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TAME REPRESENTATIONS OF THE GROUP $\mathrm{GL}(\infty, \mathbb{F}_q)$

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Abstract. A complete classification of irreducible tame representations of the group $G(\infty) = \varinjlim G(n)$ is presented. Here $G(n) = \mathrm{GL}(n, \mathbb{F}_q)$ is the group of nonsingular matrices of order n over the finite field $\mathbb{F}_q$.

Basic notation. Let $G(n) = \mathrm{GL}(n, \mathbb{F}_q)$ be the group of nonsingular matrices of order n over the finite field $\mathbb{F}_q$ ($q = p^k$, where p is a prime number). For each $n \geq 2$, we identify $G(n-1)$ with the subgroup of $G(n)$ consisting of all matrices that fix the nth vector of the basis. Let

$$G(\infty) = \mathrm{GL}(\infty, \mathbb{F}_q) = \varinjlim G(n)$$

be the direct limit of the groups $G(n)$. The group $G(\infty)$ can be regarded as the group of all infinite matrices $g = [g_{ij}]_{i,j=1}^{\infty}$ over $\mathbb{F}_q$ such that, for some $N \in \mathbb{Z}_+$,

$$g_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

whenever $i \geq N$ or $j \geq N$.

We denote by $G_n(\infty)$, $n \in \mathbb{Z}_+$, the subgroup of $G(\infty)$ consisting of all matrices of the form $\left[\begin{smallmatrix} I_n & 0 \\ 0 & * \end{smallmatrix}\right]$, where I_n is the identity matrix of order n. Sometimes we think of $G(n)$ as a subgroup of $G(\infty)$ consisting of all matrices of the form $\left[\begin{smallmatrix} *_n & 0 \\ 0 & I_\infty \end{smallmatrix}\right]$, where $*_n$ is an arbitrary nonsingular matrix of size $n \times n$.

Let T be a unitary representation of the group $G(\infty)$ in a Hilbert space H. We denote by H_n the subspace of all vectors fixed by $G_n(\infty)$,

$$H_n = \{\xi \in H : T(g)\xi = \xi,\ g \in G_n(\infty)\}.$$

Definition 1 (see [2]–[6])**.** A unitary representation of the group $G(\infty)$ in a Hilbert space H is said to be *tame* if $H = \overline{\bigcup_{n=1}^{\infty} H_n}$.

Let P_n be the orthogonal projection to H_n. We introduce some more groups. For $r, s, t \in \mathbb{Z}_+$, we put

$$G(r,s,t) = \left\{ g \in G(r+s+t) : g \text{ has the form } \begin{bmatrix} *_r & * & * \\ 0 & *_s & * \\ 0 & 0 & *_t \end{bmatrix} \right\},$$

$$G_0(r,s,t) = \left\{ g \in G(r,s,t) : g \text{ has the form } \begin{bmatrix} I_r & 0 & * \\ 0 & I_s & 0 \\ 0 & 0 & I_t \end{bmatrix} \right\},$$

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$$\widetilde{G}(r,s,t) = \left\{ g \in G(\infty) : g \text{ has the form } \begin{bmatrix} *_r & * & * & * \\ 0 & *_s & * & 0 \\ 0 & 0 & *_t & 0 \\ 0 & 0 & * & *_\infty \end{bmatrix} \right\},$$

$$\widetilde{G}_0(r,s,t) = \left\{ g \in \widetilde{G}(r,s,t) : g \text{ has the form } \begin{bmatrix} I_r & 0 & * & * \\ 0 & I_s & 0 & 0 \\ 0 & 0 & I_t & 0 \\ 0 & 0 & * & *_\infty \end{bmatrix} \right\},$$

where $*$ stands for an arbitrary matrix of the corresponding size and $*_k$ is a nonsingular matrix of size $k \times k$, $k \in \mathbb{Z}_+$.

Let $\mathrm{Mat}(m,n)$, where $m,n \in \mathbb{Z}_+ \cup \{\infty\}$, be the set of all matrices of size $m \times n$ over the finite field $\mathbb{F}_q$.

For $m,n \in \mathbb{Z}_+$, we denote by $\omega_m^{(n)}$ the matrix

$$\begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix}.$$

§0. Introduction

The main result of the present paper is a complete classification of the irreducible tame representations of $G(\infty)$. Usually, with every representation of an infinite-dimensional group one can associate a representation of its "mantle" (see [9]), more precisely, of the semigroup of its double cosets (see [2]–[6]). A careful study of this object makes it possible to explicitly construct a discrete imprimitivity system $\mathfrak{B}$ (see [1]) for each tame representation and to describe the representation in terms of this system. Every irreducible tame representation is induced by a representation of some group $D \in G(\infty)$, and the induced representation is trivial on a subgroup $D_0 \in D$. The present paper had already been prepared for publication when the author learned that the statement of the main result and an outline of its proof were contained in the survey [6] by Ol′shanskiĭ. The author's proof differs from that in the survey and involves different ideas.

The classification of irreducible tame representations is as follows.

Theorem 1. *Every irreducible tame representation of $G(\infty)$ is determined by a quadruple of invariants $(r,s,t,\pi_{r,s,t})$, where $r,s,t \in \mathbb{Z}_+$ and $\pi_{r,s,t}$ is a unitary representation of $G(r,s,t)$ trivial on $G_0(r,s,t)$.*

In §1, we present an explicit realization of irreducible tame representations of $G(\infty)$. For every irreducible representation π of $G(r,s,t)$ trivial on $G_0(r,s,t)$, we construct an irreducible representation $\widetilde{\pi}$ of $\widetilde{G}(r,s,t)$ trivial on $\widetilde{G}_0(r,s,t)$. A representation T of the group $G(\infty)$ is induced by $\widetilde{\pi}$.

Now, we outline the main ideas in more detail. The proof of the classification theorem is based on the Ol′shanskiĭ semigroup method. First, we define an associative *-semigroup structure on the set $\mathcal{G}_n = G_n(\infty) \setminus G(\infty)/G_n(\infty)$ of double cosets of $G_n(\infty)$ in $G(\infty)$ by introducing a multiplication $\circ$ acting by the rule $\gamma_n(g_1) \circ \gamma_n(g_2) = \gamma_n(g_1 \omega_m^{(n)} g_2)$ (here m is such that $g_1, g_2 \in G(n+m)$ and $\gamma_n(g)$ is the class of the matrix $g \in G(\infty)$). Let T be a tame representation of $G(\infty)$, and let n be such that $H_n \neq \{0\}$. The sequence of operators $T(\omega_m^{(n)})$ converges weakly to the projection P_n as $m \longrightarrow \infty$. We define a *-representation τ_n of the semigroup $\mathcal{G}_n$ in the space H_n by the formula $\tau_n(\gamma_n(g)) = P_n T(g) P_n$. In the

semigroup $\mathcal{G}_n$, we consider the subset

$$L_n = \left\{ \gamma \in \mathcal{G}_n : \gamma = \gamma_n(g_1) \circ \gamma_n(g_2) \text{ with } g_1 \text{ of the form } \begin{bmatrix} I_n & 0 \\ * & I_\infty \end{bmatrix} \right.$$
$$\left. \text{and } g_2 \text{ of the form } \begin{bmatrix} I_n & * \\ 0 & I_\infty \end{bmatrix} \right\}.$$

