ON EDGE-REGULAR GRAPHS WITH $k \geq 3b_1 - 3$

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Abstract. An undirected graph on $v$ vertices in which the degrees of all vertices are equal to $k$ and each edge belongs to exactly $\lambda$ triangles is said to be edge-regular with parameters $(v, k, \lambda)$. It is proved that an edge-regular graph with parameters $(v, k, \lambda)$ such that $k \geq 3b_1 - 3$ either has diameter 2 and coincides with the graph $P(2)$ on 20 vertices or with the graph $M(19)$ on 19 vertices or has at most $2k + 4$ vertices; or has diameter at least 3 and is a trivalent graph without triangles, or the line graph of a quadrivalent graph without triangles, or a locally hexagonal graph; or has diameter 3 and satisfies $|\Gamma_3(u)| \leq 1$ for each vertex $u$.

Introduction

We consider undirected graphs without loops and multiple edges. If $a$ and $b$ are vertices of a graph $\Gamma$, then we denote by $d(a, b)$ the distance between $a$ and $b$ and by $\Gamma_i(a)$ the subgraph of $\Gamma$ induced by the set of vertices that are at a distance of $i$ from $a$ in $\Gamma$. The subgraph $\Gamma_i(a)$ is called a neighborhood of $a$ and is denoted by $[a]$. We denote by $a^\perp$ the subgraph that is the unit ball centered at $a$.

A graph $\Gamma$ is said to be regular of degree $k$ if $[a]$ contains exactly $k$ vertices for each vertex $a$ in $\Gamma$. A graph $\Gamma$ is edge-regular with parameters $(v, k, \lambda)$ if $\Gamma$ has $v$ vertices and is regular of degree $k$, and each edge of $\Gamma$ lies in $\lambda$ triangles. We say that a graph $\Gamma$ is amply regular with parameters $(v, k, \lambda, \mu)$ if $\Gamma$ is edge-regular with the corresponding parameters, and the subgraph $[a] \cap [b]$ contains $\mu$ vertices whenever $d(a, b) = 2$. An amply regular graph of diameter 2 is said to be strongly regular.

We denote by $K_{m_1, \ldots, m_n}$ the complete $n$-partite graph with partite sets of orders $m_1, \ldots, m_n$. If $m_1 = \cdots = m_n = m$, then the corresponding graph is denoted by $K_{n \times m}$. The graph $K_{1,3}$ is called the 3-claw. A triangle graph $T(m)$ is a graph whose vertices are the unordered pairs of elements of $X$, $|X| = m$, and two pairs $\{a, b\}$ and $\{c, d\}$ are adjacent if and only if they have an element in common. A graph on a set $X \times Y$ of vertices is called an $(m \times n)$-lattice if $|X| = m$, $|Y| = n$, and two vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if $x_1 = x_2$ or $y_1 = y_2$. The vertices of the graph $T(19)$ are the elements of the field $F_{19}$. Two vertices are adjacent if their difference is a nonzero cube in $F_{19}$.

This is a locally hexagonal graph of diameter 2. The graph $P(m)$ with $v = 5m^2$, $k = 4m - 2$, and $b_1 = 2m - 1$ is obtained by replacing the vertices of the pentagon with pairwise disjoint $(m \times m)$-lattices (the graph $P(3)$ is depicted below). A Taylor graph is an amply regular graph $\Gamma$ of diameter 3 in which $\Gamma = u^\perp \cup w^\perp$ for any two vertices $u$ and $w$ with $d(u, w) = 3$. The Schläfli graph is a unique strongly regular graph with parameters $(27, 16, 10, 8)$. We denote by $T(k)$ the class of regular graphs of degree $k$ without triangles, and by $E(k)$ the class of line graphs for the graphs in $T(k)$. The number of vertices in a subgraph $\Delta$ will be denoted by $|\Delta|$. $\square$

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The graph $P(3)$.

Suppose the distance between two vertices $u$ and $w$ in an edge-regular graph $\Gamma$ is equal to 2. We say that the pair $(u, w)$ is good if $\mu(u, w) = k - 2b_1 + 1$, and almost good if $\mu(u, w) = k - 2b_1 + 2$. By Lemma 1.1, the $\mu$-subgraph corresponding to a good pair is a clique.

If the distance between $u$ and $w$ in $\Gamma$ is $i$, then we denote by $b_i(u, w)$ (respectively, $c_i(u, w)$) the number of vertices in the intersection of $\Gamma_{i+1}(u)$ (respectively, $\Gamma_{i-1}(u)$) with $[w]$. We note that, in an edge-regular graph with parameters $(v, k, \lambda)$, the value of $b_1(u, w)$ does not depend on the choice of an edge \{u, w\} and is equal to $k - \lambda - 1$. For edge-regular graphs with $k \geq f(b_1)$ and some specific functions $f$, it is possible to obtain an estimate $v \leq g(k)$ (or to describe the graphs for which this estimate fails). Thus, in [1, Lemma 1.4.2], it was proved that if $\Gamma$ is a connected incomplete edge-regular graph with parameters $(v, k, \lambda)$ such that $k \geq 3b_1$, then the diameter of $\Gamma$ is 2 and $v \leq 2k - 2$. In fact, it was proved that $v < k - 2 + 3b_1 + 3/(b_1 + 1)$. To sharpen the upper bound for the number of vertices, we need to describe the graphs with small values of $b_1$ (see Lemmas 1.2 and 1.3 below) and the graphs saturated by good pairs of vertices. In a corollary in [2], it was proved that if $\Gamma$ is a connected edge-regular graph with parameters $(v, k, \lambda)$ where $k \geq 3b_1 - 2$, then either $\Gamma$ is a polygon, or the icosahedron graph, or $\Gamma \in E_3$, or $\Gamma$ is a graph of diameter 2 with at most $2k$ vertices, or the pentagon, or a $(3 \times 3)$-lattice, or the triangle graph $T(7)$. The next step is the study of edge-regular graphs with $k \geq 3b_1 - 3$. The graphs of diameter 2 with $k \geq 3b_1 - 3$ were studied in [3].

**Theorem.** Let $\Gamma$ be a connected edge-regular graph with parameters $(v, k, \lambda)$, let $b_1 = k - \lambda - 1$, and let $k \geq 3b_1 - 3$. Then one of the following statements is true:

1. the diameter $\Gamma$ is at most 2, and either the number $v$ of vertices does not exceed $2k + 4$, or $\Gamma$ coincides with the graph $P(2)$, or $\Gamma$ coincides with the locally hexagonal graph on 17 or 19 vertices;
2. the diameter of $\Gamma$ is at least 3, and either $\Gamma \in T(3) \cup E(3) \cup E(4)$, or $\Gamma$ is a locally hexagonal graph;
3. the graph $\Gamma$ has diameter 3 and $|\Gamma_3(u)| \leq 1$ for each vertex $u$. 

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Example. Let \( \Phi_n \) be the graph in which the vertices are the 4-cycles of the symmetry group \( S_n \), and two vertices \( a \) and \( b \) are adjacent if \( ab \) is a 5-cycle. Then \( \Phi_n \) is a 6-extension of the Johnson graph \( J(n, 4) \) and \( \Phi_5 \) is an edge-regular graph of diameter 3 with parameters \( (30, 12, 6) \) such that \( k = 3b_1 - 3 \) and every vertex in \( [(1342)] \) forms a good pair with the vertex \( (1234) \) (if an edge-regular graph with \( k \geq 3b_1 - 1 \) is neither a polygon nor an icosahedron graph, then at most 2 vertices in \( \Gamma_2(u) \) form good pairs with \( u \); see [5]).

We do not know examples of graphs with \( k \geq 3b_1 - 3 \) that have vertices \( u \) and \( v \) such that \( |\Gamma_3(u)| = 1 \) and \( |\Gamma_3(v)| = 0 \).

Corollary. Let \( \Gamma \) be a connected amply regular graph of diameter greater than 2 with parameters \( (v, k, \lambda, \mu) \), where \( k \geq 3b_1 - 3 \). Then one of the following statements is true:

1. \( \Gamma \in \mathcal{E}(4) \) and \( \mu = b_1 - 2 = 1 \);
2. \( \Gamma \in \mathcal{T}(3) \cup \mathcal{E}(3) \) and \( \mu = b_1 - 1 = 1 \);
3. \( \mu = b_1 \) and \( \Gamma \) is either an \( n \)-gon with \( n \geq 6 \), or a complete bipartite graph \( K_{4, 4} \) with the maximal matching removed, or the icosahedron graph, or the Johnson graph \( J(6, 3) \), or the locally Taylor graph \( T(6) \) on 32 vertices, or a locally Schlafli graph on 56 vertices.

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§1. Auxiliary results

Lemma 1.1. Let \( \Gamma \) be an edge-regular graph with parameters \( (v, k, \lambda) \), and let \( b_1 = k - \lambda - 1 \). If the distance between vertices \( u \) and \( w \) in \( \Gamma \) is 2, then:

1. the degree of each vertex in a \( \mu \)-subgraph of \( \Gamma \) is at least \( k - 2b_1 \);
2. a vertex \( d \) has degree \( \alpha \) in the graph \( [u] \cap [w] \) if and only if \( |d| \) has \( \alpha - (k - 2b_1) \) vertices outside \( u^\perp \cup w^\perp \);
3. if \( \mu(u, w) = k - 2b_1 + 1 \), then the subgraph \( [u] \cap [w] \) is a clique and \( |d| \subset u^\perp \cup w^\perp \) for each vertex \( d \in [u] \cap [w] \);
4. if \( \Gamma - (u^\perp \cup w^\perp) \) has a unique vertex \( z \), then \( \mu(u, z) = \mu(w, z) \).

Proof. Let \( d \in [u] \cap [w] \). Then \( |d| - |u| = |d| - |w| = b_1 \). Therefore, at least \( k - 2b_1 \) vertices of \( |d| \) belong to \( [u] \cap [w] \). Statement (1) is proved.

Let \( d \in [u] \cap [w] \), and let the degree of \( d \) in this \( \mu \)-subgraph be equal to \( \alpha \). Then \( k = \alpha + 2b_1 - |d| - (u^\perp \cup w^\perp) \). Therefore, \( |d| \) contains \( \alpha - (k - 2b_1) \) vertices outside \( u^\perp \cup w^\perp \). Statement (2) is proved.

Statement (3) follows from (1) and (2).

Let \( \{z\} = \Gamma - (u^\perp \cup w^\perp) \). Since the number of edges between \([u] - [w] \) and \([w] - [u] \) is equal to \( b_1 |[u] - [w]| - \mu(u, z) \), we obtain \( \mu(u, z) = \mu(w, z) \). The lemma is proved. \( \square \)

Lemma 1.2. Let \( \Gamma \) be a connected edge-regular graph with parameters \( (v, k, \lambda) \), and let \( b_1 = 2 \). Then either \( \Gamma \in \mathcal{T}(3) \cup \mathcal{E}(3) \), or \( \Gamma \) is one of the following graphs:

1. the complete multipartite graph \( K_{r \times 3} \);
2. the \((3 \times 3)\)-lattice graph, or the triangle graph \( T(5) \), or the Petersen graph;
3. the icosahedron graph.

Proof. This is Proposition 1 in [4]. \( \square \)

Lemma 1.3. Let \( \Gamma \) be a connected edge-regular graph with parameters \( (v, k, \lambda) \) and with \( b_1 = 3 \). Then either \( \Gamma \in \mathcal{T}(4) \cup \mathcal{E}(4) \), or \( \Gamma \) is one of the following graphs:

1. a locally hexagonal graph (including the Paley graph with parameters \( (13, 6, 2, 3) \) and the Shrikhande graph);
2. the complete multipartite graph \( K_{r \times 4} \);
3. the triangle graph \( T(6) \), or the Clebsch graph.
Proof. This is Proposition 2 in [4].

Lemma 1.4. Let \( \Gamma \) be an edge-regular graph, and let \( \mu(u, w) = k - 2b_1 + 1 \) and \( \mu(u, z) = k - 2b_1 + 2 \) for some vertices \( w \) and \( z \) in \( \Gamma_2(u) \). Then \( |[u] \cap [w] \cap [z]| < 2 \).

Proof. This statement follows from Lemmas 4 and 5 in [5].

Let \( w, z \in \Gamma_2(u) \). We say that a triple \( (u, w, z) \) of vertices is good if \( \mu(u, w) + \mu(u, z) \leq 2k - 4b_1 + 3 \), and almost good if \( \mu(u, w) + \mu(u, z) = 2k - 4b_1 + 4 \). We have considered the case of a good triple in Lemma 1.4. The case of an almost good triple will be treated in Lemma 1.5.

Lemma 1.5. Let \( \Gamma \) be an edge-regular graph with \( k \geq 3b_1 - 3 \), let \( \mu(u, w) + \mu(u, z) = 2k - 4b_1 + 4 \) for two vertices \( w \) and \( z \) of \( \Gamma_2(u) \), let \( \Delta = [u] \cap [w] \cap [z] \), and let \( \delta = |\Delta| \).

Then one of the following statements is true:

1. the vertices \( w \) and \( z \) are not adjacent and \( \delta \leq 1 \);
2. \( \Delta \) contains two nonadjacent vertices and \( \delta \leq 2 \);
3. the vertices \( w \) and \( z \) are adjacent, \( \Delta \) is a clique, and if \( \delta > 1 \), then either
   i. the subgraph \( \Delta \) contains two nonadjacent vertices and \( \delta \leq 2 \), and, for \( e \in \Delta(d) \), the subgraph \( [d] \cup [e] \) contains \( \Delta \) and \( [d] \cap [e] \) is contained in \( \{u, w, z\} \cup ([u] \cap [w] \cup [z]) \cup ([w] \cap [z] \cup [u]) \), or
   ii. the subgraph \( \Delta \) contains no vertices adjacent to a vertex outside \( u \cup w \cup z \), and, for any two adjacent vertices \( d, e \in \Delta(d) \), the subgraph \( [d] \cap [e] \) contains a vertex of \( u, w, z \) and \( \{u, w, z\} \cup ([u] \cap [w] \cup [z]) \cup ([w] \cap [z] \cup [u]) \), where \( \gamma = |\Delta - ([d] \cup [e])| \).

Proof. If the vertices \( w \) and \( z \) are not adjacent, or \( \Delta \) contains two nonadjacent vertices, or a subgraph of \( \Delta \) contains a vertex adjacent to a vertex outside \( u \cup w \cup z \), then the lemma follows from Theorem 1 in [5]. Therefore, we may assume that the vertices \( w \) and \( z \) are adjacent and \( \Delta \) is a clique that does not contain vertices adjacent to a vertex outside \( u \cup w \cup z \).

We say that a vertex \( d \) of \( [u] \cap [w] \cap [z] \) has type \( (j) \) if \( [d] \) contains \( j \) vertices of \( ([w] - [u] \cup [z]) \cup ([z] - [u] \cup [w]) \). Obviously, \( 0 \leq j \leq 2 \). If \( \mu(u, w) \neq \mu(u, z) \), then, without loss of generality, we may assume that \( \mu(u, w) = k - 2b_1 + 1 \) and \( \mu(u, z) = k - 2b_1 + 3 \).

We prove that, for any two vertices \( d, e \in \Delta(d) \), the subgraph \( [d] \cap [e] \) contains \( \gamma = |\Delta - ([d] \cup [e])| \). This follows from analysis of all possible cases. We consider two cases in detail.

Suppose \( \mu(u, w) = k - 2b_1 + 1 \) and \( \mu(u, z) = k - 2b_1 + 3 \), and let \( d \) and \( e \) be vertices of type \( (1) \). Then \( [d] \cap [e] \) contains \( u, w, z, k - 2b_1 - 1 \) vertices of \( [u] \cap [w] \), \( k - 2b_1 + 1 - \delta \) vertices of \( [u] \cap [z] - [w] \), and at least \( 2b_1 - 6 - (k - b_1 - 1 - \delta - \gamma) \) vertices of \( [w] \cap [z] - [u] \). Altogether, we have \( k - b_1 - 2 + \gamma \) vertices.

