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ON THE DISTRIBUTION OF VALUES OF $L(1, \mathrm{sym}^2 f)$

O. M. FOMENKO

Dedicated to Yuriĭ Vladimirovich Linnik's 90th birthday anniversary

Abstract. Let $S_k(\mathrm{SL}(2,\mathbb{Z}))^+$ be the set of holomorphic Hecke eigencuspforms f of weight k with respect to $\mathrm{SL}(2,\mathbb{Z})$. Let $L(s, \mathrm{sym}^2 f)$ be the symmetric square of the Hecke L-function of a cusp form f. The moments of $L(1, \mathrm{sym}^2 f)$, $f \in S_k(\mathrm{SL}(2,\mathbb{Z}))^+$, are computed for a pure imaginary order. The limiting distribution of $\log L(1, \mathrm{sym}^2 f)$, $f \in S_k(\mathrm{SL}(2,\mathbb{Z}))^+$, is studied in the weight aspect. Namely, the limiting distribution function, the limiting characteristic function and its Euler product are investigated, and the rate of convergence of frequencies to the limiting distribution is measured.

As a consequence, new facts on the limiting distribution of $\mathrm{SL}(1, \mathrm{sym}^2 f)$ are obtained not only in the case of the holomorphic Hecke eigencuspforms f, but also in the case of the Hecke–Maass eigencuspforms f.

§1. Introduction. The results

Consider the space $S_k(\Gamma)$ of holomorphic cusp forms

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz)$$

of even weight $k \geq 12$ for the group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, where $\mathbb{Z}$ is the ring of rational integers, $z = x + iy$, $y > 0$, and $e(\xi) := e^{2\pi i \xi}$. Let $S_k(\Gamma)^+$ be the set of all Hecke eigencuspforms in this space with $a_f(1) = 1$. It is known that $S_k(\Gamma) = \varnothing$ if k is odd or if k is even and $k < 12$; for even $k \geq 12$ we have

$$\dim S_k(\Gamma) = \# S_k(\Gamma)^+ = \begin{cases} [k/12] & \text{if } k \not\equiv 2 \pmod{12}, \\ [k/12] - 1 & \text{if } k \equiv 2 \pmod{12}. \end{cases}$$

From now on we assume that $f \in S_k(\Gamma)^+$. Set

$$\lambda_f(n) = a_f(n)/n^{\frac{k-1}{2}};$$

$\lambda_f(n)$ is the eigenvalue of the Hecke operator $T_n, n = 1, 2, 3, \dots$. For a prime p we have

$$\lambda_f(p) = \alpha_f(p) + \overline{\alpha_f(p)}, \quad \alpha_f(p)\overline{\alpha_f(p)} = 1$$

(Deligne); therefore,

$$\lambda_f(p) = 2\cos\varphi_f(p), \quad \varphi_f(p) \in [0, \pi].$$

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Consider the Hecke L-function

$$\begin{aligned} L(s,f) &= \sum_{n=1}^{\infty} \lambda_f(n) n^{-s} \\ &= \prod_p (1-\alpha_f(p)p^{-s})^{-1}(1-\overline{\alpha_f(p)}p^{-s})^{-1} \\ &= \prod_p (1-\lambda_f(p)p^{-s}+p^{-2s})^{-1}, \quad \operatorname{Re} s > 1. \end{aligned}$$

$L(s,f)$ is an entire function and satisfies a Riemann-type functional equation.

For $f \in S_k(\Gamma)^+$, we also consider the symmetric square $L(s, \operatorname{sym}^2 f)$ of the Hecke L-function $L(s,f)$:

$$L(s,\operatorname{sym}^2 f) = \prod_p (1-\alpha_f(p)^2 p^{-s})^{-1}(1-p^{-s})^{-1}(1-\overline{\alpha_f(p)}^2 p^{-s})^{-1}, \quad \operatorname{Re} s > 1.$$

$L(s,\operatorname{sym}^2 f)$ is an entire function and also satisfies a Riemann-type functional equation. It is easily seen that

$$L(s,\operatorname{sym}^2 f) = \zeta(2s) \sum_{n=1}^{\infty} \lambda_f(n^2) n^{-s},$$

where $\zeta(s)$ is the Riemann zeta-function.

It is known that $L(s,\operatorname{sym}^2 f) > 0$; moreover,

$$\frac{1}{\log k} \ll L(1,\operatorname{sym}^2 f) \ll \log^3 k \tag{1.1}$$

(the proof of the lower bound, which is a deep fact, can be found in [1]).

The automorphic L-functions for holomorphic cusp forms of integral weight $k \geq 2$ and level N for the group $\Gamma_0(N)$ are defined similarly. The set of new forms $S_k(\Gamma_0(N))^+$ of weight k and level N for $\Gamma_0(N)$ is an analog of the set $S_k(\Gamma)^+$.

The automorphic L-functions can also be introduced for Hecke –Maass eigencuspforms; note that in this case no analog of Deligne's result has been proved so far.

In the present paper we study the limiting distribution of the values of $L(1,\operatorname{sym}^2 f)$, $f \in S_k(\Gamma)^+$, $k \to \infty$. For the first time a similar problem for the Dirichlet L-functions was considered by Chowla and Erdös [2]. Let $L(s,\chi_D)$ be the Dirichlet L-function with the quadratic character $\chi_D(n)=(-D/n)$, where $(-D/n)$ denotes a Kronecker symbol. Chowla and Erdös proved that, as $N \to \infty$, the limit of the frequencies

$$\#\{D, D \leq N, L(1,\chi_D) \leq x\}/N$$

exists for all real values of x and is a continuous distribution function. In [3, 4], Barban applied another method to this problem. He used Yu. V. Linnik's inequality of large sieve and computed the moments of $L(1,\chi_D)$ for every integral order $k \geq 0$. In this way, Barban found the corresponding characteristic function, but he did not prove by his method that the limiting distribution function is continuous.

The moments of $L(1,\chi_D)$ for a pure imaginary order were computed by Faĭnleĭb [5, 6]. This allowed him to investigate the limiting distribution for $\log L(1,\chi_D)$, namely, to expand the limiting characteristic function in an Euler product, to study its properties and the properties of the limiting distribution function. As a result, he was able to deeper investigate the limiting distribution function of Chowla, Erdös, and Barban. In particular, the rate of convergence of frequencies to the limiting distribution was measured.

Similar results were obtained by a somewhat different method in Elliott's paper [7].

Progress in the theory of automorphic forms made it possible to obtain limit laws for the automorphic L-functions. In the case of $L(1,\mathrm{sym}^2 f)$, limit laws were proved:

1) for the Hecke–Maass eigencuspforms with large Laplacian eigenvalues [8];
2) for the holomorphic Hecke eigencuspforms with large level [9, 10];
3) for the holomorphic Hecke eigencuspforms with large weight [11].