In §2, we prove that L_n is a semigroup isomorphic to the semigroup of all pairs of subspaces $(V, V'), V \subset \mathbb{F}_q^n, V' \subset (\mathbb{F}_q^n)^t$, under the product

$$(V_1, V_1') \circ (V_2, V_2') = (V_1 \cap V_2, V_1' \cap V_2').$$

In particular, L_n is commutative and all its elements are selfadjoint idempotents. This means that the operators $P_{n,V}^{V'} = \tau_n((V, V'))$ are orthogonal projections. The group $G(n)$ acts on L_n by conjugation $\gamma_n(g) * \gamma_n(g^{-1})$ and sends the pair (V, V') to $(gV, V'g^{-1})$. We consider the space $\mathbb{F}_q^\infty$ and its dual $(\mathbb{F}_q^\infty)^t$. For any two finite-dimensional subspaces $V \subset \mathbb{F}_q^\infty$ and $V' \subset (\mathbb{F}_q^\infty)^t$ and sufficiently large n, the orthogonal projection $P_{n,V}^{V'}$ is well defined. We shall prove that this projection does not depend on n. The group $G(\infty)$ acts on the set of all projections $P_V^{V'}, V \subset \mathbb{F}_q^\infty, V' \subset (\mathbb{F}_q^\infty)^t, \dim V < \infty, \dim V' < \infty$. The orbits of this action are indexed by the parameters $\dim V, \dim V'$, and $\dim(V \cap (V')^\perp)$. Let T be an irreducible representation, and let $m = \min\{\dim V : \exists P_V^{V'} \neq 0\}$, $k = \min\{\dim V' : \exists P_V^{V'} \neq 0\}$. There exists a nonzero projection $P_{V_0}^{V_0'}$ with $\dim V_0 = m$ and $\dim V_0' = k$. We denote by $\mathfrak{B}$ the orbit containing $P_{V_0}^{V_0'}$. Then

$$\bigoplus_{P_V^{V'} \in \mathfrak{B}} P_V^{V'} = \mathbb{I}_H,$$

where $\mathbb{I}_H$ is the identity operator on the space H of the representation T. From what has been said above, it follows that T is induced by the representation $\widetilde{\pi}(g) = P_{V_0}^{V_0'} T(g) P_{V_0}^{V_0'}$ of the stabilizer of $P_{V_0}^{V_0'}$. Put $r = \dim(V_0 \cap (V_0')^\perp)$, $s = \dim V_0 - r$, and $t = \dim(V_0') - s$. Let V_0 be of the form $\left\{ \left[\begin{smallmatrix} *_{(r+s)} \\ 0_\infty \end{smallmatrix} \right] \right\}$ and V_0' be of the form $\left\{ \begin{bmatrix} 0_r & *_{(s+t)} & 0_\infty \end{bmatrix} \right\}$. The group $\widetilde{G}(r, s, t)$ is the stabilizer of $P_{V_0}^{V_0'}$. We shall check that the representation $\widetilde{\pi}$ of $\widetilde{G}(r, s, t)$ is irreducible and trivial on the subgroup $\widetilde{G}_0(r, s, t)$. Therefore, $\widetilde{\pi}$ is uniquely determined by an irreducible representation of $G(r, s, t)$ trivial on $G_0(r, s, t)$.

§1. Irreducible tame representations of the group $\mathrm{GL}(\infty, \mathbb{F}_q)$. Realization

In this section, we construct examples of irreducible tame representations of $G(\infty)$. In §4, we shall prove that there are no other representations.

Let π be an irreducible representation of $G(r, s, t)$ trivial on $G_0(r, s, t)$. We define a representation $\widetilde{\pi}$ of $\widetilde{G}(r, s, t)$ in the space H_π. For

$$\widetilde{g} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & b \\ 0 & g_{22} & g_{23} & 0 \\ 0 & 0 & g_{33} & 0 \\ 0 & 0 & c & d \end{bmatrix} \in G(\infty),$$

where $g_{11} \in G(r)$, $g_{22} \in G(s)$, $g_{33} \in G(t)$, $b \in \mathrm{Mat}\ (r, \infty)$, and $c \in \mathrm{Mat}\ (\infty, t)$, we put

$$S(\widetilde{g}) = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ 0 & g_{22} & g_{23} \\ 0 & 0 & g_{33} \end{bmatrix} \in G(r + s + t).$$

Let $\widetilde{\pi}(\widetilde{g}) = \pi(S(\widetilde{g}))$. Since the representation π is trivial on $G_0(r,s,t)$, we see that $\widetilde{\pi}$ is a representation. Indeed, it is easily seen that, for arbitrary matrices $\widetilde{g}_1$ and $\widetilde{g}_2$, there exists a matrix $h \in G_0(r,s,t)$ with $S(\widetilde{g}_1\widetilde{g}_2) = S(\widetilde{g}_1)S(\widetilde{g}_2)h$. We have

$$\widetilde{\pi}(\widetilde{g}_1\widetilde{g}_2) = \pi(S(\widetilde{g}_1\widetilde{g}_2)) = \pi(S(\widetilde{g}_1)S(\widetilde{g}_2)h) = \pi(S(\widetilde{g}_1))\pi(S(\widetilde{g}_2)) = \widetilde{\pi}(\widetilde{g}_1)\widetilde{\pi}(\widetilde{g}_2).$$

Clearly, the representation $\widetilde{\pi}$ is unitary.

We construct a representation $T(r,s,t,\pi_{r,s,t})$ of $G(\infty)$ by inducing the representation $\widetilde{\pi}$ of the subgroup $\widetilde{G}(r,s,t)$. In the case in question, the construction of the induced representation is the same as for any finite group (see [1]). Below, we describe the construction of the representation $T(r,s,t,\pi_{r,s,t})$ in detail.

First, we give several definitions.

We identify $\mathbb{F}_q^n$ with the subspace of $\mathbb{F}_q^{n+1}$ consisting of all vectors the $(n+1)$st coordinate of which is 0. Let $\mathbb{F}_q^\infty = \varinjlim \mathbb{F}_q^n$ be the direct limit of the spaces $\mathbb{F}_q^n$. We think of $\mathbb{F}_q^\infty$ as the space of all infinite column vectors in which all but a finite number of coordinates are zero,

$$\mathbb{F}_q^\infty = \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{bmatrix} : \alpha_i \in \mathbb{F}_q, i \in \mathbb{N};\ \alpha_n = 0 \right\} \text{ for } n \text{ sufficiently large.}$$

Let

$$(\mathbb{F}_q^n)^t = \left\{ \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_n \end{bmatrix} : \beta_i \in \mathbb{F}_q, 1 \le i \le n \right\}.$$

Similarly, we define

$$(\mathbb{F}_q^\infty)^t = \left\{ \begin{bmatrix} \beta_1 & \beta_2 & \dots \end{bmatrix} : \beta_i \in \mathbb{F}_q, i \in \mathbb{N};\ \beta_n = 0 \text{ for } n \text{ sufficiently large} \right\}.$$

Let n be an integer or the symbol ∞. We regard the space $(\mathbb{F}_q^n)^t$ as the conjugate to $\mathbb{F}_q^n$. If $\alpha \in \mathbb{F}_q^n, \beta \in (\mathbb{F}_q^n)^t$, then $\beta(\alpha) = \beta\alpha = \sum_{i=1}^n \beta_i\alpha_i$. Let V be a subspace in $\mathbb{F}_q^n$. We put

$$V^\perp = \left\{ \beta \in (\mathbb{F}_q^n)^t : \beta(\alpha) = 0,\ \alpha \in V \right\}.$$

Let $r,s,t \in \mathbb{Z}_+$. We denote by M_{all} (respectively, by $M_{n,\text{all}}$), where $n \in \mathbb{Z}_+$ and $n \ge r+s+t$, the set of all pairs of subspaces (V,V') such that $V \subset \mathbb{F}_q^\infty$ and $V' \subset (\mathbb{F}_q^\infty)^t$ (respectively, $V \subset \mathbb{F}_q^n$ and $V' \subset (\mathbb{F}_q^n)^t$), and put

$$M = \{(V,V') \in M_{\text{all}} : \dim(V) = r+s, \dim(V') = s+t, \dim(V \cap (V')^\perp) = r\},$$
$$M_n = \{(V,V') \in M_{n,\text{all}} : \dim(V) = r+s, \dim(V') = s+t, \dim(V \cap (V')^\perp) = r\}.$$

We identify the set M_n with the subset of M consisting of all points of the form (V,V'), where $V \subset \mathbb{F}_q^n$ and $V' \subset (\mathbb{F}_q^n)^t$.

The group $G(\infty)$ acts naturally on M, namely, $G(\infty) : M \longrightarrow M$, $g : (V,V') \longrightarrow (gV, V'g^{-1})$. We show that this action is transitive.

Lemma 2. *Let $n \in \mathbb{Z}_+$. Then for all subspaces $V_1, V_2 \in \mathbb{F}_q^n$ and $V_1', V_2' \in (\mathbb{F}_q^n)^t$ such that $\dim(V_1) = \dim(V_2), \dim(V_1') = \dim(V_2')$, and $\dim(V_1 \cap (V_1')^\perp) = \dim(V_2 \cap (V_2')^\perp)$, there exists a matrix $g \in G(n)$ with $gV_1 = V_2, V_1'g^{-1} = V_2'$.*

Remark 1. This means precisely that the action of $G(n)$ on M_n is transitive for all r,s,t, and $n \ge r+s+t$. Since each finite-dimensional subspace V of $\mathbb{F}_q^\infty$ lies in a subspace of the form $\left\{ \left[\begin{smallmatrix} v \\ 0_\infty \end{smallmatrix}\right] : v \in \mathbb{F}_q^N \right\}, N \in \mathbb{Z}_+$, the lemma implies that the action of G on M is transitive.