Suppose \( \mu(u, w) = \mu(u, z) = k - 2b_1 + 2 \), the vertex \( d \) is of type \( (1) \) (for definiteness, let \( [d] \) contain a vertex of \( [u] \cap [w] \cup [z] \)), and \( e \) is of type \( (2) \). Then \( [d] \cap [e] \) contains \( u, w, z, k - 2b_1 - 1 \) vertices of \( [u] \cap [w] \), \( k - 2b_1 + 2 - \delta \) vertices of \( [u] \cap [z] - [w] \), and at least \( 2b_1 - 7 - (k - b_1 - 1 - \delta - \gamma) \) vertices of \( [w] \cap [z] - [u] \). Altogether, we have \( k - b_1 - 2 + \gamma \) vertices. The lemma is proved.

\[ \Box \]

§2. Reduction to graphs of diameter 3

In this section, \( \Gamma \) is an edge-regular graph with \( k = 3b_1 - 3 \). If the diameter of \( \Gamma \) is 2, then the conclusion of the theorem is valid by the results of [3].

Proposition 1. Let \( \Gamma \) be a connected edge-regular graph of diameter greater than 3 and with parameters \((v, k, \lambda)\). If \( k = 3b_1 - 3 \), then \( \Gamma \in T(3) \cup E(4) \), or \( \Gamma \) is a locally hexagonal graph.
In this section, we assume that the diameter of $\Gamma$ is at most 4. For each geodesic 3-path $abcd$, the subgraph $[b] \cap [d]$ lies in $[b] - [a]$, and therefore, $\mu(b, d) \leq b_1$. We fix a geodesic 4-path $uwxyz$ and put $\Delta = x^\perp - (u^\perp \cup z^\perp)$. If $b_1 \leq 3$, then the conclusion of the proposition is valid by Lemmas 1.2 and 1.3. Thus, we may assume that $b_1 \geq 4$.

**Lemma 2.1.** The sum $\mu(u, x) + \mu(x, z)$ does not exceed $2b_1 - 3$.

**Proof.** Let $\mu(u, x) = b_1$. Then $x^\perp \cap d^\perp = x^\perp - [u]$ for every vertex $d \in [x] \cap [z]$. If $d$ and $y$ are distinct vertices of $[x] \cap [z]$, then $d^\perp \cap y^\perp = \Delta \cup ([x] \cap [z])$, which contradicts the fact that the vertex $z$ is adjacent to $d$ and $y$. Hence, $\mu(x, z) = 1$ and $b_1 \leq 3$, a contradiction.

Let $\mu(u, x) = \mu(x, z) = b_1 - 1$. Then $|\Delta| = b_1$, and for each vertex $w \in [u] \cap [x]$ there is a unique vertex of $([u] \cap [x]) \cup \Delta$ that does not lie in $w^\perp$. The number of edges between $\Delta - \{x\}$ and $([u] \cap [x]) \cup ([x] \cap [z])$ is at least $2(b_1 - 1)(b_1 - 2)$. Therefore, each vertex in $\Delta - \{x\}$ is adjacent to exactly $2b_1 - 4$ vertices in $([u] \cap [x]) \cup ([x] \cap [z])$, and the subgraph $\Delta(x)$ is a $(b_1 - 1)$-clique. Furthermore, the above-mentioned number of edges is equal to $2(b_1 - 1)(b_1 - 2)$, and the subgraphs $[u] \cap [x]$ and $[x] \cap [z]$ are cliques. For any two vertices $w, w' \in [u] \cap [x]$, the subgraph $[u] \cap [w']$ contains $u, b_1 - 3$ vertices of $[u] \cap [x]$, and exactly $b_1 - 2$ vertices of $\Delta$.

Let $d$ be a vertex of $[w] \cap [y]$ distinct from $x$. Since $|[u] \cap [x] \cap [d]| = b_1 - 2$ and each vertex of $[u] \cap [x] \cap [d]$ is adjacent to exactly $b_1 - 2$ vertices of $\Delta - \{x\}$, we have $\mu(u, d) = \mu(d, z) = b_1 - 1$. Since $b_1 = 4$, we see that $[d] \cap [x]$ contains at least 2 vertices of $[u]$. By symmetry, $[d] \cap [x]$ contains at least 2 vertices of $[z]$, which contradicts Lemma 1.5. $\square$

**Lemma 2.2.** $\mu(u, x) = \mu(x, z) = b_1 - 2$.

**Proof.** We assume that $\mu(u, x) = b_1 - 2$ and $\mu(x, z) = b_1 - 1$. Then $|\Delta| = b_1 + 1$, and for each vertex $w \in [u] \cap [x]$ there is a unique vertex of $\Delta$ that does not lie in $[w]$. Therefore, $\Delta - \{x\}$ contains two vertices $d$ and $e$ that are adjacent to all vertices of $[u] \cap [x]$. Lemma 1.4 shows that $\mu(u, d) \geq b_1$ and $\mu(u, e) \geq b_1$. By Lemma 2.1, $d, e \in \Gamma_3(z)$. Now, for each vertex $y$ of $[x] \cap [z]$, the subgraph $[x] \cap [y]$ contains $b_1 - 2$ vertices of $[x] \cap [z]$ and the same number of vertices of $\Delta$. Therefore, $[x] \cap [z]$ is a clique, and, for distinct $y, y' \in [x] \cap [z]$, the subgraph $[y] \cap [y']$ contains $z, b_1 - 3$ vertices of $[x] \cap [z]$, and $b_1 - 1$ vertices of $\Delta$, a contradiction. The lemma is proved. $\square$

Now, we complete the proof of Proposition 1. By Lemmas 1.1 and 2.2, we have $\mu(u, d) = \mu(d, z) = b_1 - 2$ for every vertex $d$ of $[u] \cap [y]$. In particular, $[u] - u^\perp$ and $[y] - z^\perp$ lie in $d^\perp$. Next, $|\Delta| = b_1 + 2$, and the number of edges between $([u] \cap [x]) \cup ([x] \cap [z])$ and $\Delta - \{x\}$ is equal to $2(b_1 - 2)(b_1 - 1)$. On the other hand, each vertex of $\Delta - \{x\}$ is adjacent to at most two vertices of $([u] \cap [x]) \cup ([x] \cap [z])$ by Lemma 1.4. Therefore, $2(b_1 - 2)(b_1 - 1) \leq 2(b_1 + 1)$ and $b_1 \leq 3$, a contradiction. Proposition 1 is proved.

## 3. Graphs of diameter 3

Suppose $\Gamma$ is a graph of diameter 3 that provides a counterexample to the theorem. Then $k = 3b_1 - 3$ and $\lambda = 2b_1 - 4$. By Lemma 1.1, the degree of each vertex in a $\mu$-subgraph of $\Gamma$ is at least $b_1 - 3$. If $b_1 = 2$, then $\Gamma \in \mathcal{T}(3)$ by Lemma 1.2. If $b_1 = 3$, then by Lemma 1.3, the neighborhoods of vertices in $\Gamma$ are either hexagons or consist of two isolated triangles. In any case, we obtain a contradiction with the choice of $\Gamma$. Therefore, $b_1 \geq 4$.

**Proposition 2.** Let $\Gamma$ be a connected edge-regular graph of diameter 3 and with parameters $(v, k, \lambda)$. If $k = 3b_1 - 3$ and $b_1 \geq 4$, then $|\Gamma_3(u)| \geq 1$ for every vertex $u$. 
Let the conditions of Proposition 2 be fulfilled. We fix a geodesic 3-path $uvw$. In Lemmas 3.1–3.11, we prove that $b_2(u, x) = 1$. We have $[y] \cap \Delta_3(w) \subset \Delta_2(u)$ (see Lemmas 3.2–3.4). For $a \in \Delta_2(u)$ and $a = [a] \cap \Delta_3(u)$, the subgraph $\Delta(a)$ is a clique in $\Delta_2(w)$ (see Lemma 3.5). Suppose that $[x]$ contains two vertices $y$ and $z$ of $\Delta_3(u)$. Then $\mu(u, x) < b_1$ (see Lemma 3.1), $[y] \cap z^+ \subset \Delta_2(u)$ (see Lemma 3.6), and each vertex $d$ of $[y] \cap \Delta_2(y)$ is adjacent to a vertex of $[y] \cap [z]$ (see Lemma 3.7). Finally, each vertex of $[u] \cap \Delta_2(y)$ is adjacent to at most one vertex of $[y] \cap z^+$ (see Lemmas 3.8–3.10).

Lemma 3.1. If $[x]$ contains a vertex $z$ of $\Delta_3(u) - \{y\}$, then $\mu(u, x) < b_1$.

Proof. Assuming that $[x]$ contains a vertex $z$ of $\Delta_3(u) - \{y\}$ and $\mu(u, x) = b_1$, we show that

(1) $x^+ \cap y^+ = x^+ \cap z^+ = y^+ \cap z^+$.

Observe that $[x] \cap [u]$ contains $b_1$ vertices outside $y^+ \cup z^+$. Therefore, $x^+ \cap y^+ = x^+ \cap z^+$, the vertices $y$ and $z$ are adjacent, and $x^+ \cap y^+ = y^+ \cap z^+$. Statement (1) is proved. Now we prove the following:

(2) if $a \in [w] \cap [y] - \{x\}$ and $\mu(u, a) = b_1$, then the vertices $x$ and $z$ are not adjacent to $a$.

Let $a \in [w] \cap [y] - \{x\}$ and $\mu(u, a) = b_1$. If $a$ is adjacent to $z$, then, by statement (1), we have $a^+ \cap y^+ = a^+ \cap z^+ = y^+ \cap z^+$, whence $a$ is adjacent to $x$, a contradiction. Thus, the vertices $x$ and $z$ are not adjacent to $a$. Statement (2) is proved. Next, we show that

(3) if $d$ is a vertex of $[u] \cap [y]$ adjacent to $x$, then $\{w\} = [u] \cap [d] \cap [x]$.

Let $d$ be a vertex of $[u] \cap [y]$ adjacent to $x$. If $\mu(u, d) = b_1 - 1$, then the triple $u, x, d$ is almost good. Since $[d] \cap [x]$ contains the vertices $y$ and $z$, which are not adjacent to any vertex of $[u] \cap [x] \cap [d]$, we see that $\{w\} = [u] \cap [d] \cap [x]$ by Lemma 1.5.

Let $\mu(u, d) = b_1 - 1$. Then $[d] - y^+$ contains $b_1 - 1$ vertices of $[u]$ and a unique vertex $c$ outside $[u]$. By symmetry, $[d] - z^+$ contains $b_1 - 1$ vertices of $[u]$ and a unique vertex $e$ outside $[u]$. If $e = c$, then $d^+ = a^+ \cap z^+ = y^+ \cap z^+$, which contradicts the fact that $w \in [d] \cap [x]$. Thus, $e \neq c$, and $y^+ \cap z^+$ contains $\lambda + 1$ vertices of $d^+$. Therefore, $x^+ \cap d^+$ contains $\lambda + 1$ vertices outside $[u]$ and a unique vertex $w$ of $[u]$. Statement (3) is proved.

The next statement to be verified is

(4) the subgraph $[u] \cap [x]$ is a clique.

Suppose the degree of $w$ in the graph $[u] \cap [x]$ is equal to $b_1 - 3$. Then $[w] - u^+ \subset x^+$, and, by statement (1), we have $[w] \cap [y] = [w] \cap [z] = [w] - u^+$.

We assume that the subgraph $[w] \cap [y]$ contains two nonadjacent vertices $d$ and $d'$ and put $\delta = [u] \cap [d] \cap [d']$. Then $\mu(u, d) = \mu(u, d') = b_1 - 1$, and, by Lemma 1.5, we obtain $\delta \leq 2$. If $\delta = 1$, then $[u] \cap [w] \cap [d] \cap [d']$ contains $b_1 - 2$ vertices in each of the subgraphs $[d]$ and $[d']$, and, by statement (3), does not intersect $[x]$. This contradicts the fact that $b_1 \leq 3$ in this case. If $\delta = 2$, then $[u] \cap [w]$ contains a vertex of $[d] \cap [d']$ and $b_1 - 3$ vertices in each of the subgraphs $[d] - [d']$ and $[d'] - [d]$ of $[x]$. Hence, $b_1 = 4$. This contradicts the fact that $[w] \cap [w]$ contains $u, d, d'$, and one vertex of each of $[u] \cap [d]$ and $[u] \cap [d']$. Thus, the subgraph $[w] \cap [y]$ is a $b_1$-clique.

Let $d, d' \in [w] \cap [y] - \{x\}$, and let $\mu(u, d) = \mu(u, d') = b_1 - 1$. Since $[d] \cap [d']$ contains the vertices $y$ and $z$ outside $[u] \cup [w']$, where $w$ and $w'$ are distinct vertices of $[u] \cap [d] \cap [d']$, Lemma 1.5 implies that $[u] \cap [d] \cap [d'] = \{w\}$ for every two vertices $d$ and $d'$ of $[u] \cap [y]$.

If $b_1 \geq 5$, then the degree of $w$ in $[u]$ is at least $4(4b_1 - 3)$, a contradiction. Thus, $b_1 = 4 = \lambda$, which is impossible because the graph $[d] \cap [x]$ contains $w, y, z$, and two vertices of $[w] \cap [y]$.

Assume that the degree of $w$ in the graph $[u] \cap [x]$ is $b_1 - 2$. Then $[u] \cap [x] - w^+$ contains a single vertex $t$, and the degree of $x$ in the graphs $[w] \cap [y]$ and $[t] \cap [y]$ is $b_1 - 2$. Since $[x]$ contains the vertices $y$ and $z$ outside $w^+ \cup t^+$, we see that the degree of $x$ in the graph $[w] \cap [t]$ is at least $b_1 - 1$, and $[w] \cap [t]$ contains a vertex $d$ of $\Delta_2(u) \cap [x]$. Since
the following statements are valid. Suppose Lemma 3.2 contains at least two vertices and 2(µ(u, d) = b_1 - 1, and [u] ∩ [d] ∩ [x] contains the vertices w and t. This contradicts (3). Thus, the degree of each vertex w of the graph [u] ∩ [x] is b_1 - 1, so that [u] ∩ [x] is a clique. Statement (4) is proved. Finally, we prove the following statement:

(5) µ(u, d) = b_1 - 2 whenever w ∈ [u] ∩ [x] and d ∈ [w] ∩ [x] ∩ [y].

If µ(u, d) = b_1 - 1 for a vertex d ∈ [u] ∩ [x] ∩ [y], then, by statement (3), the subgraph [x] − d contains b_1 - 1 vertices of [u] and a vertex of [y] ∩ [z]. Since [w] ∩ [u] contains b_1 - 1 vertices of [x] and b_1 - 3 vertices of [d], we see that [u] ∩ [d] contains a unique vertex w that does not belong to w'. Since y' ∩ z' lies in x', the triple w, y, z is almost good, d is adjacent to a vertex w' outside [w] ∪ y' ∪ z', and b_1 = 4 by Lemma 1.5. Since [d] contains the vertices y and z outside w' ∪ (w')', the degree of d in the graph [w] ∩ [w'] is equal to 3, and [w'] contains a vertex r of [w] ∩ ([y] − [z]) and a vertex s of [w] ∩ ([z] − [y]). The graph [d] contains the vertex w outside y' ∪ z' ∪ (w')', Therefore, the degree of d in the graph [y] ∩ [w'] and [z] ∩ [w'] is equal to 3, and [w'] contains a vertex t adjacent to d, y, and z. We have µ(w', y) = µ(w', z) = b_1 - 1, and d is adjacent to the vertex w outside (w')' ∪ [y] ∪ [z] and [d] ∩ [t] = {w', y, x', z}. Therefore, the vertices r and s are not adjacent to t, and t is adjacent to a vertex t' of [x] ∩ [z] − d'. Using Lemma 1.4, we obtain µ(t, w') ≥ 3, and [w] ∩ [w'] contains a vertex of [u] ∩ [x]. By Lemma 1.1, the subgraph [x] contains a vertex outside t' ∪ w', which contradicts the fact that this vertex is in [y] ∩ [z] = {x, t, t', t'}. Statement (5) is proved.

