

QUANTUM RELATIVES OF THE ALEXANDER POLYNOMIAL

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ABSTRACT. The multivariable Conway function is generalized to oriented framed trivalent graphs equipped with additional structure (coloring). This is done via refinements of Reshetikhin–Turaev functors based on irreducible representations of quantized $\mathfrak{gl}(1|1)$ and $\mathfrak{sl}(2)$. The corresponding face state sum models for the generalized Conway function are presented.

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INTRODUCTION

Alexander polynomial and Conway function. The Alexander polynomial is one of the most classical topological invariants. It was defined [1] as early as in 1928. A recent paper by Fintushel and Stern [5] has again drawn attention to the Alexander polynomial by relating it to the Seiberg–Witten invariants of 4-dimensional manifolds.

The Alexander polynomial can be thought of in many different ways. There is a homological definition via free Abelian covering spaces, a definition via Fox’s free differential calculus, a definition via Reidemeister torsion, and Conway’s diagrammatical definition. See Turaev’s survey [18]. As a matter of fact, Conway [3] enhanced the original notion by eliminating an indeterminacy. He introduced also a link invariant that encodes the same information as the enhanced Alexander polynomial, but is more convenient from

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several points of view. Conway called it a *potential function* of a link. Following Turaev [18], we call it the *Conway function*.¹

The definitions of the (enhanced) Alexander polynomial and the Conway function are given below in Subsection 7.7. Here we only recall their formal appearance. For an oriented link $L \subset S^3$ with connected components L_1, \dots, L_k , the Alexander polynomial $\Delta_L(t_1, \dots, t_k)$ is a Laurent polynomial in the variables $t_1^{\frac{1}{2}}, \dots, t_k^{\frac{1}{2}}$, and the Conway function $\nabla(L)(t_1, \dots, t_k)$ is a rational function in the variables t_1, \dots, t_k . They are related by the formulas $\nabla(L)(t_1, \dots, t_k) = \Delta_L(t_1^2, \dots, t_k^2)$ if $k > 1$, and $\nabla(L)(t) = \frac{\Delta_L(t^2)}{t-t^{-1}}$ if L is a knot, i.e., $k = 1$.

Alexander polynomial and quantum topology. The intervention of quantum field theory into low-dimensional topology, which was initiated in the mid-eighties by the discovery of the Jones polynomial, added a new point of view on the Alexander polynomial. It was included into series of other quantum polynomial link invariants such as the Jones, HOMFLY and Kauffman polynomials.

The invariants of a topological object that are studied by quantum topology are presented by explicit formulas inspired by quantum field theory. A certain versatility of these formulas allows one to generalize the invariants to wider classes of objects and find counterparts of the invariants for completely different objects. For example, the Jones polynomial was first generalized to colored links and then to trivalent framed knotted graphs; then its counterparts (Reshetikhin–Turaev–Witten invariants) for closed oriented 3-manifolds were discovered, and they were upgraded to TQFT's, i.e., functors from categories of closed surfaces and their cobordisms with appropriate additional structures to the category of finite-dimensional vector spaces. The invariant of 3-manifolds (Witten–Reshetikhin–Turaev invariant) was defined technically even as an invariant of 4-manifolds with boundary; see [20]. Although its value depends only on the boundary, the construction deals with a 4-manifold. To some extent, the same procedure works for any quantum link polynomial.

However, the quantum theory of the Alexander polynomial has not evolved to the extent achieved by the theory based on the Jones polynomial. Only the very first steps of this program have been made: it has been shown that the Alexander polynomial can be defined via representations of quantum groups. The multivariable Conway function was obtained, along the lines of the construction of link invariants related to a quantum group, by Jun Murakami [11, 12], Rozansky and Saleur [13], and Reshetikhin [15]. Rozansky, Saleur, and Reshetikhin applied this construction to the quantum supergroup $\mathfrak{gl}(1|1)$ (i.e., a quantum deformation $U_q \mathfrak{gl}(1|1)$ of the universal enveloping algebra of the superalgebra $\mathfrak{gl}(1|1)$; see Appendix A to [14]). Murakami did this with the quantum group $\mathfrak{sl}(2)$ at $q = \sqrt{-1}$. Deguchi and Akutsu [4] included the Conway function in a family of invariants related to representations of the quantum $\mathfrak{sl}(2)$ at roots of unity, generalized these invariants to colored framed trivalent graphs, and found face state sum models for these generalizations. However, the case of the Conway function has not been studied separately from the generalizations.

Thus, the multivariable Conway function was studied by the methods of quantum topology along two lines: based on the quantum supergroup $\mathfrak{gl}(1|1)$ by Reshetikhin,

¹Some authors call it the *Alexander–Conway polynomial*; see, e.g., [8, 13, 14]. However, there are two reasons not to use this term. First, this is not a polynomial even in the sense in which the Alexander polynomial is: this is not a Laurent polynomial, but rather a rational function. Say, for the unknot it is $\frac{1}{t-t^{-1}}$. Second, there is another invariant deserving the name of Alexander–Conway polynomial. It is a polynomial knot invariant, which was introduced by Conway in the same paper [3]: the Alexander polynomial normalized and represented as a polynomial in $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$, where t is the original variable. Therefore, following Turaev [18], we use the term the *Conway function*.

Rozansky, and Saleur and based of the quantum group $sl(2)$ by Jun Murakami, Deguchi, and Akutsu. The $sl(2)$ approach was developed up to explicit generalization to colored framed trivalent graphs and face state sum models, although the results were given in the form of quite complicated formulas, which were not analyzed from a geometric point of view. In the framework of the $gl(1|1)$ approach, a generalization to colored framed trivalent graphs has not been considered.

Results. Our goal in this paper is a further development of quantum invariants related to the Alexander polynomial. We take the next two steps in the $gl(1|1)$ direction: generalize the (multivariable) Conway function to trivalent graphs equipped with some additional structures (colorings) and find a face state sum presentation for this generalization (and, in particular, for the Conway function of the classical links). In the $sl(2)$ direction the corresponding steps seem to have been done in [4], but, for the sake of a subsequent development, we do them all over again from scratch, in an effort to get simpler formulas and geometric presentations. In particular, this allows us to compare the results in the $gl(1|1)$ and $sl(2)$ directions.

None of these results could be reduced to the other one. The colorings used in the $gl(1|1)$ approach are based on a larger palette. The colorings based on a portion of the $gl(1|1)$ -colors may be turned into $sl(2)$ -colorings, but not all $sl(2)$ -colorings can be obtained in this way. A larger palette makes the $gl(1|1)$ approach somehow more flexible, but leads to more complicated face state sum models. Both directions seem to deserve investigation, since they may be useful for different purposes.

The relatives of the Alexander polynomial studied in this paper are:

- the Reshetikhin–Turaev functors \mathcal{RT}^1 and \mathcal{RT}^2 based on the irreducible representations of quantum $gl(1|1)$ and $sl(2)$, respectively;
- a modification \mathcal{A}^c of \mathcal{RT}^c for $c = 1, 2$;
- the invariants $\underline{\Delta}^c$ of closed colored framed generic graphs similar to the Conway function ∇ of links and, in a sense, generalizing it.

Further steps may be a construction of state sum invariants for shadow polyhedra and, eventually, 4-manifolds. However, there are several obstructions, which make this, at least, not completely straightforward. It is not yet clear which family of generalizations of the Conway function is more suitable for that.

Although the possibility of future four-dimensional applications was the main motivation for this work, I believe the results are interesting on their own. The formulas involved in the face model appear to be much simpler than their counterparts even for the Jones polynomial.

Reshetikhin–Turaev functors. As was shown by Reshetikhin and Turaev [16], the construction of link invariants related to a quantum group is upgraded to a construction of a functor from a category whose morphisms are colored framed generic graphs to a category of finite-dimensional representations of a quantum group. A restriction of this functor to a subcategory of tangles (graphs consisting of disjoint circles and intervals) was described in [16] explicitly, while for graphs with trivalent vertices Reshetikhin and Turaev [16] leave a choice.

As far as I know, in the case of quantum $gl(1|1)$ the choice has never been made, although the Reshetikhin–Turaev functor from the category of tangles to the category of representations of quantum $gl(1|1)$ and its relationship with the Conway function was alluded to in several papers; see, e.g., Rozansky and Saleur [14].

In this paper a Reshetikhin–Turaev functor is constructed explicitly, by presenting the Boltzmann weights (see Tables 1 and 2) on the whole category of colored framed generic graphs. The choice left by Reshetikhin and Turaev [16] can be made in many

ways, but I believe the one made here deserves special attention, because the resulting functor \mathcal{RT}^1 has special nice properties. For example, the coefficient in skein relation (15) is very simple and the Boltzmann weights are polynomials in powers of q .

Special features of quantized $\mathfrak{gl}(1|1)$ make the construction interesting and not completely straightforward. Since $\mathfrak{gl}(1|1)$ is a superalgebra, all the formulas change in a peculiar way, which looks strange for anybody but an experienced supermathematician (i.e., a mathematician working with superobjects). In its turn, the unusual algebra causes unusual topological features. For example, colorings of graphs involve orientations of 1-strata in a subtler way, and at some vertices a cyclic order of the adjacent 1-strata must also be included into a coloring.

Modifications of the Reshetikhin–Turaev functor. The obvious polynomial nature of the Boltzmann weights in Tables 1 and 2 and the polynomial nature of the Alexander polynomial suggest a modification of \mathcal{RT}^1 , which gives rise to a functor \mathcal{A}^1 that acts from almost the same category of colored framed generic graphs to the category of finite-dimensional free modules over some commutative ring B . For example B may be $\mathbb{Z}[M]$, where M is a free Abelian multiplicative group. In this case B is the ring of Laurent polynomials. Thus, \mathcal{A}^1 is closer to the Alexander polynomial.

Roughly speaking, \mathcal{A}^1 is obtained from \mathcal{RT}^1 by eliminating the Planck constant q (the parameter of the quantum deformation in the quantization $U_q \mathfrak{gl}(1|1)$) by replacing the powers of q with independent variables in formulas for the Boltzmann weights. The Boltzmann weights for \mathcal{A}^1 are presented in Tables 3 and 4.

The transition from \mathcal{RT}^1 to \mathcal{A}^1 does not eliminate the quantum nature together with q . Although $U_q \mathfrak{gl}(1|1)$ does not act in the B -modules that are the values of \mathcal{A} at objects, there is a Hopf subalgebra of $U_q \mathfrak{gl}(1|1)$ such that \mathcal{A}^1 can be upgraded to functors from the same category to the categories of modules over this subalgebra.

Alexander invariant. The method used by Rozansky and Saleur in [13] and [14] for relating the Conway function to the quantum $\mathfrak{gl}(1|1)$ version of the Reshetikhin–Turaev functor for tangles, transforms \mathcal{C}^1 to the *Alexander invariant* $\underline{\Delta}$ of closed colored framed generic graphs in \mathbb{R}^3 .

To a closed colored framed generic graph Γ in \mathbb{R}^3 , $\underline{\Delta}^1$ assigns an element $\underline{\Delta}^1(\Gamma)$ of B . The coloring of Γ consists of orientation of the 1-strata of Γ , assigning to each 1-stratum a *multiplicity* taken from $\{t \in M \mid t^4 \neq 1\}$ and an integer weight, and fixing a cyclic order of the 1-strata adjacent to some vertices. The dependence of the Alexander invariant on the weights, multiplicities, and cyclic orders is completely understood and is described below. There exist universal multiplicities on Γ with $M = H^1(\Gamma; \mathbb{Z})$ such that, for given orientations, and with the Alexander invariant of Γ colored with these orientations, arbitrary cyclic orders, weights, and multiplicities can be recovered from the Alexander invariant of Γ colored with the same orientations, the universal multiplicities, and any cyclic orders and weights. The orientations constitute the most subtle part of colorings. This is demonstrated in the case of the 1-skeleton of a tetrahedron considered in Subsection 7.2. However, the Alexander invariant of this graph with any coloring is much simpler than the Jones polynomial of the same graph, which is basically the $6j$ -symbol and plays an important role in the TQFT based on quantum $su(2)$.

For a link L colored in an appropriate way, we have

$$\nabla(L)(t_1, \dots, t_k) = \underline{\Delta}^1(t_1^2, \dots, t_k^2).$$

Thus, $\underline{\Delta}^1$ can be viewed as a generalization of the Conway function to graphs. Exactly as in the transition from the Alexander polynomial to the Conway function, here we must replace the previous variables by their square roots. Hence, the variables in the Alexander

invariant are quartic roots of the variables in the Alexander polynomial. Compare with the relationship between the Kauffman brackets and the Jones polynomial.

Face models. All formulas of quantum topology can be divided roughly into two classes: vertex and face state sums. Face state sums seem to be more versatile. At least, they are more uniform for topological objects of different kinds. The formulas in the definition of the Alexander invariant are of the vertex type.

In this paper, face state sums representing the Alexander invariant are also obtained. This is done via a version of “transition to the shadow world” invented by Kirillov and Reshetikhin [7] to obtain a face state sum model for the quantum $sl(2)$ invariant of colored framed graphs, generalizing the Jones polynomial. The construction used here is another special version of a more general construction, which I found when analyzing Kauffman’s “quantum spin-network” construction [6] for the Turaev–Viro invariants [21], and presenting this in my UC San Diego lectures in the Spring quarter of 1991. I presented the general construction in several talks, but never published it, since in the full generality it looks too cumbersome, while in the special cases, which I had known before this work and in which it looked nice, the result had already been known.

Thanks. I am grateful to Lev Rozansky for his valuable consultations concerning super-mathematics, $gl(1|1)$, and the papers [13, 14], and to Alexander Shumakovitch for pointing out a mistake in a preliminary version of this paper.

§1. GEOMETRIC PRELIMINARIES ON KNOTTED GRAPHS

1.1. Generic graphs. By a *generic graph* we mean a 1-dimensional *CW-complex* such that each of its points has a neighborhood homeomorphic either to \mathbb{R} , or to the half-line \mathbb{R}_+ , or to the union of three copies of \mathbb{R}_+ meeting at their common endpoint. Each generic graph is naturally stratified. The strata of dimension 1 are the connected components of the set of points that have neighborhoods homeomorphic to \mathbb{R} . The 0-strata of a generic graph Γ are the 3-valent and 1-valent vertices of Γ . The former are called the *internal vertices* and the latter are the *endpoints* or *boundary points* of the graph. The set of boundary points of Γ is called the *boundary* of Γ and denoted by $\partial\Gamma$; the complement $\Gamma \setminus \partial\Gamma$ is called the *interior* of Γ and denoted by $\text{Int } \Gamma$. A generic graph Γ with empty boundary is said to be *closed*. The 1-strata homeomorphic to \mathbb{R} are called *edges*. (Thus, a component of a generic graph may contain no vertex.)

A generic graph Γ is said to be *properly embedded* into a 3-manifold M if $\Gamma \subset M$ and $\partial\Gamma = \Gamma \cap \partial M$.

1.2. Additional structures on generic graphs. Usually, generic graphs embedded in a 3-manifold are equipped with various additional structures. The most common of the structures are orientations and framings. An *orientation* of a generic graph Γ is an orientation of all the 1-strata of Γ .

Let Γ be a generic graph properly embedded into a 3-manifold M . An extension of Γ to an embedded compact surface F in which Γ is sitting as a deformation retract is called a *framing* of Γ (in M) if $\partial F \cap \partial\Gamma = F \cap \partial\Gamma$ and each component of $F \cap \partial M$ is an arc containing exactly one boundary point of Γ . These arcs comprise a framing of $\partial\Gamma$ in ∂M . By a framing of a discrete subset of a surface we understand a collection of disjoint arcs embedded in the surface in such a way that each arc contains exactly one point of the subset and each point is contained in one of the arcs.

By an isotopy of a framed generic graph we mean an isotopy of the graph in the ambient 3-manifold, extended to an isotopy of the framing. In the case of a properly embedded framed generic graph, the isotopy is assumed to be fixed on the intersection with the boundary of the ambient manifold.

The notion of a framed generic graph generalizes that of a framed link. Indeed, a collection of circles can be regarded as a generic graph, and if the framing surface is orientable, the framing is defined up to isotopy by a nonzero vector field normal to the circles and tangent to the surface. Recall that, usually, by a framing of a link one means a nonzero normal vector field on it. If not orientable, the framing surface is related in a similar way to a framing with a field of normal lines, rather than normal vectors.

The third additional structure involved below is *orientation at vertices*. A generic graph is said to be *oriented at a (trivalent) vertex* if the germs of the edges adjacent to the vertex are cyclically ordered. We speak here of germs rather than edges, because an edge may be adjacent to a vertex twice, and then there are only two adjacent edges (recall that a set of two elements cannot be ordered *cyclically*). If the graph is framed, then its orientation at a vertex determines a local orientation of the framing surface at the vertex.

1.3. Diagrams. To describe a generic graph in \mathbb{R}^3 up to isotopy, a natural generalization of the link diagrams is used. Let Γ be a generic graph embedded into \mathbb{R}^3 . A projection of Γ to \mathbb{R}^2 is said to be *generic* if

- (1) its restriction to each 1-stratum of Γ is an immersion;
- (2) it has no point of multiplicity ≥ 3 ;
- (3) no double point is the image of a vertex of Γ ;
- (4) at each double point the images of the corresponding 1-strata intersect each other transversally;
- (5) at the image of a vertex no two branches of 1-strata are tangent to each other.

It is clear that the term *generic* is appropriate here in the sense that the embedded generic graphs with nongeneric projection to \mathbb{R}^2 form a nowhere dense set in the space of all embedded generic graphs: by an arbitrarily small isotopy one can make any embedded generic graph have generic projection, and no generic projection can be made nongeneric by a sufficiently small isotopy.

To describe (up to isotopy) a generic graph embedded in \mathbb{R}^3 , its generic projection should be enhanced with information as to which branch is over and which is under the other one at every double point. If this is done by breaking the lower branch, the picture obtained is called a *diagram* of the graph, as in the case of links.

1.4. Strata of a diagram. The image of a generic graph Γ under a generic projection is a graph embedded in \mathbb{R}^2 . Its vertices are the double points of the projection and the images of the vertices of Γ . The former are also called the *crossing points*. The natural stratification of this graph is extended to a stratification of \mathbb{R}^2 in which the 2-strata are the connected components of the complement of the projection. The strata of this stratification are called the *strata of the diagram*; the 2-strata are also called the *faces* of the diagram.

1.5. Moves of diagrams. Diagrams of isotopic generic graphs embedded in \mathbb{R}^3 can be obtained from each other by a sequence of transformations of 5 types shown in Figure 1. Here two transformations are viewed as being of the same type if the modifications happen in disks that can be mapped to each other by a homeomorphism moving one picture to another, possibly followed by a simultaneous change of all the crossings (this change corresponds to reflection in the plane parallel to the plane of the diagram). The types correspond to the conditions in the definition of a generic projection: each of the transformations in Figure 1 can be realized by an isotopy under which exactly one of the

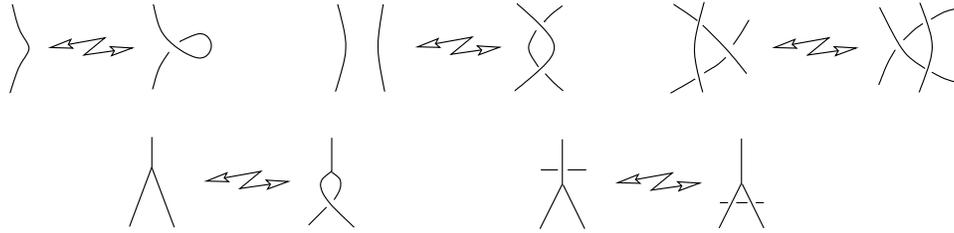


FIGURE 1. Elementary isotopies of generic graphs.

conditions is violated exactly once and in the simplest manner. The first 3 transformations are the well-known Reidemeister moves. They do not involve vertices and can be used to obtain diagrams of isotopic links from each other.

1.6. Showing additional structures on the diagram. Any generic graph with generic projection to \mathbb{R}^2 can be equipped with a framing such that the projection to \mathbb{R}^2 restricted to the framing is an immersion (recall that the framing is a surface). This framing is unique up to isotopy. It is called a *blackboard* framing.

Similarly, a generic graph generically projected to \mathbb{R}^2 can be equipped with the *blackboard* orientation at a vertex: counterclockwise cyclic ordering of the adjacent germs of the edges. An arbitrary orientation at vertices can be shown in the diagram by circular arrows around the vertices following the cyclic orders. For simplicity, we shall skip all the counterclockwise arrows.

Let Γ be a framed generic graph with a generic projection to \mathbb{R}^2 . On each 1-stratum of Γ , two framings can be compared: the blackboard and original ones. The difference is an integer or a half-integer. It is the number of (full) right twists that should be added to the blackboard framing on the 1-stratum to get the framing involved into the original framing of Γ . To describe a framed generic graph up to isotopy, it suffices to draw its diagram equipped with these numbers assigned to all 1-strata of the graph. Another way to show the difference between the framing and the blackboard framing is to supply the arc of the diagram with the fragment $\text{---}\smile\text{---}$ at a positive half-twist and with the fragment $\text{---}\frown\text{---}$ at a negative half-twist. Of course, the isotopy type of a framing is described by the total number of half-twists on each component. However, sometimes, especially when describing changes of a diagram, it is convenient to localize the places where the framing is orthogonal to the plane of the diagram and specify the local behavior of the framing at these places, as the pictures $\text{---}\smile\text{---}$ and $\text{---}\frown\text{---}$ do. We call a *diagram of a framed graph* a diagram enhanced by the differences between the framing and the blackboard framing, which are presented either by numbers, or by the pictures of half-twists.

Diagrams of isotopic framed generic graphs can be obtained from each other by a sequence of transformations of 7 types² shown in Figure 2. The first 5 of them are the transformations of Figure 1 equipped with a description of the behavior of the framing. The last 2 describe the emerging and moving of half-twists.

The notion of a diagram extends obviously to generic graphs and framed generic graphs lying in the product of a surface P by \mathbb{R} or $[0, 1]$. Such diagrams are drawn on P . To avoid situations when diagrams are not quite understandable, we agree to assume that the boundary points of graphs are over the boundary of P , that there are no double points on the boundary of P , and that the framings at the endpoints of graphs are blackboard.

²Up to the same rough equivalence as above, where isotopies of unframed graphs were considered.

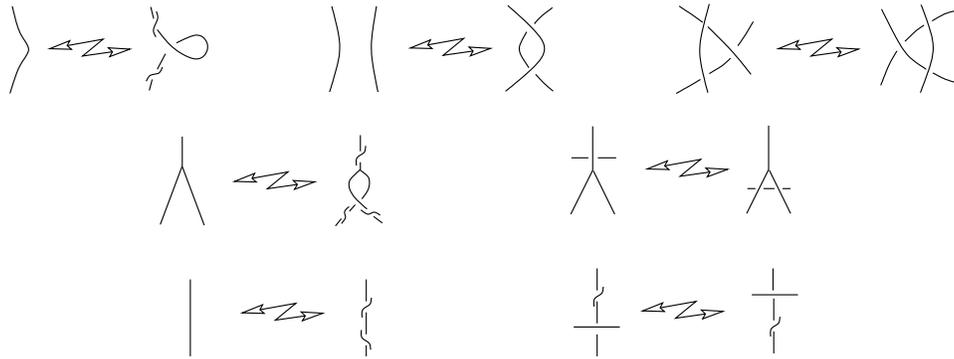


FIGURE 2. Elementary isotopies of framed generic graphs.



FIGURE 3. Generators of \mathcal{G} .

1.7. Category of graphs. The framed graphs give rise to a category \mathcal{G} , which is defined as follows. Its objects are in one-to-one correspondence with the natural numbers. The object corresponding to a natural number k is the pair consisting of \mathbb{R}^2 and the set Π_k of k points $(1, 0), (2, 0), \dots, (k, 0)$ equipped with a framing that consists of small intervals of the x -axis. A morphism from the k th to the l th object is the isotopy class of a framed generic graph Γ embedded in $\mathbb{R}^2 \times [0, 1]$, with $\partial\Gamma = \Pi_k \times 0 \cup \Pi_l \times 1 = \partial(\mathbb{R}^2 \times [0, 1]) \cap \Gamma$. A framed generic graph of this sort is called a (*framed*) (k, l) -graph. The composition of two morphisms is defined by attaching them one over the other. The identity morphism of the object (\mathbb{R}^2, Π_k) is $(\mathbb{R}^2 \times [0, 1], \Pi_k \times [0, 1])$.

There is another basic operation with framed (k, l) -graphs, which is called *tensor product*. The tensor product of a framed (k, l) -graph Γ by an (m, n) -graph Γ' is a $(k + m, l + n)$ -graph obtained by placing Γ' to the left of Γ .

1.8. Generators and relations of the category. Framed (k, l) -graphs are represented by diagrams on the strip $\mathbb{R} \times [0, 1]$. Applying general position arguments to the diagrams and their deformations under isotopies, it is easy to find the natural generators and relations for the category of framed generic graphs, as described in the following theorems.

1.8.A. *Any morphism of \mathcal{G} can be presented as a composition of morphisms each of which is the tensor product of identity morphisms (i.e., the isotopy classes of the graphs $\Pi_i \times [0, 1]$ with blackboard framing) by one of the morphisms shown in Figure 3.*

Indeed, by a small isotopy we can make the restriction of the second coordinate to each 1-stratum of the plane projection of the graph have only isolated nondegenerate critical points. Furthermore, the values of the second coordinate at these points, at the half-twist signs, and at the vertices of the projection can be made pairwise distinct. After that, we draw horizontal lines separating the critical points, the half-twist signs, and the vertices from one another. The pieces between the lines next to each other are graphs of the desired types. \square

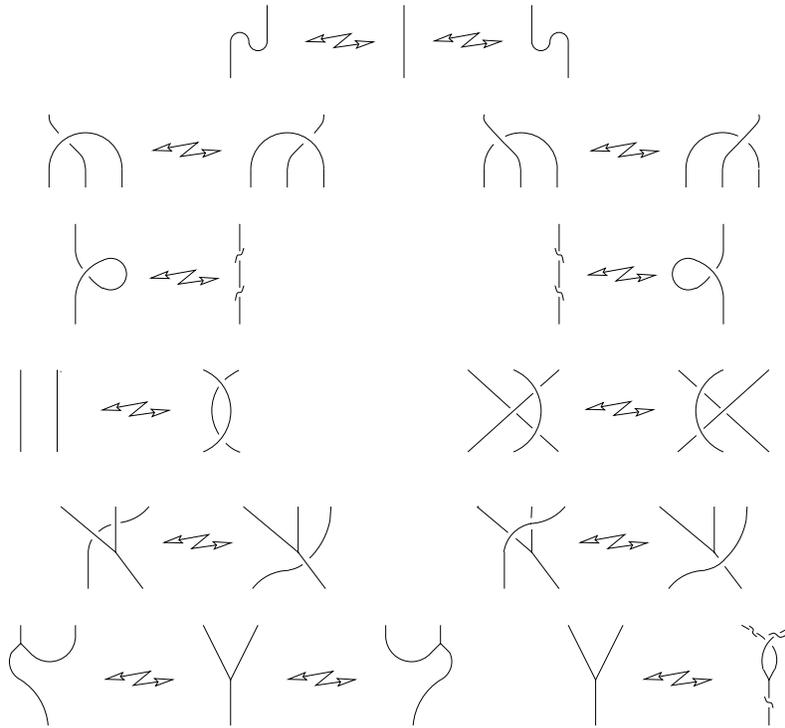


FIGURE 4. Relations of \mathcal{G} .

Similar arguments, but applied to a one-parameter family of graphs and combined with representing some of the elementary relations as consequences of other elementary relations, prove the following theorem.

1.8.B. *Two products of the graphs generating \mathcal{G} by Theorem 1.8.A are isotopic if and only if they can be transformed to each other by a sequence of elementary moves shown in Figure 4.*

§2. RESHETIKHIN–TURAEV FUNCTOR BASED ON $\mathfrak{gl}(1|1)$

2.1. Category of colored framed graphs. Objects. Recall that in Reshetikhin and Turaev’s theory [16], the invariants are constructed via a functor from a category of colored framed graphs, whose objects and morphisms are the objects and morphisms of \mathcal{G} equipped with additional structures (colored), to a category of finite-dimensional representations of some quantum group. Here, Reshetikhin and Turaev’s setup will be modified slightly.

Let h be a nonzero complex number. Below we define a category \mathcal{G}_h^1 . It is denoted also by \mathcal{G}^1 , when the dependence on h is not emphasized. Its objects are finite sequences of triples consisting of two complex numbers, say j and J , and a sign δ (i.e., $\delta = +$ or $-$). The first number must be such that $2jh\sqrt{-1}/\pi$ is not an integer. The first number is called the *multiplicity*, and the second is the *weight*.

It is convenient to associate the triples $(j_1, J_1, \delta_1), \dots, (j_k, J_k, \delta_k)$ comprising an object with points $(1, 0), \dots, (k, 0)$ of the plane \mathbb{R}^2 . The object $\{(j_1, J_1, \delta_1), \dots, (j_k, J_k, \delta_k)\}$ should be thought of as the set Π_k whose points are colored with the triples (j_1, J_1, δ_1) ,

$\dots, (j_k, J_k, \delta_k)$. The signs $\delta_1, \dots, \delta_k$ are interpreted as orientations of the corresponding points.

In fact, the triples stand for irreducible $(1|1)$ -dimensional $U_q \mathfrak{gl}(1|1)$ -modules. We recall (see Appendix 1, Subsection 11.4) that the irreducible $(1|1)$ -dimensional $U_q \mathfrak{gl}(1|1)$ -modules are enumerated by a sign \pm and a pair of complex parameters (j, J) with $j \in \mathbb{C} \setminus \{\pi n \sqrt{-1}/2h : n \in \mathbb{Z}\}$ and $J \in \mathbb{C}$. Here h is the parameter of quantization (the Planck constant).