Proof. Let $n \in \mathbb{Z}_+$, and let V_1, V_1', V_2, and V_2' satisfy the conditions of the lemma. Obviously, for each matrix $g \in G(n)$, we have $(V_1'(g^{-1}))^\perp = g(V_1')^\perp$. Thus, we need to find a matrix $g \in G(n)$ for which $gV_1 = gV_2$ and $g((V_1')^\perp) = (V_2')^\perp$. Such a matrix

exists because $\dim(V_1) = \dim(V_2), \dim(V_1 \cap (V_1')^{\perp}) = \dim(V_2 \cap (V_2')^{\perp})$, and $\dim((V_1')^{\perp}) = \dim((V_2')^{\perp})$. □

Let $(\cdot,\cdot)$ be a Hermitian quadratic form (on a space H_π) with respect to which the representation π is unitary. We denote by $\mathfrak{F}_{\mathrm{all}}$ the space of all functions $f : M \longrightarrow H_\pi$ and put

$$\mathfrak{F} = \Big\{ f \in \mathfrak{F}_{\mathrm{all}} : \sum_{p \in M} \|f(p)\|^2 < \infty \Big\},$$

where $\|f(p)\|^2 = (f(p), f(p))$. We see that $\mathfrak{F}$ is a Hilbert space with respect to the Hermitian quadratic form $\langle \cdot, \cdot \rangle$:

$$\langle f, l \rangle = \sum_{p \in M} (f(p), l(p)),$$

where $f, l \in \mathfrak{F}$. Let p_0 denote the point $(V_0, V_0') \in M$, where $V_0 = \left\{ \left[\begin{smallmatrix} * \\ 0_\infty \end{smallmatrix}^{(r+s)} \right] \right\}$ and $V_0' = \{0_r \ \ *_{(s+t)} \ \ 0_\infty\}$. It is easily seen that the subgroup $\widetilde{G}(r,s,t)$ is the stabilizer of $p_0 \in M$. Since the action of $G(\infty)$ on M is transitive, the set of left cosets of $\widetilde{G}(r,s,t)$ in $G(\infty)$ can naturally be identified with the set M. Here, the class containing an element g is identified with the point gp_0. We regard the space $\mathfrak{F}$ as a space of functions acting from the set of left cosets of $\widetilde{G}(r,s,t)$ in $G(\infty)$ to the representation space of $\widetilde{\pi}$ on $\widetilde{G}(r,s,t)$. Then the representation of $G(\infty)$ induced by the representation $\widetilde{\pi}$ of the subgroup $\widetilde{G}(r,s,t)$ acts on $\mathfrak{F}$ (see [1]). We denote this representation by T and introduce the following notation. Let $p \in M, p \neq p_0$, and let $n = \min\{N \in \mathbb{Z}_+ : p \in M_N\}$. We denote by h_p a matrix in $G(n)$ for which $h_p p = p_0$ (such a matrix exists by Lemma 2). Also, we put $h_{p_0} = e_G$, where e_G is the identity element of $G(\infty)$. Then the representation T acts as follows:

$$(T(g)f)(p) = \widetilde{\pi}(h_p g h^{-1}_{g^{-1}p}) f(g^{-1}p), \tag{1}$$

where $f \in \mathfrak{F}$ and $g \in G(\infty)$.

Remark 2. The representation T is well defined by (1), because the point p_0 is fixed under the action of the matrix $h_p g h^{-1}_{g^{-1}p}$, so that $h_p g h^{-1}_{g^{-1}p}$ belongs to the subgroup $\widetilde{G}(r,s,t)$.

Theorem 3. *T is an irreducible tame representation of $G(\infty)$.*

Proof. We describe the subspaces $\mathfrak{F}_n$ of vectors in $\mathfrak{F}$ fixed under the action of $G_n(\infty) \subset G(\infty)$.

Lemma 4. *The subspace $\mathfrak{F}_n \subset \mathfrak{F}$ consists of all functions equal to $0 \in H_\pi$ off M_n,*

$$\mathfrak{F}_n = \{ f \in \mathfrak{F} : f(p) = 0, \ p \in M \backslash M_n \}.$$

Proof. Let $f \in \mathfrak{F}$. Consider an arbitrary point $p = (V_1, V_1') \in M$. If p does not belong to M_n, then either V does not lie in $\mathbb{F}_q^n$, or V' does not lie in $(\mathbb{F}_q^n)^t$. Therefore, the set of points gp such that $g \in G_n(\infty)$ is infinite. If a function f belongs to $\mathfrak{F}_n$, then the norm of its values at all points of the form $gp, g \in G_n(\infty)$, is the same, and therefore, is zero at these points. Conversely, let f be nonzero only at points of M_n. Suppose $g \in G_n(\infty)$ and $p \in M_n$. Then $g^{-1}p = p$ and $\widetilde{\pi}(g) = 1_{H_\pi}$, where 1_{H_π} is the identity operator H_π. The definition of the matrix h_p implies that h_p is an element of $G(n)$; therefore, it commutes with g. We have

$$\widetilde{\pi}(h_p g h^{-1}_{g^{-1}p}) f(g^{-1}p) = \widetilde{\pi}(h_p h_p^{-1} g) f(p) = f(p).$$

Thus, f belongs to $\mathfrak{F}_n$. □

Remark 3. The argument used in the proof of the lemma is well known (see, e.g., [10]).

Now, it can easily be seen that $\mathfrak{F} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{F}_n}$.

We prove that the representation T is irreducible. Let $P_V^{V'}$, where $(V, V') \in M$, denote the orthogonal projection onto the subspace $H_V^{V'} = \{f \in \mathfrak{F} : f(p) = 0 \text{ for all } p \neq (V, V')\}$. For an arbitrary matrix $g \in G(\infty)$, we have $T(g)P_V^{V'}T(g^{-1}) = P_{gV}^{V'g^{-1}}$. Since the action of the group $G(\infty)$ on M is transitive, we see that, for each pair of projections $P_{V_1}^{V_1'}$ and $P_{V_2}^{V_2'}$, there exists a matrix $g \in G(\infty)$ with

$$T(g)P_{V_1}^{V_1'}T(g^{-1}) = P_{V_2}^{V_2'}. \tag{2}$$

Lemma 5. *All projections $P_V^{V'}$ belong to the algebra $\mathfrak{U}$ that is the closure of the algebra of operators of the representation T.*

Proof. The set $H_{V_0}^{V_0'}$ is the subspace of H consisting of all vectors fixed under the action of the subgroup $\widetilde{G}(r, s, t)$. Consequently, $P_{V_0}^{V_0'}$ lies in $\mathfrak{U}$. By property (2), all projections $P_V^{V'}$ lie in $\mathfrak{U}$. □

We consider the subalgebras of $\mathfrak{U}$ of the form $P_V^{V'}\mathfrak{U}P_V^{V'}$, where $V \subset \mathbb{F}_q^{\infty}$ and $V' \subset (\mathbb{F}_q^{\infty})^t$ are such that $(V, V') \in M$. By (2), all these algebras are isomorphic. The algebra $P_{V_0}^{V_0'}\mathfrak{U}P_{V_0}^{V_0'}$ is an irreducible algebra of operators in the space

$$P_{V_0}^{V_0'}\mathfrak{F} = \{f \in \mathfrak{F} : f(p) = 0 \text{ for all } p \neq p_0\}.$$

Indeed, for each matrix $g \in \widetilde{G}(r, s, t)$ and each function $f \in P_{V_0}^{V_0'}\mathfrak{F}$, we have $(T(g)f(p_0)) = \widetilde{\pi}(g)f(p_0)$. If we restrict the representation T from $G(\infty)$ to the subgroup $\widetilde{G}(r, s, t)$ and from the space H to the subspace $P_V^{V'}\mathfrak{F}$, we obtain a representation unitarily equivalent to $\widetilde{\pi}$ but irreducible, because π is irreducible.