In [12], a limit law was obtained in the case of $L(1,f)$ for holomorphic Hecke eigencuspforms with large level. The authors of [8]–[12] used Barban's approach, i.e., the asymptotic evaluation of the integral moments of $L(1,\mathrm{sym}^2 f)$ and $L(1,f)$. In its turn, this required proving large-sieve-type inequalities for Fourier coefficients of cusp forms.

In [13], Golubeva used the approach of [6] (also requiring inequalities of large sieve) to obtain new results on the distribution of the values of $L(1,f)$ with large level. Furthermore, an interesting relationship was observed in [13] between the p-factor of the limiting characteristic function associated with the random variable $\log L(1,f)$ and the p-adic Plancherel measure (a p-analog of the Sato–Tate measure); see below for more details.

In the present paper, we use the approaches of [6] and especially of [13] to study the distribution of the values of $\log L(1,\mathrm{sym}^2 f)$, $f\in S_k(\Gamma)^+$, $k\to\infty$. As a corollary, we obtain more complete (in comparison with [11]) information concerning the distribution of the values of $L(1,\mathrm{sym}^2 f)$, $f\in S_k(\Gamma)^+$, $k\to\infty$.

Before stating the results, for an arbitrary complex z we define the general divisor function $d_z(n)$ (see [14]) by the relation

$$\sum_{n=1}^{\infty} d_z(n)n^{-s} := \zeta^z(s) = \prod_p (1-p^{-s})^{-z} \quad (\mathrm{Re}\, s>1),$$

where a branch of $\zeta^z(s)$ is distinguished by the condition

$$\zeta^z(s) = \exp\{z\log\zeta(s)\} = \exp\left(z\sum_p\sum_{j=1}^{\infty} j^{-1}p^{-js}\right) \quad (\mathrm{Re}\, s>1).$$

This definition shows that $d_z(n)$ is a multiplicative function of n, which generalizes the classical divisor function $d_k(n)$ ($k\ge 2$ is a fixed integer); we recall that $d_2(n) =: d(n)$.

Theorem 1. *For $|w|\le(\log k/\log^2\log k)^{1/3}$, $w\in\mathbb{R}$, we have*

$$\widehat{G}_1(w) := \frac{1}{\#S_k(\Gamma)^+}\sum_{f\in S_k(\Gamma)^+} L^{iw}(1,\mathrm{sym}^2 f) = \widehat{G}(w) + O(\exp(-c_1\log k/\log\log k)),$$

where

$$\widehat{G}(w) = \prod_p \widehat{G}_p(w),$$

$$\widehat{G}_p(w) = \sum_{\substack{m>k\ge 0\\ l\ge 0}} \frac{d_{iw}(p^m)d_{iw}(p^k)d_{iw}(p^l)}{p^{m+k+l}}\Big(\frac{1}{p^{m-k}}-\frac{1}{p^{m-k-1}}\Big) + \sum_{m\ge 0, l\ge 0}\frac{d_{iw}^2(p^m)d_{iw}(p^l)}{p^{2m+l}},$$

and c_1 is a positive constant.

Theorem 2. 1) *The relation*

$$\Big|\frac{\#\{f, f\in S_k(\Gamma)^+, \log L(1,\mathrm{sym}^2 f)\le x\}}{\#S_k(\Gamma)^+} - G(x)\Big| \ll (\log^2\log k/\log k)^{1/3}$$

is fulfilled uniformly for all real numbers x, where $G(x)$ is the distribution function determined by the characteristic function $\widehat{G}(w)$.

2) *The p-factor $\widehat{G}_p(w)$ of the characteristic function $\widehat{G}(w)$ can be written as*

$$\widehat{G}_p(w) = \int_0^{\pi} \Big(1 - \frac{2\cos 2\varphi}{p} + \frac{1}{p^2}\Big)^{-iw} \Big(1 - \frac{1}{p}\Big)^{-iw} \mu_p(\varphi),$$

where the measure $\mu_p(\varphi)$ has the form

$$\mu_p(\varphi) := \frac{1}{\pi}\Big(1 + \frac{1}{p}\Big) \frac{1 - \cos 2\varphi}{1 - \frac{2\cos 2\varphi}{p} + \frac{1}{p^2}} d\varphi.$$

3) *The distribution function $G(x)$ has density $p_G(x)$ continuous on the entire real axis*; *moreover*, *$G(x)$ admits analytic continuation to some half-plane $\operatorname{Im} z > -\delta$, $\delta > 0$.*

Similar results are valid for $L(1, \operatorname{sym}^2 f)$ in the level aspect.

Earlier, the measure $\mu_p(\varphi)$ arose in the study of the distribution of the angles $\varphi_f(p)$ of the pth coefficients $\lambda_f(p)$ of Hecke eigenforms f [15]–[18]. Consider the k-aspect. In [16, 17] it was shown that if $f \in S_k(\Gamma)^+$, where $\lambda_f(p) = 2\cos\varphi_f(p)$ and $\varphi_f(p) \in [0, \pi]$, then, as $k \to \infty$, the set

$$\{\varphi_f(p), f \in S_k(\Gamma)^+\}$$

becomes uniformly distributed with respect to the measure $\mu_p(\varphi)$.

Corollary 1. 1) *For $x > 0$, the relation*

$$\Big|\frac{\#\{f, f \in S_k(\Gamma)^+, L(1, \operatorname{sym}^2 f) \le x\}}{\# S_k(\Gamma)^+} - F(x)\Big| \ll (\log^2 \log k / \log k)^{1/3} \tag{1.2}$$

is fulfilled uniformly in x, where $F(x) = G(\log x)$.

2) *The distribution function $F(x)$ is continuous on the entire real axis*; *moreover*, *it has a density*

$$p_F(x) = \begin{cases} p_G(\log x)\frac{1}{x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

The limit law (1.2) was proved in [11], but without estimation of the rate of convergence to the limiting distribution. The convergence of the frequencies to $F(x)$ as $k \to \infty$ was obtained in [11] only for the points of continuity of $F(x)$; moreover, the continuity of $F(x)$ on the entire real axis $\mathbb{R}$ was not proved.