2.2. Category of colored framed graphs. Morphisms. A morphism of \mathcal{G}_h^1 ,

$$\{(i_1, I_1, \epsilon_1), \dots, (i_k, I_k, \epsilon_k)\} \rightarrow \{(j_1, J_1, \delta_1), \dots, (j_l, J_l, \delta_l)\},$$

is a morphism of \mathcal{G} (see Subsection 1.7) equipped with some additional structures described below. Recall that a morphism of \mathcal{G} is a framed generic graph Γ (see Subsection 1.1) embedded in $\mathbb{R}^2 \times [0, 1]$ with $\partial\Gamma = \Pi_k \times 0 \cup \Pi_l \times 1 = \partial(\mathbb{R}^2 \times [0, 1]) \cap \Gamma$, considered up to ambient isotopy fixed on $\partial\Gamma$. Its framing at the endpoints is parallel to the x -axis.

The additional structures on Γ all together are called the \mathcal{G}^1 -coloring. Each 1-stratum is colored with an orientation and a pair of complex numbers, the first of which does not belong to $\{\pi n \sqrt{-1}/2h : n \in \mathbb{Z}\}$. These numbers are also called the *multiplicity* and the *weight*, respectively. The orientations of the edges adjacent to the boundary points are determined by the signs of the points' colors: the edge adjacent to a point with $+$ is oriented at the point upwards (along the standard direction of the z -axis); otherwise it is oriented downwards. The multiplicity and weight of the edge adjacent to an endpoint coincide with the corresponding ingredients of the color of this endpoint. At each trivalent vertex, the colors of the adjacent edges satisfy the following conditions.

2.2.A (Admissibility conditions). *Let (j_1, J_1) , (j_2, J_2) , and (j_3, J_3) be the multiplicity and weight components of the colors of the three edges whose germs are adjacent to one and the same vertex, and let $\epsilon_i = -1$ if the i th of the edges is oriented towards the vertex and $\epsilon_i = 1$ otherwise. Then*

$$(1) \quad \sum_{i=1}^3 \epsilon_i j_i = 0,$$

$$(2) \quad \sum_{i=1}^3 \epsilon_i J_i = - \prod_{i=1}^3 \epsilon_i.$$

A vertex with all the adjacent edges oriented towards it (i.e., $\sum_{i=1}^3 \epsilon_i = -3$) or in the opposite direction (i.e., $\sum_{i=1}^3 \epsilon_i = 3$) is said to be *strong*. At each strong vertex the graph Γ is oriented. The orientation of Γ at the strong vertices is viewed as part of the coloring of Γ , together with the orientations, multiplicities, and weights of the 1-strata.

Equation (1) looks like the Kirchhoff equation for currents. A more topological interpretation: the boundary of the chain $\sum j_i e_i$, where the sum is taken over all the edges e_i of the graph (with their orientations) and j_i is the first component of the color of e_i , is the sum of all the boundary points of the graph Γ taken with the coefficients that are the multiplicities of these points with appropriate signs. In particular, $\sum j_i e_i$ is a relative cycle with real coefficients of Γ modulo $\partial\Gamma$, so that it determines an element of $H_1(\Gamma, \partial\Gamma; \mathbb{C})$.

2.2.A.1. Remark. In the Reshetikhin–Turaev setup, the orientation of an edge could be reversed along with a simultaneous change of the numerical component of the color. Here we cannot do this. The orientations should be thought of as inseparable ingredients of the colors. Reversing the orientation of a single 1-stratum cannot be compensated for by changes of other components of the coloring.

Orientations at strong vertices are not that significant. Reversing one of them causes multiplication by -1 of the image under the Reshetikhin–Turaev functor; see Subsection 2.5. Therefore, many formulas below apparently do not involve orientations at vertices. Orientations at vertices are hidden by eliminating strong vertices with clockwise orientations. Recall that we have agreed not to show counterclockwise orientations at vertices in a diagram.

A vertex with an odd number of adjacent edges oriented towards it (i.e., $\prod_{i=1}^3 \epsilon_i = -1$) is said to be *odd*. If we think of weights also as of currents, then (2) means that each odd vertex is a source of capacity 1 and each even vertex is a sink of capacity 1. In other words, $\partial \sum J_i e_i$ involves each odd vertex with coefficient -1 and each even vertex with coefficient 1.

The composition of two morphisms is defined by attaching them one over the other. There is another basic operation with morphisms, which is called *tensor product*. The tensor product of a morphism Γ by a morphism Γ' is a morphism obtained by placing Γ' to the left of Γ .

There is a natural forgetting functor $\mathcal{G}_h^1 \rightarrow \mathcal{G}$. The morphisms of \mathcal{G}_h^1 mapped by this functor to the generators of \mathcal{G} generate \mathcal{G}_h^1 . Therefore, Theorem 1.8.A implies the following.

2.2.B (Generators of \mathcal{G}_h^1). *Any morphism of \mathcal{G}_h^1 can be presented as a composition of morphisms each of which is the tensor product of identity morphisms (i.e., the isotopy classes of graphs $\Pi_i \times [0, 1]$ blackboard framed and appropriately colored) by one of the morphisms whose underlying graph is shown in Figure 3.*

Similarly, Theorem 1.8.B gives the relations of \mathcal{G}_h^1 .

2.3. Refined Reshetikhin–Turaev functor on objects. Denote by \mathcal{R}_q the category of finite-dimensional modules over the q -deformed universal enveloping algebra $U_q \mathfrak{gl}(1|1)$ of the superalgebra $\mathfrak{gl}(1|1)$. Our goal is to construct a functor $\mathcal{RT}^1 : \mathcal{G}_h^1 \rightarrow \mathcal{R}_q$.

For an object $\{(j_1, J_1, \epsilon_1), \dots, (j_k, J_k, \epsilon_k)\}$ of \mathcal{G}_h^1 , the image under \mathcal{RT}^1 is defined to be the tensor product $(j_1, J_1)_{\epsilon_1} \otimes \dots \otimes (j_k, J_k)_{\epsilon_k}$ of irreducible $(1|1)$ -dimensional $U_q \mathfrak{gl}(1|1)$ -modules.

Each of the factors $(j_i, J_i)_{\epsilon_i}$ is a 2-dimensional vector space with a canonical basis. The vectors e_0 and e_1 comprising the basis are distinguished by their nature: e_0 is a boson, e_1 a fermion.³ Thus, $(j_1, J_1)_{\epsilon_1} \otimes \dots \otimes (j_k, J_k)_{\epsilon_k}$ is a complex vector space of dimension 2^k . Its basis consists of the vectors $e_{i_1} \otimes \dots \otimes e_{i_k}$, so the basis vectors may be identified with sequences of zeros and ones placed at the points of Π_k .

2.4. Digression: Boltzmann weights. To describe the images of morphisms under \mathcal{RT}^1 , a geometric interpretation borrowed from statistical mechanics can be used. Recall how this works.

The image under \mathcal{RT}^1 of an arbitrary morphism of \mathcal{G}_h^1 is a linear map between the corresponding vector spaces. This map can be presented by a matrix. The elements of the matrix are complex numbers corresponding to the pairs consisting of basis vectors of the two vector spaces. We recall that the basis vectors are interpreted as sequences of zeros and ones. These zeros and ones are placed at the endpoints of the colored framed graph representing the morphism. Hence, to define the image of this morphism under \mathcal{RT}^1 , we need to associate a complex number with any distribution of zeros and ones at the endpoints of the graph.

Of course, it suffices to define the images of all the generators of \mathcal{G}_h^1 (see Figure 3) in such a way that the images satisfy the relations shown in Figure 4.

³Whatever this means! See Appendix 1 for brief explanations.

For a graph underlying a generator, any distribution of zeros and ones at the endpoints extends naturally to the adjacent 1-strata of the diagram. The matrix entry of the morphism that is the image of the generator of \mathcal{G}_h^1 corresponds to such a coloring of the 1-strata of the diagram.

Now we deal with two colorings: the first is a \mathcal{G}^1 -coloring, the second associates zero or one with each 1-stratum of the diagram. These colors stand for basis vectors of the module associated with the \mathcal{G}^1 -color of the 1-stratum. To distinguish these colorings from each other, we call the first of them (i.e., the \mathcal{G}^1 -coloring) the *primary coloring*, and the second the *secondary coloring*.

Hence, to define the image under \mathcal{RT}^1 of the generator of \mathcal{G}^1 , with any such pair of colorings of the elementary graphs shown in Figure 3 we need to associate an appropriate complex number. In statistical mechanics (where this graphical interpretation comes from) these numbers are called the *Boltzmann weights*.

Once the Boltzmann weights are known for all the pairs of colorings of elementary graphs, the matrix element for the image of an arbitrary morphism of \mathcal{G}_h^1 can be calculated as follows: choose a diagram for the corresponding graph, and consider all the extensions of the given distributions of zeros and ones at the endpoints of the graph to secondary colorings of the entire diagram. For each of the secondary colorings, consider the product of all the Boltzmann weights of its elementary fragments. The sum of the products over all these secondary colorings is the required matrix element. In the terminology of statistical mechanics, a secondary coloring would be named a *state* (of the graph). Thus, the sum of the products is a sum over all the states, or a *state sum*.

2.5. Refined Reshetikhin–Turaev functor on morphisms. The main theorem of the Reshetikhin–Turaev paper [16] suggests the images for the generators with the six underlying graphs shown on the left in Figure 3. For the right two, this should be the Clebsch–Gordan morphisms scaled so that the relations corresponding to the bottom row of Figure 4 be satisfied.

The choice of scaling is not prescribed by the Reshetikhin–Turaev theorem, and moreover, is not unique. However, the properties of the functor depend on the choice made here, and justify it.

The images of the generators are given below in Tables 1 and 2 by the Boltzmann weights. The secondary colors are shown there as follows. The arcs colored with the boson basis vector e_0 are shown as dotted arcs, the arcs colored with the fermion basis vector e_1 are shown in solid. (We follow Kauffman and Saleur [8].)

The leftmost graph in Figure 3 can be oriented in two ways. If it is oriented from right to left, its image under \mathcal{RT}^1 should map $(j, J)_- \otimes (j, J)_+$ to \mathbb{C} . The Reshetikhin–Turaev theorem suggests⁴ the pairing \curvearrowright , which acts by $(e_a \otimes e_b) \mapsto (-1)^a q^{-2ja} \delta_{ab}$; see Subsection 11.5. For the same graph with the opposite orientation, the Reshetikhin–Turaev theorem suggests the “quantum transposed” pairing \curvearrowleft , which acts by $(e_a \otimes e_b) \mapsto q^{2j(1-a)} \delta_{ab}$; see Subsection 11.5.

The next graph in Figure 3, oriented from right to left, is mapped by \mathcal{RT}^1 to $\mathbb{C} \rightarrow (j, J)_- \otimes (j, J)_+$. The Reshetikhin–Turaev theorem suggests the copairing \curvearrowright : $1 \mapsto e_0 \otimes e_0 - e_1 \otimes q^{2j} e_1$ (the minus sign is again due to the superenvironment). For the same graph with the opposite orientation, the formula is

$$\curvearrowleft : \mathbb{C} \rightarrow (j, J)_+ \otimes (j, J)_- : 1 \mapsto q^{-2j} e_0 \otimes e_0 + e_1 \otimes e_1.$$

⁴This differs by the sign from Reshetikhin–Turaev due to the framework of supermathematics; see Appendix 1.

TABLE 1.

	+1	$-q^{2j}$	q^{-2j}	1
	+1	$-q^{-2j}$	q^{2j}	1
	q^{-jJ}	q^{-jJ}	q^{-jJ}	q^{-jJ}
	q^{jJ}	q^{jJ}	q^{jJ}	q^{jJ}
	$q^{-iJ-jI}q^{i+j}$	$q^{-iJ-jI}q^{i+j}$	$q^{-iJ-jI}q^{i+j}$	$q^{-iJ-jI}q^{i+j}$
	$q^{-iJ-jI}q^{j-i}$	$q^{-iJ-jI}q^{j-i}$	$q^{-iJ-jI}q^{j-i}$	$q^{-iJ-jI}q^{j-i}$
	$q^{-iJ-jI}q^{i-j}$	$q^{-iJ-jI}q^{i-j}$	$q^{-iJ-jI}q^{i-j}$	$q^{-iJ-jI}q^{i-j}$
	$-q^{-iJ-jI}q^{-i-j}$	$-q^{-iJ-jI}q^{-i-j}$	$-q^{-iJ-jI}q^{-i-j}$	$-q^{-iJ-jI}q^{-i-j}$
	0	$\frac{(1-q^{-4i})}{q^{iJ+jI-i-j}}$	0	0
	$\frac{(q^{4i}-1)}{q^{iJ+jI+i+j}}$	0	0	0
	0	0	$\frac{(1-q^{-4j})}{q^{iJ+jI+i-j}}$	0
	0	0	0	$\frac{(1-q^{4j})}{q^{iJ+jI-i+j}}$
	$q^{iJ+jI}q^{-i-j}$	$q^{iJ+jI}q^{-i-j}$	$q^{iJ+jI}q^{-i-j}$	$q^{iJ+jI}q^{-i-j}$
	$q^{iJ+jI}q^{i-j}$	$q^{iJ+jI}q^{i-j}$	$q^{iJ+jI}q^{i-j}$	$q^{iJ+jI}q^{i-j}$
	$q^{iJ+jI}q^{j-i}$	$q^{iJ+jI}q^{j-i}$	$q^{iJ+jI}q^{j-i}$	$q^{iJ+jI}q^{j-i}$
	$-q^{iJ+jI}q^{i+j}$	$-q^{iJ+jI}q^{i+j}$	$-q^{iJ+jI}q^{i+j}$	$-q^{iJ+jI}q^{i+j}$
	$\frac{(1-q^{4j})}{q^{-iJ-jI+i+j}}$	0	0	0
	0	$\frac{(q^{-4j}-1)}{q^{-iJ-jI-i-j}}$	0	0
	0	0	0	$\frac{(1-q^{-4i})}{q^{-iJ-jI-i+j}}$
	0	0	$\frac{(1-q^{4i})}{q^{-iJ-jI+i-j}}$	0

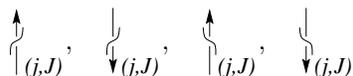
TABLE 2.

	0	1	$-q^{2i}$	1
	0	-1	q^{2i}	-1
	0	$q^{2k} - q^{-2k}$	$q^{2j} - q^{-2j}$	$(q^{-2i} - q^{2i})q^{2j}$
	0	$q^{-2k} - q^{2k}$	$q^{-2j} - q^{2j}$	$(q^{2i} - q^{-2i})q^{2j}$
	$q^{2k} - q^{-2k}$	0	$(q^{2i} - q^{-2i})q^{-2j}$	$q^{2j} - q^{-2j}$
	1	$-q^{2i}$	1	0
	$q^{2j} - q^{-2j}$	$(q^{-2i} - q^{2i})q^{2j}$	0	$q^{2k} - q^{-2k}$
	q^{-2i}	1	0	1
	$(q^{2i} - q^{-2i})q^{-2j}$	$q^{2j} - q^{-2j}$	$q^{2k} - q^{-2k}$	0
	1	0	1	q^{-2i}

	0	$q^{2k} - q^{-2k}$	$q^{2j} - q^{-2j}$	$(q^{-2i} - q^{2i})q^{-2j}$
	0	$q^{-2k} - q^{2k}$	$q^{-2j} - q^{2j}$	$(q^{2i} - q^{-2i})q^{-2j}$
	0	1	$-q^{-2i}$	1
	0	-1	q^{-2i}	-1
	1	0	1	q^{2i}
	$(q^{2i} - q^{-2i})q^{2j}$	$q^{2j} - q^{-2j}$	$q^{2k} - q^{-2k}$	0
	q^{2i}	1	0	1
	$q^{2j} - q^{-2j}$	$(q^{-2i} - q^{2i})q^{-2j}$	0	$q^{2k} - q^{-2k}$
	1	$-q^{-2i}$	1	0
	$q^{2k} - q^{-2k}$	0	$(q^{2i} - q^{-2i})q^{2j}$	$q^{2j} - q^{-2j}$

The next two graphs in Figure 3 are diagrams of neighborhoods of crossing points. At a crossing point where the strings are colored with modules M and N , the Reshetikhin–Turaev theorem suggests the composition of the action of the universal R -matrix on the tensor product $M \otimes N$ and the transposition $M \otimes N \rightarrow N \otimes M$. This composition is calculated in Appendix 1, Subsection 11.6. The (nonzero) Boltzmann weights are shown in Table 1.

Each of the next two graphs in Figure 3 consists of a single string with a half-twist of the framing. We need to associate with them the action of a square root of v^{-1} and v , respectively, in the module corresponding to the color of the string. v acts both in $(j, J)_+$ and $(j, J)_-$ as multiplication by q^{2jJ} ; see Subsection 11.4. Therefore, the Boltzmann weights for



are

$$q^{-jJ}, \quad q^{-jJ}, \quad q^{jJ}, \quad q^{jJ},$$

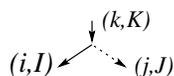
respectively.

The last two graphs in Figure 3 are regular neighborhoods of a trivalent vertex. They admit many orientations, and the Boltzmann weights heavily depend on them. As was mentioned above, the weights are the Clebsch–Gordan coefficients scaled so as to make the state sum invariant with respect to the moves shown in the last line of Figure 4. The Clebsch–Gordan morphisms are calculated in Subsection 11.6.

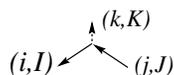
The proof that the Boltzmann weights give rise to a well-defined functor can be either a reference to the Reshetikhin–Turaev theorem [16] and a routine check of invariance with respect to the moves of the bottom row of Figure 4, or a more extensive routine check for the entire set of moves shown in Figure 4. Since either of these proofs is long and straightforward, they are left to the reader.

2.6. Symmetry of Boltzmann weights. The Boltzmann weights of Tables 1 and 2, which involve neither weight components of colors, nor clockwise orientations at strong vertices, satisfy a remarkable symmetry. If we reverse the orientations of two edges in an entry of Table 2 and switch simultaneously the bosons and fermions on them, in the Boltzmann weight each expression of the form q^{2j} , where j is the multiplicity of the arc involved in the change, becomes replaced with $-q^{-2j}$.

For example, since the Boltzmann weight at



is $(q^{2i} - q^{-2i})q^{2j}$, the Boltzmann weight at



is $(q^{2i} - q^{-2i})(-q^{-2j}) = (q^{-2i} - q^{2i})q^{-2j}$.

This symmetry makes it possible to recover the entire Table 2 by its first column. The requirement to change two edges can be replaced by the requirement to preserve the parity of a trivalent vertex.

The Boltzmann weights of Table 1 at points of maxima and minima satisfy a symmetry of the same sort: since the Boltzmann weight at $\curvearrowright_{(j, J)}$ is $-q^{2j}$, the Boltzmann weight at $\curvearrowleft_{(j, J)}$ is q^{-2j} .

2.7. How to eliminate the Planck constant. All Boltzmann weights of Tables 1 and 2 are linear combinations of powers of q . The exponents are linear combinations of multiplicities and products of a multiplicity by a weight. The number q never appears without an exponent of this sort.

The Boltzmann weights of Tables 1 and 2 satisfy equations that express the invariance of the state sums with respect to the moves of Figure 4. These equations are integral polynomial equations in powers of q with exponents $\pm j_i$ and $J_k j_l$, where j_1, \dots, j_n are

the multiplicities and J_1, \dots, J_n are the weights of the colors involved. The equations are satisfied identically for the exponents obeying the admissibility conditions. The latter appear only in the equations corresponding to the moves of two bottom lines in Figure 4. Each of these moves involves a unique triple vertex. Hence, the admissibility conditions allow us to eliminate the multiplicity and weight of one of the edges. After that, the equations are satisfied identically by powers of q with algebraically independent exponents.

In these equations, we replace each q^{j_i} with t_i and restrict ourselves to integral J_i 's. Since the original equations are satisfied identically by j_i 's and J_i 's satisfying the admissibility conditions, the new equations are satisfied identically provided the t_i 's and J_i 's satisfy the identities corresponding to the admissibility conditions.

This suggests eliminating q by replacing its powers with new quantities. The substitutes for q^{j_i} can be taken from a multiplicative group, but must be different from the fourth roots of 1, and must be as independent as the admissibility condition allows. We need not fix the group once and forever, but can choose it each time according to our needs. To keep everything polynomial, the range for weights narrows to integers. We shall follow a more general (but less polynomial) solution: take weights from an Abelian additive subgroup W of B equipped with a pairing $M \times W \rightarrow M : (t, T) \mapsto t^T$ linear in each variable. For the admissibility conditions to be fulfilled, we need to assume that, first, W contains the unity $1 \in B$, and, second, $t^1 = t$ for $t \in M$.

The functors obtained from \mathcal{RT}^1 in this way will be described from scratch below, in the next three sections.

2.8. Categories of colored framed graphs. By a *1-palette* we shall mean a quadruple P consisting of a commutative associative unitary ring B , a subgroup M of the multiplicative group of B , a subgroup W of the additive group B^+ of B , and a (bilinear) pairing $M \times W \rightarrow M : (m, w) \mapsto m^w$. We assume that W contains the unity $1 \in B$ and that the pairing sends $(m, 1)$ to m , i.e., $m^1 = m$ for any $m \in M$.

In this subsection, for each 1-palette P we define a category \mathcal{G}_P^1 . Its objects are finite sequences of triples, each consisting of $t \in M$ with $t^4 \neq 1$, $T \in W$, and a sign. The first and second elements of the triple have the same names as in the case of \mathcal{G}_h^1 : *multiplicity* and *weight*. The triples comprising an object are placed at the corresponding points on the line \mathbb{R}^1 ; cf. Subsection 2.1.

A morphism

$$\{(t_1, T_1, \epsilon_1), \dots, (t_k, T_k, \epsilon_k)\} \rightarrow \{(u_1, U_1, \delta_1), \dots, (u_l, U_l, \delta_l)\}$$

of \mathcal{G}_P^1 is a framed generic graph Γ embedded in $\mathbb{R}^2 \times [0, 1]$ with $\partial\Gamma = \Pi_k \times 0 \cup \Pi_l \times 1 = \partial(\mathbb{R}^2 \times [0, 1]) \cap \Gamma$ and with additional structures to be described below. This graph is considered up to ambient isotopy fixed on $\partial\Gamma$. Its framing at the endpoints is parallel to the x -axis.

The additional structures on Γ all together are called the \mathcal{G}_P^1 -*coloring*. Each of the 1-strata is colored with an orientation and a pair consisting of an element of $\{t \in M \mid t^4 \neq 1\}$ and an element of W . The former is called *multiplicity*, the latter *weight*. The orientations of the edges adjacent to the boundary points are determined by the signs of the points' colors as in \mathcal{G}_h^1 : the edge adjacent to a point with $+$ is oriented at the point upwards (along the standard direction of the z -axis); otherwise it is oriented downwards. The multiplicity and weight of the edge adjacent to an endpoint coincide with the corresponding ingredients of the color of this point. At each of the trivalent vertices the colors of the adjacent edges satisfy the following conditions.

2.8.A (Admissibility conditions). *Let (t_1, T_1) , (t_2, T_2) , and (t_3, T_3) be the multiplicity and weight components of the colors of three edges whose germs are adjacent to one and*

the same vertex, and let $\epsilon_i = -1$ if the i th of the edges is oriented towards the vertex and $\epsilon_i = 1$ otherwise. Then

$$(3) \quad \prod_{i=1}^3 t_i^{\epsilon_i} = 1,$$

$$(4) \quad \sum_{i=1}^3 \epsilon_i T_i = - \prod_{i=1}^3 \epsilon_i.$$

At each strong vertex the graph Γ is oriented. The orientation of Γ at strong vertices is viewed as a part of the \mathcal{G}_P^1 -coloring of Γ , together with the orientations, multiplicities and weights of 1-strata.

Equation (3) means that $\sum t_i e_i$, where the sum is taken over all the edges e_i of the graph with their orientations and t_i is the first component of the color of e_i , is a relative 1-cycle with real coefficients of Γ modulo $\partial\Gamma$, so that it determines an element of $H_1(\Gamma, \partial\Gamma; P)$. This element is called the *multiplicity homology class* of the coloring.

2.9. Functors. Let $P = (B, M, W, M \times W \rightarrow M)$ be a 1-palette (see the preceding subsection). We denote by \mathcal{M}_B the category of finitely generated free B -modules. Our goal is to construct a functor $\mathcal{A}^1 : \mathcal{G}_P^1 \rightarrow \mathcal{M}_B$, which will be called the *Alexander functor*.

Let Λ denote the B -module $B \oplus B$, and let e_0 and e_1 be the natural basis elements of Λ . For an object $\{(t_1, T_1, \epsilon_1), \dots, (t_k, T_k, \epsilon_k)\}$ of \mathcal{G}_P^1 , the image under \mathcal{A}^1 is defined to be the tensor product

$$\Lambda \otimes_B \cdots \otimes_B \Lambda$$

with k factors Λ . This is a free B -module of rank 2^k . It has a canonical basis consisting of $e_{i_1} \otimes \cdots \otimes e_{i_k}$. So, the basis elements may be identified with sequences of zeros and ones placed at the points of Π_k .

The generating morphisms of \mathcal{G}_P^1 are merely the colored generators of \mathcal{G} , as in \mathcal{G}_h^1 . Therefore, the functor \mathcal{A}^1 on morphisms can also be described by presenting Boltzmann weights, as was done for \mathcal{RT}^1 in Tables 1 and 2. The Boltzmann weights for \mathcal{A}^1 are given in Tables 3 and 4. In these tables (as in Tables 1 and 2) a string with e_0 is shown as a dotted arc, and a string with e_1 as a solid arc. The entries of Tables 3 and 4 are obtained from the corresponding entries of Tables 1 and 2: if an edge colored in a table of Subsection 2.5 with (j, J) has a counterpart in the corresponding table of this subsection colored with (t, T) , then q^j is replaced with t and J with T , as was promised in Subsection 2.7. Therefore, the isotopy invariance of the state sums based on Boltzmann weights of Tables 1 and 2 implies the isotopy invariance of the state sums based on Tables 3 and 4.

If $B = \mathbb{Z}[M]$ and $W = \mathbb{Z}$, then the morphism obtained with the help of \mathcal{A}^1 from a colored framed graph is represented by a matrix whose entries are Laurent polynomials in the multiplicities attached to the strings of the graph. Thus, this morphism looks closer to the Alexander polynomial than the morphisms provided by \mathcal{RT}^1 . It is even too nice, since we are not allowed to divide by polynomials, which is quite desirable (cf. (14)). Therefore, we are more interested in the case where M is torsion free and B is the field $Q(M)$ of quotients for $\mathbb{Z}[M]$.

2.10. Still representations of the Hopf algebra. In the definition of \mathcal{RT}^1 given in Subsections 2.3 and 2.5, the action of $U_q \mathfrak{gl}(1|1)$ is completely hidden behind the Boltzmann weights, and at first glance is not needed. However, should we forget it completely, the arguments based on the irreducibility of representations, like those used in the proofs of 5.1.A, 5.1.B, and 5.1.C, would be impossible.

The target of \mathcal{A}^1 is not yet equipped with a structure which would be a counterpart of $U_q \mathfrak{gl}(1|1)$ -representations. In fact, this is impossible. However, $U_q \mathfrak{gl}(1|1)$ contains a

TABLE 3.

	1	$-t^2$	t^{-2}	1
	1	$-t^{-2}$	t^2	1
	t^{-T}	t^{-T}	t^{-T}	t^{-T}
	t^T	t^T	t^T	t^T
	$t^{1-U}u^{1-T}$	$t^{1-U}u^{1-T}$	$t^{1-U}u^{1-T}$	$t^{1-U}u^{1-T}$
	$t^{-1-U}u^{1-T}$	$t^{-1-U}u^{1-T}$	$t^{1-U}u^{-1-T}$	$t^{1-U}u^{-1-T}$
	$t^{1-U}u^{-1-T}$	$t^{1-U}u^{-1-T}$	$t^{-1-U}u^{1-T}$	$t^{-1-U}u^{1-T}$
	$-t^{-1-U}u^{-1-T}$	$-t^{-1-U}u^{-1-T}$	$-t^{-1-U}u^{-1-T}$	$-t^{-1-U}u^{-1-T}$
	0	$\frac{(1-t^{-4})}{t^{-1+U}u^{-1+T}}$	0	0
	$\frac{(t^4-1)}{t^{1+U}u^{1+T}}$	0	0	0
	0	0	$\frac{(1-u^{-4})}{t^{1+U}u^{-1+T}}$	0
	0	0	0	$\frac{(1-u^4)}{t^{-1+U}u^{1+T}}$
	$t^{-1+U}u^{-1+T}$	$t^{-1+U}u^{-1+T}$	$t^{-1+U}u^{-1+T}$	$t^{-1+U}u^{-1+T}$
	$t^{1+U}u^{-1+T}$	$t^{1+U}u^{-1+T}$	$t^{-1+U}u^{1+T}$	$t^{-1+U}u^{1+T}$
	$t^{-1+U}u^{1+T}$	$t^{-1+U}u^{1+T}$	$t^{1+U}u^{-1+T}$	$t^{1+U}u^{-1+T}$
	$-t^{1+U}u^{1+T}$	$-t^{1+U}u^{1+T}$	$-t^{1+U}u^{1+T}$	$-t^{1+U}u^{1+T}$
	$\frac{(1-u^4)}{t^{1-U}u^{1-T}}$	0	0	0
	0	$\frac{(u^{-4}-1)}{t^{-1-U}u^{-1-T}}$	0	0
	0	0	0	$\frac{(1-t^{-4})}{t^{-1-U}u^{1-T}}$
	0	0	$\frac{(1-t^4)}{t^{1-U}u^{-1-T}}$	0

Hopf-subalgebra U^1 (see Subsection 11.7) such that \mathcal{A}^1 can be upgraded (up to a functor) to the category of modules over this subalgebra.

Now we can redefine the Alexander functor $\mathcal{A}^1 : \mathcal{G}_p^1 \rightarrow \mathcal{M}_B$ introduced in Subsection 2.9 as a functor from the same category \mathcal{G}_p^1 , but targeted at the category of finite-dimensional modules over $U^1 \otimes_{\mathbb{Z}} B$, that assigns to an object $\{(t_1, T_1, \epsilon_1), \dots, (t_k, T_k, \epsilon_k)\}$

TABLE 4.