Let A be an operator belonging to the commutator subalgebra of $\mathfrak{U}$. Then A commutes with the projections $P_V^{V'}$. Since the algebra $P_V^{V'}\mathfrak{U}P_V^{V'}$ is irreducible in the subspace $P_V^{V'}\mathfrak{F}$, we see that, in each subspace $P_V^{V'}\mathfrak{F}$ with $(V, V') \in M$, A acts as multiplication by a scalar depending on (V, V'). By (2), this scalar is one and the same for all subspaces $P_V^{V'}\mathfrak{F}$. Therefore, $A = \lambda\mathbb{I}$ for some $\lambda \in \mathbb{C}$, where $\mathbb{I}_H$ is the identity operator in the space $\mathfrak{F}$. We have proved that the algebra $\mathfrak{U}$ is irreducible; therefore, the representation T is irreducible. □

In conclusion, we note that the representations $T(r,s,t,\pi_{r,s,t})$ corresponding to distinct quadruples $(r, s, t, \pi_{r,s,t})$ are pairwise disjoint. We prove this statement in §4.

§2. Semigroups of double cosets $G_n \setminus G/G_n$

In this section, we introduce a $*$-semigroup structure on the set of double coset $\mathcal{G}_n = G_n(\infty) \setminus G(\infty)/G_n(\infty)$ and describe the properties of this semigroup.

We fix $n \in \mathbb{Z}_+$ and denote by $\gamma_n(g)$, $g \in G(\infty)$, the class of the matrix g. Let $\omega_m^{(n)}$ denote the matrix

$$\begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix}.$$

Proposition 6. *The formula $\gamma_n(g_1) \circ \gamma_n(g_2) = \gamma_n(g_1\omega_m^{(n)}g_2)$, where m is such that $g_1, g_2 \in G_{n+m}(\infty)$, yields a $*$-semigroup structure on $\mathcal{G}_n$.*

Proof. First, we prove that the operation is well defined, i.e., that $\gamma_n(g_1\omega_m^{(n)}g_2)$ does not depend on m and on the choice of representatives g_1 and g_2 of double cosets by $G_n(\infty)$ in $G(\infty)$. Let $g_1, g_2 \in G(n+m)$. We write the matrices in a block form,

$$g_i = \begin{bmatrix} A_i & B_i & 0 \\ C_i & D_i & 0 \\ 0 & 0 & I_\infty \end{bmatrix}, \quad i = 1, 2, \tag{3}$$

where A_i is a block of size $n \times n$ and D_i is a block of size $m \times m$. We obtain

$$g_1\omega_m^{(n)}g_2 = \begin{bmatrix} A_1A_2 & A_1B_2 & B_1 & 0 \\ C_1A_2 & C_1B_2 & D_1 & 0 \\ C_2 & D_2 & 0 & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix}.$$

For every $s > 0$, the matrix

$$g_1\omega_{m+s}^{(n)}g_2 = \begin{bmatrix} A_1A_2 & A_1B_2 & 0 & B_1 & 0 & 0 \\ C_1A_2 & C_1B_2 & 0 & D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_s & 0 \\ C_2 & D_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_\infty \end{bmatrix}$$

lies in the double coset of $G_n(\infty)$ in $G(\infty)$ containing $g_1\omega_m^{(n)}g_2$. It follows that the class $\gamma_n(g_1\omega_m^{(n)}g_2)$ does not depend on m (provided $g_1, g_2 \in G(n+m)$). Let $h \in G_n(\infty)$. We fix a number m such that $g_1, g_2, h \in G(n+m)$. Suppose h has the form

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & I_\infty \end{bmatrix},$$

where $K \in G(m)$. Obviously,

$$\gamma_n(hg_1\omega_m^{(n)}g_2) = \gamma_n(g_1\omega_m^{(n)}g_2h) = \gamma_n(g_1\omega_m^{(n)}g_2).$$

Next, we have

$$g_1h\omega_m^{(n)}g_2 = \begin{bmatrix} A_1A_2 & A_1B_2 & B_1K & 0 \\ C_1A_2 & C_1B_2 & D_1K & 0 \\ C_2 & D_2 & 0 & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix} = g_1\omega_m^{(n)}g_2\omega_m^{(n)}h\omega_m^{(n)},$$

and $\gamma_n(g_1h\omega_m^{(n)}g_2) = \gamma_n(g_1\omega_m^{(n)}g_2\omega_m^{(n)}h\omega_m^{(n)}) = \gamma_n(g_1\omega_m^{(n)}g_2)$. Similarly, $\gamma_n(g_1\omega_m^{(n)}hg_2) = \gamma_n(g_1\omega_m^{(n)}g_2)$. Thus, we have proved that the semigroup operation is well defined.

Now, we prove that the multiplication $\circ$ is associative. We assume that g_1, g_2, and g_3 belong to $G(\infty)$ and that $m \in \mathbb{Z}_+$ is such that g_1, g_2, and g_3 are in $G(n+m)$. The definition of $\circ$ implies that $(\gamma_n(g_1) \circ \gamma_n(g_2)) \circ \gamma_n(g_3) = \gamma_n(g_1\omega_m^{(n)}g_2\omega_{2m}^{(n)}g_3)$ and $\gamma_n(g_1) \circ (\gamma_n(g_2) \circ \gamma_n(g_3)) = \gamma_n(g_1\omega_{2m}^{(n)}g_2\omega_m^{(n)}g_3)$. We must prove that the matrices $g_1\omega_m^{(n)}g_2\omega_{2m}^{(n)}g_3$ and $g_1\omega_{2m}^{(n)}g_2\omega_m^{(n)}g_3$ lie in the same double coset of $G_n(\infty)$ in $G(\infty)$. For this, we prove the following matrix relation:

$$\omega_m^{(n+m)}g_1\omega_{2m}^{(n)}g_2\omega_m^{(n)}g_3\omega_m^{(n+2m)} = \omega_m^{(n+2m)}g_1\omega_m^{(n)}g_2\omega_{2m}^{(n)}g_3\omega_m^{(n+m)}. \tag{4}$$

Since the matrix $\omega_m^{(n+2m)}$ belongs to $G_{n+2m}(\infty)$, it commutes with g_2, g_3, and $\omega_m^{(n+m)}$. Similarly, $\omega_m^{(n+m)}$ commutes with g_1. We put

$$h = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_\infty \end{bmatrix}.$$

The following relations are verified easily:

$$\omega_m^{(n+2m)} = \omega_{2m}^{(n)}\omega_m^{(n)}\omega_{2m}^{(n)}, \quad \omega_m^{(n+m)}\omega_m^{(n)}\omega_{2m}^{(n)} = \omega_m^{(n)}h.$$

Taking the aforesaid into account, we obtain

$$\begin{aligned} \omega_m^{(n+m)}g_1\omega_{2m}^{(n)}g_2\omega_m^{(n)}g_3\omega_m^{(n+2m)} &= g_1\omega_m^{(n+m)}\omega_{2m}^{(n)}\omega_m^{(n+2m)}g_2\omega_m^{(n)}g_3 \\ &= g_1\omega_m^{(n+m)}\omega_m^{(n)}\omega_{2m}^{(n)}g_2\omega_m^{(n)}g_3 = g_1\omega_m^{(n)}hg_2\omega_m^{(n)}g_3. \end{aligned}$$

Similarly,

$$\omega_m^{(n+2m)}g_1\omega_m^{(n)}g_2\omega_{2m}^{(n)}g_3\omega_m^{(n+m)} = g_1\omega_m^{(n)}g_2h\omega_m^{(n)}g_3.$$

Since the matrix h belongs to $G_{n+m}(\infty)$, it commutes with g_2. Formula (4) is proved.

The $*$-structure on $\mathcal{G}_n$ is inherited from $G(\infty)$; namely, we put $\gamma_n^*(g) = \gamma_n(g^{-1})$. This structure is well defined because the class containing g^{-1} depends only on the class containing g. □

In what follows, we only need elements of $\mathcal{G}_n$ of a specific form. Let

$$L_{\text{upper}} = \left\{\gamma \in \mathcal{G}_n : \gamma = \gamma_n(g),\ g \in G(\infty) \text{ has the form } \begin{bmatrix} I_n & * \\ 0 & I_\infty \end{bmatrix}\right\},$$

$$L_{\text{lower}} = \left\{\gamma \in \mathcal{G}_n : \gamma = \gamma_n(g),\ g \in G(\infty) \text{ has the form } \begin{bmatrix} I_n & 0 \\ * & I_\infty \end{bmatrix}\right\},$$

and let L be the semigroup of $\mathcal{G}_n$ generated by the elements of L_{upper} and L_{lower}.