Now we cite the results of Luo [8]. Let $\{f_j(z)\}_{j\ge1}$ be the orthonormalized Hecke basis of the space $L_0^2(\Gamma \setminus \mathbb{H})$, $\mathbb{H} = \{z, z = x + iy, y > 0\}$, consisting of Maass cusp forms $f_j(z)$ with the Laplacian eigenvalue $\lambda_j = 1/4 + t_j^2$ ($t_j \ge 0$), with nth Fourier coefficient $a_j(n)$, and with the eigenvalue $\lambda_j(n)$ of the Hecke operator T_n, $a_j(n) = a_j(1)\lambda_j(n)$. The Fourier expansion of $f_j(z)$ has the form

$$f_j(z) = \cosh^{1/2}(\pi t_j) y^{1/2} \sum_{n\neq0} a_j(n) K_{it_j}(2\pi|n|y) e(nx),$$

where

$$K_\nu(z) = \int_0^\infty e^{-z\cosh\tau} \cosh(\nu\tau)\, d\tau$$

is the modified Bessel function. Luo proved that, for $x > 0$, the relation

$$\lim_{T\to\infty} \frac{\#\{j, t_j \le T, L(1, \operatorname{sym}^2 f_j) \le x\}}{\#\{j, t_j \le T\}} = F(x) \tag{1.3}$$

is valid at every point of continuity of $F(x)$, where $F(x)$ is a distribution function.

With the help of our results, the limiting distribution (1.3) can be described more precisely. The following statement is true.

Corollary 2. *The distribution function F occurring in Luo's limit law* (1.3) *coincides with the distribution function F in Corollary* 1. *Therefore, statement* 2) *of Corollary* 1 *is valid for the function F in* (1.3).

Indeed, the two collections of moments of integral positive order $m = 1, 2, 3, \dots$ that determine the limiting distribution functions F in (1.2) and F in (1.3) coincide completely and admit proper upper estimates [11, 8]. Then the classical theorem on moments (see, e.g., Lemmas 5.1 and 5.7 in Barban's survey [4]) applies, which guarantees the uniqueness of F.

It seems difficult to refine the limit law (1.3) directly, without resorting to the holomorphic case.

Notation. We add some notation to that introduced above. $\mathbb{R}$ is the field of real numbers; $\mathbb{C}$ is the field of complex numbers; $w \in \mathbb{R}$; $c', c'', c_2, c_3, c_4, \dots$ are positive constants; ε is an arbitrarily small positive fixed number; $s = \operatorname{Re} s + i \operatorname{Im} s =: \sigma + it$; and

$$S(k \le K) := \bigcup_{\substack{12 \le k \le K \\ k \text{ is even}}} S_k(\Gamma)^+; \quad N_0 := \# S_k(\Gamma)^+; \quad \sum_f \cdots := \sum_{f \in S_k(\Gamma)^+} \cdots .$$

§2. Proof of Theorem 1. Part 1

In this section we collect some lemmas needed in the proof of Theorem 1. For $\sigma \ge 1/2$, we put

$$N(\sigma, H, \mathrm{sym}^2 f) := \#\{\rho = \beta + i\gamma, L(\rho, \mathrm{sym}^2 f) = 0, \beta \ge \sigma, |\gamma| \le H\}.$$

Lemma 1. *For $\sigma \ge 3/4$, $K \ge 12$, and some positive constant b, we have*

$$\sum_{f \in S(k \le K)} N(\sigma, \log^3 K, \mathrm{sym}^2 f) \ll_\varepsilon K^{b(1-\sigma)+\varepsilon}.$$

We do not present the proof of this lemma here, restricting ourselves to some comments. This lemma is modelled on a similar fact in Yu. V. Linnik's theory of large sieve (see, e.g., the book [19]). In [8], Luo adapted the classical proof to the case of $L(s, \mathrm{sym}^2 f)$, where f is a Hecke–Maass eigencuspform. In the proof of Lemma 1, the inequality of large sieve for $\{\lambda_f(n^2)\}_{f \in S(k \le K)}$ plays a key role. Here we cite this inequality, referring the reader to the author's paper [11]:

$$\sum_{f \in S(k \le K)} \Big| \sum_{n \le N} a_n \lambda_f(n^2) \Big|^2 \ll_\varepsilon \big(N (\log K)^{c_2} + K^5 N^{\frac{1}{2}+\varepsilon} \big) \sum_{n \le N} |a_n|^2,$$

where $\{a_n\}_{n \le N}$ is an arbitrary sequence of complex numbers.

Below we choose the branch of $\log L(s, \mathrm{sym}^2 f)$ fixed by the relation

$$\log L(s, \mathrm{sym}^2 f) = \sum_p \sum_{j=1}^\infty \frac{(\alpha_f(p)^{2j} + 1 + \overline{\alpha_f(p)}^{2j})}{j p^{js}} \quad (\sigma > 1).$$

Similarly, the branch of the function $L^z(s, \mathrm{sym}^2 f)$, $z \in \mathbb{C}$, is fixed by the relation

$$L^z(s, \mathrm{sym}^2 f) = \exp\{z \log L(s, \mathrm{sym}^2 f)\}.$$

Lemma 2. *Suppose $0 < \eta < \frac{1}{2}$, $K \ge 12$. If $L(s, \mathrm{sym}^2 f)$ $(f \in S(k \le K))$ has no zeros in the rectangle*

$$1 - \eta \le \sigma < 1, \quad |t| \le \log^3 K,$$

then in the rectangle

$$1-\eta/2 \le \sigma < 1, \quad |t| \le \log^2 K,$$

we have the estimate

$$\log L(s, \operatorname{sym}^2 f) = O(\log^{c_3(1-\sigma)} K \cdot \log\log K).$$

The proof is based on the Borel–Carathéodory theorem and the Hadamard three circles theorem. See a similar argument in [3, Lemma 3].

We introduce the coefficients $\nu(n,w,f)$ by the following expansion:

$$L^{iw}(s, \operatorname{sym}^2 f) = \sum_{n=1}^{\infty} \frac{\nu(n,w,f)}{n^s} \quad (\sigma > 1). \tag{2.1}$$

Lemma 3. *Suppose that* $L(s, \operatorname{sym}^2 f)$, $f \in S_k(\Gamma)^+$, *has no zeros in the rectangle*

$$1-\eta \le \sigma < 1, \quad |t| \le \log^3 k.$$

Then for $|w| \le \log k/\log^2\log k$ *we have*

$$L^{iw}(1, \operatorname{sym}^2 f) = \sum_{n=1}^{\infty} \frac{\nu(n,w,f)}{n} \exp(-n/k) + O\Big(\exp\Big(\frac{-c'\log k}{\log\log k} + c''|w|\log\log k\Big)\Big).$$

Proof. We have

$$\sum_{n=1}^{\infty} \frac{\nu(n,w,f)}{n} \exp(-n/k) = \frac{1}{2\pi i}\int_{(1)} L^{iw}(s+1, \operatorname{sym}^2 f)\Gamma(s)k^s\,ds.$$