Now, we complete the proof of the lemma. By statement (5) and Lemma 1.4, we have [w] = [u] ∩ [d] ∩ [e] for all vertices d, e ∈ [w] ∩ [x] ∩ [y]. If b_1 ≥ 5, then [w] ∩ [x] ∩ [y] contains at least two vertices and 2(b_1 - 3) + b_1 - 1 ≤ 2b_1 - 4, a contradiction. Thus, b_1 = 4, and, for each vertex w ∈ [u] ∩ [x], we have a unique vertex of [w] ∩ [z] adjacent to x. Since x is adjacent to three vertices of [y] ∩ [z], we see that there is a vertex d of [x] ∩ [y] ∩ [z] adjacent to two vertices w and w' of [u] ∩ [x]. But this is impossible because [w] ∩ [w'] contains u, x, d, and 2 vertices of [u] ∩ [x]. The lemma is proved.

In Lemmas 3.2–3.3, we assume that [y] contains a vertex z of Γ_3(u) ∩ Γ_3(u). For a ∈ [u] ∩ [y], let Σ_a = a + (([u] ∩ [z])]. We put Σ = Σ_a. Let b ∈ ([u] ∩ [Γ_2(z)]) ∪ ([z] ∩ [Γ_2(u)]), and let [b] contain i vertices of [Γ_2(z)] ∩ [Γ_2(u)]. We say that the vertex b is strong if i = 2, and weak if i = 1.

Lemma 3.2. Suppose [u] ∩ [x] contains a vertex adjacent to the vertices of [x] ∩ [z]. Then the following statements are valid:

1. if µ(u, x) = b_1, then [Σ] = 2b_1 - 2 - µ(x, z), x' = [u] = x' ∩ y', and each vertex of [x] ∩ [z] − {y} is adjacent to a vertex of [u] ∩ [x];
2. if a ∈ [x] ∩ [z] ∩ [Γ_2(u)] and the degree of a in the graph [x] ∩ [z] is equal to b_1 - 3, then µ(u, x) = b_1, and the subgraph [u] ∩ [u] is a clique;
3. if e is a strong vertex of [x] ∩ [y] ∩ [z] and the degree of e in [x] ∩ [z] is b_1 - 3, then µ(x, z) = b_1 - 2;
4. each vertex of [x] ∩ [z] ∩ [Γ_2(u)] is adjacent to y.

Proof. We assume that some vertex of [u] ∩ [x] is adjacent to a vertex of [x] ∩ [z]. Let µ(u, x) = b_1. Then [x] ∩ [u] contains b_1 vertices outside x' ∩ y'. Therefore, x' − [u] = x' ∩ y'. Next, [Σ] = 2b_1 - 2 − µ(x, z) and x' ∩ y' = Σ ∪ ([x] ∩ [z]) for y ∈ [x] ∩ [Γ_2(u)]. If a vertex a of ([x] ∩ [z]) − {y} is not adjacent to a vertex of [u] ∩ [x], then y' ∩ u' = x' − [u], Σ ⊂ [a], and a' ∩ y' = Σ ∪ ([x] ∩ [z]). This contradicts the fact that z is adjacent to a and y. Statement (1) is proved.

Suppose a ∈ [x] ∩ [z] ∩ [Γ_2(u)] and the degree of a in [x] ∩ [z] is b_1 - 3. By Lemma 1.1, the subgraph [a] − z' lies in x'. In particular, [u] ∩ [a] ⊂ [x]. If µ(u, a) = b_1 - 2, then
\[ \mu(u, x) = b_1 \] by Lemma 1.4. Now, assume that \( \mu(u, x) < b_1 \). Then \( \mu(u, a) = \mu(u, x) = b_1 - 1 \) and \( [u] \cap [a] = [u] \cap [x] \). This contradicts the fact that any vertex \( w \) of \([u] \cap [x] \) is not adjacent to \( a \). Thus, \( \mu(u, x) = b_1 \).

If the subgraph \([u] \cap [a] \) is not a clique, then \( a \) is a weak vertex, and \( [u] \cap [a] = [u] \cap [x] - \{w\} \) contains two nonadjacent vertices \( d \) and \( d' \). Since \([a] \cap [x] \) contains the vertex \( d'' \) outside \( d^1 \cup z^1 \), the degree of \( a \) in the graph \([d] \cap [z] \) is \( b_1 - 2 \), \( \mu(d, z) = b_1 - 1 \), and \([d] \cap [x] \) contains \( b_1 - 2 \) vertices of each of \([u] \) and \([z] \). This contradicts the fact that \( a^1 \) contains only \( b_1 - 3 \) vertices of \([x] \cap [z] - \{y\} \). Statement (2) is proved.

Let \( e \in [x] \cap [z] \), and let \( e \) be adjacent to a vertex \( x' \) of \( \Gamma_2(u) \cap \Gamma_2(z) - \{x\} \). If the degree of \( e \) in the graph \([x] \cap [z] \) is \( b_1 - 3 \), then \( [e] \cap z^1 \subset x^2 \), and \( \mu(u, x) = b_1 \) by Lemma 1.4. If \( \mu(x, z) > b_1 - 2 \), then the subgraph \([x] \cap [z] - e^1 \) contains a vertex \( e' \). First, we assume that \( e' \in \Gamma_2(u) \). By Lemma 1.1, the subgraph \([u] \cap [e'] \) does not intersect \([e] \).

If \( e' \) is a weak vertex, then \([e'] \cap [x] \) contains at most \( b_1 - 2 \) vertices of \([z] \) and at most \( b_1 - 2 \) vertices of \([u] \); but \( 2(b_1 - 2) \leq b_1 - 1 \), a contradiction. Thus, the vertex \( e' \) is strong and \([e'] \cap [x] \) contains a vertex \( x'' \) of \( \Sigma \), \( b_1 - 2 \) vertices of \([z] \), and \( b_1 - 3 \) vertices of \([u] \). Therefore, \( 2b_1 - 5 \leq b_1 - 1 \), \( b_1 = 4 \), and \( \mu(u, x) = \mu(x, z) = 4 \). Hence, \( |\Sigma| = 2 \) and \( x' = x'' \). Since the degree of \( x \) in \([e'] \cap [w] \) is at least \( 2 \), we have \( x' \in [w] \). Similarly, the degree of \( x' \) in \([e] \cap [w] \) is at least \( 2 \). Consequently, \([e] \cap [u] \cap [x'] \) contains \( d \), and the subgraph \([x'] \cap [x'] \) contains \( d, u, e, \) and \( e' \). Therefore, the vertex \( x'' \) is not adjacent to \( y \).

However, \([x] - (x'' \cup y'') \) contains a vertex of each of \([e] \cap [z] \) and \([e'] \cap [z] \), and the degree of \( x \) in the graph \([x'] \cap [y] \) is at least \( 3 \), which is impossible because \( |[x] \cap [x']| \geq 5 \) in this case.

Thus, \([x] \cap [z] - e^1 = \{y\}, \mu(x, z) = b_1 - 1, \) and \( |\Sigma| = b_1 - 1 \). This contradicts the fact that \([x] \cap [y] \) contains \( b_1 - 2 \) vertices of \( \Sigma \) and at most \( b_1 - 3 \) vertices of \([z] \). Statement (3) is proved.

Suppose \( a \) belongs to \([x] \cap [z] \cap \Gamma_2(u) \) and is not adjacent to \( y \). Then \( \mu(x, z) \geq b_1 - 1 \).

If the degree of \( a \) in the graph \([x] \cap [z] \) is equal to \( b_1 - 3 \), then the vertex \( a \) is weak by (3), and the subgraph \([u] \cap [a] \) is a \((b_1 - 1)\)-clique by (2). In particular, \( \mu(u, x) = b_1 \). We note that \([x] - (w^1 \cup y^1) \) contains \( a \), so that the degree of \( x \) in the graph \([w] \cap [y] \) is at least \( b_1 - 2 \), \( |\Sigma| = b_1 - 1 = \mu(x, z) \), and \([w] \cap [y] \) contains \( \Sigma \). This contradicts the fact that \([x] \cap [y] \) contains \( b_1 - 2 \) vertices of \( \Sigma \) and at most \( b_1 - 3 \) vertices of \([x] \cap [z] \). Thus, the degree of \( a \) in the graph \([x] \cap [z] \) is equal to \( b_1 - 2 \), and \( \mu(x, z) = b_1 \).

If the vertex \( a \) is weak, then, by (2), the subgraph \([u] \cap [a] \) is a \((b_1 - 1)\)-clique and \([u] \cap [a] \cap [x] = b_1 - 2 \). As above, \( |\Sigma| = b_1 - 1 = \mu(x, z) \) and \([w] \cap [y] \) contains \( \Sigma \). Hence, \([x] \cap [w] \) contains \( b_1 - 2 \) vertices in \( \Sigma \) and in \([u] \cap [x] \); it follows that \([u] \cap [x] \) is a clique.

If \( d \) and \( d' \) are vertices in \([u] \cap [a] \cap [x] \), then \([d] \cap [d'] \) contains \( a, u, w, x, b_1 - 4 \) vertices of \([u] \cap [a] \cap [x] \), and a vertex of \([u] \cap [a] \cap [x] \); in particular, we have \( b_1 > 4 \). If some vertex of \([u] \cap [a] \cap [x] \) is adjacent to a vertex outside \([u] \cup [a] \cup [x] \), then, by Lemma 1.5, we obtain \( b_1 - 2 = 2 \), a contradiction. Therefore, each vertex of \([u] \cap [a] \cap [x] \) is adjacent to a unique vertex of each of subgraphs \([x] - (a^1 \cup [u]) \) and \([a] - (x^1 \cup [u]) \). However, \([x] - (a^1 \cup [u]) = \Sigma(x) \cup \{y\} \), so that each vertex of \([a] \cap [x] \) is strong. Next, the degree of \( a \) in the graph \([z] \cap (\bigcup_{d \in [u] \cup [a] \cap [x]} [d]) \) is at least \( 3(b_1 - 3) \) and \( b_1 - 5 \). We put \([d_1, d_2, d_3] = [u] \cap [a] \cap [x] \) and \([x_1, x_2, x_3] = \Sigma(x) \); the vertices are numbered so that \( d_1 \) is adjacent to \( x_1 \). The vertex \( d_2 \) adjacent to a unique vertex \( e_1 \) of \([a] \cap [x] \) - \([u] \), and \([e_1, e_2, e_3] \cup \Sigma(x) = [x] \cap [y] \). Since \([e_1] \cap [x] \) contains \( a, d_1, y, \) and, possibly, also \( e_2, e_3 \) and at most one vertex of \( \Sigma(x) \), we see that \( e_2, e_3, x_1 \in [e_1] \) and \([d_1] \cap [y] = \{x_1, x_1, x_1\} \).

We put \([d_1, d'_1, d''_1] = [u] \cap [e_1] \). Then \([d_1] \cap [w] \) contains \( u, x_1, d_2, d_3, \) and a vertex of \([u] \cap [e_1] \), say, \( d''_1 \). The subgraph \([x] \cap [x_1] \) contains \( u, y, d_1, e_1, \) and the vertices \( x_1 \) and \( x_2 \); in particular, \( \Sigma \) is a 4-clique. We note that \( |x_1 - y^1| \) lies in \( d''_1 \) and \( \mu(u, x_1) \geq 4, \).
whence \([x_1] \cap [u] = \{w, d, d', d''\}\). Replacing the triple \(x, a, y\) by \(x_1, d', w\), we obtain a contradiction with the fact that the degree of \(d'\) in the graph \([u] \cap [x_1] = 2 = b_1 - 3\).

If \(a\) is a strong vertex, then \([a]\) contains two vertices \(x'\) and \(x''\) of \(\Gamma_2(u) \cap \Gamma_2(z)\), \(\mu(u, a) = b_1 - 2\), and the degree of \(x\) in the graph \([w] \cap [a]\) is at least \(b_1 - 2\). Therefore, a single vertex \(b\) of \([a] - (x' \cup x'')\) lies either in \([u]\) or in \(\Gamma_2(u) \cap \Gamma_2(z)\). In the latter case, we have \([w] \cap [u] \cap [x] = b_1 - 2\) and \(\mu(u, x) = b_1 - 1\), which contradicts Lemma 1.4. Thus, \(b \in [u] \cap [a]\) and \(x' \in [w] \cap [x]\), and \(b_1 = 4\) by Lemma 1.4. Since \([x]\) contains a vertex \(a\) outside \(w'\) and \(y'\), the degree of \(x\) in the graph \([w] \cap [y]\) is at least \(2\), and \(\Sigma\) contains at least three vertices of \([w] \cap [y]\).

We put \([d, b] = [u] \cap [a]\). If the degree of \(d\) in the graph \([u] \cap [x]\) is equal to \(1\), then \([d] - u' \subset x' \cap a'\). In the case where the vertex \(d\) is weak, we obtain a clique \([d] \cap [z] = \{a, e, e'\}\), where \([e] \cap [e']\) contains \(a, d, x, z\), and either \(x'\) or a vertex of \([u] \cap [d] = \{w, d', x'\}\), a contradiction. Thus, the vertex \(d\) is strong and, using (3), we obtain \(\mu(u, x) = 2\) and \(\Sigma = [w] - u'\). Let \([x] \cap [z] = \{a, e, e', y\}\), where \(e\) is adjacent to \(d\). If \(e\) is not adjacent to \(y\), then \(\Sigma \subset [y]\). Since \(\Sigma \subset [y]\), the vertex \(e\) is strong, and \([x] \cap [x']\) contains \(d, a, x, y\), and \(y\) is a contradiction. Thus, \(e\) is adjacent to \(y\), \(\Sigma \cap [y] = 3\), and \([x] \cap [x']\) contains \(a, d, y\), and a vertex of \([e'] \cap [x']\). By Lemma 1.4, the vertex \(e\) is not adjacent to \(b\). If \(e\) is adjacent to \(e'\), then \(e\) is weak and \([u] \cap [e] = \{d, c, e'\}\). In this case, \([d] \cap [x']\) contains \(a, w, x\), and a vertex of \([c, e']\), say, \(c\). Since \(c\) is not adjacent to \(x\) and has degree \(2\) in the graph \([u] \cap [e]\), we conclude that \(c\) is weak and \([e] \cap [z] = \{e, x'\}\) contains vertices of \([e'] \cap [z]\) - \(\{a, y\}\). This contradicts the fact that, in this case, \([e] \cap [e']\) contains \(a, e, x, y\), and \(z\). Thus, \(e \in [x'] - \{e'\}\), \([x] \cap [x'] = \{a, d, e, w\}\), \(x'\) is not adjacent to \(y\), and \([d] \cap [y] = \{e, x\}\). By Lemma 1.4, we have \(\mu(x', z) = 4\). Observe that \([d] - u' \subset x' \cap (x')^{-1}\), whence the degree of \(d\) in the graph \([u] \cap [x'] = 1\) and \(b, c \notin [x']\). Hence, \(\mu(b, z) = \mu(c, z) = 4\), and Lemma 3.1 shows that the vertices \(b\) and \(c\) are not adjacent to \(w\).