	$(u,U) \begin{array}{c} \diagup \quad \diagdown \\ \vdots \\ (t,T) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \vdots \\ (v,V) \end{array}$	$(u,U) \begin{array}{c} \diagdown \quad \diagup \\ \vdots \\ (t,T) \end{array} \begin{array}{c} \diagup \quad \diagdown \\ \vdots \\ (v,V) \end{array}$	$(u,U) \begin{array}{c} \diagdown \quad \diagup \\ \vdots \\ (t,T) \end{array} \begin{array}{c} \diagup \quad \diagdown \\ \vdots \\ (v,V) \end{array}$	$(u,U) \begin{array}{c} \diagup \quad \diagdown \\ \vdots \\ (t,T) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \vdots \\ (v,V) \end{array}$
	0	1	$-u^2$	1
	0	-1	u^2	-1
	0	$t^2 - t^{-2}$	$v^2 - v^{-2}$	$(u^{-2} - u^2)v^2$
	0	$t^{-2} - t^2$	$v^{-2} - v^2$	$(u^2 - u^{-2})v^2$
	$t^2 - t^{-2}$	0	$(u^2 - u^{-2})v^{-2}$	$v^2 - v^{-2}$
	1	$-u^2$	1	0
	$v^2 - v^{-2}$	$(u^{-2} - u^2)v^2$	0	$t^2 - t^{-2}$
	u^{-2}	1	0	1
	$(u^2 - u^{-2})v^{-2}$	$v^2 - v^{-2}$	$t^2 - t^{-2}$	0
	1	0	1	u^{-2}

	$(u,U) \begin{array}{c} \diagup \quad \diagdown \\ \vdots \\ (t,T) \\ \diagdown \quad \diagup \\ \vdots \\ (v,V) \end{array}$	$(u,U) \begin{array}{c} \diagdown \quad \diagup \\ \vdots \\ (t,T) \\ \diagup \quad \diagdown \\ \vdots \\ (v,V) \end{array}$	$(u,U) \begin{array}{c} \diagdown \quad \diagup \\ \vdots \\ (t,T) \\ \diagup \quad \diagdown \\ \vdots \\ (v,V) \end{array}$	$(u,U) \begin{array}{c} \diagup \quad \diagdown \\ \vdots \\ (t,T) \\ \diagdown \quad \diagup \\ \vdots \\ (v,V) \end{array}$
	0	$t^2 - t^{-2}$	$v^2 - v^{-2}$	$(u^{-2} - u^2)v^{-2}$
	0	$t^{-2} - t^2$	$v^{-2} - v^2$	$(u^2 - u^{-2})v^{-2}$
	0	1	$-u^{-2}$	1
	0	-1	u^{-2}	-1
	1	0	1	u^2
	$(u^2 - u^{-2})v^2$	$v^2 - v^{-2}$	$t^2 - t^{-2}$	0
	u^2	1	0	1
	$v^2 - v^{-2}$	$(u^{-2} - u^2)v^{-2}$	0	$t^2 - t^{-2}$
	1	$-u^{-2}$	1	0
	$t^2 - t^{-2}$	0	$(u^2 - u^{-2})v^2$	$v^2 - v^{-2}$

of \mathcal{G}_P^1 the tensor product

$$U(t_1, T_1)_{\epsilon_1} \otimes_B \cdots \otimes_B U(t_k, T_k)_{\epsilon_k};$$

see Subsection 11.8. On morphisms, the functor is defined by the same Boltzmann weights as before.

§3. RESHETIKHIN–TURAEV FUNCTOR BASED ON $\mathfrak{sl}(2)$

In this section we construct a functor \mathcal{RT}^2 similar to the functor \mathcal{RT}^1 defined in §2. It is a modified special case of a functor introduced by Deguchi and Akutsu [4, §5]. The functor constructed in [4] depends on a natural number N . Our modification is related to the case of $N = 2$.

The choice of scaling of Clebsch–Gordan morphisms used by Deguchi and Akutsu in [4] was based on orthogonality relations. Its disadvantage is that at each vertex one of the adjacent edges is distinguished and must be directed upwards.

Here I use another scaling. It does not satisfy the orthogonality relations, but eliminates the choice of an edge at each trivalent vertex. Together with a different choice of basis vectors in the representations, this makes formulas much simpler than in [4].

3.1. Category of colored framed graphs. The source category for \mathcal{RT}^2 is denoted by \mathcal{G}^2 and is defined as follows.

An object of \mathcal{G}^2 is a class of finite sequences of pairs. Each of the pairs consists of a complex number and a sign. The complex numbers are not allowed to be odd integers. In a pair, the complex number may be multiplied by -1 with a simultaneous change of the sign component of the pair. An object of \mathcal{G}^2 consists of all sequences that can be obtained from one another by a sequence of operations like this.

A morphism

$$\{(A_1, \epsilon_1), \dots, (A_k, \epsilon_k)\} \rightarrow \{(B_1, \delta_1), \dots, (B_l, \delta_l)\}$$

of \mathcal{G}^2 is a morphism $\Gamma : k \rightarrow l$ of category \mathcal{G} (see Subsection 1.7) equipped with the additional structure described below.

Each 1-stratum is colored with an orientation and a complex number. The latter, called a *weight*, is prohibited to be an odd integer. The orientation of a 1-stratum may be changed with simultaneous multiplication of the weight by -1 . In other words, this is an element C of $H_1(\Gamma, \{\text{vertices}\}; \mathbb{C})$, that is a 1-chain of Γ with complex coefficients. This chain must satisfy the following condition.

3.1.A (Admissibility condition). *The boundary ∂C of C must involve each trivalent vertex of Γ with coefficient ± 1 , each boundary vertex $(i, 0)$, $i = 1, \dots, k$, with coefficient $-\epsilon_i A_i$, and each boundary vertex $(i, 1)$, $i = 1, \dots, l$, with coefficient $\delta_i B_i$, respectively.*

To the properties of the weight chain C , we should add that each 1-stratum is involved in C with a coefficient that cannot be an odd integer.

An internal vertex is called a *source* if it is involved in ∂C with coefficient -1 ; otherwise it is called a *sink*. In figures a sink is shown by a small light disk.

3.2. Constructing the Reshetikhin–Turaev functor. Our goal is to construct a functor \mathcal{RT}^2 from \mathcal{G}^2 to the category of finite-dimensional representations of the Hopf algebra $U_{\sqrt{-1}}\mathfrak{sl}(2)$. For a brief summary on this quantum algebra and its representations to be used below, see Appendix 2. The image of an object $\{(A_1, \epsilon_1), \dots, (A_k, \epsilon_k)\}$ of \mathcal{G}^2 is defined as the tensor product $I(\epsilon_1 A_1) \otimes \dots \otimes I(\epsilon_k A_k)$; see Subsection 12.4.

Each of the factors $I(\epsilon A)$ is a 2-dimensional vector space with a canonical basis e_0, e_1 . Thus, $I(\epsilon_1 A_1) \otimes \dots \otimes I(\epsilon_k A_k)$ has a canonical basis consisting of $e_{i_1} \otimes \dots \otimes e_{i_k}$. The basis vectors of $I(\epsilon_1 A_1) \otimes \dots \otimes I(\epsilon_k A_k)$ may be identified with sequences of zeros and ones. These zeros and ones are associated with orientation of the corresponding points.

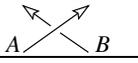
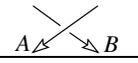
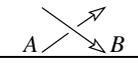
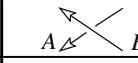
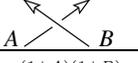
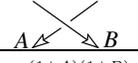
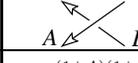
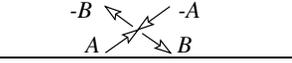
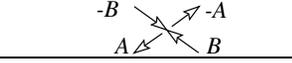
For presentation of the images of morphisms via Boltzmann weights, we need to consider diagrams of graphs that represent the morphisms of \mathcal{G}^2 , and also a coloring of their 1-strata by basis vectors of the representations associated with the colors of the 1-strata. It is convenient to identify these latter colors with orientations of 1-strata: if a 1-stratum is oriented upwards, it is colored with e_0 ; otherwise it is colored with e_1 . The same

orientations can be used for presentation of the (underlying) \mathcal{G}^2 -coloring, i.e., a coloring with pairs (orientation, weight). Recall that the orientation in this pair can be reversed with simultaneous multiplication of the weight by -1 . Therefore, any orientation may be used to specify a \mathcal{G}^2 -color, and it is natural to use the same orientation for two purposes: describing both the representation attached to the string and a basis vector chosen in the representation. The weight is shown at the string. The same picture can be used also for a \mathcal{G}^2 -colored graph. To emphasize the double usage of orientations, we employ light arrowheads.

Thus an edge with upward light arrowhead and a number $A \in \mathbb{C}$, and an edge with downward light arrowhead and the number $-A$ denote a string colored with the same primary color (upward orientation, A), which is associated with the same representation $I(A)$. The secondary colors are the vectors e_0 and e_1 , respectively, in the first and the second case. The same edge with the same orientations and multiplicities, but shown with a dark arrowhead is colored with the same primary color (associated with $I(A)$), but with no secondary color, i.e., with no basis vector of $I(A)$ specified.

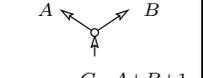
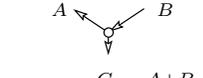
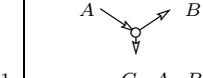
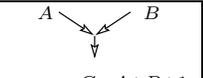
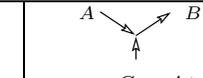
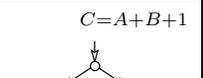
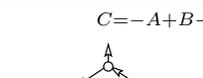
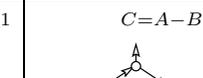
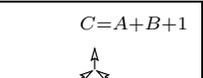
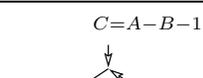
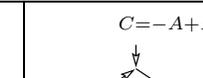
The image of a morphism is defined via the Boltzmann weights shown in Tables 5 and 6. In these tables and below, i denotes $\sqrt{-1}$, while i^x denotes $\exp(\frac{x\pi i}{2})$. The Boltzmann weights are found on the basis of the Reshetikhin–Turaev theorem [16], much as the Boltzmann weights of Tables 1 and 2. The relevant information about the Hopf algebra $U_i \mathfrak{sl}(2)$ and its representations is placed in Appendix 2. It can be checked directly that the Boltzmann weights of Tables 5 and 6 determine a functor from \mathcal{G}^2 to the category of modules over $U_i \mathfrak{sl}(2)$.

TABLE 5.

			
1	i^{-A-1}	1	i^{A+1}
			
$i^{\frac{(1+A)(1+B)}{2}}$	$i^{\frac{(1+A)(1+B)}{2}}$	$i^{\frac{(1+A)(1+B)}{2}}$	$i^{\frac{(1+A)(1+B)}{2}}$
			
$i^{\frac{-(1+A)(1+B)}{2}}$	$i^{\frac{-(1+A)(1+B)}{2}}$	$i^{\frac{-(1+A)(1+B)}{2}}$	$i^{\frac{-(1+A)(1+B)}{2}}$
			
$i^{\frac{-(1-A)(1-B)}{2}}(i^{1+A} + i^{1-A})$		$i^{\frac{(1+A)(1+B)}{2}}(i^{-1+B} + i^{-1-B})$	
			
$i^{\frac{A^2-1}{4}}$	$i^{\frac{A^2-1}{4}}$	$i^{\frac{1-A^2}{4}}$	$i^{\frac{1-A^2}{4}}$

3.3. Homological meaning of colorings. Presentation of a secondary coloring of a generic graph by orientations of 1-strata suggests a homology interpretation of the secondary coloring as an integral 1-chain in which any 1-stratum has coefficient 1 if it is taken with the orientation specified by the light arrow. We denote this chain by Z .

TABLE 6.

sink vertex	 $C = A + B + 1$	 $C = -A + B - 1$	 $C = A - B - 1$
	$i^{1+C} + i^{1-C}$	$i^{1+B} + i^{1-B}$	$i^{A-B} + i^{-A-B}$
source vertex	 $C = A + B + 1$	 $C = A - B - 1$	 $C = -A + B - 1$
	1	1	i^{-1-A}
sink vertex	$C = A + B + 1$  $C = A + B + 1$	$C = -A + B - 1$  $C = -A + B - 1$	$C = A - B - 1$  $C = A - B - 1$
	$i^{1+C} + i^{1-C}$	$i^{1+B} + i^{1-B}$	$-i^{B+A} - i^{B-A}$
source vertex	$C = A + B + 1$  $C = A + B + 1$	$C = A - B - 1$  $C = A - B - 1$	$C = -A + B - 1$  $C = -A + B - 1$
	1	1	i^{1+A}

An easy analysis of Table 6 shows that Z has a curious property: it completes the weight chain C of the corresponding primary coloring to a cycle modulo the boundary of the underlying graph. In other words, $\partial(C + Z)$ involves no internal vertices.

This suggests the following: to a secondary color (i.e., a basis vector of $I(A)$) we can relate the coefficient with which it appears in $C + Z$. For a basis vector e_i of the representation $I(A)$, the complex number $a = (-1)^i A + 1$ is called the *charge*. A secondary \mathcal{G}^2 -coloring of a generic graph Γ can be presented by a 1-chain $c = C + Z$ with complex coefficients on Γ : each 1-stratum of Γ is equipped with the orientation associated with the secondary color and the charge. Hence, a secondary coloring determines an element of $H_1(\Gamma, \partial\Gamma; \mathbb{C})$. However, the secondary coloring is not determined by this homology class: the orientations of the 1-strata should be added. These orientations are encoded in Z .

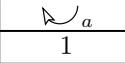
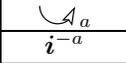
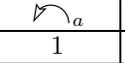
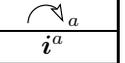
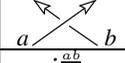
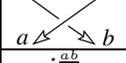
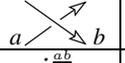
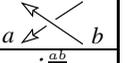
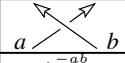
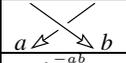
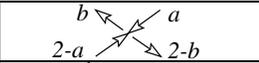
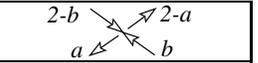
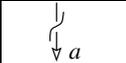
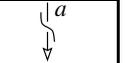
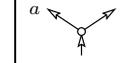
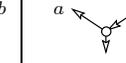
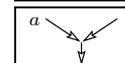
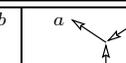
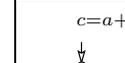
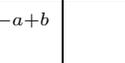
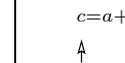
Replacing the weights by charges makes Tables 5 and 6 simpler. See Table 7. The difference between the sink and source vertices almost vanishes. Most formulas expressing the Boltzmann weights become simpler. This can be explained by the algebraic nature of the charge: it is an eigenvalue of the operator corresponding to $K \in V_{\sqrt{-1}} \mathfrak{sl}(2)$.

The homology class of a secondary coloring reduced modulo 2 depends only on the primary coloring. It belongs to $H_1(\Gamma; \mathbb{C}/2\mathbb{Z})$ and can be described directly in terms of the primary coloring by adding 1 to each weight and reducing modulo 2.

It may happen that a diagram of a generic graph with a primary \mathcal{G}^2 -coloring admits no secondary coloring. In Figure 5 such a closed generic graph with a primary \mathcal{G}^2 -coloring is shown on the left-hand side. This property depends on the embedding of the graph. On the right-hand side of Figure 5, the same graph, but embedded in a different way, is shown with a secondary coloring (to keep evident the relationship with the primary coloring, the secondary coloring is presented not by the charges, but by the weights).

Of course, the absence of secondary colorings implies that \mathcal{RT}^2 maps the (primarily) colored framed graph to the zero morphism.

TABLE 7.

			
1	i^{-a}	1	i^a
			
$i^{\frac{ab}{2}}$	$i^{\frac{ab}{2}}$	$i^{\frac{ab}{2}}$	$i^{\frac{ab}{2}}$
			
$i^{-\frac{ab}{2}}$	$i^{-\frac{ab}{2}}$	$i^{-\frac{ab}{2}}$	$i^{-\frac{ab}{2}}$
			
$i^{-\frac{ab}{2}}(i^a - i^{-a})$		$i^{\frac{ab}{2}}(i^{-b} - i^b)$	
			
$i^{\frac{a(a-2)}{4}}$	$i^{\frac{a(a-2)}{4}}$	$i^{\frac{a(2-a)}{4}}$	$i^{\frac{a(2-a)}{4}}$
			
$i^c - i^{-c}$	$i^b - i^{-b}$	$i^{-b}(i^a - i^{-a})$	
			
1	1	i^{-a}	
			
$i^c - i^{-c}$	$i^b - i^{-b}$	$i^b(i^a - i^{-a})$	
			
1	1	i^a	

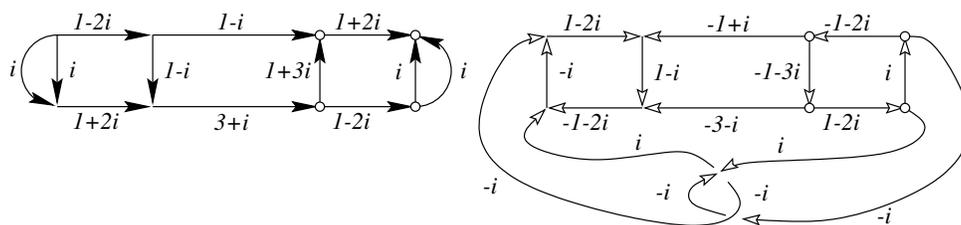


FIGURE 5.

3.4. Algebraization. In this subsection we do with \mathcal{RT}^2 what was done with \mathcal{RT}^1 in Subsections 2.7 – 2.10: we eliminate formal power series. As in the case of \mathcal{RT}^1 , the most obvious obstruction for this is the presence of products in exponents. In the case of \mathcal{RT}^1 , all the products in exponents are products of a multiplicity and a weight. In formulas this is said as follows: the Boltzmann weights involve terms of the form q^{jI} , where j is the multiplicity component of a color and I is the weight component of a (possibly different) color.

In Table 5, the quadratic part of the exponents is made of values of a single parameter (weight), which are complex numbers different from odd integers. Thus, the solution that is used in the case of \mathcal{RT}^1 cannot be applied here literally. In order to rewrite the Boltzmann weights of Tables 5 and 6 in a formal algebraic way without exponential function, the following approach can be used.

The complex parameters are substituted with parameters taken from a commutative ring W , while the Boltzmann weights are taken from a larger ring B . Besides the inclusion, W and B are related by a group homomorphism of the additive group W^+ of W to the multiplicative group B^\times of B .

In the original setup of \mathcal{RT}^2 , both W and B are \mathbb{C} , and the homomorphism $W^+ \rightarrow B^\times$ is defined by $A \mapsto i^A$.

In our generalization, we use the same exponential notation for the homomorphism $W^+ \rightarrow B^\times$. This is justified by the condition that $4 \in W$ must be mapped by the homomorphism to $1 \in B$, i.e., $i^4 = 1$. This condition implies that B contains a root of unity of degree 4, namely, this is i^1 . Indeed, $(i^1)^4 = i^{(4 \cdot 1)} = i^4 = 1$. This element of B is denoted by i .

By a *2-palette* we shall mean a triple P consisting of W , B , and the map $W^+ \rightarrow B^\times : A \rightarrow i^A$ as above:

- B is a commutative ring with unity;
- W is a subring containing the unity;
- $W^+ \rightarrow B^\times : A \rightarrow i^A$ is a homomorphism with $i^4 = 1$.

The category \mathcal{G}_P^2 of colored framed graphs replacing \mathcal{G}^2 looks like this. Every object is a class of finite sequences of pairs. Each pair consists of an element of W and a sign. The former is called the *weight* of the color. It is not allowed to belong to $\{x \in W \mid i^{2x+2} = 1\}$. In a pair, the weight may be multiplied by -1 with a simultaneous change of the sign. An object is a class of all sequences that can be obtained from each other by operations like this applied to the pairs comprising the sequence.

A morphism

$$\{(A_1, \epsilon_1), \dots, (A_k, \epsilon_k)\} \rightarrow \{(B_1, \delta_1), \dots, (B_l, \delta_l)\}$$

of \mathcal{G}_P^2 is a morphism $\Gamma : k \rightarrow l$ of category \mathcal{G} equipped with an additional structure called a \mathcal{G}_P^2 -coloring.

Each 1-stratum is colored with an orientation and with an element of W called the *weight* of the color. It must not belong to $\{x \in W \mid i^{2x+2} = 1\}$. The orientation of a 1-stratum may be changed with simultaneous multiplication of the weight by -1 . In other words, this is an element C of $H_1(\Gamma, \{\text{vertices}\}; W)$, i.e., a 1-chain of Γ with coefficients in W . This chain must satisfy the following condition.

3.4.A (Admissibility condition). *The boundary ∂C must involve each internal (i.e., trivalent) vertex of Γ with coefficient ± 1 , each boundary vertex $(i, 0)$, $i = 1, \dots, k$, with coefficient $-\epsilon_i A_i$, and each boundary vertex $(i, 1)$, $i = 1, \dots, l$, with coefficient $\delta_i B_i$, respectively.*

The generalization \mathcal{A}^2 of \mathcal{RT}^2 acts from \mathcal{G}_P^2 to the category of finitely generated free modules over B . To a class $\{(\pm A_1, \pm), \dots, (\pm A_k, \pm)\}$ of sequences, it assigns the vector

space of dimension 2^k generated by all the representatives of the class. This is the tensor product (over B) of k copies of a free B -module of rank 2.

This is not merely a B -module. It admits a natural action of a Hopf algebra U^2 , which can be defined precisely as for $U_{\mathbf{i}}\mathfrak{sl}(2)$ (see Subsection 12.3), but without the relation $K = \mathbf{i}^H$. Namely, U^2 is the Hopf algebra generated by H, K, X , and Y satisfying the relations

$$(5) \quad \begin{aligned} [Y, X] &= K - K^{-1}, \\ [H, X] &= -2X, \quad [H, Y] = 2Y, \\ KX &= -XK, \quad KY = -YK, \end{aligned}$$

with the coproduct defined by

$$(6) \quad \begin{aligned} \Delta(H) &= H \otimes 1 + 1 \otimes H, \\ \Delta(K) &= K \otimes K, \\ \Delta(K^{-1}) &= K^{-1} \otimes K^{-1}, \\ \Delta(X) &= X \otimes K^{-1} + 1 \otimes X, \\ \Delta(Y) &= Y \otimes 1 + K \otimes Y, \end{aligned}$$

with the counit defined by

$$(7) \quad \epsilon : U^2 \rightarrow \mathbb{C}[[\hbar]] : \quad \begin{aligned} \epsilon(K) &= \epsilon(K^{-1}) = 1, \\ \epsilon(H) &= \epsilon(X) = \epsilon(Y) = 0, \end{aligned}$$

and with the antipode

$$(8) \quad \begin{aligned} s(H) &= -H, \\ s(K) &= K^{-1}, \quad s(K^{-1}) = K, \\ s(X) &= -XK \quad (= KX), \\ s(Y) &= -K^{-1}Y \quad (= YK^{-1}). \end{aligned}$$

In the case of the free B -module of rank 2 corresponding to $\{(\pm A, \pm)\}$, the generators H, K, X , and Y act by the following formulas:

$$(9) \quad \begin{aligned} X : (A, +) &\mapsto (\mathbf{i}^{A+1} + \mathbf{i}^{-A+1})(-A, -), & (-A, -) &\mapsto 0, \\ Y : (A, +) &\mapsto 0, & (-A, -) &\mapsto (A, +), \\ K : (A, +) &\mapsto \mathbf{i}^{A+1}(A, +), & (-A, -) &\mapsto \mathbf{i}^{A-1}(-A, -), \\ H : (A, +) &\mapsto (A+1)(A, +), & (-A, -) &\mapsto (A-1)(-A, -). \end{aligned}$$

We denote this $U^2 \otimes_{\mathbb{Z}} B$ -module by $I_B(A)$.

In the case of a longer sequence $\{(A_1, \epsilon_1), \dots, (A_k, \epsilon_k)\}$, the corresponding B -module is regarded as the tensor product of the sequence of $U^2 \otimes_{\mathbb{Z}} B$ -modules of rank 2 corresponding to the elements of the sequence, and the action is defined via the coproduct of U^2 .

A morphism Γ is mapped to the morphism of the corresponding modules that is determined by the matrix whose entries are constructed as follows. An entry of the matrix corresponds to the choice of signs at the boundary points of Γ . Consider all the extensions of these signs to orientations of 1-strata of a diagram of Γ . As in the case of \mathcal{RT}^2 , the extensions are called secondary colorings. In a diagram, the orientations comprising a secondary coloring are shown by light arrowheads, as in the case of \mathcal{RT}^2 . For each of the extensions, take the product of the Boltzmann weights at the vertices and the extremal points of the height function in the diagram, in accordance with Tables 5 and 6, and sum up such products over all extensions. If a local picture at a vertex

of the diagram does not appear in the tables, assign 0 to the vertex, i.e., disregard the extension.

Of course, \mathcal{RT}^2 is a special case of \mathcal{A}^2 with $W = B = \mathbb{C}$.

§4. RELATIONSHIP BETWEEN THE RESHETIKHIN–TURAEV FUNCTORS

Although at first glance \mathcal{G}_h^1 and \mathcal{G}^2 seem to be quite different, a more careful analysis shows deep similarities and relationships between them.

4.1. Things to recolor. There is a subcategory $S\mathcal{G}^1$ of $\mathcal{G}_{\pi i/2}^1$ and a natural transformation of $\mathcal{RT}^1|_{S\mathcal{G}^1}$ to \mathcal{RT}^2 .

An object $\{(j_1, J_1, \epsilon_1), \dots, (j_k, J_k, \epsilon_k)\}$ of $\mathcal{G}_{\pi i/2}^1$ belongs to $S\mathcal{G}^1$ if $j_r + J_r = 1$ for $r = 1, \dots, k$. A morphism Γ of $\mathcal{G}_{\pi i/2}^1$ belongs to $S\mathcal{G}^1$ if for each 1-stratum of Γ the sum of its multiplicity and weight is 1.

This choice is motivated as follows. Between the \mathcal{G}^1 - and \mathcal{G}^2 -colorings, we want to have a correspondence that would preserve the Boltzmann weights. Comparison of the first lines of Tables 1 and 5 suggests assuming $q = i$ and $-2j = -A - 1$, i.e., $j = \frac{A+1}{2}$. The Boltzmann weights at half-twists suggest $-jJ = \frac{A^2-1}{4}$, which together with $j = \frac{A+1}{2}$ implies $J = \frac{1-A}{2}$ and $j + J = 1$. Of course, this does not pretend to be a proof, but rather a strong indication in favor of the choice made above.

4.2. Recoloring. We are going to define a functor $\Phi : S\mathcal{G}^1 \rightarrow \mathcal{G}^2$. An object

$$\{(j_1, J_1, \epsilon_1), \dots, (j_k, J_k, \epsilon_k)\}$$

of $S\mathcal{G}^1$ is turned by Φ to $\{(2j_1 - 1, \epsilon_1), \dots, (2j_k - 1, \epsilon_k)\}$. To a morphism Γ of $S\mathcal{G}^1$ it assigns the isotopy class of the same oriented framed graphs with the weights on 1-strata obtained from the original multiplicities by the same formula: if the multiplicity (in the \mathcal{G}^1 -coloring) is j , the weight (in the \mathcal{G}^2 -coloring) is $2j - 1$.

4.2.A (Consistency in admissibility conditions). *The recoloring converts any morphism of $S\mathcal{G}^1$ to a morphism of \mathcal{G}^2 . In other words, Φ turns a coloring of a generic graph satisfying admissibility conditions 2.2.A and the condition defining $S\mathcal{G}^1$ (the sum of multiplicity and weight for each 1-stratum is 1) into a coloring satisfying Admissibility condition 3.1.A.*

4.2.B (Lemma). *Any morphism of $S\mathcal{G}^1$ has no strong vertices.*

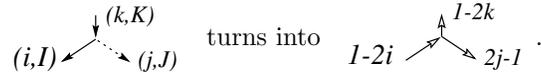
Proof. Recall that a vertex is said to be strong if the adjacent edges are oriented either all towards it or all in the opposite direction. In the notation of Admissibility conditions 2.2.A, this means that $\epsilon_1 = \epsilon_2 = \epsilon_3$. By (1) and (2), at a strong vertex we have $j_1 + j_2 + j_3 = 0$ and $J_1 + J_2 + J_3 = 1$. Hence, $j_1 + j_2 + j_3 + J_1 + J_2 + J_3 = 1$. On the other hand, $j_r + J_r = 1$ in $S\mathcal{G}^1$, whence $j_1 + j_2 + j_3 + J_1 + J_2 + J_3 = 3$. □

Proof of 4.2.A. By the lemma above, at each internal vertex at least one of the adjacent edges is directed towards the vertex and at least one is directed outwards. Therefore, $j_1 + j_2 - j_3 = 0$, where j_1, j_2 , and j_3 are the multiplicities of the edges adjacent to a vertex and enumerated appropriately. Then $|(2j_1 - 1) + (2j_2 - 1) - (2j_3 - 1)| = 1$, and, consequently, the vertex is involved with coefficient ± 1 into the boundary of the weight chain of the coloring provided by Φ . □

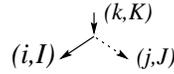
4.3. Invariance of Boltzmann weights under recoloring. The categories \mathcal{G}_h^1 , SG^1 , and \mathcal{G}^2 can be enriched by incorporating secondary colorings into objects and morphisms. The enriched categories are denoted by \mathcal{G}_h^{1b} , SG^{1b} , and \mathcal{G}^{2b} , respectively.