By the residue theorem,

$$\begin{aligned} &\frac{1}{2\pi i}\int_{(1)} L^{iw}(s+1, \operatorname{sym}^2 f)\Gamma(s)k^s\,ds \\ &\qquad = L^{iw}(1, \operatorname{sym}^2 f) + \frac{1}{2\pi i}\int_{\gamma} L^{iw}(s+1, \operatorname{sym}^2 f)\Gamma(s)k^s\,ds, \end{aligned}$$

where γ is the oriented polygonal path with the vertices

$$\begin{array}{ll} (\log\log k)^{-1} - i\infty, & (\log\log k)^{-1} - i\log^2 k, \\ -(\log\log k)^{-1} - i\log^2 k, & -(\log\log k)^{-1} + i\log^2 k, \\ (\log\log k)^{-1} + i\log^2 k, & (\log\log k)^{-1} + i\infty. \end{array}$$

The last integral is estimated using Lemma 2 and the known estimates for $\Gamma(s)$. ☐

Lemma 4. *Let* $f \in S_k(\Gamma)^+$. *For* $|w| \le \log k/\log^2\log k$ *we have*

$$L^{iw}(1, \operatorname{sym}^2 f) = \sum_{n=1}^{\infty} \frac{\nu(n,w,f)}{n} \exp(-n/k) + O(\exp(c_4 \log k/\log\log k)).$$

The proof of this lemma is fairly easy; see a similar argument in [13, Lemma 4].

Lemma 5. *Let* $\delta(n) = n^{1/2}\log^2 n$. *We have*

$$\sum_f \lambda_f(n) = \begin{cases} N_0 \frac{1}{n^{1/2}} + O(\delta(n)) & \text{if } n = n_1^2, \\ O(\delta(n)) & \text{if } n \ne n_1^2. \end{cases}$$

The proof involves the Eichler–Selberg trace formula and can be found in [16, 17].

§3. PROOF OF THEOREM 1. PART 2

We compute the coefficients $\nu(n,w,f)$ (see (2.1)). For $\sigma>1$ we have

$$L^{iw}(s,\mathrm{sym}^2 f)=\prod_p \sum_{\substack{m\ge0,k\ge0\\ l\ge0}} \frac{d_{iw}(p^m)d_{iw}(p^k)d_{iw}(p^l)\alpha_f(p)^{2m}\overline{\alpha_f(p)}^{2k}}{p^{(m+k+l)s}}.$$

This product can be written in the form

$$\prod_p\Bigl\{\sum_{\substack{m>k\ge0\\ l\ge0}}\frac{d_{iw}(p^m)d_{iw}(p^k)d_{iw}(p^l)(\alpha_f(p)^{2(m-k)}+\overline{\alpha_f(p)}^{2(m-k)})}{p^{(m+k+l)s}} + \sum_{m\ge0,l\ge0}\frac{d^2_{iw}(p^m)d_{iw}(p^l)}{p^{(2m+l)s}}\Bigr\}.$$

The known formula

$$\lambda_f(p^\nu)=\frac{\alpha_f(p)^{\nu+1}-\overline{\alpha_f(p)}^{\nu+1}}{\alpha_f(p)-\overline{\alpha_f(p)}}$$

shows that, for $m-k\ge2$,

$$\alpha_f(p)^{m-k}+\overline{\alpha_f(p)}^{m-k}=\lambda_f(p^{m-k})-\lambda_f(p^{m-k-2}).$$

Thus,

$$\begin{aligned}L^{iw}(s,\mathrm{sym}^2 f)&=\prod_p\Bigl(\sum_{r=0}^{\infty}\frac{\nu(p^r,w,f)}{p^{rs}}\Bigr)\\ &=\prod_p\Bigl\{\sum_{\substack{m>k\ge0\\ l\ge0}}\frac{d_{iw}(p^m)d_{iw}(p^k)d_{iw}(p^l)}{p^{(m+k+l)s}}\bigl(\lambda_f(p^{2(m-k)})-\lambda_f\bigl(p^{2(m-k)-2}\bigr)\bigr)\\ &\qquad+\sum_{m\ge0,l\ge0}\frac{d^2_{iw}(p^m)d_{iw}(p^l)}{p^{(2m+l)s}}\Bigr\}.\end{aligned}$$

Therefore, we have

$$\nu(n,w,f)=\prod_{p^r\| n}\nu(p^r,w,f),$$

where

$$\begin{aligned}&\nu(p^r,w,f)=1 && \text{if } r=0;\\ &\nu(p^r,w,f)=d_{iw}(p)\lambda_f(p^2) && \text{if } r=1;\\ &\nu(p^r,w,f)=\sum_{\substack{m>k\ge0,l\ge0\\ m+k+l=r}}d_{iw}(p^m)d_{iw}(p^k)d_{iw}(p^l)\bigl(\lambda_f(p^{2(m-k)})-\lambda_f(p^{2(m-k)-2})\bigr)\\ &\qquad+\sum_{\substack{m\ge0,l\ge0\\ 2m+l=r}}d^2_{iw}(p^m)d_{iw}(p^l) && \text{if } r\ge2.\end{aligned}$$

Lemma 1 implies that the number of $L(s,\mathrm{sym}^2 f)$, $f\in S_k(\Gamma)^+$, that have a zero in the rectangle

$$1-\eta\le\sigma<1,\quad |t|\le\log^3 k,$$

is estimated as $O(k^{1-\theta})$ if η is sufficiently small; here θ is some positive number. Therefore, by Lemmas 3 and 4, for $|w| \le c_5 \log k / \log^2 \log k$ we have

$$\sum_f L^{iw}(1, \operatorname{sym}^2 f) = \sum_f \sum_{n=1}^{\infty} \frac{\nu(n,w,f)}{n} \exp(-n/k) + O(k \exp(-c_6 \log k / \log\log k)). \tag{3.1}$$

