groups.** As has already been mentioned, any formal group law is strictly isomorphic to the formal group given by the $p$-typical part of its logarithm. Therefore, we define $D_F$ for an arbitrary formal group to be $D_F$ for the corresponding $p$-typical group $F_p$.

**Proposition 4.6.1.** 1. *$D_F$ satisfies Proposition* 2.1.2.

2. *For $h(x) \in K[[x]]$, we have $\exp_F(h) \in \mathfrak{O}_K[[x]]$ if and only if $\sum_s a_{p^s i}\Delta^s \in D_F$ for each $i$ with $p \nmid i$. Here $a_i$ is the coefficient of $x^i$ in $\lambda$.*

3. *$\lambda'$ is the logarithm of a formal group strictly isomorphic to $F$ if and only if $\lambda \equiv x \bmod \deg 2$ and $\lambda'$ satisfies the condition on $h$ in item* 2.

The proof can be found in [2].

By using these results, it can easily be checked whether or not a given $p$-typical formal group is isomorphic to a fixed nice group.

## §5. Comparison of the fractional part of the logarithm functor with Fontaine's functor

Since in this paper we only consider one-dimensional formal groups, comparison of functors will be done mainly for such groups. However, we note that most of the statements below can be extended to the case of arbitrary dimension in an obvious way.

**5.1. The description of Fontaine's functor.** The fractional part invariant takes its values in the module $K[[\Delta]]/R$. This module is torsion free and embeds injectively in the corresponding module for any $K' \supset K$. The invariant of Fontaine, which we describe below (note that the description differs from those in the book [4]) takes its values in certain modules that (in the original definition) depend on the reduction of the formal group and have torsion for $e \geq p$. In general, the module for the ground field is mapped to the module for an extension not injectively; if the ramification index is sufficiently large, then the entire torsion is taken to 0. Note also that torsion can be added to the fractional part invariant by extending the arguments of [2] (similar to those in [4]); yet it seems that all known applications of the torsion in Fontaine's functor are covered by [2] (without considering the torsion). For that reason, we shall compare the fractional part invariant with Fontaine's invariant factorized by the torsion.

Recall that we deal with one-dimensional formal groups of finite height $h$.

Fountaine's invariant can be split into the following two parts:

1) the reduction of $F$;

2) the module $\{c\Lambda : c \in \mathfrak{O}_K\}$ viewed as a submodule of $K[[\Delta]]/M$, where $M = \{\sum a_i\Delta^i : a_i \in \sum_{j\leq i} \mathfrak{M}^{p^j - je}\}$.

In [4], the properties of this invariant that turn it into a functor were considered. Certainly, these properties are similar to those for the fractional part invariant described above (see Theorem 3.1.1). Since in this section we only demonstrate that the fractional part of the logarithm contains more information about formal groups than Fontaine's invariant, we shall not need these properties. It will suffice to check that formal groups that have equal Fontaine's invariants can have nonequivalent fractional parts of the logarithm. Therefore, we shall not consider the isomorphism classes of Fontaine's invariants (with respect to the morphisms described in [4]).

**5.2. Comparison of functors.** First, we prove that the fractional part of the logarithm describes formal groups not worse than Fontaine's invariant. As was mentioned above, formal groups having equal fractional parts of the logarithm are isogeneous. The behavior of the fractional part of the logarithm under isogenies was described completely.

Now, let $\Lambda \equiv \Lambda' \bmod R$, where $\Lambda$ and $\Lambda'$ correspond to formal groups $F$ and $F'$. We may assume that $h > 1$, because isogeneous formal groups of height 1 are isomorphic. Then the coefficients of $\Delta^i$ in $\Lambda$ and $\Lambda'$ are divisible by $\pi/p^i$. Moreover, for some $s$ we have $\pi^s(\Lambda - \Lambda') \in \mathfrak{O}_K[[\Delta]]$ (actually, it suffices to take $s = \max_{j>0}(ej - p^j)$). We see that if the ramification index of $K$ is sufficiently large, then $\Lambda - \Lambda' \in M$, which implies that $F$ and $F'$ have equal Fontaine's invariants.

**5.3. Examples.** Now we show that formal groups with equal Fontaine's invariants but nonequivalent fractional parts of the logarithm exist indeed.

By Proposition 4.2.4, for this it suffices to construct a nice formal group $F$ and subgroups $T_1, T_2 \subset F(\mathfrak{M}_{\mathbb{K}})$ such that $\#T_1 = \#T_2$ and

$$\sum_{t\in T_1\setminus\{0\}} v(t) = \sum_{t\in T_2\setminus\{0\}} v(t), \qquad \prod_{t\in T_1\setminus\{0\}} r(t) \neq \prod_{t\in T_2\setminus\{0\}} r(t). \tag{9}$$

Indeed, a morphism $f \equiv ax \bmod \deg 2$ multiplies $r(F)$ by $a$ from the left and by $\varepsilon \in W$ from the right, where $\varepsilon \equiv a \bmod (\Delta, \pi)$. Hence, the isomorphisms can only multiply $r(F)$ from the left by $a \equiv 1 \bmod \pi$.

The situation (9) is quite frequent. Certainly, the resulting formal groups will be defined over the extension of $K$ whose absolute Galois group maps $T_1$ and $T_2$ into themselves.

In order to make this field (i.e., the minimal field of definition for the formal groups we want to construct) easy to describe, for the role of $F$ we take the Lubin–Tate formal group of the field $K$ that is an unramified extension of $\mathbb{Q}_p$ of degree 2. We denote $p^2$ by $q$.

For $0 < i < q$ we obtain $v(t_i) = 1/(q-1)$. Choosing $t_{i_1}, t_{i_2}$ $(0 < i_1, i_2 \le q)$ such that $r(t_{i_1}/t_{i_2})$ is a primitive root of 1 of degree $q-1$, we consider the group $T_1$ generated by $t_{i_1}$, and the group $T_2$ generated by $t_{i_2}$. Then $\#T_1 = \#T_2 = p$ and $\sum_{t\in T_1\setminus\{0\}} v(t) = \sum_{t\in T_2\setminus\{0\}} v(t) = 1/(p+1)$. On the other hand,

$$\prod_{t\in T_1\setminus\{0\}} r(t) = r(t_{i_1}/t_{i_2})^{p-1} \prod_{t\in T_2\setminus\{0\}} r(t) \neq \prod_{t\in T_2\setminus\{0\}} r(t),$$

so that $T_1$ and $T_2$ are as required. Observe that the formal groups corresponding to $T_1$ and $T_2$ are defined over a totally ramified extension of $K$ of degree $p+1$ (hence, its absolute ramification index is $p+1$).

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