Any $B \in \operatorname{Mat}(n, \infty)$ determines the following linear operator:

$$B: \quad (\mathbb{F}_q^n)^t \longrightarrow (\mathbb{F}_q^\infty)^t, \ \ v' \longrightarrow v'B.$$

Similarly, $C \in \operatorname{Mat}(\infty, n)$ gives rise to the linear operator

$$C: \quad \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^\infty, \ \ v \longrightarrow Cv.$$

We denote by $\operatorname{Ker} B$ and $\operatorname{Ker} C$ the kernels of the corresponding operators,

$$\operatorname{Ker} B = \{v' \in (\mathbb{F}_q^n)^t : v'B = 0\}, \quad \operatorname{Ker} C = \{v \in \mathbb{F}_q^n : Cv = 0\}.$$

Lemma 7. *The elements of L_{upper} and L_{lower} are parameterized by the subspaces of $(\mathbb{F}_q^n)^t$ and $\mathbb{F}_q^n$, respectively, in the following sense*:

$$\gamma_n\left(\begin{bmatrix} I_n & 0 \\ C_1 & I_\infty \end{bmatrix}\right) = \gamma_n\left(\begin{bmatrix} I_n & 0 \\ C_2 & I_\infty \end{bmatrix}\right) \iff \operatorname{Ker} C_1 = \operatorname{Ker} C_2,$$

$$\gamma_n\left(\begin{bmatrix} I_n & B_1 \\ 0 & I_\infty \end{bmatrix}\right) = \gamma_n\left(\begin{bmatrix} I_n & B_2 \\ 0 & I_\infty \end{bmatrix}\right) \iff \operatorname{Ker} B_1 = \operatorname{Ker} B_2.$$

Proof. Let $g_1 = \left[\begin{smallmatrix} I_n & B_1 \\ 0 & I_\infty \end{smallmatrix}\right]$ and $g_2 = \left[\begin{smallmatrix} I_n & B_2 \\ 0 & I_\infty \end{smallmatrix}\right]$, where $B_1, B_2 \in \operatorname{Mat}(n, \infty)$. The class $\gamma_n(g_1)$ consists of the matrices $g = \left[\begin{smallmatrix} I_n & B_1D \\ 0 & D_1D \end{smallmatrix}\right]$ with $D, D_1 \in G(\infty)$. It can easily be proved that, on the following list, any two consecutive statements are equivalent:

1) $\gamma_n(g_1) = \gamma_n(g_2)$;

2) there exists $D \in G(\infty)$ such that $B_1 D = B_2$;

3) the subspaces spanned by the column vectors of the matrices B_1 and B_2, respectively, coincide;

4) $\operatorname{Ker} B_1 = \operatorname{Ker} B_2$.

As a consequence, we see that $\gamma_n(g_1) = \gamma_n(g_2)$ if and only if $\operatorname{Ker} B_1 = \operatorname{Ker} B_2$. The second statement of the lemma is proved similarly. □

The last statement enables us to introduce the following notation: for every $V \subset \mathbb{F}_q^n$, we put

$$\gamma_{n,V} = \gamma_n\left(\begin{bmatrix} I_n & 0 \\ C & I_\infty \end{bmatrix}\right),$$

where C is a matrix in $\mathrm{Mat}(\infty, n)$ such that $\operatorname{Ker} C = V$, and for every $V' \subset (\mathbb{F}_q^n)^t$ we put

$$\gamma_n^{V'} = \gamma_n\left(\begin{bmatrix} I_n & B \\ 0 & I_\infty \end{bmatrix}\right),$$

where B is a matrix in $\mathrm{Mat}(n, \infty)$ such that $\operatorname{Ker} B = V'$.

The next statement shows why it is easier to work with subspaces as parameters rather than with classes of matrices.

Lemma 8. *The sets L_{upper} and L_{lower} are commutative semigroups in $\mathcal{G}_n$, their elements are self-conjugate, and for every $V_1, V_2 \subset \mathbb{F}_q^n$ and $V_1', V_2' \subset (\mathbb{F}_q^n)^t$ we have*

$$\gamma_{n,V_1} \circ \gamma_{n,V_2} = \gamma_{n,V_1 \cap V_2}, \quad \gamma_n^{V_1'} \circ \gamma_n^{V_2'} = \gamma_n^{V_1' \cap V_2'}.$$

Proof. We prove the statements concerning L_{lower}. Let

$$g_1 = \begin{bmatrix} I_n & 0 & 0 \\ C_1 & I_m & 0 \\ 0 & 0 & I_\infty \end{bmatrix}, \quad g_2 = \begin{bmatrix} I_n & 0 & 0 \\ C_2 & I_m & 0 \\ 0 & 0 & I_\infty \end{bmatrix},$$

where $m \in \mathbb{Z}_+$, $C_1 \in \mathrm{Mat}(m, n)$ and $C_2 \in \mathrm{Mat}(m, n)$ are arbitrary. By the definitions of the multiplication $\circ$, we obtain

$$\gamma_n(g_1) \circ \gamma_n(g_2) = \gamma_n(g_1 \omega_m^{(n)} g_2) = \gamma_n(g_1 \omega_m^{(n)} g_2 \omega_m^{(n)}).$$

The matrix $g_1 \omega_m^{(n)} g_2 \omega_m^{(n)}$ has the form

$$\begin{bmatrix} I_n & 0 & 0 & 0 \\ C_1 & I_m & 0 & 0 \\ C_2 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix}.$$

We put $V_1 = \{v \in \mathbb{F}_q^n : C_1 v = 0\}$ and $V_2 = \{v \in \mathbb{F}_q^n : C_2 v = 0\}$. By definition, $\gamma_{n,V_1} = \gamma_n(g_1), \gamma_{n,V_2} = \gamma_n(g_2)$, and $\gamma_{n,V_1 \cap V_2} = \gamma_n(g_1 \omega_m^{(n)} g_2 \omega_m^{(n)})$. Therefore, for all $V_1, V_2 \subset \mathbb{F}_q^n$ we have $\gamma_{n,V_1} \circ \gamma_{n,V_2} = \gamma_{n,V_1 \cap V_2}$. It follows that L_{lower} is a commutative semigroup. The self-conjugacy follows from the fact that the matrix

$$g_1^{-1} = \begin{bmatrix} I_n & 0 & 0 \\ -C_1 & I_m & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$$

belongs to the class $\gamma_n(g_1)$.

The statements concerning L_{upper} are proved similarly. □

In what follows, we use the notation $\gamma_{n,V}^{V'} = \gamma_{n,V} \circ \gamma_n^{V'}$. We say that an element $\gamma \in \mathcal{G}_n$ is *idempotent* if $\gamma \circ \gamma = \gamma$.

Lemma 9. *The elements of the semigroups L_{upper} and L_{lower} commute.*

Proof. Let $V \subset \mathbb{F}_q^n, V' \subset (\mathbb{F}_q^n)^t$, and let

$$g_1 = \begin{bmatrix} I_n & 0 & 0 \\ C & I_n & 0 \\ 0 & 0 & I_\infty \end{bmatrix}, \quad g_2 = \begin{bmatrix} I_n & B & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_\infty \end{bmatrix}$$

be such that $\gamma_n^{V'} = \gamma_n(g_2)$ and $\gamma_{n,V} = \gamma_n(g_1)$. Then $\gamma_{n,V} \circ \gamma_n^{V'} = \gamma_n(g_1) \circ \gamma_n(g_2)$ and $\gamma_n^{V'} \circ \gamma_{n,V} = \gamma_n(g_2) \circ \gamma_n(g_1)$. We must prove that $g_1 \omega_n^{(n)} g_2$ and $g_2 \omega_n^{(n)} g_1$ lie in the same double coset of $G_n(\infty)$ in $G(\infty)$. By (3), we have

$$g_1 \omega_n^{(n)} g_2 = \begin{bmatrix} I_n & B & 0 & 0 \\ C & CB & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix},$$

$$g_2 \omega_n^{(n)} g_1 = \begin{bmatrix} I_n & 0 & B & 0 \\ 0 & 0 & I_n & 0 \\ C & I_n & 0 & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix}.$$

We put

$$h = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & -CB & I_n & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix}.$$

Then $\omega_n^{(n)} g_1 \omega_n^{(n)} g_2 h \omega_n^{(n)} = g_2 \omega_n^{(n)} g_1$. The required statement is proved. □

Combining Lemmas 7, 8, and 9, we obtain the following important statement.