Using the values of the coefficients $\nu(n,w,f)$ and Lemma 5, we obtain

$$\sum_f \sum_{n=1}^{\infty} \frac{\nu(n,w,f)}{n} \exp(-n/k) = N_0 \sum\nolimits_1 + O\Big(\sum\nolimits_2\Big), \tag{3.2}$$

where

$$\begin{aligned} \sum\nolimits_1 &:= \sum_{n=1}^{\infty} \frac{A(n,w)}{n} \exp(-n/k), \\ \frac{A(n,w)}{n} &= \prod_{p^r \| n} \frac{A(p^r,w)}{p^r} \\ &= \prod_{p^r \| n} \Bigg(\sum_{\substack{m>k\ge 0, l\ge 0 \\ m+k+l=r}} \frac{d_{iw}(p^m) d_{iw}(p^k) d_{iw}(p^l)}{p^{m+k+l}} \Big(\frac{1}{p^{m-k}} - \frac{1}{p^{m-k-1}} \Big) \\ &\qquad\qquad + \sum_{\substack{m\ge 0, l\ge 0 \\ 2m+l=r}} \frac{d_{iw}^2(p^m) d_{iw}(p^l)}{p^{2m+l}} \Bigg), \end{aligned}$$

$$\sum\nolimits_2 := \sum_{n=1}^{\infty} \frac{\delta(n) d_3^2(n)}{n} \exp(-n/k) \prod_{p^r \| n} \Bigg(\sum_{\substack{m\ge k\ge 0, l\ge 0 \\ m+k+l=r}} d_{|w|}(p^m) d_{|w|}(p^k) d_{|w|}(p^l) \Bigg), \tag{3.3}$$

where $\delta(n) = n^{1/2} \log^2 n$. We prove that the series

$$\widehat{G}(w) = \sum_{n=1}^{\infty} \frac{A(n,w)}{n} \tag{3.4}$$

converges. Its p-factor $\widehat{G}_p(w)$ is of the form

$$\widehat{G}_p(w) = \sum_{r=0}^{\infty} \frac{A(p^r,w)}{p^r} = 1 + \frac{d_{iw}(p)p^{-1}}{p} + \frac{A(p^2,w)}{p^2} + \sum_{r\ge 3} \frac{A(p^r,w)}{p^r} = \sum_{r=0}^{\infty} \frac{A'(p^r,w)}{p^r},$$

where

$$A'(p^0,w) = 1, \qquad A'(p,w) = 0, \qquad A'(p^2,w) = d_{iw}(p) + A(p^2,w),$$
$$A'(p^r,w) = A(p^r,w) \quad (r \ge 3).$$

We recall the formula for $d_z(n)$ [14]:

$$d_z(p^\alpha) = \frac{z(z+1)\cdots(z+\alpha-1)}{\alpha!}$$

and, by multiplicativity,

$$d_z(n) = \prod_{p^\alpha \| n} \frac{z(z+1)\cdots(z+\alpha-1)}{\alpha!}.$$

With the help of this formula, we obtain

$$\begin{aligned}|A'(p^2,w)| &\le |w| + |A(p^2,w)|\\ &\le \begin{cases}\sum_{m+k+l=2} d_{|w|}(p^m)d_{|w|}(p^k)d_{|w|}(p^l) = d_{3|w|}(p^2) & \text{if } |w| \ge 1,\\ \sum_{m+k+l+f=2} d_{|w|}(p^m)d_{|w|}(p^k)d_{|w|}(p^l)d_{|w|}(p^f) = d_{4|w|}(p^2) & \text{if } |w| < 1,\end{cases}\end{aligned}$$

and for $r \ge 3$ we have

$$|A'(p^r,w)| \le \sum_{m+k+l=r} d_{|w|}(p^m)d_{|w|}(p^k)d_{|w|}(p^l) = d_{3|w|}(p^r) \quad \text{if } |w| \ge 0.$$

Consider the series

$$\sum_{n=1}^{\infty} \frac{A'(n,w)}{n} = \prod_p \Big(\sum_{r=0}^{\infty} \frac{A'(p^r,w)}{p^r}\Big). \tag{3.5}$$

By what was proved above,

$$|A'(n,w)| \le \begin{cases} d_{3|w|}(n) & \text{if } |w| \ge 1,\\ d_{4|w|}(n) & \text{if } |w| < 1.\end{cases}$$

It follows that

$$A'(n,w) \underset{\varepsilon,|w|}{\ll} n^{\varepsilon}.$$

On the left-hand side in (3.5), summation is actually taken over the squarefull numbers n, i.e., over the integers n with the property $p|n \implies p^2|n$ for all p. The number of such integers in the interval $n \le x$ is equal to

$$\frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + O(x^{1/3}).$$

Consequently, the series

$$\sum_{n=1}^{\infty} \frac{A'(n,w)}{n}$$

converges, and so does the series (3.4).

For definiteness, in what follows we assume that $|w| \ge 1$. Also, so far it is assumed that $|w| \le \log k/\log^2 \log k$.

Dividing the summation interval for $\sum_1$ into the parts $n \le k^{1/2}$ and $n > k^{1/2}$, we get

$$\begin{aligned}\sum\nolimits_1 &= \widehat{G}(w) + O\Big(\sum_{n>k^{1/2}} \frac{|A(n,w)|}{n}\Big) + O\Big(\frac{1}{k^{1/2}} \sum_{n \le k^{1/2}} \frac{|A(n,w)|}{n}\Big)\\ &=: \widehat{G}(w) + O\Big(\sum\nolimits_3\Big) + O\Big(\sum\nolimits_4\Big).\end{aligned}$$

We estimate the sum $\sum_3$:

$$\sum\nolimits_3 \le \frac{1}{k^{\frac12 \eta'}} \sum_{n>k^{1/2}} \frac{|A(n,w)|}{n^{1-\eta'}} \le \frac{1}{k^{\frac12 \eta'}} \sum_{n=1}^{\infty} \frac{|A(n,w)|}{n^{1-\eta'}},$$

where $\eta' = \log^{-1} \log k$. Since

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{|A(n,w)|}{n^{1-\eta'}} &= \prod_p \Big(1 + \frac{|A(p,w)|}{p^{1-\eta'}} + \frac{|A(p^2,w)|}{p^{2(1-\eta')}} + \cdots\Big)\\ &\le \prod_p \Big(1 + \frac{|w| + |A(p^2,w)|}{p^{2(1-\eta')}} + \frac{|A(p^3,w)|}{p^{3(1-\eta')}} + \cdots\Big),\end{aligned}$$

we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{|A(n,w)|}{n^{1-\eta'}} &\le \prod_{p}\Big(\Big(1-\frac{1}{p^{1-\eta'}}\Big)^{-3|w|}-\frac{3|w|}{p^{1-\eta'}}\Big)\\ &= \prod_{p<|w|\log\log k}\Big(\Big(1-\frac{1}{p^{1-\eta'}}\Big)^{-3|w|}-\frac{3|w|}{p^{1-\eta'}}\Big)\\ &\quad\times\prod_{p\ge|w|\log\log k}\Big(\Big(1-\frac{1}{p^{1-\eta'}}\Big)^{-3|w|}-\frac{3|w|}{p^{1-\eta'}}\Big)\\ &=: \prod\nolimits_1\cdot\prod\nolimits_2.\end{aligned}$$