Now, \(\mu(e, w) = \mu(a, w) = 3\) and \([e] \cap [w] = [a] \cap [w] = \{d, x, x'\}\). Since \([d, x, x']\) is a clique, we see that, by Lemma 1.5, the subgraph \([a] - e'\) contains the vertices \(b, e'\), and \(f\), and the graph \([e] - a'\) contains the vertices \(c, y\), and \(g\) adjacent to \(d, x, x'\), respectively. We put \([r, s] = ([w] \cap [x']) - \{d, x\}\). Then \([x'] \cap [r]\) contains the vertices \(w\) and \(s\) of \([u]\) and the vertices \(f\) and \(g\) outside \(u'\). By symmetry, \([x''] \cap [s]\) contains \(f\) and \(g\). This contradicts the fact that, in this case, \([r] \cap [s]\) contains \(u, w, x, f, g\), and \(g\).

Thus, the degree of \(d\) in \([u] \cap [x]\) is equal to \(2\), and \([d] - u' \subset \{x'\} \cap [z]\) and the vertex \(e\) of \([x'] - [z]\). Therefore, the degree \(d\) is weak and \([d] \cap [y] = \{x, e\}\). By Lemma 1.4, the vertex \(b\) is not adjacent to \(e\), the triple \(u, x, e\) is almost good, and \([u] \cap [x] \cap [e]\) contains exactly two vertices \(d\) and \(d'\). We note that \(x'\) is adjacent to \(b\), otherwise \(d'\) is adjacent to \(e'\) and \([a] \cap [x']\) contains \(x\) and three vertices of \([a] \cap [x']\) - \(\{e\}\); in particular, \(x'\) is adjacent to \(e'\). This contradicts the fact that \(\mu(u, e') = 3\) and \([u] \cap [a] \subset [e']\).

We put \([a] \cap [x] \cap [z] = \{e, f\}\) and assume that \(f\) is adjacent to \(x'\). Then \([x] \cap [f]\) contains \(a, x', y\), and \(d'\). However, if \(f\) is adjacent to \(d'\), then \([e] \cap [f]\) contains \(a, d', x, y\), and \(z\). Therefore, the vertices \(e\) and \(f\) are not adjacent. Since the degrees of the vertices \(e\) and \(f\) in the graph \([d'] \cap [z]\) are equal to \(1\), we see that the subgraphs \([u] \cap [e]\) and \([u] \cap [f]\) are contained in \((d')^{-1}\). Now, \([d']\) contains a vertex \(w\) outside \([e] \cup [f]\), so that \([u] \cap [e] \cap [f]\) contains the adjacent vertices \(d'\) and \(d''\). Hence, the triple \(u, e, f\) is almost good, \([u] \cap [e] \cap [f]\) contains exactly two vertices \(d'\) and \(d''\), and \([e] \cap [f]\) contains two vertices \(y\) and \(z\), none of which is adjacent to \(d\) or \(d'\). This contradicts Lemma 1.5.

Thus, \(x'\) and \(d'\) are not adjacent to \(f\) and \([x] \cap [f]\) contains \(a, e, y\), and a vertex \(x''\) of \(\Sigma\). Furthermore, \([x] \cap [x']\) contains \(a, w, y\), and the vertex \(x''\). Finally, \([x] \cap [d']\) contains \(d, e, w\), and \(x''\), which is impossible because \([x] \cap [x''] = \{d', x', w, f, y\}\). The lemma is proved. \(\square\)
Lemma 3.3. The subgraph [u] ∩ [x] contains no vertices adjacent to vertices of [x] ∩ [z], and \( \mu(u, x) < b_1 \).

Proof. We assume that some vertex of [u] ∩ [x] is adjacent to a vertex of [x] ∩ [z]. We prove that

(a) \( \mu(x, z) > b_1 - 2 \).

Suppose \( \mu(x, z) = b_1 - 2 \). Statement (3) of Lemma 3.2 implies that \( \mu(u, x) = b_1 \). By Lemma 3.1, we obtain \( \Sigma \subset \Gamma_2(u) \). If \( e \) and \( e' \) are vertices of \([x] \cap [z] \setminus \{y\}\), then the subgraphs \([u] \cap [e] \) and \([u] \cap [e']\) are contained in \([x]\), and they do not intersect because otherwise, for \( d \) in \([u] \cap [e] \cap [e']\), the subgraph \([x] \cap [d]\) contains two vertices \( e \) and \( e' \) of \([z]\), and \( \mu(d, z) < b_1 \). This contradicts Lemma 1.4. Then \( 1 + 2(b_1 - 2) \leq b_1 \) and \( b_1 \leq 3 \), a contradiction. Thus, \([x] \cap [z] = \{e, y\}, b_1 = 4, \) and \( \Sigma \subset [y] \).

If the vertex \( e \) is weak, then \( e \) is adjacent to no vertex of \( \Sigma(x) \) and \( \mu(u, e) = 3 \). By statement (2) of Lemma 3.2, the subgraph \([u] \cap [e] \) is a clique, and statement (4) of the same lemma shows that the subgraph \([u] \cap [e] \) lies in \([w]\). Therefore, for any two vertices \( d, d' \in [u] \cap [e] \), the subgraph \([d] \cap [d']\) contains \( e, u, w, x \), and a vertex of \([u] \cap [e] \), a contradiction.

Thus, the vertex \( e \) is strong, \( e \) is adjacent to a vertex \( x' \) of \( \Sigma(x) \), and \( \mu(u, e) = 2 \). We have \( \mu(x', z) = 4 \) by Lemma 1.4. The subgraph \([e] \cap [x']\) contains \( y, x', \) and two vertices \( d \) and \( d' \) of \([u] \cap [e] \). Applying statement (4) of Lemma 3.2 to the path \( yxywu \), we see that the vertices \( d \) and \( d' \) are adjacent to \( w \) and \([d] \cap [d'] = \{u, w, x, e\}\). By Lemma 1.4, we have \( \mu(w, e) = \mu(u, x) = 4 \); in particular, \( w \) is adjacent to \( x' \). If the vertices \( d \) and \( d' \) are not adjacent to \( x' \), then the degree of \( e \) in each of the subgraphs \([d] \cap [z] \) and \([d'] \cap [z] \) is equal to 2. Since \(|[e] \cap [z] \setminus \{y\}| = 3 \), the triple \( z, d, d' \) is almost good, \([d] \cap [d'] \cap [z] \) contains at least 2 vertices, and \([d] \cap [d'] \cap [z] \) is not adjacent to vertices of \([d] \cap [d'] \cap [z] \). This contradicts Lemma 1.5. Thus, we may assume that \( x' \in [d] \setminus [d'] \), so that \([x] \cap [x'] = \{d, e, w, y\}\).

Let \( c \in [u] \cap [x] \setminus \{w, d, d'\} \). If \( c \) is not adjacent to \( w \), then the degree of \( c \) in the graph \([u] \cap [x] \) is 1 and \( c - u^{-1} = \Sigma \), which contradicts the fact that the vertex \( x' \) is not adjacent to \( c \). Thus, \( c \) is adjacent to \( w \) and \([w] \cap [x] = \{c, d, d', x'\}\). In particular, \([w] \cap [y] = \{x, x'\}\). Now, for \( \{r, s\} = \Sigma - \{x, x'\} \), the subgraph \([r] \cap [x] \) contains \( c, d', y \), and the vertex \( s \). By symmetry, \([s] \cap [x] \) contains \( c, d', y \), and the vertex \( r \). This is impossible because \( d' \) is not adjacent to three vertices of \( \Sigma \). Statement (a) is proved. Now, we show that

(b) \([u] \cap [x] \) contains no weak vertices.

Let \( d \) be a weak vertex of \([u] \cap [x] \). We prove that the subgraph \([d] \cap [x] \cap [z] \) consists of strong vertices, the degree of \( d \) in the graph \([u] \cap [x] \) is equal to \( b_1 - 1 \), \( \mu(u, x) = b_1 \), and \( b_1 = 4 \).

If the subgraph \([d] \cap [x] \cap [z] \) contains a weak vertex \( e \), then, by statement (3) of Lemma 3.2 and statement (a) above, the degree of \( e \) in the graph \([d] \cap [z] \) is equal to \( b_1 - 2 \). By symmetry, the degree of \( d \) in the graph \([u] \cap [e] \) is equal to \( b_1 - 2 \). Therefore, \([d] \cap [e] \) contains \( x, b_1 - 2 \) vertices of \([u] \), and \( b_1 - 2 \) vertices of \([z] \), a contradiction. Thus, \([d] \cap [x] \cap [z] \) consists of strong vertices.

The subgraph \([x] \cap [d] \) contains at most \( b_1 - 1 \) vertices of \([u] \) and at least \( b_1 - 3 \) vertices \([z] \). If \([x] \cap [d] \) contains \( b_1 - 2 \) vertices of \([z] \), then the degree of \( d \) in the graph \([u] \) is at least \( 3(b_1 - 3) + 1 \) given \( b_1 \geq 5 \), a contradiction. If \( b_1 = 4 \), then the degree of each of the vertices \( e, e' \in [d] \cap [z] \cap [x] \) in the graph \([d] \cap [z] \) is equal to 2, and \([e] \cap [e'] \) contains \( d, x, y, z \), and the third vertex of \([d] \cap [z] \). Thus, the degree of \( d \) in \([u] \cap [x] \) is \( b_1 - 1 \), and \( \mu(u, x) = b_1 \).

Suppose \([x] \cap [d] \) contains \( b_1 - 3 \) vertices of \([z] \) and \( b_1 - 1 \) vertices of \([u] \). If \( b_1 \geq 6 \), then the degree of \( d \) in \([u] \) is at least \( 3(b_1 - 3) + 1 \), a contradiction. If \( b_1 = 5 \), then we put \([d] \cap [z] = \{e, e', a, a'\} \), where \( e \) and \( e' \) are adjacent to \( x \). Then \([e] \cap [x] \) contains \( d, y, a \).
Thus, \(\Sigma\) contains the vertex \(d\) in each of \(\Gamma_1\) of \(\Sigma\). Therefore, \(\{e,d\}\) contains both vertices of \(\{u\}\), and three vertices of \(\{u\}\). By Lemma 1.4, the degree of \(d\) in \(\{u\}\) is at most 1, a contradiction. Thus, \(\mu(u,a) = \mu(u,a') = 5\), and, by Lemma 3.1, the vertices \(a\) and \(a'\) are not adjacent to \(y\).

Now, \(\{e\} \neq \{d\}\) contains \(a, a'\), and three vertices of \(\{u\}\). Therefore, \(\{y\}\) contains \(a, d, e\), two vertices adjacent to \(e\), and two vertices adjacent to \(e'\), which implies that a vertex of \(\{x\}\) is adjacent to \(y\) and to one of the vertices \(e\) and \(e'\). In this case, \(\{u\} \neq \{x\}\) contains \(f\), and if \(f\) is not adjacent to \(a\), then \(\mu(u,y) < 5\), and \(\{u\} \neq \{w\}\) contains both \(u\), \(f\), \(x\), and three vertices of \(\{u\}\) and \(\{x\}\). Thus, \(\mu(u,z) = 4\), and \(\{x\}\) contains \(z\), a vertex of \(\Sigma\), and three vertices of \(\{u\}\). We put \(\{u\} \neq \{z\}\) of \(\{y\}\) and \(\{z\} \neq \{u\}\) are almost good, and \(\{u\} \neq \{y\} \neq \{z\}\) contains \(u\) and \(w\) that are adjacent to none of the vertices \(\{e\}, \{a\}, \{a'\}\). Applying Lemma 1.5 to the above triples, we see that \(\{g\} \neq \{a\}\) contains both vertices of \(\{e\} \neq \{z\}\) of \(\{a\}, \{a'\}, \{e\}\). This contradicts Lemma 1.5 applied to the almost good triple \(z, d, g'\).

Finally, if \(b_1 = 4\), then the subgraph \(\{d\} \not\subset \{z\}\) contains a vertex of \(\{x\}\) and two vertices, \(a\) and \(a'\), \(\{a\}\) of \(\{z\}\) outside \(\{x\}\). Then either \(\mu(a, a') = 4\), or \(\{a\} \not\subset \{a\}\) contains \(a, a'\), and two vertices of \(\{a\}\) do not exceed 3, then \(\{a\} \neq \{a\}\) contains \(d\), \(e\), \(z\), and two vertices of \(\{a\} \neq \{a\}\) of \(\{d\} \not\subset \{x\}\) of \(\{e\}\). Thus, we may assume that \(\{a\} \neq \{a\}\) contains \(\{a\} \neq \{e\}\), and, in particular, \(\mu(a, a') = 4\). We put \(\{b\} \neq \{a\}\), \(\{d\} \neq \{d'\}\). If \(d'\) and \(d\) are adjacent, then \(\{d\} \neq \{d'\}\) contains \(u, w, x, a, e, x\), and \(a\). But this contradicts \(d'\) is not adjacent to \(x\), and \(\{a\}\) contains \(d, e\), and a vertex of \(\{u\}\) not adjacent to \(e\). Since \(f\) contains the vertex \(u\) which lies outside \(a^+ \cup x^+\), we see that \(\{a\} \neq \{a\}\) contains a vertex \(g\) of \(\{z\}\) of \(\{z\}\) adjacent to \(f\), \(\mu(x, z) = 4\), and \(\Sigma\) is weak, which is impossible because \(\{u\} \neq \{r\} \neq \{x\}\) contains a strong vertex. If \(s \in \{u\}\), then \(\{x\} \neq \{u\}\) contains two weak vertices lying in \(\{e\}\), a contradiction. Statement (b) is proved.

Now, we complete the proof of the lemma. Let a vertex \(d\) of \(\{u\} \neq \{x\}\) be adjacent to a vertex \(e\) of \(\{x\} \neq \{z\}\) and to a vertex \(x'\) of \(\Gamma_2(u) \not\subset \Gamma_2(z)\). Since \(|\{d\} \neq \{z\}\| \leq 2\), we see that \(\{d\} \neq \{z\}\) contains at least \(b_1 - 4\) vertices of \(\{z\}\). If \(\{d\} \neq \{z\}\) contains three vertices, then the degree of \(d\) in \(\{u\}\) is at least \(3(b_1 - 3) + 1\), and \(b_1 = 4\). This contradicts the fact that, in this case, \(\{d\} \neq \{z\}\) contains five vertices. Suppose \(\{d\} \neq \{z\}\) contains two vertices \(e\) and \(e'\). Then \(\{z\} \neq \{x\}\) contains \(d, e, e', w, y, a\), and \(b_1 \geq 5\). If \(b_1 = 6\), then, by Lemma 1.1, the subgraph \(\{d\}\) contains five vertices of \(\{u\} \neq \{x\}\) and two vertices in each of \(\{u\} \neq \{e\}\) of \(\{x\}\) and \(\{u\} \neq \{e\}\) of \(\{x\}\). This contradicts the fact that \(\lambda < 8\).

Thus, \(b_1 = 5\). If each of the subgraphs \(\{e\} \neq \{x\}\) contains one vertex, then \(\{d\} \neq \{u\}\) contains three vertices of \(\{x\}\) and two vertices of each of \(\{u\} \neq \{e\}\) of \(\{x\}\) and \(\{u\} \neq \{e\}\) of \(\{x\}\). This contradicts the fact that \(\lambda < 6\). Thus, we may assume that \(\{u\} \neq \{x\}\) contains two vertices \(d, e\). By Lemma 1.4, the vertices \(d' = d, e' = e\) are not adjacent. Therefore, \(\{e'\} \neq \{e\}\) contains two vertices outside \(x^+ \cup z^+\), whence \(\mu(x, z) = 5\) and \(\{x\} \neq \{z\}\) of \(\{e'\}\). By symmetry, \(\{d\}\) contains two vertices outside \(y^+ \cup u^+\), whence \(\mu(u, x) = 5\) and \(\{u\} \neq \{x\}\) of \(\{d\}\) of \(\{e\}\). In this case \(\{x\} \neq \{x\}\) contains \(d, d', e, e', w, y, x\), and a vertex of \(\{u\} \neq \{x\}\), a contradiction.