Proposition 10. *The set L is a commutative subsemigroup of $\mathcal{G}_n$ consisting of all elements of the form $\gamma_{n,V}^{V'}$. The elements of L are self-conjugate idempotents.*

Now, we consider the subset $\mathcal{G}(n) = \{\gamma_n(g) : g \in G(n)\}$.

Proposition 11. *The subset $\mathcal{G}(n)$ is a subgroup of the semigroup $\mathcal{G}_n$, and the mapping $\gamma_n : G(n) \longrightarrow \mathcal{G}_n$ is an isomorphism from $G(n)$ onto the subgroup $\mathcal{G}(n)$.*

Proof. This follows directly from the definitions of γ_n and $\circ$. □

It is easy to check that $\mathcal{G}(n)$ consists of all invertible elements of $\mathcal{G}_n$. We shall not use this property of $\mathcal{G}(n)$ and leave the proof of it to the reader as an exercise.

The group $G(n)$ acts on the semigroup $\mathcal{G}_n$ by conjugation.

Property 4. *For every matrix $g \in G(n)$ and every element $\gamma_{n,V}^{V'} \in \mathcal{G}_n$, we have*

$$\gamma_n(g) \gamma_{n,V}^{V'} \gamma_n(g^{-1}) = \gamma_{n,gV}^{V'g^{-1}}.$$

§3. Associated family of representations

In this section, for each tame representation T, we construct a family $\{\tau_n\}$ of $*$-representations of the semigroups $\mathcal{G}_n$ and describe some properties of this family; these properties will help us to prove the fact that any irreducible tame representation T admits an imprimitivity system.

Proposition 12. *The operators $T(\omega_m^{(n)})$ converge weakly to P_n as $m \longrightarrow \infty$.*

Proof. We use the following general fact.

Lemma 13. *Let U be a unitary representation of an arbitrary discrete group G in a Hilbert space H. We denote by P the orthogonal projection to the subspace of vectors in H fixed under the action of the representation operators. Then, for all $\epsilon > 0$ and $\xi \in H$, there exists $p \in \mathbb{Z}_+$ and $\alpha_1, \alpha_2, \dots, \alpha_p \in \mathbb{R}_+, \alpha_1 + \alpha_2 + \dots + \alpha_p = 1$, such that for some $g_1, g_2, \dots, g_p \in G$ we have*

$$\left\| \sum_{k=1}^{p} \alpha_k U(g_k)\xi - P\xi \right\| < \epsilon.$$

This statement is well known (see [11, 7]). For the reader's convenience, we present the proof here.

Proof. In the closure of the convex hull of the U-orbit of ξ we can find a vector ξ_0 with minimal norm (see [8, p. 330]). Obviously, ξ_0 is U-invariant. Take $\eta \in PH$ with $\|\eta\| = 1$. For every $\epsilon > 0$, there are $\beta_1, \beta_2, \dots, \beta_k > 0, \sum_{i=1}^{k} \beta_i = 1$, and $g_1, g_2, \dots, g_k \in G$ such that

$$\left\| \sum_{i=1}^{k} \beta_i U(g_i)\xi - \xi_0 \right\| < \epsilon.$$

Then

$$|(\xi_0 - \xi, \eta)| < \epsilon + \left| \left(\sum_{i=1}^{k} \beta_i U(g_i)\xi - \xi_0, \eta \right) \right| = \epsilon,$$

because ξ_0 is U-invariant. Thus, $(\xi_0, \eta) = (\xi, \eta)$ for every vector $\eta \in PH$. Since $\xi_0 \in PH$, we have $\xi_0 = P\xi$. □

Suppose $m \in \mathbb{Z}_+$, $\eta \in H$, and $\epsilon > 0$. Applying Lemma 13 to $G = G_n(\infty)$ and to the restriction of T to $G_n(\infty)$ (in the role of U), and putting $P = P_n$ and $\xi = P_{n+m}\eta$, we find $p \in \mathbb{Z}_+$, $\alpha_1, \alpha_2, \dots, \alpha_p \in \mathbb{R}_+$, $\alpha_1 + \alpha_2 + \dots + \alpha_p = 1$, and $g_1, g_2, \dots, g_p \in G$ such that

$$\left\| \sum_{k=1}^{p} \alpha_k T(g_k) P_{n+m}\eta - P_n\eta \right\| < \epsilon.$$

Let $g \in G$. Applying the operator $P_{n+m}T(g)$ to the vector $\sum_{k=1}^{p} \alpha_k T(g_k) P_{n+m}\eta - P_n\eta$ and recalling that the representation T is unitary, we obtain

$$\left\| \sum_{k=1}^{p} \alpha_k P_{n+m} T(gg_k) P_{n+m}\eta - P_n\eta \right\| < \epsilon.$$

Let $N \in \mathbb{Z}_+$ be such that $N \geq m$ and $g_k \in G(N+n)$ for $k = 1, 2, \dots, p$. We put

$$g = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & I_N & 0 & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix}.$$

Then, for $k = 1, 2, \dots, p$, the matrix gg_k has the form

$$\begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & *_N & 0 & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix}$$

and lies in the same class in $G_{n+m}(\infty) \setminus G(\infty)/G_{n+m}$ as $\omega_m^{(n)}$. We have

$$\|P_{n+m}T(\omega_m^{(n)})P_{n+m}\eta - P_n\eta\| < \epsilon.$$

Since $\epsilon > 0$ and $\eta \in H$ are arbitrary, we obtain $P_{n+m}T(\omega_m^{(n)})P_{n+m} = P_n$. Consequently, the operators $T(\omega_m^{(n)})$ converge weakly to P_n as $m \longrightarrow \infty$. □

We put $\mathrm{cond}(T) = \min\{n \in \mathbb{Z}_+ : H_n \neq \{0\}\}$. Let $n \geq \mathrm{cond}(T)$. For a given tame representation T of the group G in the space H, we define a representation τ_n of the semigroup $\mathcal{G}_n$ in the space H_n. We put $\tau_n(\gamma_n(g)) = P_nT(g)P_n$ for every $g \in \mathcal{G}_n$. Obviously, the representation τ_n is well defined.

Theorem 14. *For each $n \geq \mathrm{cond}(T)$, the mapping τ_n is a $*$-representation of the semigroup $\mathcal{G}_n$ in the class of contractions.*

Proof. We only need to prove that the mapping τ_n is multiplicative. Let $g_1, g_2 \in G$. Let $N \in \mathbb{Z}_+$ be such that $g_1, g_2 \in G(n+N)$. Then, for $m \geq N$, the element $\gamma_n(g_1\omega_m^{(n)}g_2)$ of the semigroup $\mathcal{G}_n$ does not depend on m. The operator $P_nT(g_1)P_nT(g_2)P_n$ is the weak limit of the sequence $P_nT(g_1\omega_m^{(n)}g_2)P_n$ as $m \longrightarrow \infty$. Since for $m \geq N$ the operator $P_nT(g_1\omega_m^{(n)}g_2)P_n$ does not depend on m, we have $P_nT(g_1)P_nT(g_2)P_n{=}P_nT(g_1\omega_N^{(n)}g_2)P_n$, which means that $\tau_n(\gamma_n(g_1))\tau_n(\gamma_n(g_2)) = \tau_n(\gamma_n(g_1) \circ \gamma_n(g_2))$. □

The family $\{\tau_n\}_{n\geq \mathrm{cond}(T)}$ of representations of semigroups of double cosets will be called the *associated family* of the representation T.

Now, the results obtained in §2 for the semigroups $\mathcal{G}_n$ will be applied to the representation operators τ_n.

For all $V \subset \mathbb{F}_q^n$ and $V' \subset (\mathbb{F}_q^n)^t$, we put $P_{n,V} = \tau_n(\gamma_{n,V})$, $P_n^{V'} = \tau_n(\gamma_n^{V'})$, and $P_{n,V}^{V'} = \tau_n(\gamma_{n,V}^{V'})$.