4.3.A (Consistency in Boltzmann weights). *The functor Φ can be enhanced to a functor Φ^b from SG^{1b} to \mathcal{G}^{2b} in such a way that the Boltzmann weight of \mathcal{RT}^1 be equal to the Boltzmann weight of \mathcal{RT}^2 at the same point after the application of Φ^b .*

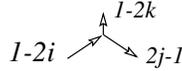
Proof. The construction looks like this: the dotted arcs (i.e., the arcs colored with bosons) get a light arrowhead with the same orientation, while the solid arcs (colored with fermions) change orientation and the \mathcal{G}^2 -weight. For example,



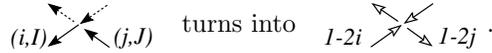
Notice that the \mathcal{RT}^1 -Boltzmann weight at



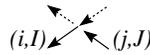
is $(q^{2i} - q^{-2i})q^{2j}$, while the \mathcal{RT}^2 -Boltzmann weight at



is $-i^{2j-1}(i^{1-2i} + i^{2i-1}) = i^{2j}(i^{2i} - i^{-2i})$. Thus, after the transformation, the \mathcal{RT}^2 -Boltzmann weight coincides with the \mathcal{RT}^1 -Boltzmann weight of the original graph. It is easy to check this for each of the Boltzmann weights of Tables 1 and 2, except for the weights at strong vertices, which do not appear in SG^1 by Lemma 4.2.A. Here is another example. This is one of the most complicated entries in Table 1:



(Here we present a secondary \mathcal{G}^{2b} -coloring by weights.) The \mathcal{RT}^1 -Boltzmann weight at



is

$$q^{iJ+jI+i-j}(1 - q^{-4i}) = q^{i(1-j)+j(1-i)+i-j}(1 - q^{-4i}) = q^{-2ij}(q^{2i} - q^{-2i}),$$

while the \mathcal{RT}^2 -Boltzmann weight at



is

$$\begin{aligned} & i^{\frac{1}{2}(-(-1-2i)(1-2j)+(1-2i)+(1-2j)+1)}(i^{1-2i} + i^{2i-1}) \\ &= i^{\frac{1}{2}(-4ij+2)}i(i^{-2i} - i^{2i}) = i^{-2ij}(i^{2i} - i^{-2i}). \end{aligned}$$

The other entries of Tables 1 and 2 can be treated similarly, but the number of necessary checkings can be reduced if one takes into account the behavior of the Boltzmann weights under the moves of Figure 4. \square

4.4. Functorial reformulation. Let $\text{Vect}(\mathbb{C})$ denote the category of complex vector spaces, and let $(\mathcal{RT}^1)^\sharp$ and $(\mathcal{RT}^2)^\sharp$ be the compositions of \mathcal{RT}^1 and \mathcal{RT}^2 , respectively, with the appropriate forgetting functor to $\text{Vect}(\mathbb{C})$.

4.4.A (Corollary to 4.3.A). *The functor Φ that maps the subcategory SG^1 of $\mathcal{G}_{\pi i/2}^1$ to \mathcal{G}^2 comprises, together with the identity functor of the category $\text{Vect}(\mathbb{C})$, a natural transformation of $(\mathcal{RT}^1)^\sharp|_{SG^1}$ to $(\mathcal{RT}^2)^\sharp$. In other words,*

$$\begin{array}{ccc} SG^1 & \xrightarrow{(\mathcal{RT}^1)^\sharp|_{SG^1}} & \text{Vect}(\mathbb{C}) \\ \downarrow \Phi & & \parallel \\ \mathcal{G}^2 & \xrightarrow{(\mathcal{RT}^2)^\sharp} & \text{Vect}(\mathbb{C}) \end{array}$$

is a commutative diagram. In particular, for any morphism Γ of SG^1 we have

$$(\mathcal{RT}^2)^\sharp \Phi(\Gamma) = (\mathcal{RT}^1)^\sharp(\Gamma). \quad \square$$

Probably, this is a manifestation of a deeper relationship between $U_i \mathfrak{gl}(1|1)$ and $U_i \mathfrak{sl}(2)$, and the forgetting functors to $\text{Vect}(\mathbb{C})$ can be eliminated; cf. [10]. However, we do not elaborate upon this, since it does not seem to have immediate topological consequences.

4.5. Generalization. In this subsection we generalize the results of Subsections 4.1–4.4, replacing the Reshetikhin–Turaev functors \mathcal{RT}^1 and \mathcal{RT}^2 by their generalizations \mathcal{A}^1 and \mathcal{A}^2 .

Let P be a 1-palette (see Subsection 2.8, where the \mathcal{G}_P^1 -colored graphs were also defined). Recall that P consists of a commutative ring B , a subgroup M of the multiplicative group of B , a subgroup W of the additive group B^+ of B , and a (bilinear) pairing $M \times W \rightarrow M : (m, w) \mapsto m^w$. We assume that W contains the unity 1 of B and that the pairing takes the value m at $(m, 1)$.

Furthermore, suppose that

- W is a subring of B ;
- the pairing $M \times W \rightarrow M : (m, w) \mapsto m^w$ is an action of the multiplicative monoid of W ; and
- M contains an element i such that $i^4 = 1$, and $i^2 = -1 \neq 1$.

The second assumption means that $(m^u)^v = m^{uv}$ for any $u, v \in W$.

We denote by Q the triple consisting of the rings B and W and the map $W \rightarrow B : w \mapsto i^w$. The latter is a homomorphism of the additive group of W to the multiplicative group of B , because $M \times W \rightarrow M : (m, w) \mapsto m^w$ is bilinear. Consequently, Q is a 2-palette (see Subsection 3.4), and we can consider the category \mathcal{G}_Q^2 of \mathcal{G}_Q^2 -colored graphs.

This construction of Q can be reversed. Given any 2-palette $Q = (B, W, W^+ \rightarrow B^\times : A \mapsto i^A)$ (i.e., Q consists of a commutative unitary ring B , its unitary subring W , and a homomorphism $W^+ \rightarrow B^\times : A \mapsto i^A$ such that $i^4 = 1$), one can construct a quadruple P consisting of

- the same ring B ;
- the image M of the homomorphism $W^+ \rightarrow B^\times$ determined by the formula $A \mapsto i^A$;
- the subring $W \subset B$; and
- the pairing $M \times W \rightarrow M : (i^A, B) \mapsto i^{AB}$.

It is clear that this P is a 1-palette and satisfies all the conditions imposed on P at the beginning of this subsection. Moreover, applying the construction described above to this P , we arrive at the initial 2-palette Q .

Let P be as at the beginning of Subsection 4.5. We define a subcategory $S_i\mathcal{G}_P^1$ of \mathcal{G}_P^1 . An object $\{(t_1, T_1, \epsilon_1), \dots, (t_k, T_k, \epsilon_k)\}$ belongs to $S_i\mathcal{G}_P^1$ if $i^{1-T_r} = t_r$ for $r = 1, \dots, k$. A morphism Γ of \mathcal{G}_P^1 belongs to $S_i\mathcal{G}_P^1$ if for each 1-stratum of Γ its multiplicity t and weight T are related in the same way: $i^{1-T} = t$.

Now we define a functor $\Phi_P : S_i\mathcal{G}_P^1 \rightarrow \mathcal{G}_Q^2$. It turns an object $\{(t_1, T_1, \epsilon_1), \dots, (t_k, T_k, \epsilon_k)\}$ of $S_i\mathcal{G}_P^1$ to $\{(1 - 2T_1, \epsilon_1), \dots, (1 - 2T_k, \epsilon_k)\}$. To a morphism Γ of $S_i\mathcal{G}_P^1$ it assigns the isotopy class of the same oriented framed graphs with charges on the 1-strata obtained from the original weights by the same formula: if the original weight (in the \mathcal{G}_P^1 -coloring) is T , then the new weight (in the \mathcal{G}_Q^2 -coloring) is $1 - 2T$.

Obviously, the functor $\Phi_P : S_i\mathcal{G}_P^1 \rightarrow \mathcal{G}_Q^2$ generalizes the functor $\Phi : S\mathcal{G}^1 \rightarrow \mathcal{G}^2$ defined in Subsection 4.2: if $B = \mathbb{C}$, $W = \mathbb{Z}$, $M = \mathbb{C} \setminus 0$, and $M \times W \rightarrow M$ is defined by $(m, w) \mapsto m^w$, then $S_i\mathcal{G}_P^1 = S\mathcal{G}^1$, $\mathcal{G}_Q^2 = \mathcal{G}^2$, and $\Phi_P = \Phi$.

4.5.A (Consistency in admissibility conditions). *Φ_P transforms a coloring of a generic graph satisfying Admissibility conditions 2.8.A and the conditions defining $S_i\mathcal{G}_P^1$ into a coloring satisfying Admissibility condition 3.4.A.*

4.5.B (Lemma). *Any morphism of $S\mathcal{G}_P^1$ has no strong vertices.*

Proof. Recall that a vertex is said to be strong if the adjacent edges are oriented either all towards it or all in the opposite direction. In the notation of Admissibility conditions 2.8.A, this means that $\epsilon_1 = \epsilon_2 = \epsilon_3$. By (3) and (4), at a strong vertex we have $t_1 t_2 t_3 = 1$ and $T_1 + T_2 + T_3 = 1$. Belonging to $S_i\mathcal{G}_P^1$ means that $t_i = i^{1-T_i}$. Hence, $t_1 t_2 t_3 = i^{3-(T_1+T_2+T_3)} = i^2 \neq 1$, which contradicts the fact that $t_1 t_2 t_3 = 1$. \square

Proof of 4.5.A. By the lemma above, at each internal vertex at least one of the adjacent edges is directed towards the vertex and at least one is directed outwards. Therefore, $|T_1 + T_2 - T_3| = 1$, where T_1, T_2 , and T_3 are the weights of the edges adjacent to a vertex and enumerated appropriately. Then $|(1 - 2T_1) + (1 - 2T_2) - (1 - 2T_3)| = |1 - 2| = 1$, so that the vertex is involved with coefficient ± 1 into the boundary of the weight chain of the coloring provided by Φ . \square

Denote by \mathcal{G}_P^{1b} , $S_i\mathcal{G}_P^{1b}$, and \mathcal{G}_Q^{2b} the categories \mathcal{G}_P^1 , $S_i\mathcal{G}_P^1$, and \mathcal{G}_Q^2 , respectively, enhanced by incorporating secondary colorings into objects and morphisms.

4.5.C (Consistency in Boltzmann weights). *The functor Φ_P can be enhanced to a functor Φ_P^b from $S_i\mathcal{G}_P^{1b}$ to \mathcal{G}_Q^{2b} in such a way that the Boltzmann weight of \mathcal{A}^1 be equal to the Boltzmann weight of \mathcal{A}^2 at the same point after the application of Φ_P^b .*

This is proved much as Theorem 4.3.A. The functor Φ_P^b turns each dotted arc colored with (i^{1-T}, T) into a solid one colored with $1 - 2T$ without changing orientation, but the arrowhead becomes light. Each solid arc colored with (i^{1-T}, T) changes its orientation and receives weight $2T - 1$. The proof is completed by a straightforward comparison of the Boltzmann weights. \square

4.5.D (Corollary to 4.5.C). *The functor Φ_P that maps the subcategory $S_i\mathcal{G}_P^1$ of \mathcal{G}_P^1 to \mathcal{G}_Q^2 comprises, together with the identity functor of the category \mathcal{M}_B , a natural transformation of $\mathcal{A}^1|_{S\mathcal{G}^1}$ to \mathcal{A}^2 . In other words,*

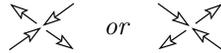
$$\begin{array}{ccc} S_i\mathcal{G}_P^1 & \xrightarrow{\mathcal{A}^1|_{S\mathcal{G}^1}} & \mathcal{M}_B \\ \downarrow \Phi_P & & \parallel \\ \mathcal{G}_Q^2 & \xrightarrow{\mathcal{A}^2} & \mathcal{M}_B \end{array}$$

is a commutative diagram. In particular, for any morphism Γ of $S_i\mathcal{G}_P^1$ we have

$$\mathcal{A}^2 \Phi_P(\Gamma) = \mathcal{A}^1(\Gamma). \quad \square$$

4.6. Nonsurjectivity and surjectivity of recoloring. As was observed in Subsection 3.3, there exist \mathcal{G}^2 -colorings that admit no secondary coloring and, therefore, give rise to morphisms of \mathcal{G}^2 mapped by \mathcal{RT}^2 to the zero morphisms. Obviously, if a morphism of \mathcal{G}^2 admits no secondary coloring, then it does not belong to the image of Φ . However, morphisms of this kind are not interesting, especially if we compare \mathcal{RT}^2 and \mathcal{RT}^1 .

4.6.A (Partial reversing of Φ_P^b). Any morphism of \mathcal{G}_P^{2b} with a diagram involving no fragment of the form



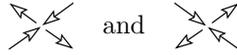
belongs to the image of Φ_P^b .

Proof. Graphically, the construction that converts a morphism of this kind can be described as follows: it replaces the solid arcs by dotted ones and the light arrowheads by solid ones; each weight A is replaced by the pair made of the multiplicity $i^{\frac{A+1}{2}}$ and the weight $\frac{1-A}{2}$. This construction is inverse to the construction which proves 4.5.C. It can be checked that the fragments of Table 6 are turned to fragments of the first column of Table 4, with the multiplicities of the adjacent edges satisfying Admissibility conditions 2.8.A. Similarly, all entries of Table 5, besides



are turned into the corresponding entries of Table 3. □

The condition that



are absent cannot be skipped. It can be proved that the right-hand side graph of Figure 5 does not belong to the image of Φ^b . Indeed, it is impossible to find appropriate weights. The existence of weights is equivalent to the existence of a secondary coloring for the graph on the left-hand side of Figure 5.

4.6.B (Reversing Φ_P^b on graphs without internal vertices). A morphism of \mathcal{G}_Q^{2b} with a diagram having no internal vertices belongs to the image of Φ_P^b . The \mathcal{G}_P^b -coloring is uniquely determined by the underlying orientations of the components.

Proof. Each arc of the diagram with weight A where the orientation of the secondary color and the orientation of the component coincide, must be replaced by a dotted arc with the same orientation, multiplicity $i^{\frac{A+1}{2}}$, and weight $\frac{1-A}{2}$. Each arc with weight A and the orientations opposite to each other must be equipped with the orientation of the component, multiplicity $i^{\frac{1-A}{2}}$, and weight $\frac{A+1}{2}$. An easy verification shows that each entry of Table 5 transforms to an entry of Table 3. □

4.7. Conclusion. The results of this section show that \mathcal{RT}^1 and \mathcal{RT}^2 are closely related, but do not reduce completely to each other. The source category \mathcal{G}_h^1 seems to be wider in several respects than the source category \mathcal{G}^2 of \mathcal{RT}^2 . First, the colors have more components. Second, the admissibility conditions allow more kinds of local pictures. The orientations at vertices in \mathcal{G}^1 have no counterpart in \mathcal{G}^2 . Finally, the restriction of \mathcal{RT}^1 to a comparatively small part of $\mathcal{G}_{\pi i/2}^1$ can be factored through a map onto an essential part of \mathcal{G}^2 .

On the other hand, \mathcal{G}^2 and \mathcal{RT}^2 are substantially simpler than \mathcal{G}^1 and \mathcal{RT}^1 . I have not been able to find a complete reduction of the former categories to the latter, and I doubt if such a reduction exists. A simpler theory has obvious advantages. Thus, we need to consider both.

The generalizations \mathcal{A}^1 and \mathcal{A}^2 of \mathcal{RT}^1 and \mathcal{RT}^2 are also related to each other. However, the conditions on the initial algebraic data under which the relating natural transformation exists are quite hard. This provides an additional reason for treating both theories.

§5. SKEIN PRINCIPLES AND RELATIONS

5.1. Skein principles. Since \mathcal{A}^1 and \mathcal{A}^2 map a colored point to an irreducible representation, the Schur lemma implies the following.

5.1.A (Colors' simplicity). *Let c be 1 or 2, and let Γ be a morphism of the category $\mathcal{G}_{\mathcal{P}}^c$ with $\partial\Gamma = \Pi_1 \times 0 \cup \Pi_1 \times 1$. Assume that B is a field. Then $\mathcal{A}^c(\Gamma)$ acts as multiplication by a constant. This constant is zero, unless the endpoints of Γ are colored with the same color.* □

5.1.B (Restricted colors' completeness). *Let c be 1 or 2, and let Γ be a morphism of $\mathcal{G}_{\mathcal{P}}^c$ with $\partial\Gamma = \Pi_2 \times 0 \cup \Pi_2 \times 1$. Suppose that the coloring of the endpoints of Γ extends to a coloring of the graph $\begin{array}{c} \diagup \\ \diagdown \end{array}$ satisfying the admissibility conditions. Assume that B is a field. Then $\mathcal{A}^c(\Gamma)$ is a linear combination of morphisms that are the images under \mathcal{A}^c of the graph $\begin{array}{c} \diagup \\ \diagdown \end{array}$ equipped with colorings that extend the same coloring of the endpoints and satisfy the admissibility conditions.*

In fact, there are at most two terms in the linear combination mentioned above.

Proof of 5.1.B. Consider the case where $c = 1$. The case of $c = 2$ differs only in notation. The morphism under consideration is a morphism

$$(u_0, U_0)_{\epsilon_0} \otimes (v_0, V_0)_{\delta_0} \rightarrow (u_1, U_1)_{\epsilon_1} \otimes (v_1, V_1)_{\delta_1}.$$

The assumptions on the colors imply that each of the tensor products is a direct sum of two irreducible modules over the algebra U^1 ; see Appendix 1, Subsection 11.8. The morphism under consideration must preserve these decompositions. Therefore, it is a linear combination of at most two isomorphisms between the corresponding summands. (If there are no isomorphic summands in the decompositions, then the original morphism is trivial.) An isomorphism between the summands of the decompositions can be realized as the image of the graph $\begin{array}{c} \diagup \\ \diagdown \end{array}$ colored in an appropriate way. □

The restriction imposed on the colors of $\partial\Gamma$ in 5.1.B underlies a profound difference from the case of quantum invariants studied in [16] and [17]. However, it is necessary. Indeed, by Admissibility conditions 2.8.A, $\begin{array}{c} \diagup \\ \diagdown \end{array}$ does not admit any coloring such that the two bottom endpoints are colored with $(t, T_1, +)$ and $(t^{-1}, T_2, +)$.

It is impossible to enlarge the palette so as to eliminate the restrictions occurring in 5.1.B without sacrificing 5.1.A. Indeed, the tensor product of the modules $(t, T_1)_+$ and $(t^{-1}, T_2)_+$ is not a direct sum of irreducible U^1 -modules.

5.1.C (Killing by a splittable closed part). *Let $c = 1$ or $c = 2$, and let $\Gamma \subset \mathbb{R}^2 \times [0, 1]$ be a colored framed graph representing a morphism of $\mathcal{G}_{\mathcal{P}}^c$ and containing a closed nonempty subgraph Σ separated from the rest of Γ by a 2-sphere disjoint with Γ and embedded in $\mathbb{R}^2 \times [0, 1]$. Then $\mathcal{A}^c(\Gamma) = 0$.*

Proof. First, let $\Sigma = \Gamma$. Then Γ is an endomorphism of the empty sequence. Since $\Sigma \neq \emptyset$, Γ can be decomposed as $\curvearrowright \circ \Phi \circ \curvearrowleft$, where Φ is an endomorphism of $(t, T)_+ \otimes (t, T)_-$ or $I(A) \otimes I(-A)$ for $c = 1$ and $c = 2$, respectively. In this composition, the first operator \curvearrowleft maps B to the invariant irreducible submodule which can be described as $\text{Ker } X \cap \text{Ker } Y$ in both cases. Whatever Φ is, it takes this subspace to itself (since Φ commutes with X and Y). The operator \curvearrowright annihilates this subspace. Therefore, the entire composition vanishes.

If $\Gamma \neq \Sigma$, the splitting provides the representation $\Gamma = \Sigma \otimes (\Gamma \setminus \Sigma)$. Thus, $\mathcal{A}^c(\Gamma) = \mathcal{A}^c(\Sigma \otimes (\Gamma \setminus \Sigma)) = \mathcal{A}^c(\Sigma) \otimes \mathcal{A}^c(\Gamma \setminus \Sigma) = 0 \otimes \mathcal{A}^c(\Gamma \setminus \Sigma) = 0$. \square

While Theorems 5.1.A and 5.1.B have counterparts in the cases of other Reshetikhin–Turaev functors, Theorem 5.1.C is more special. It implies other specific features of theories related to the Alexander polynomial.

5.2. First skein relations. Although 5.1.A and 5.1.B do not look like skein relations, they obviously imply the existence of skein relations. Of course, these relations can be proved independently by a straightforward calculation.

For example, Theorem 5.1.A implies that

$$\mathcal{A}^1 \left(\begin{array}{c} \uparrow \\ \updownarrow \\ \uparrow \\ (t, T) \end{array} \right) \quad \text{and} \quad \mathcal{A}^1 \left(\begin{array}{c} \uparrow \\ \uparrow \\ (t, T) \end{array} \right)$$

coincide up to a multiplicative constant. The constant is obvious from the construction of \mathcal{A}^1 : it is t^{-T} . Hence, we have

$$(10) \quad \mathcal{A}^1 \left(\begin{array}{c} \uparrow \\ \updownarrow \\ \uparrow \\ (t, T) \end{array} \right) = t^{-T} \mathcal{A}^1 \left(\begin{array}{c} \uparrow \\ \uparrow \\ (t, T) \end{array} \right).$$

This relation can be used to replace any graph by the same graph with the blackboard framing.

5.2.A (Removing loop relation for \mathcal{A}^1). *We have*

$$(11) \quad \mathcal{A}^1 \left(\begin{array}{c} \uparrow (t, T) \\ (u, U) \circlearrowleft (v, V) \\ \uparrow (t, T) \end{array} \right) = (t^2 - t^{-2}) \mathcal{A}^1 \left(\begin{array}{c} \uparrow \\ \uparrow \\ (t, T) \end{array} \right)$$

for any admissible colorings of the arcs in the middle of the left-hand side such that at the strong vertices all the orientations are of the same sign.

The proof of 5.2.A is a straightforward calculation. It is facilitated by the following observation. It suffices to calculate both sides with any color at both endpoints, because the morphism is scalar. However, a choice of the color may simplify calculations. For example, if the arcs in the middle of the left-hand side of (11) are oriented upwards, then the left-hand side admits only one coloring with bosonic colors at the endpoints, and two colorings with fermionic colors. In the bosonic case, the Boltzmann weight at the upper vertex is 1 and at the bottom we have $t^2 - t^{-2}$; see Table 4. Similar calculations prove (11) for the other orientations of these arcs. If the arcs constitute an oriented cycle, the choice of the fermionic color at the endpoints reduces the state sum to a single summand; otherwise the bosonic color does this. \square

5.2.B (Junction relations for \mathcal{A}^1). If $u^4v^4 \neq 1$, then

$$\begin{aligned}
 & \mathcal{A}^1 \left(\begin{array}{c} \uparrow \quad \uparrow \\ (u,U) \quad (v,V) \end{array} \right) \\
 (12) \quad &= \frac{1}{u^2v^2 - u^{-2}v^{-2}} \mathcal{A}^1 \left(\begin{array}{c} \swarrow \quad \searrow \\ (u,U) \quad (v,V) \\ \uparrow \\ (uv, U+V-I) \\ \swarrow \quad \searrow \\ (u,U) \quad (v,V) \end{array} \right) \\
 &+ \frac{1}{u^{-2}v^{-2} - u^2v^2} \mathcal{A}^1 \left(\begin{array}{c} \swarrow \quad \searrow \\ (u,U) \quad (v,V) \\ \downarrow \\ (u^{-1}v^{-1}, -U-V-I) \\ \swarrow \quad \searrow \\ (u,U) \quad (v,V) \end{array} \right).
 \end{aligned}$$

If $u^4v^{-4} \neq 1$, then

$$\begin{aligned}
 & \mathcal{A}^1 \left(\begin{array}{c} \uparrow \quad \downarrow \\ (u,U) \quad (v,V) \end{array} \right) \\
 (13) \quad &= \frac{1}{u^2v^{-2} - u^{-2}v^2} \mathcal{A}^1 \left(\begin{array}{c} \swarrow \quad \searrow \\ (u,U) \quad (v,V) \\ \uparrow \\ (uv^{-1}, U-V+I) \\ \swarrow \quad \searrow \\ (u,U) \quad (v,V) \end{array} \right) \\
 &+ \frac{1}{u^{-2}v^2 - u^2v^{-2}} \mathcal{A}^1 \left(\begin{array}{c} \swarrow \quad \searrow \\ (u,U) \quad (v,V) \\ \downarrow \\ (u^{-1}v, V-U+I) \\ \swarrow \quad \searrow \\ (u,U) \quad (v,V) \end{array} \right).
 \end{aligned}$$

This is proved by a direct calculation. □

Relations (12) and (13) are so similar that it would be natural to unite them. The only tool we need for this is the following

Agreement on hiding inessential.

- If the orientations of some edges are not shown in a diagram, then the edges are oriented, but in an unspecified way.
- If several diagrams are involved in the same identity and have endpoints, then the edges adjacent to the equally positioned endpoints are colored (and, in particular, oriented) in the same way in all the diagrams.
- If only the multiplicity of a color is shown, then the other ingredients of the color are ignored because they are not important. They may be recovered in any way respecting the admissibility conditions. (However, missing orientations at strong vertices are counterclockwise.)
- If a formula involves a sum over colors of an edge without a specified orientation, then the sum runs over all the colors (and, in particular, orientations) making admissible triples at the endpoints of the edge with the colors of the adjacent edges.

Under this agreement, (12) and (13) unite as follows:

$$(14) \quad \mathcal{A}^1 \left(\begin{array}{c} | \quad | \\ u \quad v \end{array} \right) = \sum_{(w,W)} \frac{1}{w^2 - w^{-2}} \mathcal{A}^1 \left(\begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ (w,W) \\ \diagup \quad \diagdown \\ u \quad v \end{array} \right).$$

Relation (11), which is already written partially under the agreement, can be rewritten in a more concise form:

$$(15) \quad \mathcal{A}^1 \left(\begin{array}{c} | \\ \text{---} \circ \text{---} \\ | \\ s \\ u \quad v \\ | \\ t \end{array} \right) = \delta_{s,t}(t^2 - t^{-2}) \mathcal{A}^1 \left(\begin{array}{c} | \\ | \\ | \\ t \end{array} \right).$$

5.3. First skein relations for the second functor. In a similar but simpler way one can prove the following counterparts of (10), 5.2.A, and 5.2.B for \mathcal{A}^2 :

$$(16) \quad \mathcal{A}^2 \left(\begin{array}{c} \uparrow \\ | \\ \uparrow \\ (t,T) \end{array} \right) = i^{\frac{A^2-1}{4}} \mathcal{A}^2 \left(\begin{array}{c} \uparrow \\ | \\ \uparrow \\ A \end{array} \right).$$

5.3.A (Removing loop relation for \mathcal{A}^2). *We have*

$$(17) \quad \mathcal{A}^2 \left(\begin{array}{c} \uparrow A \\ \text{---} \circ \text{---} \\ \uparrow A \\ B \quad C \end{array} \right) = (i^{1+A} + i^{1-A}) \mathcal{A}^2 \left(\begin{array}{c} \uparrow \\ | \\ \uparrow \\ A \end{array} \right)$$

for any admissible colorings of the arcs in the middle of the left-hand side. □

5.3.B (Junction relations for \mathcal{A}^2). *If $A + B \notin 2\mathbb{Z}$, then*

$$(18) \quad \begin{aligned} & \mathcal{A}^2 \left(\begin{array}{c} \uparrow \\ | \\ \uparrow \\ A \quad B \end{array} \right) \\ &= \frac{1}{i^{A+B} - i^{-A-B}} \mathcal{A}^2 \left(\begin{array}{c} \nearrow A \quad \searrow B \\ \text{---} \circ \text{---} \\ \nwarrow A \quad \nearrow B \\ A+B-1 \end{array} \right) \\ &+ \frac{1}{-i^{A+B} + i^{-A-B}} \mathcal{A}^2 \left(\begin{array}{c} \nearrow A \quad \searrow B \\ \text{---} \circ \text{---} \\ \nwarrow A \quad \nearrow B \\ A+B+1 \end{array} \right). \end{aligned} \quad \square$$

§6. ALEXANDER INVARIANT OF CLOSED COLORED FRAMED GRAPHS

6.1. Definition of the Conway function via the Reshetikhin–Turaev functor.

An isotopy class of a \mathcal{G}^1 -colored framed link can be viewed as a morphism of \mathcal{G}_h^1 from the empty set to itself. The tensor product of the empty family of modules is naturally identified with the ground field \mathbb{C} , because this is the unity for the tensor multiplication of modules. Thus, the isotopy class of a colored framed link determines, via the Reshetikhin–Turaev functor, a complex number depending on q . For other polynomial link invariants, such as the Jones polynomial, this is a way to relate the polynomial and the functor: a similar function of q determined by the functor is the corresponding polynomial.

However, Theorem 5.1.C shows that in the case of $U_q \mathfrak{gl}(1|1)$ this does not work: the number constructed in this way is identically zero. This is well known; see [8, 13]. The standard way to obtain the Conway function from the Reshetikhin–Turaev functor involves an auxiliary geometric construction. One of the strings in the link should be pulled and cut, to convert the link into a tangle connecting $\Pi_1 \times 0$ with $\Pi_1 \times 1$ in $\mathbb{R}^2 \times [0, 1]$. By 5.1.A, the corresponding morphism is multiplication by a number depending on q and the colors of the components. This number divided by $q^{2j} - q^{-2j}$, where j is the multiplicity of the cut string’s color, is the value of the Conway function of the

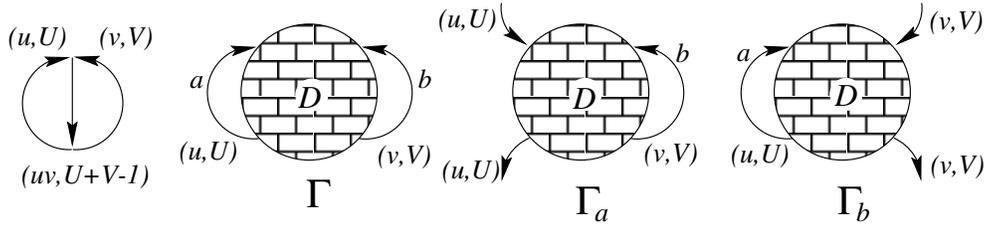


FIGURE 6.

original link evaluated at q^{j_1}, \dots, q^{j_n} , where j_1, \dots, j_n are the multiplicities of the link components' colors, provided the weights of all the colors are zero. See Rozansky and Saleur [13, 14], Murakami [11], Deguchi and Akutsu [4].

6.2. Why it does not matter where to cut. Here I show that, in a similar way, the Conway functor determines an invariant for \mathcal{G}_P^1 -colored framed graphs in R^3 . The proof is based on the following lemma.