Thus, \(\{d\} \neq \{x\}\) contains a unique vertex \(e\), in particular, \(b_1 \leq 5\). If \(b_1 = 5\), then, as above, \(\{e\}\) contains two vertices outside \(x^+ \cup z^+\), so that \(\mu(x, z) = 5\) and \(\{x\} \neq \{z\}\) of \(\{e\}\). By symmetry, \(\{d\}\) contains two vertices outside \(u^+ \cup x^+\), hence \(\mu(u, x) = 5\) and \(\{u\} \neq \{x\}\) of \(\{d\}\). Similarly, \(\mu(u, x) = 5\), \(\{u\} \neq \{x\}\) of \(\{x\}\), and \(\mu(x, z) = 5\), \(\{x\} \neq \{z\}\) of \(\{e\}\). In this case,
Lemma 1.5. The lemma is proved.

Thus, \( b_1 = 4 \), \(|x| = |x'|\), and the degree of \( d \) in each of the graphs \([u] \cap [x]\) and \([u] \cap [x']\) is equal to 3. We put \([d] \cap [z] = \{e, l'e'\}\) and \([u] \cap [e] = \{d, d'\}\). Since \([d] \cap [e']\) contains \( e \) and three vertices \( d', f, g \), and \( g \) of \([d] \cap [u]\), the vertices \( f, g \) and \( d \) are adjacent to \( w \). This contradicts the fact that \([d] \cap [u]\) contains \( u, f, g, x \), and \( x' \).

Lemma 3.4. For every geodesic 3-path \( u w f y z \), the subgraph \([y] \cap \Gamma_3(w)\) lies in \( \Gamma_2(u) \).

Proof. We assume that \([y]\) contains a vertex \( z \) of \( \Gamma_3(u) \cap \Gamma_3(w) \). By Lemma 3.3, both \( u(w, x) \) and \( \mu(u, x) \) are less than \( b_1 \).

Now, suppose that \( \mu(u, x) = b_1 - 2 \). Then \(|\Sigma|\) is either \( b_1 + 1 \) or \( b_1 + 2 \). Let \( w \) and \( e \) be distinct vertices of \([u] \cap [x]\). Then \([e] = \{u, \} \) contains \( b_1 \) vertices of \( \Sigma \). Therefore, \([w] \cap [e]\) contains at least \( b_1 - 2 \) vertices of \( \Sigma \). However, for \( f \in \{w\} \cap [e] \cap \Sigma(x) \), we obtain a path \( u w f y z \), the subgraph \([f]\) contains two vertices of \([u] \cap [x]\), and \( \mu(u, f) < b_1 \), a contradiction.

Thus, we have \( \mu(u, x) = \mu(x, z) = b_1 - 1 \) for each vertex \( x \in [w] \cap [y] \), and \(|\Sigma| = b_1 \).

Therefore, for any \( a \in [u] \cap [x] \), there is a unique vertex of \( ([u] \cap [x]) \cup \Sigma \) that does not lie in \( a^2 \). By symmetry, \(|\Sigma| = b_1 \). If \([u] \cap [x]\) contains two nonadjacent vertices \( a \) and \( u \), then \([u] \cap [x] \subset [d] \) for some vertex \( d \in \Sigma(x) \cap [y] \). By Lemma 1.5, we obtain \(|[u] \cap [x] \cup [d] = 2\), which contradicts the fact that \( b_1 > 4 \). Thus, each vertex of \([u] \cap [x]\) is not adjacent to the only vertex of \( \Sigma \); in particular, the subgraph \([u] \cap [x]\) is a clique.

If two distinct vertices \( a \) and \( b \) of \([u] \cap [x]\) are not adjacent to one and the same vertex \( e \) of \( \Sigma \), then \([a] \cap [b] \) contains \( u, x, b_1 - 3 \) vertices of \([u] \cap [x]\), and \( b_1 - 2 \) vertices of \( \Sigma(x) \), a contradiction. Thus, every vertex of \( \Sigma(x) \) is adjacent to \( b_1 - 2 \) vertices of each of the graphs \([u] \cap [x]\) and \([x] \cap [z]\). In particular, the subgraph \( \Sigma(x) \) is a \((b_1 - 1)\)-coclique.

If \( b_1 > 4 \), then \( \Sigma(x) \) contains two vertices \( d \) and \( e \) of \([w] \cap [y]\), \( \mu(u, d) = \mu(d, e) = b_1 - 1 \), and \([u] \cap [d] \cap [e]\) is a clique, which contradicts Lemma 1.5. Thus, \( b_1 = 4 \). For \( d \in \Sigma(x) \cap [w] \cap [y] \), the subgraph \([u] \cap [d] \cap [x]\) contains two vertices \( a \) and \( w \), and \([d] \cap [x] \cap [w] \) contains two vertices of \([x] \cap [z]\) and do not lie in \([a] \cap [w]\). This contradicts Lemma 1.5. The lemma is proved.

For \( a \in \Gamma_2(u) \), we put \( \Delta(a) = [a] \cap \Gamma_3(u) \).

Lemma 3.5. The subgraph \( \Delta(x) \) is a clique in \( \Gamma_2(w) \).

Proof. By Lemma 3.4, we have \( \Delta(x) \subset \Gamma_2(w) \). Let \( y \) and \( z \) be two nonadjacent vertices of \( \Delta(x) \). By Lemma 1.1, \( \mu(w, y) \) and \( \mu(w, z) \) do not exceed \( b_1 - 1 \). Next, \(|x| = y^2 + z^2 + b_1 - 1 \) contains \( z \) and at most \( b_1 - 1 \) vertices of \( [u] \).

If \( \mu(w, y) = b_1 \), then \([w] = u^2 \) contains \([u] \cap [z]\), and \([w] \cap [z]\) lies in \([y]\). Assume that \( \mu(w, z) = b_1 - 1 \). Let \( d \in ([w] \cap [y]) - [z]\). By Lemma 1.1, the subgraph \([u] \cap [z]\) is a clique, and the subgraph \([x] \cap [y]\) lies in \([y] \cup u^2 \cup z^2 \) for each vertex \( x \) of \([w] \cap [z]\). We observe that if \( d \) is adjacent to a vertex \( e \) of \([w] \cap [z]\), then \([e] = z^2 \) contains \( d, e, y, z \), and \( b_1 - 2 \) vertices of \([u] \). Suppose \( d \) is adjacent to \( i \) vertices of \([w] \cap [z]\). If \( b_1 \geq 5 \), then \( i \geq 2 \), and the degree of \( w \) in \([u]\) is at least \( 2(b_1 - 3) + (b_1 - 2) \), a contradiction. Thus, \( b_1 = 4 \) and \( i = 1 \), because otherwise, for vertices \( e, e' \in [w] \cap [z] \) adjacent to \( d \), the subgraph \([e] \cap [e']\) contains \( d, w, y, z, \) and a vertex of \([w] \cap [z] - [d]\). We put \([e, a, a'] = [w] \cap [z]\), where only \( e \) is adjacent to \( d \). Again by Lemma 1.1, the degree of \( w \) is 2 both in \([u] \cap [a]\) and \([u] \cap [a']\), and \( [a] \cap [a']\) contains \( e, y, z, \) and two vertices of \([u]\), a contradiction.

Thus, if \( \mu(w, y) = b_1 \), then \([w] \cap [y] = [w] \cap [z]\). Now, we assume that \([w] \cap [y]\) contains two nonadjacent vertices \( e \) and \( e' \). Then the degree of \( w \) is equal to \( b_1 - 2 \) in each of the graphs \([u] \cap [e]\) and \([u] \cap [e']\), and \([u] \cap [e] \cap [e'] = [y]\) by Lemma 1.5. Let \( a \in [w] \cap [y] - \{e, e'\} \). Then either \([u] \cap [e]\), or \([u] \cap [e']\) contains a vertex \( d \) of \([w] \cap [a]\). For
definiteness, let $d \in \{e\}$. It follows that $[a] \cap [e]$ contains two vertices $y$ and $z$ that do not lie in $[w] \cup [d]$. This contradicts Lemma 1.5. Thus, $[u] \cap [y]$ is a clique. If $[u] \cap [y] \cap [b] \geq 2$ for distinct vertices $a$ and $b$ of the graph $[u] \cap [y]$, then $[a] \cap [e]$ contains two vertices $y$ and $z$ that do not belong to the union of neighborhoods of two vertices in $[u] \cap [a] \cap [b]$, which contradicts Lemma 1.5. Thus, $4(b_1 - 2) \leq 2b_1 - 4$, a contradiction.

Therefore, we have $\mu(w, y) = \mu(w, z) = b_1 - 1$ for every vertex $w \in [u] \cap [x]$. In particular, this implies that the subgraph $[u] \cap [x]$ is a clique. Indeed, otherwise $[u] \cap [x]$ contains nonadjacent vertices $w$ and $w'$, the vertex $x$ is adjacent to two vertices $w'$ and $z$ outside $w^+ \cup y^+$, and $\mu(w, y) = b_1$ by Lemma 1.1. By Lemma 1.5, we obtain $[w] \cap [y'] \cap [z] = \{x\}$, and $[w] \cap [x]$ contains $2b_1 - 4$ vertices of $[y] \cup [z]$, which contradicts the fact that $\mu(u, x) = 1$. The lemma is proved. \hfill \square

In Lemmas 3.6–3.8 below, we assume that $\Delta(x)$ contains distinct vertices $y$ and $z$. By Lemma 3.5, the subgraph $\Delta(x)$ is a clique. Therefore, the vertices $y$ and $z$ are adjacent. By Lemma 3.1, we have $\mu(u, x) < b_1$.

**Lemma 3.6.** The following is true:

1. if $t \in [y] \cap [z] \cap \Delta_2(u)$ and $t$ is adjacent to exactly $j$ vertices of $[y] - z^\perp$, then $t$ is not adjacent to exactly $j$ vertices of $[y] \cap [z] - \{t\}$, $j \leq 2$, and $[t] \cap ([y] - z^\perp) \subset \Gamma_2(u)$;
2. for any adjacent vertices $p$ and $q$ of $\Delta_3(u)$, we have $[u] \cap \Delta_2(p) = [u] \cap \Delta_2(q)$;
3. $[y] - z^\perp \subset \Delta_2(u)$.

**Proof.** Observe that $[t] \cap [y]$ contains $z$, $j$ vertices of $[y] - z^\perp$, and $2b_1 - 5 - j$ vertices of $[y] \cap [z]$. Therefore, $t$ is not adjacent to exactly $j$ vertices of $[y] \cap [z]$. Since $j + \mu(t, u) \leq b_1$, we have $j \leq 2$. By Lemma 3.5 (applied to $t$ in the role of $x$), we obtain $[t] \cap ([y] - z^\perp) \subset \Delta_2(u)$. Statement (1) is proved.

Suppose $[u] \cap \Delta_2(p) \neq [u] \cap \Delta_2(q)$. We may assume that $\Delta_2(p) \cap \Delta_3(g) \cap \Delta_3(d)$ contains a vertex $d$ nonadjacent to $u$. Then $u \in [d] \cap \Delta_3(p)$, and Lemma 3.4 shows that $u \in \Delta_2(q)$, a contradiction. Statement (2) is proved.

Now, assume that $r \in ([y] - z^\perp) - \Delta_2(u)$. Then $[r] \cap [z]$ lies in $\Delta_3(u)$, because otherwise, for $a \in [r] \cap [z] \cap \Delta_2(u)$, the subgraph $\Delta(a)$ is not a clique, which contradicts Lemma 3.5. By statement (2), we have $[u] \cap \Delta_2(r) = [u] \cap \Delta_2(o) = [u] \cap \Delta_2(z)$ for every vertex $o \in [r] \cap [z]$. If $d \in [u] \cap \Delta_2(r)$, then each of the graphs $[d] \cap [z]$ and $[d] \cap [r]$ contains at least $b_1 - 2$ vertices. Hence, $b_1 = 4$ and $\mu(d, z) = \mu(d, r) = 2$. We put $\{e, f\} = [d] \cap [z]$ and $\{g, h\} = [d] \cap [r]$. By Lemma 1.4, either $[d] \cap [y] = \{e, f, g, h\}$, or we may assume that $[d] \cap [y] = \{e, y\}$. Furthermore, we have $([y] - z^\perp) \cap \Delta_3(u) = \{r\}$, because otherwise, if a vertex $r'$ of this subgraph is different from $r$, then $[d] \cap [r] = [d] \cap [r']$, which contradicts Lemma 1.4. By symmetry, $([y] - r^\perp) \cap \Delta_3(u) = \{z\}$.

We prove that $[y] \cap \Delta_2(u) \subset [r] \cup [z]$, and $[u] \cap [a] \cap [b] \ni b \in [u] \cap [a] \cap [b]$, we have $b \in \Delta_2(r) \cap \Delta_2(z)$ by statement (2). This contradicts the inclusion $[b] - u^\perp \subset [r] \cup [z]$. Now, by Lemma 1.1, the degree of $y$ in the graph $[r] \cap [z]$ is equal to 1. If $y'$ is a vertex of $[r] \cap [z]$ adjacent to $y$, then we can take this vertex in place of $y$, and $\{y', z\}$ is a connected component of the subgraph $[r] \cap [z]$. The subgraph $[y] \cap [y']$ contains $r, z$, and two vertices of $\Delta_2(u)$. Similarly, $[r] \cap [y]$ contains $y'$ and three vertices of $\Delta_2(u)$. Thus, each of the vertices $y$ and $y'$ is adjacent to six vertices of $\Delta_2(u)$. Therefore, $\Delta_2(u)$ contains ten vertices, each being adjacent to at least two vertices of $[r, y, y', z]$. We note that $[r] - y^\perp$ does not intersect $\Delta_3(u)$; indeed, otherwise $s \in ([r] - y^\perp) \cap \Delta_3(u)$, the degree of the vertex $r$ in $[y] \cap [s]$ is equal to 1, and $s$ is adjacent to $y'$. Since $([y'] - z^\perp) \cap \Delta_3(u) = \{r\}$, we see that $s$ is adjacent to $r$, a contradiction.

Let $e \in [r] - ([y] \cup [y'])$. Then $e \in \Delta_2(u)$, and, for $d \in [u] \cap [e]$, the subgraph $[d] \cap [r]$ contains a vertex $s$ adjacent to $y$ and $y'$, and we have $\mu(d, y) = \mu(d, y') = 2$. Also, $[s] - d^\perp$ contains a vertex of $[y] \cap [y'] \cap [r]$. Therefore, both vertices of $[y] \cap [y'] \cap \Delta_2(u)$ lie in $[r]$. 


Lemma 3.7. Every vertex $d$ of $[u] \cap \Gamma_2(y)$ is adjacent to a vertex of $[y] \cap [z]$.

Proof. We assume that a vertex $d$ of $[u] \cap \Gamma_2(y)$ is adjacent to no vertex of $[y] \cap [z]$. Then each of the subgraphs $[d] \cap [y]$ and $[d] \cap [z]$ contains at least $b_1 - 2$ vertices. It follows that $b_1 = 4$ and $\mu(d, y) = \mu(d, z) = 2$. We put $\{e, f\} = [d] \cap [z]$, $\{g, h\} = [d] \cap [y]$, and $\{s_1, \ldots, s_4\} = [d] \cap [u]$. By Lemma 1.1, a neighborhood of each vertex of $\{e, f\}$ lies in $d' \cup z'$. 