Proposition 15. *The operators $P_{n,V}, P_n^{V'}$, and $P_{n,V}^{V'}$ are orthogonal projections. For all $\gamma = \gamma_n(g) \in \mathcal{G}(n)$, $V \subset \mathbb{F}_q^n$, and $V' \subset (\mathbb{F}_q^n)^t$, we have*

$$\tau_n(\gamma)P_{n,V}^{V'}(\tau_n(\gamma))^* = P_{n,gV}^{V'g^{-1}}.$$

Proof. This statement follows from Theorem 14, Proposition 10, and Property 4. □

We fix $n, m \in \mathbb{Z}_+$ and introduce the following notation:

$$S_{n,m,\mathrm{over}} = \left\{ g \in G_n(\infty) : g \text{ has the form } \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & * \\ 0 & 0 & *_\infty \end{bmatrix} \right\},$$

$$S_{n,m,\mathrm{under}} = \left\{ g \in G_n(\infty) : g \text{ has the form } \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & * & *_\infty \end{bmatrix} \right\},$$

$$o_m^{(n)} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_m & I_m & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix} \in S_{n,m,\mathrm{over}},$$

$$u_m^{(n)} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & I_m & I_m & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix} \in S_{n,m,\mathrm{under}}.$$

Let

$$H_{\text{over}} = \{\xi \in H : T(g)\xi = \xi \text{ for all } g \in S_{n,m,\text{over}}\}$$

be the subspace of H consisting of all vectors fixed under the action of the group $S_{n,m,\text{over}}$. Similarly, the subspace

$$H_{\text{under}} = \{\xi \in H : T(g)\xi = \xi \text{ for all } g \in S_{n,m,\text{under}}\}$$

of H consists of all vectors fixed under the action of the group $S_{n,m,\text{under}}$. We denote by P_{over} and P_{under} the orthogonal projections to H_{over} and to H_{under}, respectively.

We need several auxiliary statements.

Lemma 16. *We have* $P_{n+m}T(o_m^{(n)})P_{n+m} = P_{\text{over}}$ *and* $P_{n+m}T(u_m^{(n)})P_{n+m} = P_{\text{under}}$.

Proof. We use Lemma 13. Putting $G = S_{n,m,\text{over}}$, we let the restriction of T to G play the role of U. Then $P = P_{\text{over}}$. By Lemma 13, for every $\xi \in H$ and every $\epsilon > 0$, there exist positive numbers $\alpha_1, \alpha_2, \dots, \alpha_k, \sum_{i=1}^k \alpha_i = 1$, and matrices $g_1, g_2, \dots, g_k \in S_{n,m,\text{over}}$ such that $\| \sum_{i=1}^k \alpha_i T(g_i)\xi - P_{\text{over}}\xi\| < \epsilon$. The definitions of the projections P_{over} and P_{n+m} show that $P_{\text{over}}P_{n+m} = P_{n+m}P_{\text{over}} = P_{\text{over}}$, because $G_{n+m} \subset S_{n,m,\text{over}}$. Let $\xi = P_{n+m}\eta$, where η is an arbitrary vector in H. We fix an arbitrary $\epsilon > 0$ and choose $\alpha_1, \alpha_2, \dots, \alpha_k > 0$, $\sum_{i=1}^k \alpha_i = 1$, and $g_1, g_2, \dots, g_k \in S_{n,m,\text{over}}$ such that

$$\left\| \sum_{i=1}^k \alpha_i T(g_i) P_{n+m}\eta - P_{\text{over}}\eta \right\| < \epsilon. \tag{5}$$

Let g be an arbitrary matrix in $S_{n,m,\text{over}}$. Then $T(g)P_{\text{over}} = P_{\text{over}}$. Since the representation T is unitary, we have

$$\begin{aligned} &\left\| \sum_{i=1}^k \alpha_i P_{n+m} T(gg_i) P_{n+m}\eta - P_{\text{over}}\eta \right\| \\ &\quad = \left\| P_{n+m}T(g)\left(\sum_{i=1}^k \alpha_i T(g_i) P_{n+m}\eta - P_{\text{over}}\eta \right) \right\| \\ &\quad \le \left\| \sum_{i=1}^k \alpha_i T(g_i) P_{n+m}\eta - P_{\text{over}}\eta \right\|. \end{aligned}$$

Using (5), we obtain

$$\left\| \sum_{i=1}^k \alpha_i P_{n+m} T(gg_i) P_{n+m}\eta - P_{\text{over}}\eta \right\| < \epsilon. \tag{6}$$

Let $N \in \mathbb{Z}_+$ be such that the matrices g_i lie in $G(n+m+N)$ for all $1 \le i \le k$, i.e., the g_i have the form

$$\begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_m & * & 0 \\ 0 & 0 & *_N & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix}.$$

We put

$$g = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & I_m & 0 \\ 0 & 0 & I_N & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & I_\infty \end{bmatrix}.$$

Then the matrices gg_i are of the form

$$g = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & I_m & * & I_m & 0 \\ 0 & 0 & *_N & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & I_\infty \end{bmatrix}$$

for all $1 \le i \le k$, i.e., they belong to the same double coset of $G_{n+m}(\infty)$ in $G(\infty)$ as the matrix $o_m^{(n)}$. By (6), we have

$$\|P_{n+m}T(o_m^{(n)})P_{n+m}\eta - P_{\text{over}}\eta\| < \epsilon.$$

Since $\eta \in H$ and $\epsilon > 0$ are arbitrary, we see that

$$P_{n+m}T(o_m^{(n)})P_{n+m} = P_{\text{over}}.$$

The second relation in the statement of the lemma is proved similarly. □

Corollary 17. *Suppose* $V = \left\{ \left[\begin{smallmatrix} *_n \\ 0_m \end{smallmatrix}\right] \right\} \subset \mathbb{F}_q^{n+m}$ *and* $V' = \left\{ \begin{bmatrix} *_n & 0_m \end{bmatrix} \right\} \subset (\mathbb{F}_q^{n+m})^t$. *Then* $P_{n+m,V} = P_{\text{under}}$ *and* $P_{n+m}^{V'} = P_{\text{over}}$.

Proof. By the definition of the classes γ_n and τ_n and the projections $P_{k,V}$ and $P_k^{V'}$, we have $P_{n+m,V}T(u_m^{(n)})P_{n+m,V} = P_{n+m,V}$ and $P_{n+m}^{V'}T(o_m^{(n)})P_{n+m}^{V'} = P_{n+m}^{V'}$. It remains to use Lemma 16. □

Now, we prove a lemma that will imply the relation

$$P_{\text{over}}P_{\text{under}} = P_n. \tag{7}$$

Lemma 18. *The set* $S_{n,m,\text{over}} \cup S_{n,m,\text{under}}$ *of matrices generates the subgroup* $G_n(\infty) \subset G(\infty)$.

Proof. It suffices to check that $G(\infty)$ is generated by the set $S_{m,\text{over}} \cup S_{m,\text{under}}$, where $S_{m,\text{over}} = S_{0,m,\text{over}}$ and $S_{m,\text{under}} = S_{0,m,\text{under}}$. We prove this by induction on m. Let S_m denote the subgroup generated by the set $S_{m,\text{over}} \cup S_{m,\text{under}}$. For each vector $v \in \mathbb{F}_q^\infty \setminus \{0\}$, in the set $S_{1,\text{under}}S_{1,\text{over}}S_{1,\text{under}}$ there is a matrix, the first column of which is v. Since $S_1 S_{1,\text{over}} = S_1$, we have $S_1 = G(\infty)$. In particular, if $m \ge 2$, then all matrices of the form

$$\begin{bmatrix} I_{m-1} & 0 & * \\ 0 & *_1 & * \\ 0 & * & *_\infty \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} I_{m-1} & 0 & 0 \\ 0 & *_1 & * \\ * & * & *_\infty \end{bmatrix}$$

lie in S_m. Now, it is easy to show that $S_m \supset (S_{m-1,\text{over}} \cup S_{m-1,\text{under}})$, i.e., $S_m \supset S_{m-1}$, which implies the required statement. □

Theorem 19. *Suppose* $n, m \in \mathbb{Z}_+, V_1 \subset \mathbb{F}_q^n, V_1' \subset (\mathbb{F}_q^n)^t$, $V_2 = \left\{ \left[\begin{smallmatrix} v \\ 0_m \end{smallmatrix}\right] : v \in V_1 \right\} \subset \mathbb{F}_q^{n+m}$, *and* $V_2' = \left\{ \begin{bmatrix} v' & 0_m \end{bmatrix} : v' \in V_1' \right\} \subset (\mathbb{F}_q^{n+m})^t$. *Then* $P_{n,V_1}^{V_1'} = P_{n+m,V_2}^{V_2'}$.