We estimate $\prod_1$:

$$\begin{aligned}\prod\nolimits_1 &\le \prod_{p<|w|\log\log k}\exp\Big(-3|w|\log\Big(1-\frac{1}{p^{1-\eta'}}\Big)\Big)\\ &\le \prod_{p<|w|\log\log k}\exp(c_7|w|/p^{1-\eta'})\\ &\ll \exp\Big(c_7|w|\sum_{p<|w|\log\log k}1/p^{1-\eta'}\Big)\\ &\ll \exp\Big(c_8|w|\sum_{p<|w|\log\log k}1/p\Big)\\ &\ll \exp(c_9|w|\log\log\log k)\\ &\ll \exp(c_9\log k\cdot\log\log\log k/\log^2\log k).\end{aligned}$$

To estimate $\prod_2$, we apply the inequality

$$d_{|w|}(p^l)\le|w|^l\quad(|w|\ge1),$$

yielding

$$\begin{aligned}\prod\nolimits_2 &\le \prod_{p\ge|w|\log\log k}\Big(1+O\Big(\frac{|w|^2}{p^{2(1-\eta')}}\Big)\Big)\\ &\ll \prod_{p\ge|w|\log\log k}\exp(c_{10}|w|^2/p^{2(1-\eta')})\\ &\ll \exp\Big(c_{10}|w|^2\sum_{p\ge|w|\log\log k}1/p^{2(1-\eta')}\Big)\\ &\ll \exp(c_{11}|w|^{1+2\eta'}/\log\log k)\\ &\ll \exp(c_{12}|w|/\log\log k)\ll\exp(c_{12}\log k/\log^3\log k).\end{aligned}$$

Using the above estimates for $\prod_1$ and $\prod_2$, we see that

$$\sum\nolimits_3\ll\exp(-c_{13}\log k/\log\log k).$$

The sum $\sum_4$ will be estimated under the assumption

$$|w|\le\Big(\frac{\log k}{\log^2\log k}\Big)^{1/3}.$$

The term

$$\frac{|A(n,w)|}{n}=\prod_{p\|n}\frac{|A(p,w)|}{p}\cdot\prod_{\substack{p^r\|n\\ r\ge2}}\frac{|A(p^r,w)|}{p^r}$$

can be written as

$$\frac{|A(n,w)|}{n} = \prod_{p\|n} \frac{|w|p^{-1}}{p} \cdot \prod_{\substack{p^r\|n\\ r\geq 2}} \frac{|A(p^r,w)|}{p^r},$$

whence it follows that

$$\sum\nolimits_4 \leq \frac{1}{k^{1/2}} \sum_{n_0\leq k} \frac{C(n_0,w)}{n_0}.$$

The summation $\sum_{n_0\leq k}\cdots$ is over the squarefull $n_0 \leq k$, and $C(n_0,w) \leq 2d_{3|w|}(n_0)$.

Every squarefull number n_0 is representable in the form $n_0 = a^3b^2$, where $\mu^2(a) = 1$. We have

$$\begin{aligned}
\sum\nolimits_4 &\ll k^{-1/2} \sum_{n_0\leq k^{1/2}} \frac{d_{3|w|}(n_0)}{n_0} + k^{-1/2} \sum_{k^{1/2}<n_0\leq k} \frac{d_{3|w|}(n_0)}{n_0} \\
&\ll k^{-1/2} \sum_{n_0\leq k^{1/2}} d_{3|w|}(n_0) + k^{-1} \sum_{n_0\leq k} d_{3|w|}(n_0) \\
&\ll k^{-1/2} \Bigg(\sum_{n\leq k^{1/6}} d^3_{3|w|}(n) \Bigg) \Bigg(\sum_{n\leq k^{1/4}} d^2_{3|w|}(n) \Bigg) \\
&\quad + k^{-1} \Bigg(\sum_{n\leq k^{1/3}} d^3_{3|w|}(n) \Bigg) \Bigg(\sum_{n\leq k^{1/2}} d^2_{3|w|}(n) \Bigg) \\
&=: k^{-1/2} I_1 \cdot I_2 + k^{-1} I_3 \cdot I_4.
\end{aligned}$$

We need the following inequality proved by Mardzhanishvili in [20]: for $x \geq 1$ and integers $l \geq 1$ and $\alpha \geq 2$ we have

$$x^{-1} \sum_{1\leq m\leq x} d^l_\alpha(m) < A^{(l)}_\alpha (\log x + \alpha^l - 1)^{\alpha^l - 1},$$

where

$$A^{(l)}_\alpha := \frac{\alpha^l}{(\alpha!)^{(\alpha^l-1)/(\alpha-1)}}.$$

Using this inequality, we obtain

$$\begin{aligned}
I_1 &\ll k^{1/6} (\log k + |3w|^3)^{|3w|^3} \ll k^{1/6} \exp(c_{14} \log k / \log\log k), \\
I_1 I_2 &\ll k^{\frac{1}{6}+\frac{1}{4}} \exp(c_{15} \log k / \log\log k), \\
I_3 I_4 &\ll k^{\frac{1}{3}+\frac{1}{2}} \exp(c_{16} \log k / \log\log k).
\end{aligned}$$

Therefore,

$$\sum\nolimits_4 \ll k^{-1/12} \exp(c_{17} \log k / \log\log k).$$

Consequently, if $|w| \leq (\log k / \log^2 \log k)^{1/3}$, then

$$\sum\nolimits_1 = \widehat{G}(w) + O(\exp(-c_{13} \log k / \log\log k)). \tag{3.6}$$

Returning to the sum $\sum_2$ (see (3.3)), we may assume that $|w| \le \log k/\log^2\log k$. We have

$$\begin{aligned}
&\prod_{p^r\|n}\Big(\sum_{\substack{m\ge k\ge 0, l\ge 0\\ m+k+l=r}} d_{|w|}(p^m)d_{|w|}(p^k)d_{|w|}(p^l)\Big)\\
&\qquad\le d_{3|w|}(n),\\
&\sum\nolimits_2 \ll \sum_{n\ge 1} n^{-\frac12}(\log n)^2 d_3^2(n) d_{3|w|}(n)\exp(-n/k)\\
&\qquad\ll_\varepsilon \sum_{n\le k\log^2 k} n^{-\frac12+\varepsilon} d_{3|w|}(n) + k^{-1}\sum_{n\ge 1}\frac{d_{3|w|}(n)}{n^2}\\
&\qquad =: S_1+S_2;\\
&S_1 = \sum_{n\le k\log^2 k} n^{\frac12+\varepsilon}\frac{d_{3|w|}(n)}{n} \ll k^{\frac{51}{100}}\prod_{p\le k\log^2 k}\Big(1-\frac1p\Big)^{-3|w|}\\
&\qquad\ll k^{\frac{51}{100}}\exp\Big(c_{18}|w|\sum_{p\le k\log^2 k} 1/p\Big) \ll k^{\frac{51}{100}}\exp(c_{19}\log k/\log\log k);\\
&S_2 \ll k^{-1}c_{20}^{|w|}.
\end{aligned}$$

Consequently,

$$\sum\nolimits_2 \ll k^{\frac{51}{100}}\exp(c_{19}\log k/\log\log k). \tag{3.7}$$

Combining (3.1), (3.2), (3.6), and (3.7), we prove Theorem 1 in the case where $|w|\ge 1$. For $|w|<1$ the proof is similar, but easier.