Suppose that $g$ and $h$ are not adjacent to $e$. Then $[d] \cap [e]$ contains $f$ and three vertices of $\{s_1, \ldots, s_4\}$. In particular, $\mu(e, u) = 4$, and we may assume that $e$ is adjacent to $s_1, s_2,$ and $s_3$. Since $d$ is adjacent to the vertex $e$ outside $u^\perp \cup g^\perp$, the degree of $d$ in the graph $[u] \cap [g]$ is at least 2 by Lemma 1.1. We note that each vertex $s_i$, $i = 1, 2, 3,$ is adjacent to at most one vertex of $\{g, h\}$ (otherwise, $[s_i] \cap [d]$ contains $u, e, g, h,$ and two vertices of $[u] \cap [e]$). If each vertex $s_i$, $i = 1, 2, 3,$ is adjacent to $g$ or to $h$ (for definiteness, let $s_1, s_2 \in [g]$ and $s_3 \in [h]$), then the degree of $s_i$ in the graph $[e] \cap [u] - \{d\}$ is at least 1. Therefore, some vertex $s_i$ has degree 2 in the graph $\{s_1, \ldots, s_3\}$. This contradicts the fact that the subgraph $[d] \cap [s_i]$ contains $u, e$, two vertices of $[e] \cap [u]$, and a vertex of $\{g, h\}$. Thus, one of the vertices $\{s_1, \ldots, s_3\}$ does not belong to $[g] \cup [h]$.

If $f$ is not adjacent to $g$, then, repeating the argument for $g$ as $e$, we see that $[g]$ contains $\{s_1, s_2, s_4\}$, $s_3$ is not adjacent to $h$, and $s_4$ is not adjacent to $f$. However, in this case $[d] \cap [h]$ contains $f, g, s_4$, and a vertex of $\{s_1, s_2\}$, a contradiction. Thus, if $e$ is not adjacent to $g$ and to $h$, then $[f]$ contains $g$ and $h$. Since $[d] \cap [g]$ contains $f, h$, and two vertices of $\{s_1, \ldots, s_4\}$, and $[d] \cap [h]$ contains two vertices of $\{s_1, \ldots, s_4\}$, we may assume that $g$ is adjacent to $s_1$ and to $s_4$ and $h$ is adjacent to $s_2$ and to $s_4$. Observe that $[u] \cap [g] \cap [h]$ contains two vertices $d$ and $s_3$ such that $d$ is adjacent to $e$ outside $u^\perp \cup [g] \cup [h]$. By Lemma 1.5, the subgraph $[d] \cup [s_4]$ contains $[g] \cap [h]$, which contradicts the fact that the vertex $y$ of $[g] \cap [h]$ does not belong to $[d] \cup [s_4]$.

Thus, every vertex of $\{e, f\}$ is adjacent to a vertex of $\{g, h\}$. We assume that $g, h \in [e]$. There is no loss of generality in assuming that $f$ is adjacent to $g$. Then $[d] - u^\perp$ lies in $e^\perp \cup g^\perp$. Therefore, $\mu(u, e) = \mu(u, g) = 2$, and the subgraph $[e] \cap [g]$ contains $d, f, h$, and either a vertex $t$ of $[y] \cap [z]$ or a vertex $s_i$. In the latter case, we obtain a contradiction with Lemma 1.4.

Thus, the subgraph $[e] \cap [g]$ contains a vertex $t$ of $[y] \cap [z]$. For definiteness, assume that $e$ is adjacent to $s_1$ and $g$ is adjacent to $s_2$. If $s_1$ is adjacent to $f$, then the degrees of $e$ and $f$ in the graph $[g] \cap [z]$ are at least 2 (because $s_1 \notin g^\perp \cup z^\perp$). However, the vertices $e$ and $f$ are not adjacent to the vertex $y$ of $[g] \cap [z]$. Therefore, $[e] \cap [f]$ contains $d, g, s_1, z$, and a vertex of $[g] \cap [z]$, a contradiction. Thus, $s_1$ is not adjacent to $f$ and, by symmetry, $s_2$ is not adjacent to $h$.

If $s_1$ is adjacent to $h$, then, by Lemma 1.4, we have $\mu(u, h) = 4$. Since $[h] - y^\perp \subset d^\perp$, we see that $[d] \cap [h]$ contains $e, g, s_1, s_3,$ and $s_4$, a contradiction. Thus, $s_1$ is not adjacent to $h$ and $s_2$ is not adjacent to $f$. The subgraph $[s_1] - u^\perp$ contains three vertices $p, p',$ and $t$ of $\{f, g, h, z\}$. By symmetry, $[s_2] - u^\perp = \{q, q', t\}$. Furthermore, $[t]$ contains a vertex of $[s_1] \cap [g]$ and a vertex of $[s_2] \cap [z]$. This allows us to assume that $p, q \in [t]$. Since $[t] \cap [e] = \{g, p, s_1, z\}$, the degree of $t$ in the graph $[s_1] \cap [g]$ is equal to 1. However, $[s_1]$ contains a vertex off $u^\perp \cup t^\perp$. Therefore, the subgraph $[u] \cup [t]$ is a 3-clique $\{s_1, s_2, r\}$. Since $[p] \cap [t] = \{e, s_1, y, z\}$ and $[g] \cap [t] = \{e, s_2, y, z\}$, the graph $[r] \cap [t]$ contains only $s_1$ and $s_2$, a contradiction.

Thus, we may assume that $e \in [g] - [h]$, $f \in [h] - [g]$, and $|[d] \cap [u]| = 3$ for every vertex $o$ of $\{e, f, g, h\}$. Suppose that $e$ is adjacent to three vertices of $\{e, f, g, h\}$: say, $s_1$ is adjacent to $e, f,$ and $g$. Then the vertices $s_2, s_3, s_4,$ and $h$ are not adjacent to
Lemma 3.8. Suppose a vertex $d$ of $[u] \cap \Gamma_2(y)$ is adjacent to $\alpha$ vertices $g_1, \ldots, g_\alpha$ of $[y] - z^\perp$. Then $\alpha \le 2$, and the following is true if $\alpha = 2$:

1. each vertex $t$ of $[d] \cap [y] \cap [z]$ is adjacent to a vertex of $\{g_1, g_2\}$;
2. each of the vertices $g_1$ and $g_2$ is not adjacent to at most one vertex of $[d] \cap [y] \cap [z]$;
3. at most one vertex of $[d] \cap [y] \cap [z]$ does not belong to $\{g_1, g_2\}$.

Proof. Since $\mu(d, z) \le b_1 - \alpha$, we have $\alpha \le 2$. Let $\alpha = 2$. Then $d, z$ is a good pair. By Lemma 3.7, the vertex $d$ is adjacent to a vertex $t$ of $[y] \cap [z]$.

If $\mu(d, y) < b_1$, then, by Lemma 1.4, we obtain $\{t\} = [d] \cap [y] \cap [z]$, whence $\mu(d, y) + b_1 - 3 \le b_1$. In particular, $b_1 \le 5$, and $b_1 \le 4$ if $d, y$ is an almost good pair. In this case, all statement of the lemma are true. Thus, we may assume that $\mu(d, y) = b_1$ and $[d] \cap [z] \subset [y]$.

If $t$ is not adjacent to the vertices of $\{g_1, g_2\}$, then $\{t\}$ contains $b_1 - 3$ vertices of $[d] \cap [y]$ and $b_1 - 1$ vertices of $[d] - y^\perp$ (and the latter vertices lie in $[u]$). Now, $[u] \cap \{t\}$ contains $d$ and $b_1 - 1$ vertices of $[d] - [y]$. This contradicts Lemma 3.1. Statement (1) is proved.

Suppose $t$ is not adjacent to $g_1$. Let $[d] \cap [z]$ contain a vertex $e$ distinct from $t$ and not adjacent to $g_1$. Then $[g_2]$ contains the vertices $t$ and $e$ of $[d] \cap [z]$, so that $\mu(g_2, z) \ge b_1$. By Lemma 3.1, the triple $(u, t, e)$ is almost good, and $[t] \cap [e]$ contains two vertices $y$ and $z$ not adjacent to the vertices of $[u] \cap \{t\} \cap [e]$. By Lemma 1.5, we have $[u] \cap \{t\} \cap [e] = \{d\}$. Observe that $[t] \cup [e]$ contains $[d] - ([u] \cup [y])$. If $b_1 > 4$, then $[d] \cap [z]$ contains a third vertex $f$. We may assume that $[f] \cap [e]$ contains at least $(b_1 - 3)/2$ vertices of $[d] \cap [u]$, and $b_1 \le 3$ by Lemma 1.4, a contradiction. Thus, $b_1 = 4$, and $[d] \cap [g_1]$ contains $g_2$ and three vertices of $[u]$. In particular, $\mu(u, g_1) = 4$. Now we restore $[y]$. This subgraph contains the $K_{1,1,2}$-subgraph $\{t, e, g_2, z\}$ and two vertices of $[y] - ([d] \cup [z])$ belonging to $X_0(\{t, e, g_2, z\})$. It follows that $[y] - \{t, e, g_2, z\}$ contains the $K_{1,1,3}$-subgraph (which, possibly, is not induced). However, $[g_1] \cap [g_2]$ contains $d, y$, at most one vertex of $[d] \cap [u]$, and a vertex of $[y]$. Therefore, $X_0(\{t, e, g_2, z\})$ contains a $4$-clique $K$, and the vertex $g_2$ of $[y] - K$ is adjacent to two vertices of $K$. But $[y] - ([g_2] \cup K)$ is a $4$-clique $L$, and $[t] \cap [e]$ contains $d, y, z, g_2$, and one more vertex of $L$, a contradiction. Statement (2) is proved.

Now, we assume that (3) fails. Then $[d] \cap [y] \cap [z]$ contains vertices $t$ and $e$ not adjacent to $g_1$ and $g_2$, respectively. As above, $[t] \cap [e]$ contains a unique vertex of $[d] \cap [u]$, and every vertex of $[d] \cap [u]$ is adjacent to $t$ or to $e$.

Suppose $[d] \cap [y] \cap [z]$ contains a vertex $f$ distinct from $t$ and $e$. Then $f$ is adjacent to $g_1$ and $g_2$, and $[u] \cap [f] = b_1 - 2 = 1$, a contradiction.

Thus, $[d] \cap [y] \cap [z] = \{t, e\}$ and $b_1 = 4$. Suppose that the vertices $g_1$ and $g_2$ are not adjacent. Then the degree of $g_1$ in the graph $[d] \cap [y]$ is equal to $b_1 - 3$, and $[g_1] \cap [g_2]$ contains a vertex $r$ of $[d] \cap [u]$. There is no loss of generality in assuming that $r$ is adjacent to $t$; then $[d] \cap [r]$ contains $u, t, g_1, g_2$, and a vertex of $[u] \cap [g_1]$, a contradiction. Thus, the vertices $g_1$ and $g_2$ are adjacent. If $[g_1] \cap [g_2]$ contains a vertex $r$ of $[d] \cap [u]$ (for definiteness, let $r$ be adjacent to $t$), then $[d] \cap [r]$ contains $u, t, g_1, g_2$, and a vertex of $[u] \cap [t]$, a contradiction. Thus, $[g_1] \cap [g_2]$ does not intersect $[d] \cap [u]$.

Let $\{r, s\} = [d] \cap [u] \cap [r]$, and let $f$ and $h$ be vertices of $([y] \cap [z]) - [d]$ and $[z] - y^\perp$, respectively, adjacent to $t$. If $r$ is adjacent to $g_1$, then $[d]$ contains the $K_{3,3}$-subgraph $\{e, g_2, r; t, g_1, r'\}$. Then the degree of $r$ in the graph $[t] \cap [u]$ is equal to 2, and $[d] \cap [r]$ contains $u, t, s, g_1, r$, a contradiction. Thus, $\{r, s\} = [d] \cap [u] \cap [g_2]$ and $[t] \cap [g_2] = \{d, r, s, y\}$. If the vertices $r$ and $s$ are not adjacent, then the degree of $r$ in the graph
[t] ∩ [u] is equal to 1 and [r] ∩ [t] contains two vertices of [z] − [g_2]. By Lemma 1.1, the degree of r in the graph [u] ∩ [g_2] is equal to 3, which contradicts the fact that r is not adjacent to s. Thus, the vertices r and s are adjacent and [r] ∩ [s] = {d, u, t, g_2}. Now, at least one of the vertices r or s (for definiteness, let it be r) is not adjacent to f, the degree of t in the graph [r] ∩ [g_2] is equal to 1, and r is adjacent to h. Hence, [t] ∩ [s] = {d, r, f, g_2}. Since [t] is a regular graph of degree 4, the vertex h is adjacent to f and to e.

Let o and h be vertices of ([y] ∩ [z]) − [d] and [z] − y^−, respectively, adjacent to e. As above, h is adjacent to o and µ(o, u) = 3. It follows that there is a vertex p of [z] − y^− adjacent to no vertex of [y] ∩ [z]. This contradicts the fact that, in this case, the vertex z is isolated in [p] ∩ [y].

**Lemma 3.9.** Suppose a vertex d of [u] ∩ \( \Gamma_2(y) \) is adjacent to two vertices g_1 and g_2 of [y] − z^±. Then every vertex of [d] ∩ [y] ∩ [z] is adjacent to both g_1 and g_2.

*Proof.* By assumption, the pair d, z is good. Suppose that [d] ∩ [y] ∩ [z] contains a unique vertex t outside [g_1] ∩ [g_2]. Without loss of generality, we may assume that t is not adjacent to g_1.

Let \( \mu(d, y) = b_1 \). By Lemma 3.8, every vertex of [d] ∩ [y] ∩ [z] − {t} is adjacent to g_1 and g_2 and forms a good pair with u. Furthermore, the pair u, t is almost good, and the degree of d in the graph [u] is at least \((b_1 - 2) + (b_1 - 3)^2\). Hence, \( b_1^2 - 5b_1 + 7 \leq 2b_1 - 4 \) and \( b_1 = 4 \). We put [d] ∩ [z] = {e, t}. Then [e] ∩ [y] = \( \{g_1, g_2, t, z\} \). If the vertices g_1 and g_2 are not adjacent, then the degree of g_1 in the graph [d] ∩ [y] is equal to \( b_1 - 3 = 1 \). Therefore, \( [g_1] ∩ [e] ∩ [z] = \{y\} \), which is impossible because e is adjacent to the vertex g_2 lying outside \( g_1^+ \cup z^± \). Thus, the vertices g_1 and g_2 are adjacent. Then [e] ∩ [z] contains two vertices h and h' outside y^−. On the other hand, [e] ∩ [g_1] contains the vertices d, g_2, y and at most one vertex of \{h, h'\}. For definiteness, let g_1 be not adjacent to h. Since [e] ∩ [g_2] = \( \{d, g_1, t, y\} \) and [e] ∩ [t] = \( \{d, g_2, y, z\} \), we see that [e] − h^− contains d, g_1, g_2, t, and y, a contradiction.

Let \( \mu(d, y) = b_1 - 1 \). By Lemma 1.4, we have [d] ∩ [y] ∩ [z] = {t} and \( b_1 = 4 \). Let [d] ∩ [z] = \{e, t\}. If e is adjacent to g_1 and g_2, then u, e is a good pair. The graph [d] ∩ [t] = \{e, g_1, g_2, t\} contains two vertices belonging to [u]. Therefore, u, t is an almost good pair. Finally, [d] ∩ [g_1] contains two vertices of [u], and u, g_1 is an almost good pair. This contradicts the fact that, by Lemma 1.4, the degree of d in the graph [u] is equal to 5. If g_1 and g_2 are not adjacent to e, then [d] ∩ [e] ∩ [g_1] contains at least two vertices lying in [u]. First, suppose that [d] ∩ [e] ∩ [g_1] = \{r, s\}. Then [u] ∩ [e] = \( \{d, r, s, e'\} \) and [u] ∩ [g_1] = \( \{d, r, s, g_1'\} \). By Lemma 1.1, the degrees of r and s in the graphs [u] ∩ [e] and [u] ∩ [g_1] are at least 2. If the vertices r and s are not adjacent, then [d] ∩ [r] contains u, e, e', g_1, and g_1', a contradiction. Thus, the vertices r and s are adjacent and [d] ∩ [g_2] = \{t, e', g_1, g_1'\}, which contradicts the fact that, in this case, the degree of e' in the graph [u] ∩ [e] is at least 2 by Lemma 1.1. Now, let [d] ∩ [e] ∩ [g_1] = \{q, r, s\}. Then [u] ∩ [e] = \( \{d, q, r, s\} = [u] ∩ [g_1] \). By Lemma 1.1, the degrees of the vertices q, r, and s in the graph [u] ∩ [e] − {d} are at least 1. This contradicts the fact that none of the vertices q, r, or s has degree 5 in the graph [d].