Proof. Let V and V' be such as in Corollary 17, let $V = \left\{ \left[\begin{smallmatrix} *_n \\ 0_m \end{smallmatrix}\right] \right\} \subset \mathbb{F}_q^{n+m}$ and let $V' = \left\{ \begin{bmatrix} *_n & 0_m \end{bmatrix} \right\} \subset (\mathbb{F}_q^{n+m})^t$. Suppose that V_1, V_1', V_2, and V_2' satisfy the assumptions of the theorem, $V_1 \subset \mathbb{F}_q^n, V_1' \subset (\mathbb{F}_q^n)^t$, $V_2 = \left\{ \left[\begin{smallmatrix} v \\ 0_m \end{smallmatrix}\right] : v \in V_1 \right\} \subset \mathbb{F}_q^{n+m}$, and $V_2' = \left\{ \begin{bmatrix} v' & 0_m \end{bmatrix} : v' \in V_1' \right\} \subset (\mathbb{F}_q^{n+m})^t$. Corollary 17 and Lemma 18 yield the relation $P_{n+m,V}^{V'} = P_n$. Thus,

$$P_{n+m,V_2}^{V_2'} = P_{n+m,V}^{V'} P_{n+m,V_2}^{V_2'} P_{n+m,V}^{V'} = P_n P_{n+m,V_2}^{V_2'} P_n. \tag{8}$$

Let $B, C \in \mathrm{Mat}(n,n)$ be such that $\mathrm{Ker}\begin{bmatrix} B & 0_{n\times\infty}\end{bmatrix} = V_1'$ and $\mathrm{Ker}\begin{bmatrix} C \\ 0_{\infty\times n}\end{bmatrix} = V_1$. We put

$$g_1 = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ C & 0 & I_n & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix}, \quad g_2 = \begin{bmatrix} I_n & 0 & B & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix}, \quad \widetilde{g} = \begin{bmatrix} I_n & 0 & 0 & B & 0 \\ 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 \\ C & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & I_\infty \end{bmatrix}.$$

Then $g_2\omega_n^{(n+m)}g_1 = \widetilde{g}$, whence $\gamma_{n+m}(g_2)\gamma_{n+m}(g_1) = \gamma_{n+m}(\widetilde{g}), \gamma_{n+m,V_2}^{V_2'} = \gamma_{n+m}(\widetilde{g})$, and

$$P_{n+m,V_2}^{V_2'} = P_{n+m}T(\widetilde{g})P_{n+m}. \tag{9}$$

Let

$$g = \begin{bmatrix} I_n & B & 0 & 0 \\ C & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_\infty \end{bmatrix}.$$

Arguing as in the proof of formula (9), we obtain

$$P_nT(g)P_n = P_{n,V_1}^{V_1'}. \tag{10}$$

The matrices g and $\widetilde{g}$ lie in one and the same double coset of $G_n(\infty)$ in $G(\infty)$. By (9), we have

$$P_{n+m,V_2}^{V_2'} = P_nT(\widetilde{g})P_n = P_nT(g)P_n = P_{n,V_1}^{V_1'}. \quad \square$$

Proposition 20. *For every projection $P_V^{V'}$ and every matrix $g \in G(\infty)$, there exists a pair of subspaces $V \subset \mathbb{F}_q^\infty$ and $V' \subset (\mathbb{F}_q^\infty)^t$ such that $\dim V = \dim V_1$, $\dim V' = \dim V_1'$, $\dim(V \cap (V')^\perp) = \dim(V_1 \cap (V_1')^\perp)$, and $T(g)P_V^{V'} = P_{V_1}^{V_1'}T(g)$.*

Proof. We choose an integer $N \in \mathbb{Z}_+$ so that $V \subset \mathbb{F}_q^N$, $V' \subset (\mathbb{F}_q^N)^t$, and $g \in G(N)$. By Lemma 2, there exists a pair of subspaces $V \subset \mathbb{F}_q^N$ and $V' \subset (\mathbb{F}_q^N)^t$ such that $\dim V = \dim V_1$, $\dim V' = \dim V_1'$, $\dim(V \cap (V')^\perp) = \dim(V_1 \cap (V_1')^\perp)$, and $\tau_N(g)P_V^{V'} = P_{V_1}^{V_1'}\tau_N(g)$. Since $P_V^{V'}$ and $P_{V_1}^{V_1'}$ are orthogonal projections to subspaces in H_N, we see that $T(g)P_V^{V'} = P_{V_1}^{V_1'}T(g)$. $\square$

§4. Irreducible tame representations of the group $\mathrm{GL}(\infty,\mathbb{F}_q)$. Classification

Let T be an irreducible tame representation of the group $G(\infty)$. We say that a projection $P_V^{V'}$ is *minimal* if $P_{V_1}^{V_1'} = 0$ for all $V_1 \subset V$ and $V_1' \subset V'$ such that $(V,V') \neq (V_1,V_1')$.

Theorem 21. *Every irreducible tame representation of $G(\infty)$ is unitarily equivalent to exactly one of the representations constructed in §1.*

Proof. Let $P_V^{V'}$ be a minimal projection. We put $r = \dim(V \cap (V')^\perp)$, $s = \dim V - r$, and $t = \dim(V') - s$. Let M be the set defined in §1,

$$M = \left\{(V,V') \in M_{\mathrm{all}} : \dim(V) = r+s,\ \dim(V') = s+t,\ \dim(V \cap (V')^\perp) = r\right\}.$$

Since the action of $G(\infty)$ on M is transitive, for every pair V_1, V_1' and some matrix $g \in G(\infty)$ we have $T(g)P_V^{V'}T(g^{-1}) = P_{V_1}^{V_1'}$. It is easily seen that all projections $P_{V_1}^{V_1'}, (V_1,V_1') \in M$ are minimal. Any two minimal projections are orthogonal. Proposition 20 shows that

$\bigoplus_{(V,V')\in M}(P_V^{V'}H)$ is an invariant subspace in H; therefore, it coincides with H. The set of minimal projections forms an imprimitivity system of the representation T. Let (V_0, V_0') be the point defined in §1, $V_0 = \left\{\left[\begin{smallmatrix} *_{(r+s)} \\ 0_\infty \end{smallmatrix}\right]\right\}$, $V_0' = \left\{\left[0_r \quad *_{(s+t)} \quad 0_\infty\right]\right\}$. The group $\widetilde{G}(r,s,t)$ is the stabilizer of the point (V_0, V_0'). From the aforesaid, it follows that the representation T is induced by the representation $\widetilde{\pi}(g) = P_{V_0}^{V_0'}T(g)$ of the subgroup $\widetilde{G}(r,s,t)$ in the space $P_{V_0}^{V_0'}H$. We prove that $\widetilde{\pi}$ is irreducible and trivial on the subgroup $\widetilde{G}(r,s,t)$.

The algebra $\mathfrak{U}$ of operators of the representation T decomposes into the direct sum of the subalgebras $P_V^{V'}\mathfrak{U}P_V^{V'}, (V,V') \in M$. They are irreducible (in the corresponding spaces), because the algebra $\mathfrak{U}$ is irreducible. In particular, we see that the representation $\widetilde{\pi}$ is irreducible. For every projection $P_V^{V'}, \mathbb{F}_q^{r+s+t} \supset V \supset V_0, (\mathbb{F}_q^{r+s+t})^t \supset V' \supset V_0'$, we have $P_V^{V'}P_{V_0}^{V_0'} = P_{V_0}^{V_0'}P_V^{V'} = P_{V_0}^{V_0'}$. It follows that the representation $\widetilde{\pi}$ is trivial on the set of matrices of the form

$$\begin{bmatrix} I_r & 0 & 0 & * \\ 0 & I_s & 0 & 0 \\ 0 & 0 & I_t & 0 \\ 0 & 0 & 0 & *_\infty \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & I_s & 0 & 0 \\ 0 & 0 & I_t & 0 \\ 0 & 0 & * & *_\infty \end{bmatrix},$$

and this set generates the subgroup $\widetilde{G}_0(r,s,t)$.

If two irreducible tame representations T_1 and T_2 are unitarily equivalent, then the sets of their minimal projections, the corresponding groups $\widetilde{G}(r,s,t)$, and the representations $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ coincide. Therefore, the representations π_1 and π_2 coincide, and we conclude that the quadruples of invariants of the representations T_1 and T_2 coincide. □

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