6.2.A (Lemma). *Let $P = (B, M, W, M \times W \rightarrow M)$ be a 1-palette (see Subsection 2.8). Assume that B is a field. Let Γ be a \mathcal{G}_P^1 -colored framed closed generic graph embedded in \mathbb{R}^3 such that its projection to \mathbb{R}^2 is generic. Let $D \subset \mathbb{R}^2$ be a disk containing the entire projection of Γ except for two arcs, a and b , colored with (u, U) and (v, V) , respectively. Let Γ_a and Γ_b be the colored framed graphs obtained from Γ by cutting at a and b , respectively, and moving the endpoints upwards and downwards to a position such that one of them becomes the uppermost and the other one lowermost. While moving, the framing at the endpoints is kept blackboard. See Figure 6. Then $\mathcal{A}^1(\Gamma_a)$ is multiplication by $(u^2 - u^{-2})\Delta$ and $\mathcal{A}^1(\Gamma_b)$ is multiplication by $(v^2 - v^{-2})\Delta$, with one and the same $\Delta \in B$.*

Proof. First, assume that these arcs are colored in such a way that the coloring can be extended to an admissible coloring of a planar theta graph containing a and b . By 5.1.B, the common part of all three graphs Γ , Γ_a , and Γ_b , i.e., the part whose projection is covered by D , is mapped by \mathcal{A}^1 to a linear combination of the images of two copies of χ colored differently. This linear combination looks like this:

$$\alpha \mathcal{A}^1 \left(\begin{array}{c} \diagup \quad \diagdown \\ \uparrow \end{array} \right) + \beta \mathcal{A}^1 \left(\begin{array}{c} \diagup \quad \diagdown \\ \downarrow \end{array} \right).$$

Then

$$\mathcal{A}^1(\Gamma_a) = \alpha \mathcal{A}^1 \left(\begin{array}{c} \diagup \quad \diagdown \\ \uparrow \end{array} \right) + \beta \mathcal{A}^1 \left(\begin{array}{c} \diagup \quad \diagdown \\ \downarrow \end{array} \right).$$

By (11),

$$\mathcal{A}^1 \left(\begin{array}{c} \diagup \quad \diagdown \\ \uparrow \end{array} \right) = \mathcal{A}^1 \left(\begin{array}{c} \diagup \quad \diagdown \\ \downarrow \end{array} \right) = (u^2 - u^{-2}) \mathcal{A}^1 \left(\begin{array}{c} \diagup \quad \diagdown \\ \uparrow \end{array} \right),$$

whence

$$\mathcal{A}^1(\Gamma_a) = (\alpha + \beta)(u^2 - u^{-2}) \mathcal{A}^1 \left(\begin{array}{c} \diagup \quad \diagdown \\ \uparrow \end{array} \right).$$

Thus, $\mathcal{A}^1(\Gamma_a)$ is multiplication by $(\alpha + \beta)(u^2 - u^{-2})$.

Similarly, $\mathcal{A}^1(\Gamma_b)$ is multiplication by $(\alpha + \beta)(v^2 - v^{-2})$. This proves the statement: take $\Delta = \alpha + \beta$.

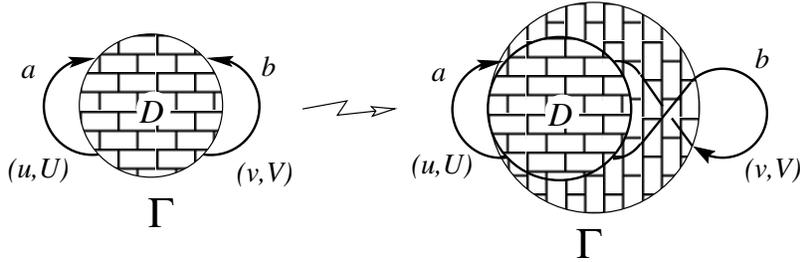


FIGURE 7.

The assumption of the extensibility of the colors over a theta graph was used at the very beginning of the proof. There are situations when it does not hold true. For example, with the orientations as in Figure 6, it can happen that $u = v^{-1}$. Then 5.1.B cannot be applied, and the fragment of Γ covered by D cannot be replaced by the linear combination as above. However, in this case one can first make a small kink on one of the arcs, say b , and then expand D to hide the new crossing point under D . See Figure 7. Outside D we see the arc b on the previous place, but oriented oppositely. Now the assumption is fulfilled, so that we can apply the arguments above.

The result is not exactly what we desire: we have proved a statement in which in place of Γ_b we have a graph isotopic to the graph symmetric to Γ_b with respect to a horizontal line. However, it is easy to show that this symmetry does not change the morphism. This follows from the isotopy invariance of the image with respect to \mathcal{A}^1 , by application of the following isotopy:



Here is a counterpart of Lemma 6.2.A concerning \mathcal{A}^2 . It is proved similarly.

6.2.B (Lemma). *Let $P = (B, W, W^+ \rightarrow B^\times)$ be a 2-palette (see Subsection 3.4). Assume that B is a field. Let Γ be a \mathcal{G}_P^2 -colored framed closed generic graph embedded in \mathbb{R}^3 such that its projection to \mathbb{R}^2 is generic. Let $D \subset \mathbb{R}^2$ be a disk containing the entire projection of Γ except for two arcs, a and b , colored with A and B , respectively. Let Γ_a and Γ_b be the colored framed graphs obtained from Γ by cutting a and b , respectively, and moving the endpoints upwards and downwards to a position such that one of them becomes the uppermost and the other one lowermost. While moving, the framing at the endpoints is kept blackboard. Then $\mathcal{A}^2(\Gamma_a)$ is multiplication by $\mathbf{i}(i^A + i^{-A})\Delta$ and $\mathcal{A}^2(\Gamma_b)$ is multiplication by $\mathbf{i}(i^B + i^{-B})\Delta$, with one and the same $\Delta \in B$. \square*

6.3. Definition of Alexander invariants. Let $P = (B, M, W, M \times W \rightarrow M)$ be a 1-palette and B a field. Let Γ be a \mathcal{G}_P^1 -colored framed closed generic graph in \mathbb{R}^3 . We cut Γ at a point on any outermost arc a of its diagram, where the framing is blackboard, and move the new endpoints up and down keeping the framing blackboard during the movement. The result represents a morphism Γ_a of \mathcal{G}_P^1 . Let the color of a be (t, T) . Then $\frac{1}{t^2 - t^{-2}} \mathcal{A}^1(\Gamma_a)$ is multiplication by some element of B . We denote this element by $\underline{\Delta}^1(\Gamma)$ and call it the $\text{gl}(1|1)$ -Alexander invariant of Γ . By Lemma 6.2.A, $\underline{\Delta}^1(\Gamma)$ does not depend of the choice of a . Of course, it depends on the coloring.

Let $P = (B, W, W^+ \rightarrow B^\times)$ be a 2-palette, and let B be a field. Let Γ be a \mathcal{G}_P^2 -colored framed closed generic graph in \mathbb{R}^3 . Cut Γ at a point on any outermost arc a of its diagram, where the framing is blackboard, and move the new endpoints up and down

keeping the framing blackboard during the movement. The result represents a morphism Γ_a of \mathcal{G}_P^2 . Let the color of a be A . Then $\frac{1}{i^{1+A} + i^{1-A}} \mathcal{A}^2(\Gamma_a)$ is multiplication by some element of B . We denote this element by $\underline{\Delta}^2(\Gamma)$ and call it the $\mathfrak{sl}(2)$ -Alexander invariant of Γ . By Lemma 6.2.B, $\underline{\Delta}^2(\Gamma)$ does not depend of the choice of a . Of course, it depends on the coloring.

When the choice is obvious from the context, we shall call the $\mathfrak{gl}(1|1)$ - and $\mathfrak{sl}(2)$ -Alexander invariants simply the *Alexander invariants*. The Alexander invariant $\underline{\Delta}^c(\Gamma)$ is invariant under an (ambient) isotopy of Γ . This follows from the isotopy invariance of $\mathcal{A}^c(\Gamma_a)$ and an obvious construction turning an isotopy of Γ into an isotopy of Γ_a .

6.4. Vertex state sum representation. The geometric operations of cutting and moving the cut points are redundant if it is only required to calculate $\underline{\Delta}^c(\Gamma)$. Instead, the following calculation can be used.

First, consider the $\mathfrak{gl}(1|1)$ -case. Let $\Gamma \subset \mathbb{R}^3$ be a closed \mathcal{G}^1 -colored graph. Choose its generic projection to \mathbb{R}^2 . On the boundary of the exterior domain of the diagram choose an arc. For this arc, choose either bosonic or fermionic color and consider all the extensions of this choice to the entire set of 1-strata of Γ . Choose a generic projection of the diagram to a line. For each distribution of bosons and fermions on the 1-strata of Γ , in accordance with Tables 3 and 4, put the Boltzmann weights at the critical points of the projection and the vertices of the diagram and take the product of them. Sum up all the products for all the distributions. If on the selected arc the fermions have been chosen, multiply the sum by -1 . If the selected arc is colored with multiplicity t , divide the result by $t^2 - t^{-2}$. If the orientation of this arc determines the clockwise direction of moving around the diagram of Γ , multiply by t^2 . Otherwise, multiply by t^{-2} . The final result is $\underline{\Delta}^1(\Gamma)$.

To prove that this is indeed $\underline{\Delta}^1(\Gamma)$, we can cut Γ at a generic point on the chosen arc, move the cut points, and calculate $\underline{\Delta}^1(\Gamma)$ in accordance with the definition applied to this cut. Moving the cut points creates two new critical points. To calculate the image of the graph under \mathcal{A}^1 , it suffices to calculate the image of one of the basis vectors, either boson or fermion, and divide the result by $t^2 - t^{-2}$. The calculation described above does this in terms of Boltzmann weights.

Now, consider the $\mathfrak{sl}(2)$ -case. Let $\Gamma \subset \mathbb{R}^3$ be a closed \mathcal{G}^2 -colored graph. Choose its generic projection to \mathbb{R}^2 . On the boundary of the exterior domain of the diagram choose an arc. On this arc, put a light arrowhead indicating the clockwise direction of moving around the diagram. Consider all the extensions of this choice of a light arrowhead to secondary colorings of the diagram. Choose a generic projection of the diagram to a line. For each distribution of orientations on the 1-strata of Γ , in accordance with Tables 5 and 6, put the Boltzmann weights at the critical points of the projection and the vertices of the diagram and take the product of them. Sum up all the products for all the distributions. If the selected arc is colored with weight A , divide the sum by $1 + i^{-2A}$. The result is $\underline{\Delta}^1(\Gamma)$.

6.5. Vanishing on splittables. Theorem 5.1.A shows that the Alexander invariant vanishes on any closed \mathcal{G}_P^c -colored framed graph Γ that can be presented as the union of two closed subgraphs separated from each other by a sphere embedded in \mathbb{R}^3 .

§7. SPECIAL PROPERTIES OF THE $\mathfrak{gl}(1|1)$ -ALEXANDER INVARIANT

7.1. First examples: unknot and the theta graph. The $\mathfrak{gl}(1|1)$ -Alexander polynomial of the unknot U with 0-framing colored with color (t, T) is $\frac{1}{t^2 - t^{-2}}$. More generally,

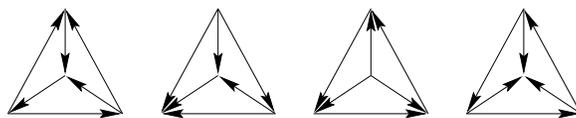


FIGURE 8.

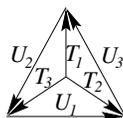


FIGURE 9.

for the unknot U_k with framing $k/2$, $k \in \mathbb{Z}$, colored with (t, T) we have

$$\underline{\Delta}^1(U_k) = \frac{t^{-kT}}{t^2 - t^{-2}}.$$

For any planar theta graph Θ with planar framing and arbitrary coloring with counterclockwise orientations at the strong vertices we have $\underline{\Delta}^1(\Theta) = 1$. This follows from 5.2.A.

7.2. The $\mathfrak{gl}(1|1)$ -Alexander invariant of a tetrahedron's 1-skeleton. Now we consider a planar 1-skeleton T of a tetrahedron with planar framing. This graph plays an important role in the face model; see §10. Contrary to the case of the theta graph, the $\mathfrak{gl}(1|1)$ -Alexander invariant of T depends on the orientation, and we begin with a classification of orientations.

7.2.A (Orientations of T). *Up to homeomorphism, there are 4 orientations on T . They are classified by the numbers of repulsing and attracting vertices. All orientations without strong vertices are homeomorphic. All orientations having both a repulsing and an attracting vertex also comprise one homeomorphism type. There is a unique type with one repulsing and no attracting vertices, and a unique type with one attracting and no repulsing vertices. See Figure 8.* \square

7.2.B (Orientations nonextendible to colorings). *If an orientation of T has only one strong vertex, then it cannot be extended to an admissible \mathcal{G}_P^1 -coloring of T .*

Proof. Indeed, consider the orientation with a repulsing and no attracting vertices. Assume that an admissible coloring exists. Let T_1, T_2, T_3 denote the weights of the edges adjacent to the repulsing vertex. Let U_1, U_2, U_3 be the weights of the edges opposite to the edges with weights I, J, K , respectively. Assume that the edge with weight L is directed towards the endpoint of the edge with weight K . See Figure 9.

Then, by Admissibility condition 4, $T_1 + T_2 + T_3 = -1$, $U_1 = T_2 + U_3 - 1$, $U_2 = T_3 + U_1 - 1$, and $U_3 = T_1 + U_2 - 1$. The sum of these four equations is $0 = -4$.

Similarly one can prove that the orientation with an attracting and no repulsing vertices cannot be extended to an admissible coloring. \square

The other two orientations can be extended to colorings.

7.2.C (Lemma on the characteristic edge). *For an orientation of T extendible to an admissible \mathcal{G}_P^1 -coloring, there exists a unique edge such that reversing its orientation turns the orientation of T to an orientation with a repulsing vertex and without attracting ones.* \square

The edge specified by Lemma 7.2.C is said to be *characteristic*. In Figure 8 the bottom edges in the two leftmost graphs are characteristic. This edge can also be defined as the only edge connecting two odd vertices; see Subsection 2.2.

7.2.D (gl(1|1)-Alexander invariant of T). *Let T be a colored planar 1-skeleton of a tetrahedron with planar framing. Then $\underline{\Delta}^1(\mathbb{T})$ is equal to $t^2 - t^{-2}$, where t is the multiplicity of the color of the characteristic edge of T.*

I am not aware of any conceptual proof of 7.2.D. This statement can be proved by a straightforward calculation, and I leave this to the reader. To save effort, I would recommend choosing an edge for the cut in such a way that there is a color extending from this edge uniquely to a coloring of T. In the case with no strong vertices, one can take the edge preceding the characteristic one in the 4-cycle and color it with boson. In the case of two strong vertices, color with boson the edge connecting the strong vertices. In both cases, all four edges meeting the chosen one must be colored with fermion and the disjoint edge with boson. \square

The calculations of this section demonstrate that the Alexander invariant is much simpler than the Jones polynomial. Indeed, the Jones polynomials for the same elementary graphs are much more complicated. See, e.g., [2].

7.3. Functorial change of colors. Let $P_1 = (B_1, M_1, W_1, M_1 \times W_1 \rightarrow M_1)$ and $P_2 = (B_2, M_2, W_2, M_2 \times W_2 \rightarrow M_2)$ be 1-palettes, let S be a subring of B_1 containing M_1 and W_1 , and let $\beta : S \rightarrow B_2$ be a ring homomorphism such that $\beta(M_1) \subset M_2$, $\beta(W_1) \subset W_2$ and $\beta(m)^{\beta(w)} = \beta(m^w)$ for any $m \in M_1$ and any $w \in W_1$.

Let Γ_1 be a morphism of $\mathcal{G}_{P_1}^1$ such that the multiplicity of each 1-stratum of Γ_1 does not belong to $\beta^{-1}\{t \in M_2 \mid t^4 = 1\}$. Then, if on each 1-stratum we replace the multiplicity and weight with their images under β (and preserve the other components of the coloring), then Γ_1 will turn to a morphism Γ_2 of $\mathcal{G}_{P_2}^1$. Each entry of the matrix representing $\mathcal{A}^1(\Gamma_2)$ in the standard bases is equal to the β -image of the corresponding entry of the matrix representing $\mathcal{A}^1(\Gamma_1)$. This follows immediately from the definition of \mathcal{A}^1 .

Assume now that B_1 and B_2 are fields. Then there is a similar relation for the Alexander invariant: if Γ_1 is a $\mathcal{G}_{P_1}^1$ -colored closed framed generic graph in \mathbb{R}^3 , then the replacement of the multiplicities and weights of the 1-strata of Γ_1 with their images under β turns Γ_1 to a $\mathcal{G}_{P_2}^1$ -colored graph Γ_2 , and $\underline{\Delta}^1(\Gamma_2)$ is equal to the image of $\underline{\Delta}^1(\Gamma_1)$ under β .

7.4. Universal colors. As we saw in Subsection 7.2, the restrictions on the orientations and weights imposed by Admissibility conditions 2.8.A are quite subtle. The multiplicities are more straightforward: for given orientations and weights, the multiplicities are, in essence, a 1-cycle with coefficients in the multiplicative group M . The only nontrivial condition on this cycle is that on each 1-stratum it takes a value that is not a fourth root of unity, and, in particular, is not 1. This follows from Admissibility condition (3). For example, a graph consisting of two disjoint circles and a segment that joins them admits no coloring, because any 1-cycle vanishes on the segment.

Among all the 1-cycles on a graph Γ with all the coefficient groups, there is a *universal 1-cycle* such that any other 1-cycle can be obtained from it by a change of coefficients (i.e., as in the preceding subsection). The coefficient group for the universal cycle is $H^1(\Gamma; \mathbb{Z})$. The universal cycle is constructed as follows. An (oriented) 1-stratum e of Γ can be viewed as a 1-cocycle of Γ . (The value of this 1-cocycle on a 1-chain c is equal to the coefficient with which e occurs in c .) It is easy to check that assigning to a 1-stratum the cohomology class defined in this way is a cycle. We denote it by u .

Let M be an Abelian group. Fix some $x \in H_1(\Gamma; M)$. The evaluation of cohomology classes on x determines a homomorphism $y \mapsto y \cap x : H^1(\Gamma; \mathbb{Z}) \rightarrow M$, and x is the image of u under the homomorphism $H_1(\Gamma; H^1(X; \mathbb{Z})) \rightarrow H^1(\Gamma; M)$ induced by $y \mapsto y \cap x : H^1(\Gamma; \mathbb{Z}) \rightarrow M$.

Let $P = (B, M, W, M \times W \rightarrow M)$ be a 1-palette with $W = \mathbb{Z}$. Let Γ be a framed closed generic graph in \mathbb{R}^3 equipped with orientations. Suppose that its diagram is equipped with weights satisfying Admissibility condition (4). If the collection of these orientations and weights extends to a \mathcal{G}_P^1 -coloring on the diagram of Γ , it extends also to the quadruple composed of

$$B = Q(H^1(\Gamma; \mathbb{Z})), \quad M = H^1(\Gamma; \mathbb{Z}), \quad W = \mathbb{Z}.$$

Moreover, $\underline{\Delta}^1(\Gamma)$ with any \mathcal{G}_P^1 -coloring such that $W = \mathbb{Z}$ can be expressed, in the way shown in Subsection 6.5, in terms of the Alexander invariant of Γ with a coloring whose multiplicity component is the universal 1-cycle.

Notice that the Alexander invariant with the universal multiplicities is the ratio of formal linear combinations of cohomology 1-classes of the graph. To avoid confusion of the addition in $H^1(\Gamma)$ and the formal addition, it makes sense to write the addition in $H^1(\Gamma)$ as multiplication.

7.5. Dependence of the Alexander invariant on framings and weights: the case of links. Relation (10) describes the dependence of $\mathcal{A}^1(\Gamma)$ and, thus, of $\underline{\Delta}^1(\Gamma)$ on a framing. By this relation, if $T = 0$, then $\underline{\Delta}^1(\Gamma)$ does not change when the framing changes. In the case of a link, a coloring can be chosen such that all the weight components of the colors are zero, which eliminates the dependence of $\underline{\Delta}^1(\Gamma)$ on a framing.

Introduction of nonzero weights and framings does not enrich the Alexander invariant: still it can be calculated if the linking matrix and the Alexander invariant of the same link with zero weights are known. By the linking matrix I mean the matrix comprised of the self-linking numbers (i.e., the framing numbers) and the pairwise linking numbers of the components.

These numbers are easy to calculate out of a link diagram. The linking number of two components is half the sum of the signs (local writhe numbers) of all the crossing points where the projections of the components intersect. The self-linking number is the sum of the signs of all the crossing points where the projection of the component intersects itself, plus half the sum of the signs of the half-twists on the projection of the component.

Since several components can be colored with the same color, we need a reduced linking matrix. For a framed colored link L , let $\text{lk}_L((u, U)(v, V))$ denote the sum of the linking numbers of all the components of L colored with (u, U) and all the components colored with (v, V) . Denote by $\text{lk}_L((u, U)(u, U))$ the sum of all self-linking numbers of the components of L colored with (u, U) plus twice the sum of all their pairwise linking numbers. To calculate these numbers for a link presented by its diagram, one can use the same rule as above: if $(u, U) \neq (v, V)$, then $\text{lk}_L((u, U)(v, V))$ is half the sum of the signs of all the crossing points where the colors (u, U) , (v, V) meet; $\text{lk}_L((u, U)(u, U))$ is the sum of the signs of the crossing points where both strings are colored with (u, U) plus half the sum of the half-twist signs on the strings colored with (u, U) .

7.5.A. *Let L be a \mathcal{G}_P^1 -colored framed link in \mathbb{R}^3 , and let L_0 be the \mathcal{G}_P^1 -colored framed link obtained from L by replacing the weight components of the color with zero. Then*

$$(19) \quad \underline{\Delta}^1(L) = \prod_{(u, U)} u^{-2 \sum_{(v, V)} V \text{lk}_L((u, U), (v, V))} \underline{\Delta}^1(L_0).$$

Proof. This follows from a straightforward calculation of the contribution made by the weights of colors to the Boltzmann weights. The weights of colors contribute only to

the Boltzmann weights at half-twist signs and crossing points. At a positive half-twist of a component colored with (u, U) this contribution is u^{-U} , and at a positive crossing point with colors (u, U) and (v, V) it is $u^{-V}v^{-U}$. The product of all these factors is $\prod_{(u,U)} u^{-2\sum_{(v,V)} V \text{lk}_L((u,U),(v,V))}$. \square

7.6. Dependence of the Alexander invariant on weights: the general case.

In the presence of trivalent vertices, it is impossible to make all the weights of colors equal to zero, because if at a trivalent vertex two of the three adjacent edges are colored with the zero weight, then, due to the admissibility condition, the third one is colored with the weight ± 1 . However, a formula similar to (19) allows us to relate the Alexander invariants of a framed graph colored in two ways with colors different only in their weight components.

Let Γ be a framed graph equipped with two \mathcal{G}_P^1 -colorings with colors different only in their weight components. Denote by T_e^k the value of the weight of the k th (with $k = 1, 2$) coloring taken on an oriented (with the orientation shared by the colorings) 1-stratum e . Then $\sum(T_e^1 - T_e^2)e$, where e runs over the set of 1-strata of Γ , is a 1-cycle with integral coefficients. This follows from the Admissibility conditions (4). This cycle is called the *weight cycle* and is denoted by Γ_W . Another cycle is formed in the same way, but with $T_e^1 - T_e^2$ replaced by the multiplicities (shared by the colors) of 1-strata. This cycle will be called the *multiplicity cycle* and will be denoted by Γ_M .

Consider a field of lines on Γ normal to the framing. On each of these lines, choose a short segment centered at the corresponding point of Γ and such that the endpoints of the segments comprise a generic graph $\tilde{\Gamma}$ which is a two-fold covering space of Γ . We equip $\tilde{\Gamma}$ with the orientation such that the covering projection maps it to the orientation of Γ incorporated in the colorings. Each 1-stratum of $\tilde{\Gamma}$ will be equipped with the multiplicity of its image under the projection. The oriented 1-strata of $\tilde{\Gamma}$ equipped with halves of these multiplicities comprise a cycle with coefficients in M . We denote this cycle by $\tilde{\Gamma}_M$. It can be thought of as the multiplicity cycle Γ_M pushed away from Γ in both directions normal to the framing. The cycle $\tilde{\Gamma}_M$ is disjoint from Γ , and hence from Γ_W . Therefore, the linking number $\text{lk}(\tilde{\Gamma}_M, \Gamma_W)$ of this cycle and the weight cycle associated with the pairing $M \times W \rightarrow M : (m, w) \mapsto m^w$ is well defined. In the case where Γ is a link and $T_e^2 = 0$, we have

$$\text{lk}(\tilde{\Gamma}_M, \Gamma_W) = \prod_{(u,U)} u^{2\sum_{(v,V)} V \text{lk}_L((u,U),(v,V))}$$

(see (19)).

Let Γ_1 and Γ_2 denote the graph Γ equipped with the colorings under consideration. Then

$$(20) \quad \underline{\Delta}^1(\Gamma_2) = \text{lk}(\tilde{\Gamma}_M, \Gamma_W)^{-1} \underline{\Delta}^1(\Gamma_1).$$

Formula (20) generalizes (19) and is proved similarly.

7.7. Relationship between the $\text{gl}(1|1)$ -Alexander invariant and the Conway function. Recall that for an oriented link L with (linearly) ordered connected components L_1, \dots, L_k , the Conway function $\nabla(L)(t_1, \dots, t_k)$ is a rational function in the variables t_1, \dots, t_k defined by the following axioms (see Turaev [18]):

7.7.A. $\nabla(L)$ does not change under an (ambient) isotopy of L .

7.7.B. If L is the unknot, then $\nabla(L)(t) = (t - t^{-1})^{-1}$.

7.7.C. If L consists of at least two connected components, then $\nabla(L)$ is a Laurent polynomial, i.e., $\nabla(L)(t_1, \dots, t_k) \in \mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$.

Denote by $\tilde{\nabla}(L)$ the function $\nabla(L)(t, t, \dots, t)$ of one variable.

7.7.D. $\tilde{\nabla}(L)(t)$ does not depend on the ordering of the connected components of L .

7.7.E (Conway skein relation). If $L \searrow \nearrow$, $L \swarrow \nwarrow$ and $L \smile \frown$ are links that coincide outside a ball and look as the subscripts in their notation in the ball, then

$$(21) \quad \tilde{\nabla}(L \searrow \nearrow) = \tilde{\nabla}(L \swarrow \nwarrow) + (t - t^{-1})\tilde{\nabla}(L \smile \frown).$$

7.7.F (Doubling axiom). If L' is a link obtained from a link $L = L_1 \cup \dots \cup L_k$ by replacing the component L_i by its $(2, 1)$ -cable, then

$$(22) \quad \nabla(L')(t_1, \dots, t_r) = (T + T^{-1})\nabla(L)(t_1, \dots, t_{i-1}, t_i^2, t_{i+1}, \dots, t_k),$$

where $T = t_i \prod_{j \neq i} t_j^{\text{lk}(L_i, L_j)}$.

7.7.G (Alexander invariant vs Conway function). Let M_k be a free Abelian group generated by t_1, \dots, t_k and written multiplicatively, let B_k be the field $Q(M_k)$ of rational functions of t_1, \dots, t_k with integral coefficients. Denote by P_k the quadruple $(B_k, M_k, \mathbb{Z}, M_k \times \mathbb{Z} \rightarrow \mathbb{Z} : (m, w) \mapsto m^w)$. Let L be a $\mathcal{G}_{P_k}^1$ -colored framed link with components L_1, \dots, L_k colored with multiplicities t_1, \dots, t_r and zero weights (i.e., $T_1 = T_2 = \dots = T_k = 0$). Then $\underline{\Delta}^1(L)$ and the Conway function of the same L (oriented with the orientation taken from the coloring) are related as follows:

$$(23) \quad \underline{\Delta}^1(L) = \nabla(L)(t_1^2, \dots, t_k^2).$$

Proof. It suffices to prove that the Alexander invariant of a \mathcal{G}_P^1 -colored framed link with zero weights satisfies the following counterparts of the axioms 7.7.A–7.7.F.

7.7.A.bis. $\underline{\Delta}^1(L)$ does not change under an (ambient) isotopy of L .

7.7.B.bis. If L is the unknot \mathcal{G}_P^1 -colored with multiplicity t , then $\underline{\Delta}^1(L)(t) = (t^2 - t^{-2})^{-1}$.

7.7.C.bis. If L consists of at least two connected components $\mathcal{G}_{P_k}^1$ -colored as in 7.7.G, then $\underline{\Delta}^1(L)$ is a Laurent polynomial in multiplicities, i.e.,

$$\underline{\Delta}^1(L) \in \mathbb{Z}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}].$$

Let L be a $\mathcal{G}_{P_k}^1$ -colored framed link as in 7.7.G. Let L^t denote the same framed link, but with all the components colored with $(t, 0)$ and with orientations taken from the coloring of L .

7.7.D.bis. $\underline{\Delta}^1(L^t)(t) = \underline{\Delta}^1(L)^1(t, \dots, t)$.

7.7.E.bis. (Conway skein relation). If $L \searrow \nearrow$, $L \swarrow \nwarrow$ and $L \smile \frown$ are links that coincide outside a ball and look as the subscripts in their notation, in the ball, then

$$(24) \quad \underline{\Delta}^1(L \searrow \nearrow^t) = \underline{\Delta}^1(L \swarrow \nwarrow^t) + (t^2 - t^{-2})\underline{\Delta}^1(L \smile \frown^t).$$

7.7.F.bis. (Doubling axiom). If L' is a link obtained from a link $L = L_1 \cup \dots \cup L_k$ by replacing the L_i by its $(2, 1)$ -cable, then

$$(25) \quad \underline{\Delta}^1(L')(t_1, \dots, t_r) = (T + T^{-1})\underline{\Delta}^1(L)(t_1, \dots, t_{i-1}, t_i^2, t_{i+1}, \dots, t_k),$$

where $T = t_i^2 \prod_{j \neq i} t_j^{2\text{lk}(L_i, L_j)}$.

Statement 7.7.C.bis is a weak counterpart of 7.7.C. The true counterpart would be $\underline{\Delta}^1(L) \in \mathbb{Z}[t_1^2, t_1^{-2}, \dots, t_k^2, t_k^{-2}]$. However, 7.7.C.bis is also sufficient for proving the uniqueness of the invariant satisfying the axioms, because, anyhow, in the proof imitating Turaev's in [18] it is used in combination with the Conway skein relation 7.7.E.bis and

the doubling axiom. The former allows one to prove that $\underline{\Delta}^1(L^t)(t) \in \mathbb{Z}[t^2, t^{-2}]$, after which the strong analog of 7.7.C can easily be deduced from 7.7.C.bis with the help of the doubling axiom 7.7.F.bis.