Let e ∈ [g_2] − [g_1], and let [e] ∩ [u] = \{d, r, s\}. Without loss of generality, we may assume that the vertex g_1 is adjacent to r. By Lemma 1.1 applied to the graph [u] ∩ [e], the vertex r is adjacent to s. Similarly, [u] ∩ [t] is a triangle containing d and yet another vertex t' adjacent to g_1 and distinct from r. If the vertices t and s are adjacent, then [d] ∩ [s] = \{e, r, t, u\}, which is impossible because s^± contains [u] ∩ [t]. Thus, the vertices t and s are not adjacent. If g_1 is not adjacent to s, then [d] ∩ [r] contains e, g_1, s, u, and a vertex of [u] ∩ [g_1], a contradiction. We have [d] ∩ [g_1] = \{g_2, r, s, t'\}, so that g_1 is
not adjacent to a vertex of $[u] \cap [t] - \{t', d\}$. Thus, $t'$ is adjacent to the vertex $t$ outside $g^+_1 \cup u^+$. By Lemma 1.1, the vertex $t'$ must be adjacent to $r$ or to $s$, a contradiction.

Let $e \in [g_1] - [g_2]$. Since $u, g_1, t$ is an almost good triple and the vertices $t$ and $g_1$ are not adjacent, we have $[u] \cap [t] \cap [g_1] = \{d\}$ by Lemma 1.5. If both vertices of $[u] \cap [g_1] - \{d\}$ are not adjacent to $e$, then, by Lemma 1.1, the degree of $g_1$ in the graph $[e] \cap [y]$ is equal to 3, which contradicts the fact that $[e] \cap [t]$ contains three vertices of $[u]$ at least two vertices of $z^+$. We put $\{r_1, r_2\} = [d] \cap [u] \cap [t]$, $\{s_1, s_2\} = [d] \cap [u] \cap [g_1]$, $\{o_1, o_2\} = [y] \cap [z] \cap [t]$, and $\{f\} = [y] \cap [z] - t^+$. Assume that the vertex $e$ is adjacent to $s_1$ and $s_2$. Then $[t] \cap [e] = \{d, o_1, o_2, z\}$, and the degree of $t$ in the graph $[r_1] \cap [y]$ is equal to 2. If $r_1$ is not adjacent to $g_2$, then $o_1, o_2 \in [r_1]$ by Lemma 1.1, and $r_2$ is adjacent to some vertex $o_i$. Since $[o_i] \cap [t]$ contains $e, r_1, r_2, y$, and $z$, we arrive at a contradiction. Thus, if the vertex $e$ is adjacent to both $s_1$ and $s_2$, then $g_2$ is adjacent to both $r_1$ and $r_2$. In this case, the vertices $r_1$ and $r_2$ are adjacent, because otherwise the degree of $t$ in the graph $[r_1] \cap [y]$ is at least 3 and again $[o_i] \cap [t] = 5$. Thus, we may assume that $r_i$ is adjacent to $o_i$. Since $[t] \subseteq r_1^+ \cup o_2^+$, we see that $o_1$ is not adjacent to $o_2$. Therefore, $[o_1] \cap [u]$ contains $r_1$ and two vertices of $([u] - d^+) \cap [r_1]$. Furthermore, $[o_i] - y^+$ contains three vertices of $[u]$. Similarly, $[o_i] - z^+$ contains three vertices of $[u]$. Thus, $[o_1] \cap [y]$ contains $f, t, z$, and a vertex $h_1$ of $[y] - z^+$. Now, $[r_1] \cap [o_1]$ contains $t$, two vertices of $[u]$, and a vertex of $\{h_1, f\}$. However, $f \notin [r_1] \cap [r_2]$, so that some vertex $r_1$ is adjacent to $h_1$. Replacing the triple $(d, g_1, g_2)$ by $(r_1, h_1, g_2)$, we obtain a contradiction with statement (3) of Lemma 3.8.

Without loss of generality, we may assume that the vertex $e$ is adjacent to $r_1, s_1$, and $o_1$. Then $\mu(r_1, y) = 3$, and $[r_1] \cap [d]$ contains $e, t, u$, and at most one vertex of $\{r_2, s_1\}$. First, suppose that the vertices $r_1$ and $r_2$ are not adjacent. Then $[r_1] \subseteq u^+ \cup t^+$. Since the degree of $t$ in the graph $[r_1] \cap [y]$ is equal to 2, we see that $[r_1]$ contains two vertices of $\{g_2, o_1, o_2\}$. Since the degree of $t$ in the graph $[r_2] \cap [y]$ is equal to 3, the graph $[r_2]$ contains the vertex $g_2, o_1, o_2$. On the other hand, $[r_1]$ contains $[d] \cap [z]$. By Lemma 1.4, we have $\mu(r_1, z) = 4$, and $r_1$ is adjacent to $o_1$ and $o_2$. This contradicts the fact that $[o_i] \cap [t]$ contains $e, r_1, r_2, y$, and $z$.

Thus, $r_1 \in [r_2] - [s_1]$, and the degree of $r_1$ in the graph $[e] \cap [u]$ is equal to 1. Since $[t] \subseteq r_1^+ \cup y^+$, the degree of $t$ in the graph $[r_1] \cap [y]$ is equal to 1. On the other hand, $[r_1]$ contains $[d] \cap [z]$. By Lemma 1.4, we obtain $\mu(r_1, z) = 4$, and $r_1$ is not adjacent to $g_2$. If $r_1$ is adjacent to a vertex $h$ of $[z] - \{e\} \cup y^+$, then, replacing the triple $(d, g_1, g_2)$ with $(r_1, h, e)$ and applying Lemma 3.8, we conclude that $o_1 \in [r_1] \cap [h]$. This case was analyzed in the second paragraph of the proof of the lemma.

Thus, $[r_1] \cap [z] = \{e, f, o_1, t\}$. Since the degree of $s_1$ in the graph $[u] \cap [e]$ is equal to 1, we have $[s_1] - u^+ \subseteq e^+$. If $r_2$ is adjacent to $o_1$, then we obtain a contradiction with the fact that $[o_1] \cap [t]$ contains $e, r_1, r_2, y$, and $z$. Therefore, $[t] \cap [r_2] = \{d, g_2, o_2, r_1\}$, and $\{f\} \cap [u]$ contains three vertices of $r_1^+$. The subgraph $[u] \cap [o_1]$ contains a unique vertex $r_1$ of $[d]$. Furthermore, the degree of $e$ in the graph $[g_1] \cap [z]$ is at least 2 ($[e]$ contains the vertex $r_1$ outside $g_1^+ \cup z^+$). Since the degrees of the vertices $g_1$ and $e$ in the graphs $[d] \cap [y]$ and $[d] \cap [z]$ (respectively) are equal to 1, we see that $[g_1] \cap [e] \cap [z]$ lies in $[y]$ and therefore, contains $f$ and $o_1$. On the other hand, $[o_1]$, contains the vertex $r_1$ and a vertex of $[u] - d^-$. Thus, the degree of $o_1$ in the graph $[g_1] \cap [z]$ is at least 3, and, in particular, $o_1$ is adjacent to $f$. This contradicts the fact that $[f] \cap [o_1]$ contains $e, g_1, r_1, y$, and $z$. The lemma is proved.

Lemma 3.10. Every vertex of $[u] \cap \Gamma_2(y)$ is adjacent to at most one vertex of $[y] - z^+$.

Proof. Suppose that a vertex $d$ of $[u] \cap \Gamma_2(y)$ is adjacent to two vertices $g_1$ and $g_2$ of $[y] - z^+$. Then $\mu(d, z) = b_1 - 2$, and $[t] \subseteq d^+ \cup z^+$ for $t \in [d] \cap [z]$. First, we consider the case
where \([d]\) contains the vertices \(e_1, \ldots, e_{b_1-2}\) of \([y] \cap [z]\). By Lemma 3.9, each vertex \(e_i\) is adjacent to \(g_1\) and \(g_2\), and \(\mu(u, e_i) = b_1 - 2\). For \(i \neq j\), the subgraphs \([u] \cap [e_i]\) and \([u] \cap [e_j]\) contain a unique vertex \(d\) in common. Therefore, \((b_1 - 3)(b_1 - 2) \leq 2b_1 - 4\) and \(b_1 \leq 5\). If \(b_1 = 5\), then \([d] \cap [z] = \{e_1, e_2, e_3\}\) is a triangle, and \([e_1] \cap [e_2] = \{e_3, g_1, g_2, d, y, z\}\). Since \(e_i \in [g_1] \cap [g_2]\), Lemma 1.1 shows that the subgraph \([e_i]\) contains two vertices of \([z] - (d \cup y^-)\). This is impossible, because \([z]\) contains seven vertices of \(y^+\) and six vertices outside \(y^-\). If \(b_1 = 4\), then \([e_1] \cap [e_2] = \{g_1, g_2, d, y, z\}\), a contradiction.

Thus, \([d]\) contains a vertex \(h\) of \([z] - y^-\). Then the pair \(d, y\) is almost good. By Lemma 1.4, we obtain \([d] \cap [y] \cap [z] \leq 1\). Hence, \(b_1 = 4\) and \([d] \cap [y] \cap [z] = \{\}\). By Lemma 3.9, the vertex \(e\) is adjacent to \(g_1\) and \(g_2\), whence \(\mu(u, e) = 2\). We put \([e] \cap [y] \cap [z] = \{o\}\). Let \(r\) and \(s\) be vertices of \([u] \cap [e] - \{d\}\) and \([e] \cap [z] - \{h, o, y\}\), respectively.

Suppose that the vertices \(g_1\) and \(g_2\) are not adjacent. Then \(g_1^+ \subset d^- \cup y^-\). If \(g_1\) and \(g_2\) are adjacent to \(h\), then \([g_1] \cap [d]\) contains two vertices of \([u]\). Therefore, the pairs \(u, g_1\) and \(u, g_2\) are almost good, and \([u] \cap [g_1] \cap [g_2] = \{d\}\) by Lemma 1.5. On the other hand, \([h] \cap [u]\) contains a vertex \(p\) of \([d]\) adjacent to \(g_1\), so that \([p, d] \subset [u] \cap [h] \cap [g_1]\). This contradicts Lemma 1.4. If \(h \in [g_1] - [g_2]\), then the degree of \(e\) in the graph \([g_1] \cap [z]\) is at least 2 \(\{e\}\) contains the vertex \(g_1\) outside \(g_2^+ \cup z^\perp\). In this case, we have \(o \in [g_1]\). Applying Lemma 1.4 to the triples \(u, g_1, g_2\) and \(u, g_1, h\), we see that \(r\) is adjacent neither to \(g_1\) nor to \(h\). Therefore, \([e] \cap [g_1]\) contains \(d, h, y, o, z, r\). Since the degree of \(e\) in the graph \([r] \cap [z]\) is at least 2 \(\{e\}\) contains \(g_1\), we see that \(r\) is adjacent to \(o\) and \(s\). This contradicts the fact that \([e] \cap [o]\) contains \(g_1, g_2, r, y, o, z\). If \(g_1\) and \(g_2\) are not adjacent to \(h\), then \([d] \cap [z]\) contains three vertices of \([u]\) for \(x \in [g_1, g_2, h]\). Therefore, \([u] \cap [d]\) contains the vertex \(r\) adjacent to \(g_1, g_2, h\). By Lemma 1.1, the degree of \(r\) in the graph \([u] \cap [h]\) is at least 3 and \([h] \cap [r] \subset d^-\). This contradicts the fact that \([d] \cap [r]\) contains \(u, g_1, g_2, h\), and two more vertices.

Thus, the vertices \(g_1\) and \(g_2\) are adjacent. We prove that \(r \in [g_1] \cup [g_2]\). Assuming the contrary, we have \([e] \cap [r] = \{d, h, o, s\}\). Replacing \(d\) with \(r\), we obtain \([r] \cap [y] \cap [z] = \{o\}\), which contradicts the statement obtained in the first paragraph of the proof. We may assume that \(r \in [g_1]\).

If \(h\) is adjacent to \(r\), then \([d] \cap [r] \cap [z] = \{e, h\}\). Since the pair \(d, z\) is good, we have \(\mu(r, z) = 4\) by Lemma 1.4. In this case, we have \([r] \cap ([y] \cup [z]) = 5\), which contradicts the fact that \([r] \cap [y] = 2\). Thus, \(h\) is not adjacent to \(r\). Now, \([e]\) contains the vertex \(h\) outside \(r^- \cup y^-\), and the degree of \(e\) in the graph \([r] \cap [y]\) is at least 2. If the degree of \(e\) in \([r] \cap [y]\) is at least 3, then \([r]\) contains two vertices of each of the graphs \([y] - z^\perp\) and \([y] \cap [z]\). Again, this contradicts the statement proved in the first paragraph of the proof. Thus, \([e] \cap [r]\) contains \(d, g_1, e\), which contradicts the statement obtained in the first paragraph of the proof. We may assume that \(r \in [g_1]\).

Assume that \(o \in [r] \cap [h]\). Then \([o]\) contains the vertex \(h\) outside \(r^- \cup y^-\), the degree of \(o\) in the graph \([r] \cap [y]\) is at least 2, and \(o\) is adjacent to \(g_1\). In this case, \([e] \cap [g_1]\) contains \(d, g_2, o, r, y\), and \(y\), a contradiction. Thus, \(o \notin [r] \cap [h]\). Again, \([e]\) contains the vertex \(y\) outside \(r^- \cup y^-\), and the degree of \(e\) in the graph \([r] \cap [h]\) is at least 2. Therefore, \([e] \cap [r] \cap [h] = \{d, s\}\) (we recall that \([e] \cap [g_1] = \{d, g_2, r, y\}\)).

If \(s\) is adjacent to \(g_2\), then \(r\) and \(h\) are not adjacent to \(g_2\) because \([e] \cap [g_2] = \{d, g_1, s, o\}\) and \([e] \cap [h] = \{d, s, o, z\}\), which is incompatible with \(o \notin [r] \cap [h]\). Thus, \(s\) is not adjacent to \(g_2\), whence \([e] \cap [s] = \{h, o, r, z\}\).

First, assume that \(r\) is adjacent to \(g_2\). Then \([u] \cap [g_1] \cap [d] = \{r, p_1\}\) and \([u] \cap [g_2] \cap [d] = \{r, p_2\}\). Here, \(p_1 \neq p_2\) since \([g_1] \cap [g_2] = \{d, e, r, y\}\). In this case, \([d] \cap [h] = \{g_1, g_2, r, u\}\), so that \(p_1, p_2, q \in [h]\), where \(q \in [d] - (e^- \cup \{u, p_1, p_2\})\). Therefore, \([d] \cap [q] = \{h, p_1, p_2, u\}\), and the graph \([d]\) is constructed. But now the vertices \(p_1\) and \(p_2\) are not adjacent, \([h]\) contains the vertices \(e\) and \(z\) outside \(p_1^- \cup p_2^-\), and the degree of \(h\) in the graph.
[p_1] \cap [p_2] is at least 3. Therefore, [p_1] \cap [p_2] \cap [h] contains d, q, and a vertex of [z] - d^2. Thus, p_1 and p_2 are nonadjacent vertices forming good or almost good pairs with z, and [p_1] \cap [p_2] \cap [z] \geq 2. This contradicts Lemma 1.5.