§4. PROOF OF THEOREM 2

First, we prove statement 2) of Theorem 2.

Lemma 6.

$$\widehat{G}_p(w) = \int_0^\pi \Big(1-\frac{2\cos 2\varphi}{p}+\frac{1}{p^2}\Big)^{-iw}\Big(1-\frac1p\Big)^{-iw}\mu_p(\varphi).$$

Proof. We reduce this integral to the form indicated in Theorem 1. We have

$$\begin{aligned}
&\frac{1+p^{-1}}{\pi}\int_0^\pi \frac{(1-\cos 2\varphi)\,d\varphi}{\big(1-\frac{2\cos 2\varphi}{p}+\frac{1}{p^2}\big)\big(1-\frac{2\cos 2\varphi}{p}+\frac{1}{p^2}\big)^{iw}\big(1-\frac1p\big)^{iw}}\\
&\quad = \frac{1+p^{-1}}{\pi}\int_0^\pi (1-\cos 2\varphi)\sum_{j=0}^\infty \frac{\exp(2ij\varphi)}{p^j}\sum_{n=0}^\infty\frac{\exp(-2in\varphi)}{p^n}\\
&\qquad\qquad \times\sum_{m=0}^\infty \frac{d_{iw}(p^m)\exp(2im\varphi)}{p^m}\sum_{k=0}^\infty\frac{d_{iw}(p^k)\exp(-2ik\varphi)}{p^k}\sum_{l=0}^\infty\frac{d_{iw}(p^l)}{p^l}d\varphi.
\end{aligned}$$

The coefficient of $d_{iw}(p^m)d_{iw}(p^k)d_{iw}(p^l)$, where $m>k\ge 0$, $l\ge 0$, is equal to

$$\begin{aligned}
&\frac{2(1+p^{-1})}{\pi p^{m+k+l}}\int_0^\pi (1-\cos 2\varphi)\cos((2m-2k)\varphi)\\
&\qquad\qquad\times\Big(2\sum_{r=1}^\infty\frac{1}{p^r}\sum_{\substack{j>n\ge 0\\ j+n=r}}\cos((2j-2n)\varphi)+\sum_{r=0}^\infty\frac{1}{p^{2r}}\Big)\,d\varphi.
\end{aligned}$$

Since

$$\int_0^\pi \cos(t\varphi)\,d\varphi = \begin{cases}\pi & \text{if } t=0,\\ 0 & \text{if } t\neq 0,\end{cases}$$

we have

$$\begin{aligned}(4.1)\quad &\frac{2(1+p^{-1})}{\pi p^{m+k+l}}\int_0^\pi 1\cdot\cos((2m-2k)\varphi)\Big(2\sum_{r=1}^\infty \frac{1}{p^r}\sum_{\substack{j>n\geq 0\\ j+n=r}}\cos((2j-2n)\varphi)+\sum_{r=0}^\infty\frac{1}{p^{2r}}\Big)\,d\varphi\\ &\qquad=\frac{2(1+p^{-1})}{p^{m+k+l}}\frac{1}{p^{m-k}}\frac{1}{1-p^{-2}}.\end{aligned}$$

Similarly,

$$\begin{aligned}(4.2)\quad &-\frac{2(1+p^{-1})}{\pi p^{m+k+l}}\int_0^\pi \cos 2\varphi\cdot\cos((2m-2k)\varphi)\Big(2\sum_{r=1}^\infty \frac{1}{p^r}\sum_{\substack{j>n\geq 0\\ j+n=r}}\cos((2j-2n)\varphi)+\sum_{r=0}^\infty\frac{1}{p^{2r}}\Big)\,d\varphi\\ &=-\frac{1+p^{-1}}{\pi p^{m+k+l}}\int_0^\pi(\cos((2m-2k+2)\varphi)+\cos((2m-2k-2)\varphi))\\ &\qquad\qquad\times\Big(2\sum_{r=1}^\infty \frac{1}{p^r}\sum_{\substack{j>n\geq 0\\ j+n=r}}\cos((2j-2n)\varphi)+\sum_{r=0}^\infty\frac{1}{p^{2r}}\Big)\,d\varphi\\ &=-\frac{1}{p^{m+k+l}}\Big(\frac{1}{p^{m-k+1}}+\frac{1}{p^{m-k-1}}\Big)\frac{1}{1-p^{-1}}.\end{aligned}$$

Adding (4.1) and (4.2), we see that the coefficient of $d_{iw}(p^m)d_{iw}(p^k)d_{iw}(p^l)$, where $m>k\geq 0$, $l\geq 0$, is equal to

$$\frac{1}{p^{m+k+l}}\Big(\frac{1}{p^{m-k}}-\frac{1}{p^{m-k-1}}\Big).$$

Similar computations show that the coefficient of $d^2_{iw}(p^m)d_{iw}(p^l)$, where $m\geq 0$, $l\geq 0$, is equal to

$$\frac{1}{p^{2m+l}}.$$

Lemma 6 is proved. ☐

We need several additional lemmas.

Lemma 7. *For* $|w|\geq C$, *where* $C>0$ *is a sufficiently large constant, we have*

$$\widehat{G}(w)=O(\exp(-c_{21}|w|/\log^2|w|)).$$

Proof. First, under the assumption that $p\geq |w|\log|w|$, we obtain the relation

$$\widehat{G}_p(w)=\exp\Big(-w^2/2p^2+O\Big(\frac{|w|}{p^2}+\frac{|w|^3}{p^3}\Big)\Big). \tag{4.3}$$

Indeed,

$$\begin{aligned}\widehat{G}_p(w) &= 1 + \frac{iw + d_{iw}(p^2)\left(\frac{1}{p^2} - \frac{1}{p}\right) + d_{iw}^2(p)(\frac{1}{p} - 1)}{p^2}\\ &\quad + \frac{d_{iw}^2(p) + d_{iw}(p^2)}{p^2} + O\Big(\sum_{r\ge 3} \frac{d_{3|w|}(p^r)}{p^r}\Big)\\ &= 1 + \frac{iw}{p^2} + \frac{iw(iw+1)}{2p^2} + O\Big(\frac{|w|^3}{p^3}\Big)\\ &= 1 - \frac{w^2}{2p^2} + \frac{3}{2}\frac{iw}{p^2} + O\Big(\frac{|w|^3}{p^3}\Big),\end{aligned}$$

whence (4.3) follows immediately.