Obviously, statements 7.7.A.bis, 7.7.B.bis, and 7.7.D.bis are true.

To prove 7.7.C.bis, observe first that for any endomorphism of a one-point object of \mathcal{G}_P^1 , the image under \mathcal{A}^1 is multiplication by a Laurent polynomial in t_1, \dots, t_k . Indeed, the Boltzmann weights for all the generators of $\mathcal{G}_{P_k}^1$ are Laurent polynomials. To calculate $\underline{\Delta}^1(L)$, we can take $\mathcal{A}^1(L_a)$ with a from different components. Let a be an arc on L_1 and b an arc on L_2 . Then both $(t_1^2 - t_1^{-2})^{-1} \mathcal{A}^1(L_a)$ and $(t_2^2 - t_2^{-2})^{-1} \mathcal{A}^1(L_b)$ are multiplication by $\underline{\Delta}^1(L)$. The first of these morphisms is multiplication by an element of $\mathbb{Z}[M_k]$ divided by $t_1^2 - t_1^{-2}$, and the second is multiplication by an element of $\mathbb{Z}[M_k]$ divided by $t_2^2 - t_2^{-2}$. Since $t_1^2 - t_1^{-2}$ and $t_2^2 - t_2^{-2}$ are relatively prime in $\mathbb{Z}[M_k]$, $\underline{\Delta}^1(L)$ is an element of $\mathbb{Z}[M_k]$.

The Conway skein relation 7.7.E.bis follows easily by comparing the Boltzmann weights at the positive and negative crossing points; see the first column of Table 3.

Now, consider the doubling axiom 7.7.F.bis. Since the framing does not matter when the weights of the colors vanish, we may assume that L_i has self-linking number $\frac{1}{2}$ and the $(2, 1)$ -cable has self-linking number 2. Under this choice, the $(2, 1)$ -cable L'_i of L can be obtained as the boundary of the framing surface F (which is a Möbius band) of L_i , and the framing surface of L'_i touches F along the whole of L'_i . Squeezing two parallel arcs of L'_i along F into one arc, we apply the junction relation 5.2.B:

$$\underline{\Delta}^1(L') = (t_i^2 - t_i^{-2})^{-1} \underline{\Delta}^1(L_+) - (t_i^2 - t_i^{-2})^{-1} \underline{\Delta}^1(L_-),$$

where L_+ is a graph obtained from L' by squeezing two parallel arcs on L'_i with the orientation induced by that of L'_i , and L_- is the same graph, but with opposite orientation on the new arc. In L_+ and L_- the new arc has framing along F and weight -1 . Its multiplicity is t_i^2 in L_+ , and t_i^{-2} in L_- .

Now we expand the new arcs along L_i and apply the removing loop relation 5.2.A:

$$\underline{\Delta}^1(L_+) = (t_i^2 - t_i^{-2}) \underline{\Delta}^1(L^1), \quad \underline{\Delta}^1(L_-) = -(t_i^2 - t_i^{-2}) \underline{\Delta}^1(L^2),$$

where L^1 is L with L_i equipped with multiplicity t_i^2 and weight -1 , and L^2 is L with L_i equipped with the opposite orientation, multiplicity t_i^{-2} , and weight -1 . Hence,

$$\underline{\Delta}^1(L') = \underline{\Delta}^1(L^1) + \underline{\Delta}^1(L^2).$$

Next, by (19), we have

$$\underline{\Delta}^1(L^1) = (t_i^2)^{2\text{lk}_L(L_i, L_i)} \prod_{j \neq i} t_j^{2\text{lk}(L_j, L_i)} \underline{\Delta}^1(L_0^1) = t_i^2 \prod_{j \neq i} t_j^{2\text{lk}(L_j, L_i)} \underline{\Delta}^1(L_0^1),$$

where L_0^1 differs from L only by the multiplicity of L_i , which is t_i^2 . Similarly, by (19),

$$\underline{\Delta}^1(L^2) = (t_i^{-2})^{2\text{lk}_L(L_i, L_i)} \prod_{j \neq i} t_j^{-2\text{lk}(L_j, L_i)} \underline{\Delta}^1(L_0^2) = t_i^{-2} \prod_{j \neq i} t_j^{-2\text{lk}(L_j, L_i)} \underline{\Delta}^1(L_0^2),$$

where L_0^2 differs from L only by the multiplicity of L_i , which is t_i^{-2} , and by the orientation of L_i .

It is easily seen that squaring the multiplicity t_i of a link component corresponds to replacing t_i with t_i^2 in all the Boltzmann weights, while simultaneously reversing orientation and reversing the multiplicity of a link component does not change the Boltzmann weights. Therefore, $\underline{\Delta}^1(L_0^1)$ and $\underline{\Delta}^1(L_0^2)$ are obtained from $\underline{\Delta}^1(L)$ by replacing t_i with t_i^2 . This corresponds to the replacement of t_i with t_i^2 in $\underline{\Delta}^1(L)$. The coefficient $t_i^2 \prod_{j \neq i} t_j^{2\text{lk}(L_j, L_i)}$ equals T . □

§8. SPECIAL PROPERTIES OF THE $\mathfrak{sl}(2)$ -ALEXANDER INVARIANT

8.1. First calculations: the unknot and theta graph. The $\mathfrak{sl}(2)$ -Alexander polynomial of the unknot U with 0-framing colored with color A is $\frac{1}{i^{1+A} + i^{1-A}}$. For the unknot U_k with framing $k/2$, $k \in \mathbb{Z}$, colored with A , we have

$$\underline{\Delta}^2(U_k) = \frac{i^{k\frac{A^2-1}{4}}}{i^{1+A} + i^{1-A}}.$$

For any planar theta graph Θ with planar framing and arbitrary coloring, $\underline{\Delta}^2(\Theta) = 1$. This follows from 5.3.A.

8.2. $\mathfrak{sl}(2)$ -Alexander invariant of a tetrahedron’s 1-skeleton. Recall that a planar 1-skeleton of a tetrahedron with planar framing is denoted by T . In a \mathcal{G}_P^2 -coloring of T , two vertices of T are sources and the other two vertices are sinks. Indeed, the augmentation of the 0-cycle stocks – sources is zero because this 0-cycle is the boundary of the weight chain.

8.2.A (The $\mathfrak{sl}(2)$ -Alexander invariant of T). *Let T be a colored planar 1-skeleton of a tetrahedron with planar framing. Then $\underline{\Delta}^2(T)$ is equal to $i^{1+A} + i^{1-A}$, where A is the weight of the edge connecting the two sink vertices of T .*

This is proved by a straightforward calculation, and I leave it to the reader. To save effort, I would recommend choosing the edge for the cut in such a way that there exists a secondary coloring, i.e., an orientation, of this edge extending uniquely to a secondary coloring of T . One can choose an edge connecting a source with a sink and orient it from the source to the sink. □

8.3. Functorial change of colors. Let

$$P_1 = (B_1, W_1, W_1^+ \rightarrow B_1^\times)$$

and

$$P_2 = (B_2, W_2, W_2^+ \rightarrow B_2^\times)$$

be 2-palettes, let S be a subring of B_1 containing W_1 , and let $\beta : S \rightarrow B_2$ be a ring homomorphism such that $\beta(W_1) \subset W_2$ and $\beta(i^w) = i^{\beta(w)}$ for any $w \in W_1$.

Let Γ_1 be a morphism of $\mathcal{G}_{P_1}^2$ such that the weight of each 1-stratum of Γ_1 does not belong to $\beta^{-1}\{x \in W_2 \mid i^{2x+2} = 1\}$. Then, if on each 1-stratum we replace the weight with its image under β (and preserve the orientation component of the coloring), then Γ_1 will turn into a morphism Γ_2 of $\mathcal{G}_{P_2}^2$. Each entry of the matrix representing $\mathcal{A}^2(\Gamma_2)$ in the standard bases is equal to the image under β of the corresponding entry of the matrix representing $\mathcal{A}^2(\Gamma_1)$. This follows immediately from the definition of \mathcal{A}^2 .

Assume now that B_1 and B_2 are fields. Then there is a similar relation for the $\mathfrak{sl}(2)$ -Alexander invariant: if Γ_1 is a $\mathcal{G}_{P_1}^2$ -colored closed framed generic graph in \mathbb{R}^3 , then the replacement of the weights of the 1-strata of Γ_1 with their images under β turns Γ_1 into a $\mathcal{G}_{P_2}^2$ -colored graph Γ_2 , and $\underline{\Delta}^2(\Gamma_2)$ is equal to the image of $\underline{\Delta}^2(\Gamma_1)$ under β .

8.4. Effect of adding numbers divisible by 4 to weights: the case of a link.

8.4.A. *Let L be a \mathcal{G}_P^2 -colored framed link with components L_1, \dots, L_n colored with weights A_1, \dots, A_n , respectively. Let p_1, \dots, p_n be integers divisible by 4, and let L' be a \mathcal{G}_P^2 -colored framed link made of the same components L_1, \dots, L_n as L but colored with the weights $p_1 + A_1, \dots, p_n + A_n$. Then*

$$(26) \quad \underline{\Delta}^2(L') = \underline{\Delta}^2(L) i^{\sum_{i=1}^n \sum_{j=1}^n (1+A_i)p_j \text{lk}(L_i, L_j)}.$$

Proof. This follows from a straightforward calculation of the contribution made by the change of the weights in the Boltzmann weights. Only the Boltzmann weights at half-twist signs and crossing points may change. At a positive half-twist of the i th component the Boltzmann weight is multiplied by $i^{\frac{A_i p_i}{2}}$, at a positive crossing point of the i th and j th components the Boltzmann weight is multiplied by $i^{\frac{(1+A_i)p_j+(1+A_j)p_i}{2}}$. The product of all these factors is $\prod_{i=1}^n i^{\sum_{j=1}^n (1+A_i)p_j \text{lk}(L_i, L_j)}$. \square

8.5. Effect of adding numbers divisible by 4 to weights: the general case. Let Γ be a framed graph equipped with two \mathcal{G}_P^2 -colorings such that on each 1-stratum of Γ their weights differ by an integer divisible by 4. Denote by A_e^k the value of the weight of the k th (with $k = 1, 2$) coloring taken on an oriented (with the orientation shared by the colorings) 1-stratum e . Then $\sum (A_e^1 - A_e^2)e$, where e runs over the set of 1-strata of Γ , is a 1-cycle with coefficients in $4\mathbb{Z}$. This follows from the Admissibility conditions 3.1.A. We denote this cycle by Γ_W . Another cycle has coefficients in $M/2\mathbb{Z}$. This is the weight cycle $\sum (1 + A_e^1)e = \sum (1 + A_e^2)e \pmod 2$. Denote it by Γ_M .

Consider a field of lines on Γ normal to the framing. On each of these lines, choose a short segment centered at the corresponding point of Γ and such that the endpoints of the segments comprise a generic graph $\tilde{\Gamma}$ that is a two-fold covering space of Γ . We equip $\tilde{\Gamma}$ with the orientation such that the covering projection maps it to the orientation of Γ incorporated into the colorings. Each 1-stratum of $\tilde{\Gamma}$ will be equipped with the multiplicity of its image under the projection. The oriented 1-strata of $\tilde{\Gamma}$ equipped with halves of these multiplicities comprise a cycle with coefficients in M . We denote this cycle by $\tilde{\Gamma}_M$. It can be thought of as the multiplicity cycle Γ_M pushed away from Γ in both directions normal to the framing. The cycle $\tilde{\Gamma}_M$ is disjoint from Γ , and hence from Γ_W . Therefore, the linking number $\text{lk}(\tilde{\Gamma}_M, \Gamma_W)$ associated with the multiplication pairing $M/2\mathbb{Z} \times 4\mathbb{Z} \rightarrow M/4\mathbb{Z}$ is well defined. In the case where Γ is the link $L = L_1 \cup \dots \cup L_n$ considered in the preceding subsection, we have

$$\text{lk}(\tilde{\Gamma}_M, \Gamma_W) = \sum_{i=1}^n \sum_{j=1}^n (1 + A_i)p_j \text{lk}(L_i, L_j) \pmod 4$$

(see (26)).

Let Γ_1 and Γ_2 denote the graph Γ equipped with the colorings under consideration. Then

$$(27) \quad \underline{\Delta}^2(\Gamma_2) = i^{\text{lk}(\tilde{\Gamma}_M, \Gamma_W)} \underline{\Delta}^2(\Gamma_1).$$

Formula (27) generalizes (26) and is proved similarly.

8.6. Relationship between the $\text{sl}(2)$ -Alexander invariant and the Conway function.

8.6.A. *Let L be an oriented framed link with components L_1, \dots, L_n equipped with \mathcal{G}_P^2 -coloring. Let L_i be colored with weight A_i . Then*

$$(28) \quad \underline{\Delta}^2(L) = \nabla(L)(i^{1+A_1}, \dots, i^{1+A_n}) i^{\sum_{i,j=1}^n \frac{A_i A_j - 1}{2} \text{lk}(L_i, L_j)}.$$

Proof. We construct (cf. Subsection 4.5) a 1-palette R . It consists of the ring B involved in P , the image M of the homomorphism $W^+ \rightarrow B^\times : A \rightarrow i^A$, the subring $W \subset B$, and the pairing $M \times W \rightarrow M : (i^A, B) \mapsto i^{AB}$. By 4.6.B, any secondary \mathcal{G}_P^2 -coloring of the diagram of L is the image of a secondary \mathcal{G}_R^1 -coloring under Φ_Q^b . Hence, by 4.5.D, $\underline{\Delta}^2(L) = \underline{\Delta}^1(L')$, where L' is the same oriented framed link L , but equipped with the corresponding \mathcal{G}_R^1 -coloring. The \mathcal{G}_R^1 -color of L_i includes the multiplicity $i^{\frac{1+A_i}{2}}$ and the

weight $\frac{1-A_i}{2}$. Let L^0 be the same link L equipped with \mathcal{G}_R^1 -coloring that has the same orientations and multiplicities but zero weights. By 7.5.A,

$$\underline{\Delta}^1(L') = \underline{\Delta}^1(L^0) \prod_{i=1}^n i^{\frac{1+A_i}{2} \left(-2 \sum_{j=1}^n \frac{1-A_j}{2} \text{lk}(L_i, L_j)\right)}.$$

By 7.7.G,

$$\underline{\Delta}^1(L^0) = \nabla(L)(\mathbf{i}^{1+A_1}, \dots, \mathbf{i}^{1+A_n}).$$

Hence,

$$\begin{aligned} \underline{\Delta}^2(L) &= \underline{\Delta}^1(L') \\ (29) \quad &= \underline{\Delta}^1(L^0) \prod_{i=1}^n i^{-(1+A_i) \sum_{j=1}^n \frac{1-A_j}{2} \text{lk}(L_i, L_j)} \\ &= \nabla(L)(\mathbf{i}^{1+A_1}, \dots, \mathbf{i}^{1+A_n}) i^{\sum_{i,j=1}^n \frac{A_i A_j - 1}{2} \text{lk}(L_i, L_j)}. \quad \square \end{aligned}$$

§9. GRAPHICAL SKEIN RELATIONS

9.1. Another look of skein relations. The Alexander invariant for colored framed generic graphs provides terms in which one can rewrite the skein relations of §5 and write similar but more complicated ones.

First, observe that, since

$$\underline{\Delta}^1 \left(\bigcirc_{(t,T)} \right) = \frac{1}{t^2 - t^{-2}},$$

the skein relations (15) and (14) can be interpreted in the following way:

$$(30) \quad \mathcal{A}^1 \left(\begin{array}{c} \uparrow(t,T) \\ \bigcirc \\ \uparrow(t,T) \end{array} \right) = \frac{1}{\underline{\Delta}^1 \left(\bigcirc_{(t,T)} \right)} \mathcal{A}^1 \left(\begin{array}{c} \uparrow \\ \uparrow(t,T) \end{array} \right),$$

$$(31) \quad \mathcal{A}^1 \left(\left| \quad \right| \right) = \sum_{(t,T)} \underline{\Delta}^1 \left(\bigcirc_{(t,T)} \right) \mathcal{A}^1 \left(\begin{array}{c} \diagup \\ \diagdown \end{array} (t,T) \right).$$

Similarly, since

$$\underline{\Delta}^2 \left(\bigcirc_A \right) = \frac{1}{\mathbf{i}^{1+A} + \mathbf{i}^{1-A}},$$

the skein relations (15) and (18) can be interpreted in the following way:

$$(32) \quad \mathcal{A}^2 \left(\begin{array}{c} \uparrow A \\ B \bigcirc C \\ \uparrow A \end{array} \right) = \frac{1}{\underline{\Delta}^2 \left(\bigcirc_A \right)} \mathcal{A}^2 \left(\begin{array}{c} \uparrow \\ \uparrow A \end{array} \right),$$

$$(33) \quad \mathcal{A}^2 \left(\left| \quad \right| \right) = \sum_A \underline{\Delta}^2 \left(\bigcirc_A \right) \mathcal{A}^2 \left(\begin{array}{c} \diagup \\ \diagdown \\ A \end{array} \right).$$

This more graphical form is used in transition to the face models.

9.2. New relations. The relations of this section are formulated in the graphical form and used in the transition to the face models, too. Should we not interpret the coefficients as the Alexander invariants of the graphs related to the graphs involved, it would be difficult even to formulate those relations. Another advantage of the graphical formulation is that we can unite the relations for \mathcal{A}^1 and \mathcal{A}^2 .

9.2.A (Flip relation). For $c = 1, 2$,

$$(34) \quad \mathcal{A}^c \left(\begin{array}{c} u \\ | \\ w \\ | \\ v \end{array} \right) \left(\begin{array}{c} s \\ | \\ r \end{array} \right) = \sum_t \underline{\Delta}^c \left(\begin{array}{c} u \quad s \\ | \quad | \\ w \quad t \\ | \quad | \\ v \quad r \end{array} \right) \underline{\Delta}^c \left(\bigcirc_t \right) \mathcal{A}^c \left(\begin{array}{c} u \quad s \\ | \quad | \\ t \\ | \\ v \quad r \end{array} \right)$$

provided the colorings satisfy the admissibility condition and the sum runs over a nonempty set of colors t (i.e., there exists a color t involved together with the given colors in an admissible coloring of the graphs on the right-hand side of (34)).

Proof. By Theorem 5.1.B, the left-hand side of (34) can be presented as the following linear combination:

$$(35) \quad \mathcal{A}^c \left(\begin{array}{c} u \\ | \\ w \\ | \\ v \end{array} \right) \left(\begin{array}{c} s \\ | \\ r \end{array} \right) = \sum_t x_t \mathcal{A}^c \left(\begin{array}{c} u \quad s \\ | \quad | \\ t \\ | \\ v \quad r \end{array} \right).$$

To find the coefficient of

$$\mathcal{A}^c \left(\begin{array}{c} u \quad s \\ | \quad | \\ p \\ | \\ v \quad r \end{array} \right),$$

we adjoin the graphs

$$\begin{array}{c} p \\ | \\ u \quad s \end{array}, \quad \begin{array}{c} v \quad r \\ | \\ p \end{array}$$

to the top and to the bottom (respectively) of each of the graphs involved in (35). The left-hand side of (35) turns into

$$\mathcal{A}^c \left(\begin{array}{c} p \\ | \\ u \quad s \\ | \quad | \\ w \quad r \\ | \quad | \\ v \quad p \end{array} \right) = \underline{\Delta}^c \left(\begin{array}{c} u \quad s \\ | \quad | \\ w \quad p \\ | \quad | \\ v \quad r \end{array} \right) \left(\underline{\Delta}^c \left(\bigcirc_p \right) \right)^{-1} \text{id}.$$

By (30), the summand corresponding to the color p on the right-hand side turns into the morphism

$$x_p \left(\mathcal{A}^c \left(\bigcirc_p \right) \right)^{-2} \text{id},$$

while the other summands annihilate. Thus, we obtain

$$\underline{\Delta}^c \left(\begin{array}{c} u \quad s \\ | \quad | \\ w \quad p \\ | \quad | \\ v \quad r \end{array} \right) \left(\underline{\Delta}^c \left(\bigcirc_p \right) \right)^{-1} = x_p \left(\underline{\Delta}^c \left(\bigcirc_p \right) \right)^{-2}$$

and

$$x_p = \underline{\Delta}^c \left(\begin{array}{c} i \quad m \\ | \quad | \\ k \quad p \\ | \quad | \\ j \quad l \end{array} \right) \underline{\Delta}^c \left(\bigcirc_p \right). \quad \square$$

The next skein relation is similar to 9.2.A and admits a similar proof. Moreover, it can be deduced from 9.2.A, (10) and (16).

9.2.B (Crossing relation). For $c = 1, 2$,

$$(36) \quad \mathcal{A}^c \left(\begin{array}{c} w \\ | \\ v \\ | \\ u \end{array} \begin{array}{c} t \\ / \\ \backslash \\ t \end{array} \begin{array}{c} s \\ \backslash \\ / \\ t \end{array} \right) = \sum_r \underline{\Delta}^c \left(\begin{array}{c} w \\ | \\ v \\ | \\ u \end{array} \begin{array}{c} t \\ / \\ \backslash \\ t \end{array} \begin{array}{c} s \\ \backslash \\ / \\ t \end{array} \begin{array}{c} r \\ / \\ \backslash \\ r \end{array} \right) \underline{\Delta}^c \left(\bigcirc \right) \mathcal{A}^c \left(\begin{array}{c} w \\ / \\ u \end{array} \begin{array}{c} s \\ \backslash \\ v \end{array} \begin{array}{c} r \\ / \\ v \end{array} \begin{array}{c} t \\ \backslash \\ u \end{array} \right)$$

provided the colorings satisfy the admissibility condition and the sum runs over a nonempty set of colors r (i.e., there exists a color r comprising together with the other colors an admissible coloring of the graphs on the right-hand side of (36)). \square

9.2.C (Triangle relation). For $c = 1, 2$, we have

$$(37) \quad \mathcal{A}^c \left(\begin{array}{c} w \\ / \\ u \end{array} \begin{array}{c} s \\ \backslash \\ t \end{array} \begin{array}{c} r \\ / \\ v \end{array} \right) = \underline{\Delta}^c \left(\begin{array}{c} s \\ / \\ u \end{array} \begin{array}{c} w \\ \backslash \\ t \end{array} \begin{array}{c} r \\ / \\ v \end{array} \right) \mathcal{A}^c \left(\begin{array}{c} w \\ / \\ u \end{array} \begin{array}{c} v \\ \backslash \\ v \end{array} \right)$$

provided the colorings satisfy the admissibility condition.

This also follows from 9.2.A, (10), and (16). \square

9.3. Skein relations for the Alexander invariants. All the skein relations for \mathcal{A}^c discussed in Subsections 9.1 and 9.2 can be thought of as skein relations for $\underline{\Delta}^c$. One should merely replace \mathcal{A}^c by $\underline{\Delta}^c$ and understand $\underline{\Delta}^c$ (the nonclosed diagram of a graph) in the usual way, i.e., as the Alexander invariant of some closed graph whose diagram contains the shown diagram as a fragment that is the only variable part of the diagram: in the other terms of the relation involving a nonclosed diagram, the unshown parts of the diagrams are assumed to be the same.

§10. FACE STATE SUMS

10.1. Colorings of a diagram. Let $\Gamma \subset \mathbb{R}^3$ be a framed generic closed graph. Fix a diagram for Γ .

Let $P = (B, M, W, M \times W \rightarrow M)$ be a 1-palette. By a \mathcal{G}_P^1 -color of a 2-stratum of the diagram⁵ we shall mean an orientation of this 2-stratum, an element t of M with $t^4 \neq 1$ (called the *multiplicity* of the 2-stratum), and an element T of W (called its *weight*).

A \mathcal{G}_P^1 -coloring of the diagram of Γ is an assignment, to each of the 1- and 2-strata of the diagram, of a color such that

- (1) at each triple vertex, the colors of the adjacent edges satisfy the Admissibility condition 2.8.A;
- (2) at each crossing point the colors of the opposite edges (which belong to the image of the same 1-stratum of Γ) coincide;
- (3) for each 1-stratum, its color and the colors of the two adjacent 2-strata satisfy the following condition 10.1.A (cf. 2.8.A):

10.1.A (Admissibility of 1-palette colors at a 1-stratum). *Let t be the multiplicity and T the weight components of a 1-stratum's color, let (t_1, T_1) and (t_2, T_2) be the corresponding components of the colors of the adjacent 2-strata, and let $\epsilon_i = 1$ if the orientation of the i th adjacent 2-stratum induces the orientation of the 1-stratum and $\epsilon_i = -1$ otherwise. Then*

$$(38) \quad tt_1^{\epsilon_1} t_2^{\epsilon_2} = 1,$$

$$(39) \quad T + \epsilon_1 T_1 + \epsilon_2 T_2 = -\epsilon_1 \epsilon_2.$$

⁵Recall that a 2-stratum of a diagram is a connected component of the complement in \mathbb{R}^2 of the projection of Γ .

A 1-stratum of a diagram is said to be *strong* with respect to a \mathcal{G}_P^1 -coloring of the diagram if the orientations of the adjacent 2-strata both induce the orientation of the 1-stratum (i.e., $\epsilon_1 = \epsilon_2 = 1$ in the notation of 10.1.A). It can be seen that the union of the strong 1-strata is a 1-submanifold of the plane. The boundary of this submanifold is the set of images of all strong vertices of the graph. The orientations of strong 1-strata comprise a natural orientation of their union.

Now, let $P = (B, W, W^+ \rightarrow B^\times)$ be a 2-palette. By a \mathcal{G}_P^2 -color of a 2-strata of the diagram we shall mean a pair formed from an orientation of this 2-stratum and an element A of $B \setminus \{x \in W \mid i^{2x+2} = 1\}$ defined up to simultaneous reversing of the orientation and multiplication of A by -1 . The component of the color belonging to B is called the *weight* of the 2-stratum.

A \mathcal{G}_P^2 -coloring of the diagram of Γ is an assignment of a color to each of the 1- and 2-strata of the diagram in such a way that

- (1) at each triple vertex the colors of the adjacent edges satisfy the Admissibility condition 3.4.A;
- (2) at each crossing point the colors of the opposite edges (which belong to the image of the same 1-stratum of Γ) coincide;
- (3) for each 1-stratum, its color and the colors of the two adjacent 2-strata satisfy the following condition 10.1.B (cf. 3.4.A):

10.1.B (Admissibility of 2-palette colors at a 1-stratum). *Let a 1-stratum be colored with weight A , and let A_1 and A_2 be the weight components of the colors of the adjacent 2-strata. Let $\epsilon_i = 1$ if the orientation of the i th adjacent 2-stratum induces the orientation of the 1-stratum and $\epsilon_i = -1$ otherwise. Then*

$$(40) \quad A + \epsilon_1 A_1 + \epsilon_2 A_2 = \pm 1.$$

The sign on the right-hand side of (40) depends on the orientation of the 1-stratum. The orientation for which the sign is plus is said to be *defined* by the \mathcal{G}_P^2 -coloring.

A coloring of Γ defines a coloring of the 1-strata of the diagram. A coloring of the diagram extending this one is called a *face extension* of the coloring of Γ . Making a face extension, one should care only about the last of the three conditions in the definitions 10.1.A or 10.1.B of \mathcal{G}_P^1 - or \mathcal{G}_P^2 -colorings of a diagram.

10.2. Alexander invariant of colored strata. Given a diagram with \mathcal{G}_P^c -coloring, we associate a \mathcal{G}_P^c -colored framed generic graph with each strata of the diagram.

For a 2-stratum colored with u , this graph is $\bigcirc u$, i.e., an unknot with zero framing colored with u :



Recall that the Alexander invariant of $\bigcirc u$ is $(t^2 - t^{-2})^{-1}$ if $c = 1$ and $u = (t, T)$, and $(i^{1+A} + i^{1-A})^{-1}$ if $c = 2$ and $u = A$.

For a 3-valent vertex whose adjacent edges are colored with u , v , and w and the adjacent 2-strata are colored with r , s , and t as shown in Figure 10, this graph is the 1-skeleton of a tetrahedron embedded in a plane, with planar framing. It is obtained by selecting small arcs of the 1-strata adjacent to the vertex and joining their free endpoints with three arcs in the plane. The 1-strata's orientations included in the colors induce orientations on the former three arcs. The adjacent 2-strata's orientations included into the colors of these 2-strata induce orientations on the latter three arcs as on the boundary

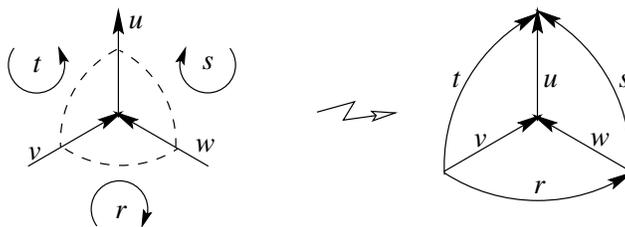


FIGURE 10.

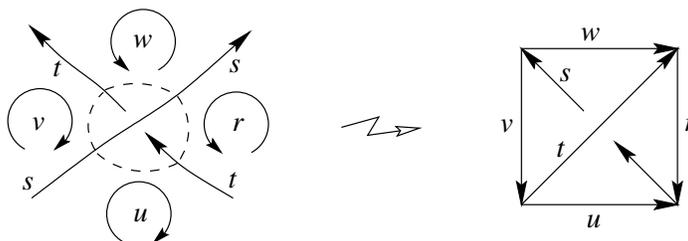
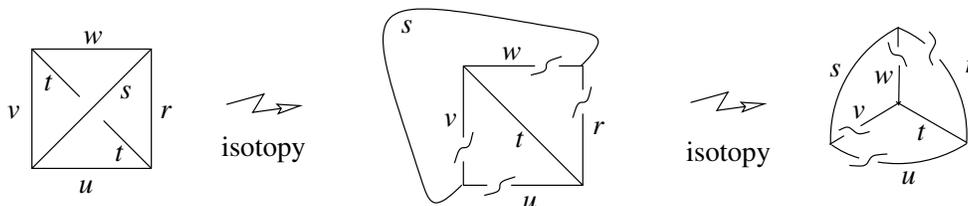


FIGURE 11.

of the parts of the 2-strata separated from the vertex by the arcs. See Figure 10. If $c = 1$ and the initial 3-valent vertex is strong, then the orientation at it is inherited by the corresponding vertex of the tetrahedron. At the other strong vertices, the tetrahedron is equipped with the counterclockwise orientation.