Consequently, r is not adjacent to g_2. Therefore, r is adjacent to o, and o is not adjacent to h. The subgraph \([r] \cap [z] = \{e, o, s\}\) is a triangle, and \([r] \cap [s] \) contains e, o, and two vertices of \([r] \cap [u]\). Similarly, \([r] \cap [o] \) contains e, s, and two vertices of \([r] \cap [u]\).

Since the vertex d of \([r] \cap [u]\) does not belong to \([o] \cup [s] \), we see that \([o] \cap [s] \) contains a vertex t of \([r] \cap [u]\). If t is adjacent to g_1, then the degree of t in the graph \([o] \cap [u]\) is at least 2. On the other hand, \([r] \cap [t] = \{g_1, o, s, u\} \), and the degree of o in the graph \([r] \cap [y]\) is equal to 1. Therefore, \([o] \cap [u] \subset [o] - y^+ \subset r^+\), which contradicts the fact that \([o] \cap [u] \cap [t] \) lies in \{r\}. Thus, t is not adjacent to g_1. By the construction of r, the subgraph \([o] \cap [g_1] \) contains a unique vertex p of \([r] \cap [u]\). Now, \([o] \cap [s] \) contains the vertex s outside \(p^+ \cup y^+\) (because \([o] \cap [s] = \{e, p, r, z\}\), and the degree of o in the graph \([p] \cap [y]\) is at least 2. However, \(g_1 \in [p] - [o]\), whence \(\mu(p, y) = 4\). If \([p] \cap [y] \cap [z] \) = 2, then, replacing d with p, we obtain a contradiction with the statement established in the first paragraph of the proof. Thus, \([p] \cap [y] \cap [z] \) = 3. Since \([o] - r^+ \subset y^+\) and the degree of o in \([r] \cap [z] \) is 2, we conclude that \([o] - (r^+ \cup z^+)\) contains a vertex q of \([y]\). Also, \([e] \cap [g_1] = \{d, g_2, r, y\}\). Therefore, p is not adjacent to e. This contradicts the fact that \([o] \cap [y] \) contains e, q, z, and two vertices of \([p] \cap [y] \cap [z]\). The lemma is proved.

\textbf{Lemma 3.11.} For any two vertices u and x at a distance of 2 in the graph \(\Gamma\), we have \(b_2(u, x) \leq 1\).

\textit{Proof.} By Lemma 3.10, each vertex of \([u] \cap \Gamma_2(y)\) is adjacent to at most one vertex of \([y] - z^+\). It follows that the number of edges between \([u] \cap \Gamma_2(y)\) and \([y] - z^+\) is at least \(b_1(b_1 - 2)\) and at most \(3b_1 - 3\). Then \(b_1^2 - 5b_1 + 3 \leq 0\) and \(b_1 = 4\). We put \(\{g_1, \ldots, g_4\} = [y] - z^+\) and \(\{h_1, \ldots, h_4\} = [z] - y^+\). Either each vertex \(g_i\) is adjacent to exactly two vertices of \([u]\) and some vertex t of \([u]\) is not adjacent to any of the vertices \(\{g_1, \ldots, g_4\}\), or each vertex \(g_1, \ldots, g_4\) is adjacent to exactly two vertices of \([u]\) and \(g_4\) is adjacent to exactly three vertices of \([u]\).

We assume that, in the first case, the vertex t belongs to \(\Gamma_2(y)\). If the vertex e of \([u] \cap [z] \) is adjacent to the vertex \(g_i\) of \([y] - z^+\), then the degree of e in the graph \([t] \cap [y]\) is equal to 2, and \([e] \cap [y] \) contains \(g_i, z\), and two vertices of \([t] \cap [y]\).

If the degrees of two distinct vertices \(e_1, e_2\) of \([t] \cap [y]\) in the graph \([t] \cap [y]\) are equal to 1, then \([e_1] \subset t^+ \cup y^+\). Consequently, \(e_i\) is not adjacent to a vertex of \([y] - z^+\), and \([e_1] \cap [e_2] \) contains t, y, z, and two vertices of \([y] \cap [z]\), a contradiction. If \(\mu(t, y) = 2\), then \(|[t] \cap ([z] - [y])| \leq 1\) by Lemma 3.10, but this contradicts Lemma 1.4. If \(\mu(t, y) = 3\), then the subgraph \([t] \cap [y] = \{e_1, e_2, e_3\}\) is a clique. In this case, \([e_1] \cap [e_2] = \{t, y, z, e_6, i, j\}\) and precisely one vertex \(w\) of \([t] - (u^+ \cup y^+)\) is adjacent to at most one vertex of \(\{e_1, e_2, e_3\}\).

There is no loss of generality in assuming that \(e_1, e_2\) are not adjacent to \(w\). Then \(\mu(u, e_1) = \mu(u, e_2) = 3\) and the degree of t in the graph \([u] \cap [e_i]\), \(i = 1, 2\), is equal to 2. By Lemma 1.5, we have \([u] \cap [t] \subset [e_1] \cup [e_2]\). However, \([u] \cap [e_3]\) contains a vertex of \([t]\) lying in \([e_1]\), and \([u] \cap [e_1] \cap [e_3]\) \(\geq 2\), which contradicts Lemma 1.5.

Thus, \(\mu(t, y) = 4\). If the subgraph \([t] \cap [y]\) contains a vertex e of degree 1, then \([e] \cap [y] \) contains z, a vertex of \([t] \cap [y]\), and two vertices of \([y] - z^+\), which contradicts Lemma 1.1. If the subgraph \([t] \cap [y]\) contains two vertices d and e of degree 3, then \([d] \cap [e] \) contains t, y, z, and two vertices of \([y] \cap [z]\). Thus, \([t] \cap [y]\) is a quadrangle. Therefore, the subgraph \([y] - z^+ = \{g_1, \ldots, g_4\}\) is a clique. Furthermore, each vertex e of \([y] \cap [z]\) is adjacent to exactly two vertices of \([t] \cap [u]\), and each vertex w of \([t] \cap [u]\) is adjacent to \(g_i, h_j\), and two vertices of \([y] \cap [z]\). On the other hand, each vertex \(g_i\) is adjacent to exactly one vertex of \([y] \cap [z]\). This contradicts the fact that \([w] - u^+ \subset [g_i]\).
Thus, \( t \in \Gamma_3(y) \cap \Gamma_3(z) \). Hence, \( \{u \cap [g_i] = \{d_i, w_i\} \), where \( d_i \in [u] - t^+, w_i \in [t] \cap [u], i = 1, \ldots, 4 \). By symmetry, we have \( \{z \cap [d_i] = \{h_i, e_i\} \), where \( e_i \in [y] \cap [z], i = 1, \ldots, 4 \). and, by Lemma 1.4, the vertex \( g_i \) is adjacent to \( e_i \) and \( h_i \). Since \( \{w_i\} \subset u^+ \cup g_i^1 \), we see that \( w_i \) is adjacent to a unique vertex \( r_i \) of \([t] - u^+ \). Now, the subgraph \( \{t, u, y, z, g_1, e_1, h_1, d_1, w_1, r_1 | i = 1, \ldots, 4\} \) is a connected 28-vertex component of the graph \( \Gamma \). But this is impossible because, in this case, for any two adjacent vertices \( e_i, e_j \), the subgraph \( [e_i] \cap [e_j] \) coincides with \( \{y, z\} \).

Thus, each of the vertices \( g_1, g_2, g_3 \) is adjacent to exactly two vertices of \( [u] \), and \( g_4 \) is adjacent to three vertices \( p_1, p_2 \), and \( p_3 \) of \( [u] \). By symmetry, each of the vertices \( h_1, h_2, h_3 \) is adjacent to exactly two vertices of \( [u] \), and \( h_4 \) is adjacent to three vertices of \( [u] \). If \( p_1 \) is adjacent to a vertex \( h_i \) for \( i \leq 3 \), then \( [p_1] \subset u^+ \cup h_i^1 \). In particular, \( h_i \) is a unique vertex of \( \{h_1, \ldots, h_4\} \) adjacent to \( g_i \). Then \( p_2, p_3 \in [h_4] \), and the vertices \( g_4 \) and \( h_4 \) are adjacent by Lemma 1.5, a contradiction.

So, \( p_1, p_2, p_3 \in [h_4] \); Lemma 1.5 shows that the vertices \( g_4 \) and \( h_4 \) are adjacent. Since \( \{u \cap [h_1] \) contains two vertices adjacent to only one vertex of \( \{g_1, g_2, g_3\} \), we see that \( |[h_1] \cap [g_1, g_2, g_3]| = 2 \); in particular, \( [h_1] \subset [u] \cup [y] \cup [z] \) for \( i = 1, \ldots, 4 \). The number of edges between \( [u] \) and \( \Gamma_2(u) \) is equal to 36, and at most 30 of these edges are incident to vertices of \( [y] \cup [z] \). However, if \( r \) is a vertex of \( \Gamma_2(u) - ([y] \cup [z]) \), then, by Lemma 1.4, the subgraph \( [u] \cap [r] \) lies in \([g_4]\). We obtain \( |[u] \cap [g_4] \cap [r]| \geq 2 \), which contradicts Lemma 1.8. The lemma is proved.

**Lemma 3.12.** For every vertex \( u \), we have \( |\Gamma_3(u)| \leq 1 \).

**Proof.** Let \( y \) and \( z \) be distinct vertices of \( \Gamma_3(u) \), and let \( uwxy \) be a geodesic 3-path. By Lemma 3.11, the subgraph \( [y] \cap \Gamma_2(u) \) does not intersect \([z] \). We assume that the vertices \( y \) and \( z \) are adjacent. We have \( z \in \Gamma_2(u) \) by Lemma 3.4. Furthermore, \( [y] \cap [z] \) contains \( 2b_1 - 4 \) vertices of \( \Gamma_3(u) \). By Lemma 3.11, \( |[u]| = 3b_1 - 3 \geq (b_1 - 2)(2b_1 - 2) \). Hence, \( b_1 \leq 3 \), a contradiction.

If the distance between the vertices \( y \) and \( z \) is 2, then \( [y] \cap [z] \) contains at least \( b_1 - 2 \) vertices of \( \Gamma_3(u) \). This contradicts the fact proved in the preceding paragraph. Thus, the distance between the vertices \( y \) and \( z \) is equal to 3, and \( kb_1 \geq 2k(b_1 - 2) \). Consequently, \( b_1 = 4 \), \( \Gamma_2(u) = [y] \cup [z] \), and \( \mu(u, r) = 2 \) for every vertex \( r \) of \( \Gamma_2(u) \). Since the subgraph \( [w] \cap u^+ \) lies in \( x^+ \), the graph \( [x] \cap ([u] \cup [z]) \) is a 4-clique. If \( x \) and \( x' \) are distinct vertices of \( [w] \cap [y] \), then \([w] \cap u^+ \) is a 4-clique, and \( [x] \cap [x'] \) contains \( y \) and four vertices of \( [w] \cap [y] \), a contradiction. The lemma and Proposition 2 are proved.

The theorem follows from [2, 3] and Propositions 1 and 2. We prove the corollary. Let \( \Gamma \) be a connected amply regular graph of diameter exceeding 2 and with parameters \((v, k, \lambda, \mu)\), and let \( k \geq 3b_1 - 3 \). Then \( b_1 - 2 \leq \mu \leq b_1 \).

**Lemma 3.13.** If \( \mu = b_1 - 2 \), then \( \Gamma \in \mathcal{E}(4) \).

**Proof.** Let \( \mu = b_1 - 2 \). Then any two vertices the distance between which is equal to 2 form a good pair. Therefore, \( \Gamma \) is a Terwilliger graph without 3-claws. By [7], either \( \mu = 1 \), or \( \Gamma \) is the icosahedron graph (and \( \mu = b_1 \)). In the case where \( \mu = b_1 - 2 = 1 \), we obtain \( b_1 = 3 \) and \( \Gamma \in \mathcal{E}(4) \).

**Lemma 3.14.** If \( \mu = b_1 \), then \( \Gamma \) is either an \( n \)-gon with \( n \geq 6 \), or the complete bipartite graph \( K_{4,4} \) with a maximal matching removed, or the icosahedron graph, or the Johnson graph \( J(6,3) \), or the locally Taylor graph \( T(6) \) on 32 vertices, or the locally Schl"afli graph on 56 vertices.

**Proof.** Let \( \mu = b_1 \). By [1, Theorem 1.5.5], the graph \( \Gamma \) is either a polygon or a Taylor graph. In the latter case, either \( \lambda = 0 \) and \( \Gamma \) is a complete bipartite graph \( K_{k+1,k+1} \)
with a maximal matching removed, or a neighborhood of each vertex of $\Gamma$ is a strongly regular graph with parameters $(v', k', \lambda', \mu')$ and $k' = 2\mu'$. If $\lambda = 0$, then $k = b_1 + 1$, and the condition $k \geq 3b_1 - 3$ implies that $b_1 \leq 2$ (if $b_1 = 1$, the graph $\Gamma$ is a hexagon).

Now, let $v' = k, k' = \lambda = 2\mu'$. Then $b_1 = 2(k' - \lambda' - 1)$. Since $k \geq 3b_1 - 3$, we have $\mu' \geq b_1 - 2$. If $\mu' = b_1 - 1$, then the subgraph $[a]$ is a strongly regular graph with parameters $(v', 2b_1 - 2, 3b_1/2 - 3, b_1 - 1)$. In the half case, we obtain $b_1 = 2$, and $\Gamma$ is the icosaedron graph. If $(\lambda' - \mu')^2 + 4(k' - \mu')$ is a square, then $b_1 = 2s$ and $s(s + 4)$ is a square, a contradiction.

If $\mu' = b_1 - 2$, then the subgraph $[a]$ is a strongly regular graph with parameters $(v', 2b_1 - 4, 3b_1/2 - 5, b_1 - 2)$. In the half case, we obtain $b_1 = 2$, the graph $[a]$ is the $(3 \times 3)$-lattice, and $\Gamma$ is the Johnson graph $J(6, 3)$. If $(\lambda' - \mu')^2 + 4(k' - \mu')$ is a square, then $n' = b_1/2 - 1$, and the nonprincipal eigenvalues of the graph $[a]$ are equal to $b_1/2 - 2$ and $-2$. If $b_1 = 6$, the graph $[a]$ is the triangular graph $T(6)$, and $\Gamma$ is the Taylor graph on 32 vertices.

Since a Seidel graph with $\mu \geq 6$ is either the Clebsch graph or a Schl"afli graph and, for $b_1 = 8$, the graph $[a]$ must have the parameters $(21, 12, 7, 6)$, we see that $\Gamma$ is the locally Taylor Schl"afli graph on 56 vertices. $\Box$

Lemma 3.15. If $\mu = b_1 - 1$, then $\mu = 1$ and $\Gamma \in T(3) \cup E(3)$.

Proof. Let $\mu = b_1 - 1$. First, we assume that $k = 3b_1 - 3$. Then $\mu = k - 2b_1 + 2$. The following statement was proved in \cite{8}.

Corollary. Let $\Gamma$ be an amply regular graph with parameters $(v, k, \lambda, \mu)$, and let $\mu = k - 2b_1 + 2$. Then $\Gamma$ is either a Seidel graph or a trivalent graph without triangles of diameter exceeding 2, and with $\mu = 1$.

Let $k \geq 3b_1 - 2$. In \cite{2} it was proved that if $\Gamma$ is a connected edge-regular graph with parameters $(v, k, \lambda)$, and if $k \geq 3b_1 - 2$, then either $\Gamma$ is a polygon, or $\Gamma$ is the icosaedron graph, or $\Gamma \in E(3)$, or $\Gamma$ is a graph of diameter 2. The lemma is proved. $\Box$

The corollary follows from Lemmas 3.13–3.15.

References


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