We have

$$|\widehat{G}(w)| = \Big|\prod_{p<|w|\log|w|} \widehat{G}_p(w)\Big| \cdot \Big|\prod_{p\ge|w|\log|w|} \widehat{G}_p(w)\Big| =: \prod{}' \cdot \prod{}'' .$$

By Lemma 6,

$$\prod{}' \le \prod_{p<|w|\log|w|} \frac{1+p^{-1}}{(1-p^{-1})^2} \cdot \frac{1}{\pi}\int_0^{\pi} (1-\cos 2\varphi)\, d\varphi = \prod_{p<|w|\log|w|} \frac{1-p^{-2}}{(1-p^{-1})^3} \ll \log^3|w|.$$

By (4.3),

$$\begin{aligned}\prod{}'' &\ll \exp\Big(-\frac{w^2}{2}\sum_{p\ge|w|\log|w|}\frac{1}{p^2} + c_{22}|w|\sum_{p\ge|w|\log|w|}\frac{1}{p^2} + c_{23}|w|^3\sum_{p\ge|w|\log|w|}\frac{1}{p^3}\Big)\\ &\ll \exp(-c_{21}|w|/\log^2|w|).\end{aligned}$$

Lemma 7 is proved. □

Lemma 8. 1) *For* $|w| \le c/\log\log k$, *where* $c > 0$ *is a sufficiently small constant, we have*

$$\widehat{G}_1(w) = \frac{1}{N_0}\sum_f L^{iw}(1, \operatorname{sym}^2 f) = 1 + O(|w|\log\log k).$$

2) *For* $|w| \le \frac{1}{8}$ *we have* $\widehat{G}(w) = 1 + O(|w|)$.

Proof. Statement 1) is proved with the help of the identity

$$L^{iw}(1, \operatorname{sym}^2 f) = 1 + iw\log L(1, \operatorname{sym}^2 f) + \cdots$$

and the estimate $\log L(1, \operatorname{sym}^2 f) = O(\log\log k)$, which follows from (1.1).

Statement 2) is an obvious consequence of the estimate

$$|\widehat{G}_p(w) - 1| \le \Big(1 - \frac{1}{p}\Big)^{-4|w|} - 1 - \frac{4|w|}{p} = \sum_{r\ge 2}\frac{d_{4|w|}(p^r)}{p^r} \le 8|w|/p^2,$$

where $p \ge 2$ is an arbitrary prime. □

Finally, we need the following lemma, in which nearness of distributions is estimated by the difference of their characteristic functions.

Lemma 9. *Let* $G^*(x)$ *and* $G(x)$ *be distribution functions, and let* $\widehat{G}^*(w)$ *and* $\widehat{G}(w)$ *be the corresponding characteristic functions. Then, for any* $T > 0$,

$$\sup_x |G^*(x) - G(x)| \ll \sup_{\xi\ge T}\frac{1}{\xi}\int_0^{\xi}|\widehat{G}(w)|\,dw + \int_0^T \frac{|\widehat{G}^*(w) - \widehat{G}(w)|}{w}\,dw.$$

The proof of this generalization of the well-known Esseen inequality can be found in [21].

Now we turn to the proof of statement 1) of Theorem 2. Consider the distribution functions (frequencies)

$$G_1(x) := \frac{\#\{f, f \in S_k(\Gamma)^+, \log L(1, \mathrm{sym}^2 f) \le x\}}{N_0}$$

and the corresponding characteristic functions

$$\widehat{G}_1(w) = \frac{1}{N_0} \sum_f L^{iw}(1, \mathrm{sym}^2 f).$$

We recall that $k \ge 12$ and $k \equiv 0 (\bmod 2)$. Theorem 1 implies that the function $\widehat{G}(w)$ is the limit of the characteristic functions $\widehat{G}_1(w)$ as $k \to \infty$. Also, from what was proved above, it follows that $\widehat{G}(w)$ is continuous. Therefore, $\widehat{G}(w)$ is the characteristic function of some distribution function $G(x)$. By Lemma 9,

$$G_1(x) = G(x) + O\biggl(\sup_{\xi \ge T} \frac{1}{\xi} \int_0^{\xi} |\widehat{G}(w)| \, dw \biggr) + O\biggl(\int_0^T \frac{|\widehat{G}_1(w) - \widehat{G}(w)|}{w} \, dw \biggr),$$

where $T := (\log k / \log^2 \log k)^{1/3}$. By Lemma 7,

$$\sup_{\xi \ge T} \frac{1}{\xi} \int_0^{\xi} |\widehat{G}(w)| \, dw \ll \frac{1}{T}.$$

We write the integral $\int_0^T \cdots$ as follows:

$$\int_0^T \frac{|\widehat{G}_1(w) - \widehat{G}(w)|}{w} \, dw = \int_0^{(\log k)^{-1}} \frac{|\widehat{G}_1(w) - \widehat{G}(w)|}{w} \, dw + \int_{(\log k)^{-1}}^T \frac{|\widehat{G}_1(w) - \widehat{G}(w)|}{w} \, dw.$$

By Lemma 8,

$$\int_0^{(\log k)^{-1}} \frac{|\widehat{G}_1(w) - \widehat{G}(w)|}{w} \, dw \ll \frac{\log \log k}{\log k}.$$

By Theorem 1,

$$\int_{(\log k)^{-1}}^T \frac{|\widehat{G}_1(w) - \widehat{G}(w)|}{w} \, dw \ll \exp(-c_{24} \log k / \log \log k).$$

This proves statement 1).

We prove statement 3) of Theorem 2. Since

$$\int_{-\infty}^{\infty} |\widehat{G}(w)| \, dw < \infty$$

by Lemma 7, the distribution function $G(x)$ has a density $p_G(x)$ continuous on the entire real axis,

$$p_G(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{G}(w) e^{-iwx} \, dw.$$

The analytic continuability of $G(x)$ to a half-plane $\mathrm{Im}\, z > -\delta$, $\delta > 0$, is proved with the help of the same arguments as in [7, p. 195]. □

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St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023, Russia

E-mail address: `fomenko@pdmi.ras.ru`

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