For a crossing point of strings colored with s and t , with the adjacent 2-strata colored with u , v , w , and r as in Figure 11, this graph is again the 1-skeleton of a tetrahedron. However, this time it is not embedded in the plane of the picture, but is presented by a diagram which looks like a square with diagonals, with blackboard framing. The diagram is obtained from a regular neighborhood of the crossing point in the same way as in the case of a 3-valent vertex: select small arcs of the 1-strata adjacent to the crossing point and join their free endpoints with four arcs surrounding the crossing point in the plane. As above, the orientations of the 1-strata induce the orientations on the former arcs, while the orientations of the adjacent 2-strata induce orientations on the latter four arcs in the same way as above: as on the boundary of the parts of the 2-strata separated from the crossing point by the arcs. See Figure 11. At strong vertices the graph is equipped with the counterclockwise orientation.

The Alexander invariant of this 1-skeleton of the tetrahedron differs from the Alexander invariant of the graph corresponding to a 3-valent vertex, because of the difference between their framings. The latter is isotopic to the former, but with a nonblackboard framing:



Thus, here the Alexander invariant involves an additional factor. If $c = 1$, the graph has strong vertices, and they do not belong to the same diagonal of the original square

diagram, then the factor -1 also appears, because the isotopy turns one strong vertex upside down.

We have not associated any graph with the 1-strata. This could be done in a natural way, but would not contribute to the state sum, which is our main goal here. The reason is that the graph associated with a 1-stratum should be a flat theta graph, and the Alexander invariant of a flat theta graph is equal to 1.

The last ingredient that we need to introduce here is the *twist factor*. It is associated with each half-twist sign on a 1-stratum. For $c = 1$ and a stratum colored with multiplicity t and weight T , this is t^{-T} if the half-twist is positive, and t^T if it is negative. For $c = 2$ and a stratum colored with weight A , this is $i^{\frac{A^2-1}{4}}$ if the half-twist is positive, and $i^{\frac{1-A^2}{4}}$ if it is negative.

10.3. Spot faces, a shadow domain, and a contour. Let $\Gamma \subset \mathbb{R}^3$ be a framed \mathcal{G}_P^c -colored generic graph. Fix several faces of its diagram. They will be referred to as *spot faces*. The colors of the spot faces will be fixed in such a way that the Admissibility conditions 10.1.A are satisfied for any 1-stratum adjacent to two spot faces.

Let C be a smooth 1-dimensional closed submanifold of \mathbb{R}^2 that is in general position with respect to the diagram of Γ . This means that C contains no vertex of the diagram, is transversal to its 1-strata, and does not pass over the signs \smile and \frown of half-twists. Let \mathcal{S} be an open subset of \mathbb{R}^2 with boundary C . We assume that each of the spot faces intersects \mathcal{S} . The set \mathcal{S} will be called the *shadow domain* and C the *contour*.

The shadow domain is decomposed into its intersections with the strata of the diagram of Γ . Some of the strata have fixed colors. These are all the 1-strata and the intersections of \mathcal{S} with the spot faces. Let $\text{Col}_P(\mathcal{S})$ denote the set of all \mathcal{G}_P^c -colorings of the 1- and 2-strata of \mathcal{S} satisfying all the conditions of Subsection 10.1 and extending the above-mentioned coloring of 1-strata and spot faces.

The contour C and the part of the diagram of Γ contained in the complement of \mathcal{S} constitute a diagram of a generic graph Γ_C . Although its embedding in \mathbb{R}^3 is not specified, it is well defined up to ambient isotopy, since its diagram is given. The blackboard framing of C together with the framing signs on the part of the diagram of Γ determine a framing of Γ_C .

The intersection points with the diagram divide C into arcs, which are 1-strata of Γ_C . Each of these 1-strata lies on the boundary of a 2-stratum of \mathcal{S} . For $c \in \text{Col}_P(\mathcal{S})$, to each arc of C we assign the color of the adjacent 2-stratum of \mathcal{S} . The coloring of Γ_C obtained in this way satisfied the admissibility condition. The skein class of this colored framed generic graph is denoted by $\Gamma_{C,c}$.

10.4. State sums. We denote by $\tau(\mathcal{S})$ the product of the twist factors of all the half-twist signs in \mathcal{S} . For $x \in \text{Col}_P(\mathcal{S})$ and a stratum Σ of \mathcal{S} , denote by $\underline{\Delta}^c(\Sigma, x)$ the Alexander invariant of the colored framed generic graph associated with Σ for the coloring x . We put

$$Z(\mathcal{S}) = \tau(\mathcal{S}) \sum_{x \in \text{Col}_P(\mathcal{S})} \underline{\Delta}^c(\Gamma_{C,x}) \prod_{\Sigma} \underline{\Delta}^c(\Sigma, x)^{\chi(\Sigma)}.$$

Here the products are taken over all the strata s of \mathcal{S} , and χ is the Euler characteristic.

10.4.A (Moving contour theorem). $Z(\mathcal{S})$ does not change under an isotopy of C in \mathbb{R}^2 during which the spot faces remain intersecting the shadow domain and $\text{Col}_P(\mathcal{S}) \neq \emptyset$.

In the simplest and most interesting cases, by an isotopy of C one can shrink the shadow domain to a subset of the spot faces. Then the state sum reduces to one term, and it is easy to relate it to the Alexander invariant of the original graph Γ .

On the other hand, by an isotopy of C one can make the shadow domain engulf most of the diagram. Then the graph Γ_C may become pretty standard, and it will be easy to calculate the factors $\Delta^c(\Gamma_{C,x})$.

Thus, Theorem 10.4.A provides a way to express the Alexander invariant of an arbitrary colored framed generic graph in terms of state sums involving the well-known Alexander invariants of standard unknotted graphs. The state sum is determined by a diagram of the original graph.

Theorem 10.4.A is similar to the Kirillov–Reshetikhin theorem [7] for the invariants based on the quantized $\mathfrak{sl}(2)$ (i.e., generalizations of the Jones polynomial). However, there are two important points where they differ. First, the terms of the $\mathfrak{sl}(2)$ state sum are more complicated functions of colors. For example, in our state sums the factors corresponding to the edges are invisible (since the Alexander invariant of a theta graph is 1). The factors corresponding to other strata are also much simpler in our case. Second, the requirement that the set of colorings be nonempty does not appear in the $\mathfrak{sl}(2)$ case, while here it is crucial.

Proof of 10.4.A. Under a generic isotopy of C , the picture changes topologically only when C passes through vertices of the diagram of Γ or through the signs of half-twists, or becomes tangent to a branch of the diagram. Therefore, we need to check the invariance of $Z(\mathcal{S})$ only under the moves shown in Figure 12.

Under the first move the set $\text{Col}_P(\mathcal{S})$ of colorings does not change (in the sense that there is a natural one-to-one correspondence between the colorings before and after the move). Consider the case where $c = 1$. By (10), $\underline{\Delta}^1(\Gamma_{C,x})$ is divided by t^{-T} for each $x \in \text{Col}_P(\mathcal{S})$, where t is the multiplicity and T the weight of the color of the arc involved in the move. On the other hand, the total twist factor $\tau(\mathcal{S})$ of the shadow domain is multiplied by t^{-T} , because an additional half-twist appeared on it. Therefore, $Z(\mathcal{S})$ does not change under the first move. Similar arguments prove this for $c = 2$.

To prove invariance with respect to the second move, we decompose the sets of colorings before and after the move to the classes of colorings coinciding outside the fragment that changes. Such a class before the move consists of a single coloring, say x , while a class after the move contains as many colorings as colors can be put in the new triangle. Denote this color by t , and the corresponding coloring by x_t . Suppose the adjacent strata are colored as shown in Figure 12. The Alexander invariants $\underline{\Delta}^c(\Gamma_{C,x})$ for the colorings in these classes before and after the move are related by (34) (with \mathcal{A}^c replaced by $\underline{\Delta}^c$; see Subsection 9.3). The applicability of (34) follows from the assumption that $\text{Col}_P(\mathcal{S}) \neq \emptyset$ during the movement. We rewrite this relation as follows:

$$\underline{\Delta}^c(\Gamma_{C,x}) = \sum_t \underline{\Delta}^c \left(\text{Diagram with vertices } u, w, s, v, r \text{ and arc } t \right) \underline{\Delta}^c \left(\text{Circle with arc } t \right) \underline{\Delta}^c(\Gamma_{C,x_t}).$$

On the other hand,

$$\prod_{\Sigma} \underline{\Delta}^c(\Sigma, x_t)^{\chi(\Sigma)} = \prod_{\Sigma} \underline{\Delta}^c(\Sigma, x)^{\chi(\Sigma)} \underline{\Delta}^c \left(\text{Diagram with vertices } u, w, s, v, r \text{ and arc } t \right) \underline{\Delta}^c \left(\text{Circle with arc } t \right),$$

because new strata appear in the shadow, and these strata are:

- (1) A new trivalent vertex, which contributes

$$\underline{\Delta}^c \left(\text{Diagram with vertices } u, w, s, v, r \text{ and arc } t \right).$$

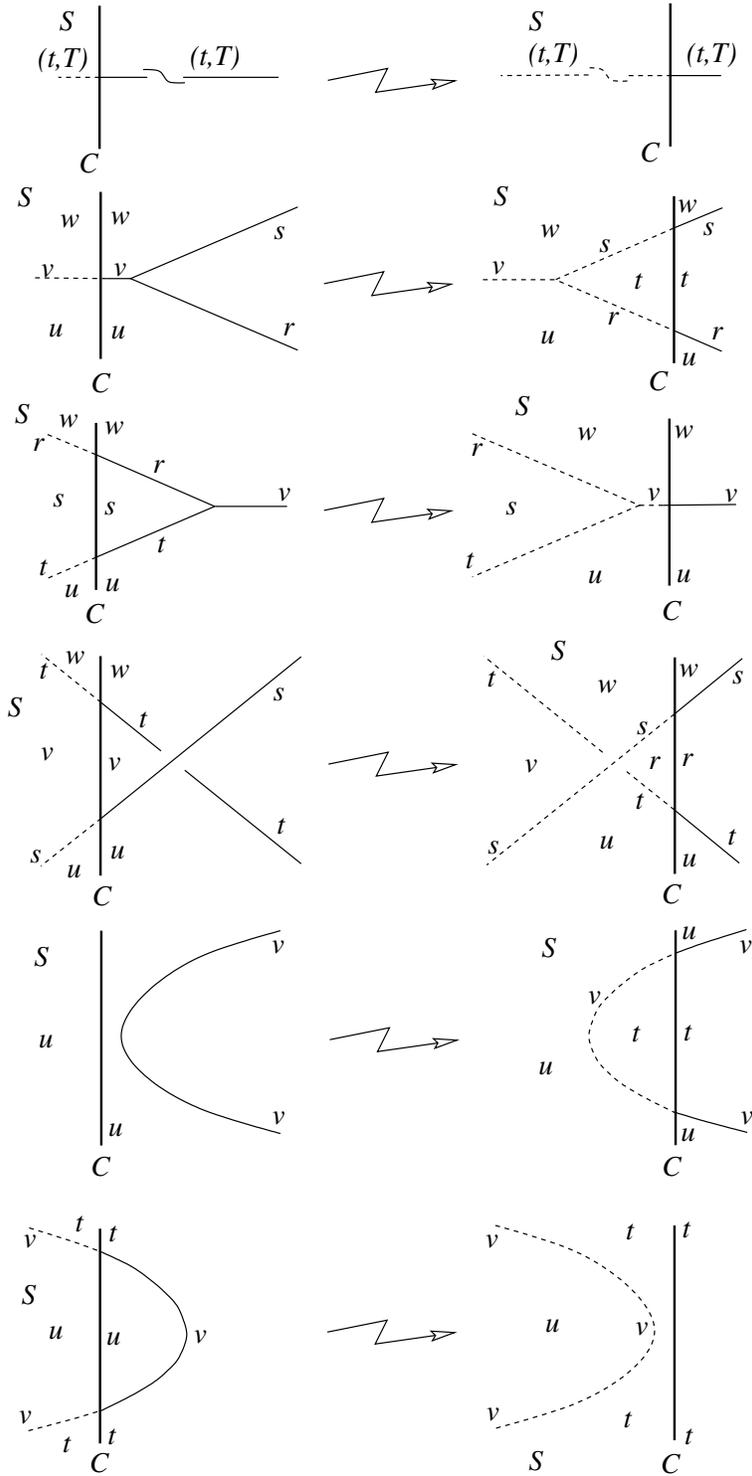


FIGURE 12. Moves of $\Gamma_{C,x}$ under generic isotopies of C . The fragments of $\Gamma_{C,x}$ are shown in solid, while the pieces of the diagram of Γ that are in the shadow domain are shown by dashed arcs.

- (2) New edges without half-twists. They do not contribute.
- (3) A new triangle colored with t . It contributes

$$\underline{\Delta}^c \left(\bigcirc t \right).$$

Therefore,

$$\begin{aligned}
 & \underline{\Delta}^c(\Gamma_{C,x}) \prod_{\Sigma} \underline{\Delta}^c(\Sigma, x)^{\chi(\Sigma)} \\
 (41) \quad &= \sum_t \underline{\Delta}^c \left(\begin{array}{c} u \quad w \quad s \\ \text{---} \quad \text{---} \quad \text{---} \\ v \quad \text{---} \quad r \quad t \end{array} \right) \underline{\Delta}^c \left(\bigcirc t \right) \underline{\Delta}^c(\Gamma_{C,x_t}) \prod_{\Sigma} \underline{\Delta}^c(\Sigma, x)^{\chi(\Sigma)} \\
 &= \sum_t \underline{\Delta}^c(\Gamma_{C,x_t}) \prod_{\Sigma} \underline{\Delta}^c(\Sigma, x_t)^{\chi(\Sigma)}.
 \end{aligned}$$

Invariance with respect to the third, fourth, fifth, and sixth moves is proved in the same way as invariance with respect to the second move, but by using, instead of (34),

- (37) in the case of the third move,
- (36) in the case of the fourth move,
- (31) and (33) in the case of the fifth move, and
- (30) and (32) in the case of the sixth move. □

10.5. Face state sums for the Alexander invariant. The moving contour theorem 10.4.A gives no explicit recipe for calculating the Alexander invariant. Here we deduce explicit formulas for this. The formulas depend on the choice of spot faces and the movement of the contour to which the moving contour theorem is applied.

Of course, the spot faces are needed for making the state sum finite. In our case they also play a role that does not emerge in the $sl(2)$ case: to make the state sum nonzero. For example, if a single spot face is chosen (as is usual in the $sl(2)$ case), then $Z(\mathcal{S})$ vanishes. Indeed, by the moving contour theorem, the contour can be made disjoint from the projection of the graph under consideration. This makes Γ_C splittable and annihilates $\underline{\Delta}^c(\Gamma_C)$ by what was said in Subsection 6.5.

The next possibility for the choice of spot faces is to choose two faces next to each other. This works as follows.

10.5.A (Face state sum at an arc). *Let $\Gamma \subset \mathbb{R}^3$ be a \mathcal{G}_P^c -colored framed closed generic graph, e a 1-stratum of its diagram, and s the color of e . Let f_1, f_2 be the 2-strata of the diagram adjacent to e . Color them with colors u and v such that the admissibility condition on e is satisfied. Denote by $\text{Col}_P(u, v)$ the set of all the admissible \mathcal{G}_P^c -colorings of the diagram of Γ that are face extensions of the coloring of Γ and have the colors described above on f_1 and f_2 . Let τ be the product of the twist factors of all the half-twist signs on the diagram. If $\text{Col}_P(u, v) \neq \emptyset$, then*

$$(42) \quad \underline{\Delta}^c(\Gamma) = \frac{\tau \underline{\Delta}^c \left(\bigcirc s \right)}{\underline{\Delta}^c \left(\bigcirc u \right) \underline{\Delta}^c \left(\bigcirc v \right)} \sum_{x \in \text{Col}_P(u, v)} \prod_{\Sigma} \underline{\Delta}^c(\Sigma, x)^{\chi(\bar{\Sigma})},$$

where $\bar{\Sigma} = \Sigma$ if Σ is bounded in \mathbb{R}^2 ; otherwise $\bar{\Sigma}$ is obtained by adding to Σ the point at infinity in the one-point compactification S^2 of \mathbb{R}^2 .

Proof. First, for the role of a contour C we take a small circle bounding in \mathbb{R}^2 a disk \mathcal{S} and intersecting the diagram only in an arc contained in e . This disk is a shadow

domain. Then $\text{Col}_P(u, v)$ consists of one coloring, say x ,

$$\underline{\Delta}^c(\Gamma_{C,x}) = \frac{\underline{\Delta}^c(\Gamma)}{\underline{\Delta}^c\left(\bigcirc_s\right)},$$

and

$$\prod_{\Sigma} \underline{\Delta}^c(\Sigma, x)^{\chi(\Sigma)} = \underline{\Delta}^c\left(\bigcirc_u\right) \underline{\Delta}^c\left(\bigcirc_v\right).$$

Therefore,

$$Z(\mathcal{S}) = \underline{\Delta}^c(\Gamma) \frac{\underline{\Delta}^c\left(\bigcirc_u\right) \underline{\Delta}^c\left(\bigcirc_v\right)}{\underline{\Delta}^c\left(\bigcirc_s\right)}.$$

Now we expand C until the moment when \mathcal{S} engulfs the entire diagram. If $\text{Col}_P(u, v) \neq \emptyset$, then we can apply Theorem 10.4.A, concluding that $Z(\mathcal{S})$ does not change. At the final moment we have

$$\underline{\Delta}^c(\Gamma_{C,x}) = \underline{\Delta}^c\left(\bigcirc_t\right),$$

where t is the color of C . Therefore,

$$Z(\mathcal{S}) = \tau \sum_{x \in \text{Col}_P(u,v)} \underline{\Delta}^c\left(\bigcirc_t\right) \prod_{\Sigma} \underline{\Delta}^c(\Sigma, x)^{\chi(\Sigma)} = \tau \sum_{x \in \text{Col}_P(u,v)} \prod_{\Sigma} \underline{\Delta}^c(\Sigma, x)^{\chi(\bar{\Sigma})}.$$

Comparing this with the initial value of $Z(\mathcal{S})$, we get the desired result. □

§11. APPENDIX 1. QUANTUM $\mathfrak{gl}(1|1)$ AND ITS IRREDUCIBLE REPRESENTATIONS

11.1. Superalgebra $\mathfrak{gl}(1|1)$. Recall that, in algebra, the prefix *super-* means that the object under consideration is \mathbb{Z}_2 -graded or, at least, is related to a \mathbb{Z}_2 -graded object.

For example, a vector superspace of (super-)dimension $(p|q)$ is a \mathbb{Z}_2 -graded vector space $V = V_0 \oplus V_1$ such that the 0-summand V_0 is p -dimensional and the 1-summand V_1 is q -dimensional. Let $\mathbb{C}^{p|q}$ denote the superspace $\mathbb{C}^p \oplus \mathbb{C}^q$. The 0-summand of a superspace is called its *bosonic part*, and the vectors belonging to it are called *bosons*, while the 1-summand is called the *fermionic part*, and its elements are called *fermions*.

The endomorphisms of a superspace $\mathbb{C}^{p|q}$ comprise a Lie superalgebra, which is denoted by $\mathfrak{gl}(p|q)$. It consists of $(p|q) \times (p|q)$ -matrices $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. This is a *super-algebra*, its bosonic part consists of $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, and the fermionic part consists of $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. On bosons and fermions, the Lie (super)brackets are defined as the supercommutator $[X, Y] = XY - (-1)^{\deg X \deg Y} YX$, where $\deg X$ is 0 if X is a boson and 1 if X is a fermion. This extends linearly to the entire $\mathfrak{gl}(p|q)$.

The *supertrace* $\text{str } M$ of $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is defined to be $\text{tr } A - \text{tr } D$. This is a natural bilinear nondegenerate form

$$\mathfrak{gl}(p|q) \times \mathfrak{gl}(p|q) \rightarrow \mathbb{C} : (x, y) \mapsto \text{str}(xy).$$

The Lie superalgebra $\mathfrak{gl}(1|1)$ is generated by the 4 elements

$$(43) \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

subject to the following relations:

$$(44) \quad \begin{aligned} \{X, Y\} (= XY + YX) &= E, \\ X^2 = Y^2 &= 0, \\ [G, X] = X, \quad [G, Y] &= -Y, \\ [E, G] = [E, X] = [E, Y] &= 0. \end{aligned}$$

In fact, these relations define $\mathfrak{gl}(1|1)$.

11.2. Quantization. The universal enveloping algebra $U\mathfrak{gl}(1|1)$ admits a deformation (see Kulish [9]) with parameter h , resulting in the quantum superalgebra $U_q \mathfrak{gl}(1|1)$. Only the first of relations (44) changes: its right-hand side E is replaced by the “quantum E ”, that is, by $\frac{q^E - q^{-E}}{q - q^{-1}}$ with $q = e^h$. To make this nonalgebraic expression meaningful, the algebra is supplemented with formal power series in h .

Denote

$$q^E = e^{hE} = \sum_{n=0}^{\infty} \frac{h^n E^n}{n!}$$

by K . Since E is central, K is also central.

The algebra A of formal power series in h generated by X, Y, E , and G and satisfying the relations

$$(45) \quad \begin{aligned} \{X, Y\} &= \frac{K - K^{-1}}{q - q^{-1}}, \\ X^2 = Y^2 &= 0, \\ [E, X] = 0, \quad [E, Y] = 0, \quad [E, G] &= 0, \\ [G, X] = X, \quad [G, Y] &= -Y \end{aligned}$$

can be equipped with a coproduct

$$(46) \quad \Delta : A \rightarrow A \otimes A : \quad \begin{aligned} \Delta(E) &= 1 \otimes E + E \otimes 1, \\ \Delta(G) &= 1 \otimes G + G \otimes 1, \\ \Delta(X) &= X \otimes K^{-1} + 1 \otimes X, \\ \Delta(Y) &= Y \otimes 1 + K \otimes Y, \end{aligned}$$

a counit

$$(47) \quad \epsilon : A \rightarrow \mathbb{C} : \quad \epsilon(X) = \epsilon(Y) = \epsilon(E) = \epsilon(G) = 0,$$

and an antipode

$$(48) \quad s : A \rightarrow A : \quad \begin{aligned} s(E) &= -E, \quad s(G) = -G, \\ s(X) &= -XK, \quad s(Y) = -YK^{-1}; \end{aligned}$$

thus, it turns into a Hopf (super)algebra. It is denoted by $U_q \mathfrak{gl}(1|1)$ and is called the *quantum supergroup* $U_q \mathfrak{gl}(1|1)$.

It is easy to check that

$$(49) \quad \begin{aligned} \Delta(K) &= K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \\ s(K) &= K^{-1}, \quad \epsilon(H) = 1. \end{aligned}$$

11.3. R -Matrix. Recall that a Hopf algebra A is said to be *quasitriangular* if it is equipped with a *universal R -matrix* R that is an invertible element of $A^{\otimes 2} = A \otimes A$ satisfying the following three conditions:

$$(50) \quad P \circ \Delta(a) = R\Delta(a)R^{-1} \quad \text{for any } a \in A,$$

$$(51) \quad (\Delta \otimes \text{id}_A)(R) = R_{13}R_{23},$$

$$(52) \quad (\text{id}_A \otimes \Delta)(R) = R_{13}R_{12}.$$

Here P is the permutation homomorphism $a \otimes b \mapsto (-1)^{\deg a \deg b} b \otimes a : A \otimes A \rightarrow A \otimes A$, and $R_{12} = R \otimes 1 \in A^{\otimes 3}$, $R_{13} = (\text{id} \otimes P)(R_{12})$, $R_{23} = 1 \otimes R$.

The quantum supergroup $U_q \text{gl}(1|1)$ is known to be quasitriangular with the universal R -matrix

$$(53) \quad R = (1 + (q - q^{-1})(X \otimes Y)(K \otimes K^{-1})) q^{-E \otimes G - G \otimes E}.$$

Each quasitriangular Hopf superalgebra has a unique element

$$u = \sum_i (-1)^{\deg \alpha_i \deg \beta_i} s(\beta_i) \alpha_i,$$

where $R = \sum_i \alpha_i \otimes \beta_i$. In $U_q \text{gl}(1|1)$ we have

$$u = q^{2EG} (1 + (q - q^{-1})K Y X).$$

Then

$$s(u) = q^{2EG} (1 - (q - q^{-1})K^{-1} X Y);$$

cf. [16, 14].

Recall that a universal twist of a quasitriangular Hopf superalgebra A is an element $v \in A$ such that $v^2 = us(u)$. In $U_q \text{gl}(1|1)$ this means that

$$v^2 = us(u) = q^{4EG} (1 + (q - q^{-1})(K Y X - K^{-1} X Y));$$

cf. [16] and [14]. It is easy to check that this equation in $U_q \text{gl}(1|1)$ has a solution

$$v = q^{2EG} (K - (q - q^{-1})X Y).$$

11.4. (1|1)-dimensional irreps. By a $U_q \text{gl}(1|1)$ -module (of dimension $(p|q)$) we shall mean a vector superspace V of dimension $(p|q)$ equipped with a homomorphism of $U_q \text{gl}(1|1)$ to the superalgebra of endomorphisms of V (for instance, to $\text{gl}(p|q)$).

It is assumed that this homomorphism respects the \mathbb{Z}_2 -grading of the superstructure. In particular, the images of the bosonic E and G map bosons to bosons and fermions to fermions, while the images of the fermions X and Y map bosons to fermions and fermions to bosons.

Recall that a module is said to be *cyclic* if it is generated as a module by one vector, and *irreducible* if it contains no proper submodule.

There are two families of irreducible (1|1)-dimensional $U_q \text{gl}(1|1)$ -modules. In each of them the modules are parametrized by two parameters, which are denoted by j and J and run over $\mathbb{C} \setminus \{\pi n \sqrt{-1}/2h : n \in \mathbb{Z}\}$ and \mathbb{C} , respectively.

The module of the first family, corresponding to (j, J) , is described by the following formulas:

$$(54) \quad (j, J)_+ : U_q \text{gl}(1|1) \rightarrow \text{gl}(1|1) :$$

$$E \mapsto \begin{pmatrix} 2j & 0 \\ 0 & 2j \end{pmatrix}, \quad G \mapsto \begin{pmatrix} \frac{J-1}{2} & 0 \\ 0 & \frac{J+1}{2} \end{pmatrix},$$

$$X \mapsto \begin{pmatrix} 0 & 0 \\ q^{2j-q^{-2j}} & 0 \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We recall that $\frac{q^n - q^{-n}}{q - q^{-1}}$ is denoted by $[n]_q$ and is called the *quantum n*. Hence, $(j, J)_+ : X \mapsto \begin{pmatrix} 0 & 0 \\ [2j]_q & 0 \end{pmatrix}$.

The standard basis vectors in $\mathbb{C}^{1|1}$ are denoted by e_0 (boson) and e_1 (fermion). On these vectors, the generators of $U_q \mathfrak{gl}(1|1)$ act via $(t, w)_+$ as follows:

$$(55) \quad Ee_0 = 2je_0, \quad Ee_1 = 2je_1;$$

$$(56) \quad Ge_0 = \frac{J-1}{2}e_0, \quad Ge_1 = \frac{J+1}{2}e_1;$$

$$(57) \quad Xe_0 = [2j]_q e_1, \quad Xe_1 = 0;$$

$$(58) \quad Ye_0 = 0, \quad Ye_1 = e_0.$$

It is easy to show that $u \in U_q \mathfrak{gl}(1|1)$ (see Subsection 11.2) acts in $(j, J)_+$ as multiplication by $q^{2j(J+1)}$ and v acts as multiplication by q^{2jJ} . Consequently, $v^{-1}u$ acts as multiplication by q^{2j} .

Another family of irreducible $U_q \mathfrak{gl}(1|1)$ -modules, denoted by $(j, J)_-$, is obtained from $(j, J)_+$ by switching the bosonic and fermionic subspaces. We change also the signs of the parameters in anticipation of 11.5.A:

$$(59) \quad (j, J)_- : U_q \mathfrak{gl}(1|1) \rightarrow \mathfrak{gl}(1|1) : \begin{aligned} E &\mapsto \begin{pmatrix} -2j & 0 \\ 0 & -2j \end{pmatrix}, & G &\mapsto \begin{pmatrix} \frac{-J+1}{2} & 0 \\ 0 & \frac{-J-1}{2} \end{pmatrix}, \\ X &\mapsto \begin{pmatrix} 0 & -[2j]_q \\ 0 & 0 \end{pmatrix}, & Y &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

The operator $u \in U_q \mathfrak{gl}(1|1)$ acts in $(j, J)_-$ as multiplication by $q^{2j(J-1)}$ and v as multiplication by q^{2jJ} , so that $v^{-1}u$ acts as multiplication by q^{-2j} .

11.4.A (Classification of $(1|1)$ -dimensional irreps). *An irreducible $U_q \mathfrak{gl}(1|1)$ -module of dimension $(1|1)$ is isomorphic to a module of one of the families (54), (59). No module of one of these families is equivalent to a module of the other.* \square

11.5. Duality. Recall that, for a finite-dimensional A -module V , where A is a Hopf superalgebra over a field k , the dual space $V^\vee = \text{Hom}_k(V, k)$ is equipped with the structure of an A -module by the formula $(Lc)(v) = c((-1)^{\deg L \deg c} s(L)(v))$, where $L \in A$, $c \in V^\vee$, $v \in V$, and s is the antipode of A . This representation is said to be *dual* to V .

11.5.A (Dual $(1|1)$ -dimensional modules). *The modules $(j, J)_+$ and $(j, J)_-$ are dual to each other. There is a canonical isomorphism $(j, J)_+^\vee \rightarrow (j, J)_-$ described by the formulas $e^0 \mapsto e_0$, $e^1 \mapsto q^{2j}e_1$, where e^i is the generator of $(j, J)_+^\vee$ from the basis superdual to the standard basis e_0, e_1 of $(j, J)_+$; this generator is defined by $e^i(e_j) = (-1)^{ij} \delta_j^i$.*

Duality allows one to define various pairings and copairings (cf. [16]). First, there is an obvious pairing

$$(60) \quad \blacktriangleright : (j, J)_- \times (j, J)_+ \rightarrow \mathbb{C} : (x, y) \mapsto x(y).$$

Here we identify $(j, J)_-$ with $((j, J)_+)^{\vee}$ by the isomorphism of 11.5.A. This pairing acts as follows:

$$\blacktriangleright : (e_a, e_b) \mapsto (-1)^a q^{-2ja} \delta_{ab}.$$

Another pairing is obtained from this one by a ‘‘quantum supertransposition’’:

$$(61) \quad \blacktriangleleft : (j, J)_+ \times (j, J)_- \rightarrow \mathbb{C} : (x, y) \mapsto (-1)^{\deg x \deg y} y(v^{-1}ux).$$

The factor $(-1)^{\deg x \deg y}$ appears here because of the transposition of x and y (this is the contribution of supermathematics). On the basis vectors, the pairing acts as follows:

$$\blacktriangleleft : (e_a, e_b) \mapsto q^{2j(1-a)} \delta_{ab}.$$

There is an obvious (super)copairing

$$(62) \quad \smile : \mathbb{C} \rightarrow (j, J)_+ \otimes (j, J)_- : 1 \mapsto e_0 \otimes e_0 - e_1 \otimes q^{2j} e_1,$$

and a “quantum supertransposed” copairing

$$(63) \quad \smile : \mathbb{C} \rightarrow (j, J)_- \otimes (j, J)_+ :$$

$$1 \mapsto \sum_{a=0}^1 (-1)^a e_a \otimes (u^{-1} v e_a) = q^{-2j} e_0 \otimes e_0 + e_1 \otimes e_1.$$

11.6. Tensor product $(i, I)_+ \otimes (j, J)_+$.

11.6.A (Lemma. Action of generators). *The generators E, G, X, Y of $U_q \mathfrak{gl}(1|1)$ act in $(i, I)_+ \otimes (j, J)_+$ as follows:*

$$(64) \quad \begin{array}{ll} E : & \begin{array}{l} e_0 \otimes e_0 \mapsto 2(i+j)(e_0 \otimes e_0), \\ e_1 \otimes e_1 \mapsto 2(i+j)(e_1 \otimes e_1), \\ e_0 \otimes e_1 \mapsto 2(i+j)(e_0 \otimes e_1), \\ e_1 \otimes e_0 \mapsto 2(i+j)(e_1 \otimes e_0); \end{array} \\ X : & \begin{array}{l} e_0 \otimes e_0 \mapsto [2j]_q(e_0 \otimes e_1) \\ \quad + q^{-2j}[2i]_q(e_1 \otimes e_0), \\ e_1 \otimes e_1 \mapsto 0, \\ e_0 \otimes e_1 \mapsto q^{-2j}[2i]_q(e_1 \otimes e_1), \\ e_1 \otimes e_0 \mapsto -[2j]_q(e_1 \otimes e_1); \end{array} \\ G : & \begin{array}{l} e_0 \otimes e_0 \mapsto \frac{I+J-2}{2}(e_0 \otimes e_0), \\ e_1 \otimes e_1 \mapsto \frac{I+J+2}{2}(e_1 \otimes e_1), \\ e_0 \otimes e_1 \mapsto \frac{I+J}{2}(e_0 \otimes e_1), \\ e_1 \otimes e_0 \mapsto \frac{I+J}{2}(e_1 \otimes e_0); \end{array} \\ Y : & \begin{array}{l} e_0 \otimes e_0 \mapsto 0, \\ e_1 \otimes e_1 \mapsto (e_0 \otimes e_1) \\ \quad - q^{2i} e_1 \otimes e_0, \\ e_0 \otimes e_1 \mapsto q^{2i}(e_0 \otimes e_0), \\ e_1 \otimes e_0 \mapsto e_0 \otimes e_0. \end{array} \end{array}$$

Proof. Recall that the action of $U_q \mathfrak{gl}(1|1)$ in the tensor product is defined via the co-product $\Delta : U_q \mathfrak{gl}(1|1) \rightarrow U_q \mathfrak{gl}(1|1) \otimes U_q \mathfrak{gl}(1|1)$. The above formulas can be checked easily by using (46). \square

11.6.B (Decomposition in the generic case). *Let i, j , and $k = i + j$ be complex numbers that are not of the form $\pi n \sqrt{-1}/2h$ with $n \in \mathbb{Z}$. Let I, J be arbitrary complex numbers. Then $(i, I)_+ \otimes (j, J)_+$ is isomorphic to*

$$(i + j, I + J - 1)_+ \oplus (-i - j, -I - J - 1)_-.$$

There is an isomorphism

$$(i + j, I + J - 1)_+ \oplus (-i - j, -I - J - 1)_- \rightarrow (i, I)_+ \otimes (j, J)_+$$

with matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{[2j]_q}{[2i+2j]_q} & 0 & 1 \\ 0 & q^{-2j} \frac{[2i]_q}{[2i+2j]_q} & 0 & -q^{2i} \end{pmatrix}$$

with respect to the basis $\{(e_0, 0), (e_1, 0), (0, e_0), (0, e_1)\}$ of $(i + j, I + J - 1)_+ \oplus (-i - j, -I - J - 1)_-$ and the basis $\{e_0 \otimes e_0, e_1 \otimes e_1, e_0 \otimes e_1, e_1 \otimes e_0\}$ of $(i, I)_+ \otimes (j, J)_+$.

Proof. Consider the submodule generated by $e_0 \otimes e_0$. Since $E(e_0 \otimes e_0) = 2(i+j)(e_0 \otimes e_0)$ and $i + j \not\equiv 0 \pmod{\pi \sqrt{-1}/2h}$, this is a cyclic $(1|1)$ -dimensional representation. Since $Y(e_0 \otimes e_0) = 0$ and $e_0 \otimes e_0$ is a boson, this is a vector representation. Since $G(e_0 \otimes e_0) = (\frac{I+J}{2} - 1)(e_0 \otimes e_0)$, this is $(i + j, I + J - 1)_+$. For its basis we choose $e_0 \otimes e_0$ and $X(1/[2i+2j]_q)(e_0 \otimes e_0) = [2j]_q/[2i+2j]_q(e_0 \otimes e_1) + q^{-2j}[2i]_q/[2i+2j]_q(e_1 \otimes e_0)$.

Now, consider the submodule generated by $e_1 \otimes e_1$. Since $E(e_1 \otimes e_1) = 2(i+j)(e_1 \otimes e_1)$ and $i+j \not\equiv 0 \pmod{(\pi\sqrt{-1}/2h)}$, this is a cyclic $(1|1)$ -dimensional representation. Since $X(e_1 \otimes e_1) = 0$ and $e_1 \otimes e_1$ is a boson, this is a covector representation. Since $G(e_1 \otimes e_1) = (\frac{I+J}{2} + 1)(e_1 \otimes e_1)$ and $(j, J)_-(G)(e_0) = (\frac{-J+1}{2})e_0$ (see (59)), this is $(-i-j, -I-J-1)_-$. Take $e_1 \otimes e_1$ and $Y(e_1 \otimes e_1) = e_0 \otimes e_1 - q^{2i}(e_1 \otimes e_0)$ for its basis. \square

11.6.C. In the representation $(i, I)_+ \otimes (j, J)_+$ with the basis $e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1$, the R -matrix is

$$q^{-iJ-jI} \begin{pmatrix} q^{i+j} & 0 & 0 & 0 \\ 0 & q^{j-i} & 0 & 0 \\ 0 & q^{i-j}(q^{2i} - q^{-2i}) & q^{i-j} & 0 \\ 0 & 0 & 0 & q^{-i-j} \end{pmatrix}.$$

Proof. Recall (see (53)) that the universal R -matrix is

$$R = (1 + (q - q^{-1})(X \otimes Y)(K \otimes K^{-1})) q^{-E \otimes G - G \otimes E}.$$

It is easily seen that $q^{-E \otimes G - G \otimes E}$ is

$$q^{-iJ-jI} \begin{pmatrix} q^{i+j} & 0 & 0 & 0 \\ 0 & q^{j-i} & 0 & 0 \\ 0 & 0 & q^{i-j} & 0 \\ 0 & 0 & 0 & q^{-i-j} \end{pmatrix}.$$

Since $Xe_1 = 0$ and $Ye_0 = 0$, among the basis vectors only $e_0 \otimes e_1$ is not annihilated by $X \otimes Y$. Next, $q^{-E \otimes G - G \otimes E}(e_0 \otimes e_1) = q^{-iJ-jI} q^{j-i}(e_0 \otimes e_1)$ and

$$(X \otimes Y)(e_0 \otimes e_1) = [2i]_q(e_1 \otimes e_0),$$

so that

$$(q - q^{-1})(X \otimes Y)(K \otimes K^{-1}) q^{-E \otimes G - G \otimes E}(e_0 \otimes e_1) = q^{-iJ-jI} q^{i-j}(q^{2i} - q^{-2i})(e_1 \otimes e_0).$$

All together, this gives the desired formula. \square

The R -matrix will always be used composed with the operator of supertransposition

$$P : (i, I)_+ \otimes (j, J)_+ \rightarrow (j, J)_+ \otimes (i, I)_+$$

with matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(Here -1 at the right bottom corner is due to the super-rule: the transposition exchanges two fermions e_1 and e_1 in $e_1 \otimes e_1$ and fermions skew commute.)

The composition is

$$(65) \quad \begin{aligned} P \circ R &= q^{-iJ-jI} \begin{pmatrix} q^{i+j} & 0 & 0 & 0 \\ 0 & q^{i-j}(q^{2i} - q^{-2i}) & q^{i-j} & 0 \\ 0 & q^{j-i} & 0 & 0 \\ 0 & 0 & 0 & -q^{-i-j} \end{pmatrix} \\ &= q^{-iJ-jI-i-j} \begin{pmatrix} q^{2i+2j} & 0 & 0 & 0 \\ 0 & q^{4i} - 1 & q^{2i} & 0 \\ 0 & q^{2j} & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

Similarly one can calculate the corresponding objects related to the tensor products $(i, I)_- \otimes (j, J)_-$, $(i, I)_+ \otimes (j, J)_-$, and $(i, I)_- \otimes (j, J)_+$.

11.7. A q -less subalgebra. Consider the supersubring U^1 of $U_q \mathfrak{gl}(1|1)$ generated by the bosons $K = q^E$, K^{-1} , $F = 2G$ and the fermions $Z = (q - q^{-1})X$, Y ; see Subsection 11.2. This is a superalgebra over \mathbb{Z} determined by the relations

$$(66) \quad \begin{aligned} \{Z, Y\} &= K - K^{-1}, \\ Z^2 &= Y^2 = 0, \\ [K, Z] &= 0, \quad [K, Y] = 0, \\ [F, Z] &= 2Z, \quad [F, Y] = -2Y. \end{aligned}$$

In U^1 , the coproduct of $U_q \mathfrak{gl}(1|1)$ induces a coproduct

$$(67) \quad \begin{aligned} \Delta(K) &= K \otimes K, \\ \Delta(K^{-1}) &= K^{-1} \otimes K^{-1}, \\ \Delta(F) &= 1 \otimes F + F \otimes 1, \\ \Delta(Z) &= Z \otimes K^{-1} + 1 \otimes Z, \\ \Delta(Y) &= Y \otimes 1 + K \otimes Y. \end{aligned}$$

U^1 has a counit

$$(68) \quad \epsilon : U^1 \rightarrow \mathbb{Z} : \quad \epsilon(F) = \epsilon(Y) = \epsilon(Z) = 0, \quad \epsilon(K) = \epsilon(K^{-1}) = 1$$

and an antipode

$$(69) \quad \begin{aligned} s : U^1 \rightarrow U^1 : \quad s(K) &= K^{-1}, \quad s(F) = -F, \\ s(Z) &= -ZK, \quad s(Y) = -YK^{-1}, \end{aligned}$$

so that U^1 is a Hopf (super)algebra over \mathbb{Z} .

11.8. Irreducible representations of U^1 . Let $P = (B, M, W, M \times W \rightarrow M)$ be a 1-palette (see Subsection 2.8). In other words, let B be a commutative ring, M a subgroup of the multiplicative group of B , and W a subgroup of the additive group of B equipped with a (bilinear) pairing $M \times W \rightarrow M : (m, w) \mapsto m^w$ such that $1 \in W$ and $m^1 = m$ for each $m \in M$.

For $t \in M$, $t^4 \neq 1$, and $T \in W$, consider the action of U^1 on $B^{(1|1)}$ defined by

$$(70) \quad \begin{aligned} K &\mapsto \begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix}, \quad F \mapsto \begin{pmatrix} T-1 & 0 \\ 0 & T+1 \end{pmatrix}, \\ Y &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Z \mapsto \begin{pmatrix} 0 & 0 \\ t^2 - t^{-2} & 0 \end{pmatrix}. \end{aligned}$$

This is an irreducible representation to be denoted by $U(t, T)_+$. The dual representation is an action defined as follows in the same space:

$$(71) \quad \begin{aligned} K &\mapsto \begin{pmatrix} t^{-2} & 0 \\ 0 & t^{-2} \end{pmatrix}, \quad F \mapsto \begin{pmatrix} -T+1 & 0 \\ 0 & -T-1 \end{pmatrix}, \\ Y &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z \mapsto \begin{pmatrix} 0 & t^{-2} - t^2 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

It is denoted by $U(t, T)_-$.

11.8.A (Duality). *The modules $(t, T)_+$ and $(t, T)_-$ are dual to each other. There is a canonical isomorphism $(t, T)_+^\vee \rightarrow (t, T)_-$ described by the formulas $e^0 \mapsto e_0$, $e^1 \mapsto t^2 e_1$, where e^i is the generator of $(t, T)_+^\vee$ from the basis superdual to the standard basis e_0, e_1 of $(t, T)_+$; this generator is defined by $e^i(e_j) = (-1)^{ij} \delta_j^i$. \square*

11.8.B (Decomposition of $(u, U)_+ \otimes_B (v, V)_+$). Suppose u, v , and $t = uv$ belong to $M \setminus \{t \in M \mid t^4 = 1\}$ and $t^2 - t^{-2}$ is invertible in B . Let $U, V \in W$. Then $(u, U)_+ \otimes (v, V)_+$ is isomorphic to $(uv, U + V - 1)_+ \oplus (u^{-1}v^{-1}, -U - V - 1)_-$. There is an isomorphism

$$(t, U + V - 1)_+ \oplus (t^{-1}, -U - V - 1)_- \rightarrow (u, U)_+ \otimes (v, V)_+$$

with matrix

$$\begin{pmatrix} t^2 - t^{-2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & v^2 - v^{-2} & 0 & 1 \\ 0 & v^{-2}(u^2 - u^{-2}) & 0 & -u^2 \end{pmatrix}$$

with respect to the basis $\{(e_0, 0), (e_1, 0), (0, e_0), (0, e_1)\}$ of the B -module

$$(t, U + V - 1)_+ \oplus (t^{-1}, -U - V - 1)_-$$

and the basis $\{e_0 \otimes e_0, e_1 \otimes e_1, e_0 \otimes e_1, e_1 \otimes e_0\}$ of $(u, U)_+ \otimes_B (v, V)_+$. \square

11.8.C (Decomposition of $(u, U)_- \otimes_B (v, V)_-$). Suppose u, v , and $t = uv$ belong to $M \setminus \{t \in M \mid t^4 = 1\}$ and $t^2 - t^{-2}$ is invertible in B . Let $U, V \in W$. Then $(u, U)_- \otimes_B (v, V)_-$ is isomorphic to $(t^{-1}, -U - V - 1)_+ \oplus (t, U + V - 1)_-$. There is an isomorphism

$$(t^{-1}, -U - V - 1)_+ \oplus (t, U + V - 1)_- \rightarrow (u, U)_- \otimes_B (v, V)_-$$

with matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ t^2 - t^{-2} & 0 & 0 & 0 \\ 0 & v^2(u^2 - u^{-2}) & 0 & u^{-2} \\ 0 & -v^2 + v^{-2} & 0 & 1 \end{pmatrix}$$

with respect to the basis $\{(e_0, 0), (e_1, 0), (0, e_0), (0, e_1)\}$ of the B -module

$$(t^{-1}, -U - V - 1)_+ \oplus (t, U + V - 1)_-$$

and the basis $\{e_0 \otimes e_0, e_1 \otimes e_1, e_0 \otimes e_1, e_1 \otimes e_0\}$ of $(u, U)_- \otimes_B (v, V)_-$.

11.8.D (Decomposition of $(u, U)_+ \otimes_B (v, V)_-$). Suppose u, v , and $t = uv^{-1}$ belong to $M \setminus \{t \in M \mid t^4 = 1\}$ and $t^2 - t^{-2}$ is invertible in B . Let $U, V \in W$. Then $(u, U)_+ \otimes_B (v, V)_-$ is isomorphic to $(t, U - V + 1)_+ \oplus (t^{-1}, V - U + 1)_-$. There is an isomorphism

$$(t, U - V + 1)_+ \oplus (t^{-1}, V - U + 1)_- \rightarrow (u, U)_+ \otimes_B (v, V)_-$$

with matrix

$$\begin{pmatrix} 1 & 0 & v^2 - v^{-2} & 0 \\ -u^2 & 0 & -v^2(u^2 - u^{-2}) & 0 \\ 0 & 0 & 0 & t^2 - t^{-2} \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

with respect to the basis $\{(e_0, 0), (e_1, 0), (0, e_0), (0, e_1)\}$ of the B -module

$$(t^{-1}, V - U - 1)_+ \oplus (t, U - V - 1)_-$$

and the basis $\{e_0 \otimes e_0, e_1 \otimes e_1, e_0 \otimes e_1, e_1 \otimes e_0\}$ of $(u, U)_+ \otimes (v, V)_-$. \square

§12. APPENDIX 2. REPRESENTATIONS OF THE QUANTUM $\mathfrak{sl}(2)$ AT $\sqrt{-1}$

12.1. Algebra $\mathfrak{sl}(2)$. Recall that the Lie algebra $\mathfrak{sl}(2)$ is generated by the three elements

$$(72) \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

subject to the relations

$$(73) \quad [X^+, X^-] = H, \quad [H, X^+] = 2X^+, \quad [H, X^-] = -2X^-.$$

In fact, these relations determine $\mathfrak{sl}(2)$.

12.2. Quantization. The universal enveloping algebra $U\mathfrak{sl}(2)$ admits a deformation resulting in the quantum superalgebra $U_q \mathfrak{sl}(2)$. Only the first of relations (73) changes: its right-hand side H is replaced by the “quantum H ”, that is, by $\frac{q^H - q^{-H}}{q - q^{-1}}$ with $q = e^h$. To make this nonalgebraic expression meaningful, the algebra is supplemented with formal power series in h . Denote

$$q^H = e^{hH} = \sum_{n=0}^{\infty} \frac{h^n H^n}{n!}$$

by K . Thus, the deformed system of relations looks like this:

$$(74) \quad \begin{aligned} K &= q^H, \quad [X^+, X^-] = \frac{K - K^{-1}}{q - q^{-1}}, \\ [H, X^+] &= 2X^+, \quad [H, X^-] = -2X^-. \end{aligned}$$

The algebra A over the field $\mathbb{C}[[h]]$ of formal power series in h that consists of formal power series in h with coefficients generated by the operators H, X^+ , and X^- subject to (74) can be equipped with a coproduct

$$(75) \quad \begin{aligned} \Delta(H) &= H \otimes 1 + 1 \otimes H, \\ \Delta : A \rightarrow A \otimes A : \quad \Delta(X^+) &= X^+ \otimes 1 + K \otimes X^+, \\ \Delta(X^-) &= X^- \otimes K^{-1} + 1 \otimes X^-, \end{aligned}$$

a counit

$$(76) \quad \epsilon : A \rightarrow \mathbb{C}[[h]] : \quad \epsilon(K) = \epsilon(K^{-1}) = 1, \quad \epsilon(H) = \epsilon(X^+) = \epsilon(X^-) = 0,$$

and an antipode

$$(77) \quad s : A \rightarrow A : \quad s(H) = -H, \quad s(X^+) = -K^{-1}X^+, \quad s(X^-) = -X^-K,$$

so that A turns into a Hopf algebra. It is denoted by $U_q \mathfrak{sl}(2)$.

12.3. Simplification at the square root of -1 . Consider $U_q \mathfrak{sl}(2)$ at a special value $q = \sqrt{-1}$ corresponding to $h = \frac{\pi\sqrt{-1}}{2}$. To abbreviate formulas, but keep letter i available for other purposes, we denote $\sqrt{-1}$ by i and $\exp(a\frac{\pi\sqrt{-1}}{2})$ by i^a .

In case $q = i$, the generators may be changed to simplify relations. Indeed, let

$$X = X^-, \quad Y = 2X^+ \sqrt{-1}.$$

Then the relations

$$(78) \quad \begin{aligned} K &= i^H, \quad [Y, X] = K - K^{-1}, \\ [H, X] &= -2X, \quad [H, Y] = 2Y, \text{ and hence } KX = -XK, \quad KY = -YK, \end{aligned}$$

determine the algebra $U_{\mathbf{i}} \mathfrak{sl}(2)$. This is a Hopf algebra with a coproduct

$$(79) \quad \begin{aligned} \Delta(H) &= H \otimes 1 + 1 \otimes H, \\ \Delta(K) &= K \otimes K, \\ \Delta(K^{-1}) &= K^{-1} \otimes K^{-1}, \\ \Delta(X) &= X \otimes K^{-1} + 1 \otimes X, \\ \Delta(Y) &= Y \otimes 1 + K \otimes Y, \end{aligned}$$

a counit

$$(80) \quad \epsilon : U_{\mathbf{i}} \mathfrak{sl}(2) \rightarrow \mathbb{C}[[h]] : \quad \epsilon(K) = \epsilon(K^{-1}) = 1, \quad \epsilon(H) = \epsilon(X) = \epsilon(Y) = 0,$$

and an antipode

$$(81) \quad \begin{aligned} s(H) &= -H, \\ s(K) &= K^{-1}, \\ s(K^{-1}) &= K, \\ s(X) &= -XK \quad (= KX), \\ s(Y) &= -K^{-1}Y \quad (= YK^{-1}). \end{aligned}$$

12.4. Family of modules of dimension 2. Relations (78) are satisfied by the matrices

$$(82) \quad \begin{aligned} H &= \begin{pmatrix} a+1 & 0 \\ 0 & a-1 \end{pmatrix}, & K &= \begin{pmatrix} \mathbf{i}^{a+1} & 0 \\ 0 & -\mathbf{i}^{a+1} \end{pmatrix}, \\ X &= \begin{pmatrix} 0 & 0 \\ \mathbf{i}^{a+1} & -\mathbf{i}^{-a-1} \end{pmatrix}, & Y &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

where $a \in \mathbb{C}$.

The space of this representation is a two-dimensional vector space over the field of quotients of the ring of formal power series $\mathbb{C}[[h]]$. Recall that this field consists of formal Laurent series, i.e., $\mathbb{C}[[h]][h^{-1}]$.

We denote this representation of $U_{\mathbf{i}} \mathfrak{sl}(2)$ by $I(a)$. The standard basis vectors in $(\mathbb{C}[[h]][h^{-1}])^2$ are denoted by e_0, e_1 . On these vectors, the generators of $U_{\mathbf{i}} \mathfrak{sl}(2)$ act via $I(a)$ as follows:

$$(83) \quad He_0 = (a+1)e_0, \quad He_1 = (a-1)e_1,$$

$$(84) \quad Ke_0 = \mathbf{i}^{a+1}e_0, \quad Ke_1 = -\mathbf{i}^{a+1}e_1,$$

$$(85) \quad Xe_0 = (\mathbf{i}^{a+1} - \mathbf{i}^{-a-1})e_1, \quad Xe_1 = 0,$$

$$(86) \quad Ye_0 = 0, \quad Ye_1 = e_0.$$

This representation is irreducible unless $\mathbf{i}^{a+1} - \mathbf{i}^{-a-1} = 0$, i.e., unless a is an odd integer.

12.5. Duality. Recall that for any finite-dimensional representation V of a Hopf algebra A over a field k , the dual space $V^\vee = \text{Hom}_k(V, k)$ is equipped with the structure of an A -module by the formula $(Tc)(v) = c(s(T)(v))$, where $T \in A$, $c \in V^\vee$, $v \in V$, and s is the antipode of A . This representation is said to be *dual* to V .

12.5.A (Duality between $I(a)$ and $I(-a)$). *The modules $I(a)$ and $I(-a)$ are dual to each other. There is an isomorphism*

$$f : I(a)^\vee \rightarrow I(-a)$$

described by the formulas

$$f : e^0 \mapsto e_1, \quad f : e^1 \mapsto i^{a-1}e_0,$$

where e^i is the generator of $I(a)^\vee$ from the basis dual to the standard basis e_0, e_1 of $I(a)$; this generator is defined by $e^i(e_j) = \delta_j^i$. \square

12.5.A.1. Remark. The dual module $I(a)^\vee$ can be continuously deformed to $I(a)$ in the family of modules $I(x)$: the complex numbers $-a$ and a can be connected by a path in $\mathbb{C} \setminus (2\mathbb{Z} + 1)$.

12.6. R-Matrix. It is known that the quantum group $U_q \mathfrak{sl}(2)$ is quasitriangular with the universal R -matrix

$$(87) \quad R = q^{H \otimes H/2} \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[n]_q!} ((q - q^{-1})X^- \otimes X^+)^n.$$

In all the representations of $U_q \mathfrak{sl}(2)$ considered in this paper, $(X^+)^2 = (X^-)^2 = 0$ and $q = i$. Factorization by these relations reduces the R -matrix to

$$(88) \quad R = i^{H \otimes H/2} (1 + X \otimes Y).$$

12.6.A. In the representation $I(a) \otimes I(b)$ with the basis $e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1$, the R -matrix is

$$i^{\frac{ab+1}{2}} \begin{pmatrix} i^{\frac{a+b}{2}} & 0 & 0 & 0 \\ 0 & i^{\frac{-a+b}{2}+1} & 0 & 0 \\ 0 & i^{\frac{a-b}{2}}(i^a + i^{-a}) & i^{\frac{a-b}{2}-1} & 0 \\ 0 & 0 & 0 & i^{\frac{-a-b}{2}} \end{pmatrix}. \quad \square$$

The R -matrix will always be used composed with the operator of transposition

$$P : I(a) \otimes I(b) \rightarrow I(b) \otimes I(a)$$

with the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The composition is

$$P \circ R = i^{\frac{ab+1}{2}} \begin{pmatrix} i^{\frac{a+b}{2}} & 0 & 0 & 0 \\ 0 & i^{\frac{a-b}{2}}(i^a + i^{-a}) & i^{\frac{a-b}{2}-1} & 0 \\ 0 & i^{\frac{-a+b}{2}+1} & 0 & 0 \\ 0 & 0 & 0 & i^{\frac{-a-b}{2}} \end{pmatrix}.$$

12.7. Tensor product $I(a) \otimes I(b)$.

12.7.A (Action of generators). *Let a, b be complex numbers. The generators $H, K, X,$ and Y act in $I(a) \otimes I(b)$ as follows:*

$$(89) \quad \begin{aligned} H : \quad & e_0 \otimes e_0 \mapsto (a + b + 2)(e_0 \otimes e_0), \\ & e_1 \otimes e_1 \mapsto (a + b - 2)(e_1 \otimes e_1), \\ & e_0 \otimes e_1 \mapsto (a + b)(e_0 \otimes e_1), \\ & e_1 \otimes e_0 \mapsto (a + b)(e_1 \otimes e_0); \\ K : \quad & e_0 \otimes e_0 \mapsto -i^{a+b}(e_0 \otimes e_0), \\ & e_1 \otimes e_1 \mapsto -i^{a+b}(e_1 \otimes e_1), \\ & e_0 \otimes e_1 \mapsto i^{a+b}(e_0 \otimes e_1), \\ & e_1 \otimes e_0 \mapsto i^{a+b}(e_1 \otimes e_0); \\ X : \quad & e_0 \otimes e_0 \mapsto i(i^b + i^{-b})(e_0 \otimes e_1) + (i^a + i^{-a})i^{-b}(e_1 \otimes e_0), \\ & e_1 \otimes e_1 \mapsto 0, \\ & e_0 \otimes e_1 \mapsto (-i^a - i^{-a})i^{-b}(e_1 \otimes e_1), \\ & e_1 \otimes e_0 \mapsto i(i^b + i^{-b})(e_1 \otimes e_1); \\ Y : \quad & e_0 \otimes e_0 \mapsto 0, \\ & e_1 \otimes e_1 \mapsto e_0 \otimes e_1 - i^{a+1}e_1 \otimes e_0, \\ & e_0 \otimes e_1 \mapsto i^{a+1}(e_0 \otimes e_0), \\ & e_1 \otimes e_0 \mapsto e_0 \otimes e_0. \end{aligned}$$

Proof. Recall that the action of $U_i \mathfrak{sl}(2)$ in the tensor product is defined via the coproduct $\Delta : U_i \mathfrak{sl}(2) \rightarrow U_i \mathfrak{sl}(2) \otimes$. The formulas above can be checked easily by using (79). \square

12.7.B (Decomposition in the generic case). *Let a and b be complex numbers such that neither a , nor b , nor $a + b + 1$ is an odd integer. Then $I(a) \otimes I(b)$ is isomorphic to $I(a + b + 1) \oplus I(a + b - 1)$. There is an isomorphism*

$$I(a + b + 1) \oplus I(a + b - 1) \rightarrow I(a) \otimes I(b)$$

with matrix

$$\begin{pmatrix} i^{-a-b} - i^{a+b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & i(i^b + i^{-b}) & 1 & 0 \\ 0 & i^{-b}(i^a + i^{-a}) & i^{a-1} & 0 \end{pmatrix}$$

with respect to the basis

$$\{(e_0, 0), (e_1, 0), (0, e_0), (0, e_1)\} \text{ of } I(a + b + 1) \oplus I(a + b - 1)$$

and the basis

$$\{e_0 \otimes e_0, e_1 \otimes e_1, e_0 \otimes e_1, e_1 \otimes e_0\} \text{ of } I(a) \otimes I(b). \quad \square$$

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