ON THE STRUCTURE OF THE SET OF PERIODS FOR PERIODIC SOLUTIONS OF SOME LINEAR INTEGRO-DIFFERENTIAL EQUATIONS ON THE MULTIDIMENSIONAL SPHERE

DANG KHANH HOI

Abstract. The problem of periodic solutions for the family of linear differential equations

\[(L - \lambda)u \equiv \left(\frac{1}{i} \frac{\partial}{\partial t} - a\Delta - \lambda\right)u(x, t) = \nu G(u - f)\]

is considered on the multidimensional sphere \(x \in S^n\) under the periodicity condition \(u|_{t=0} = u|_{t=b}\). Here \(a\) and \(\lambda\) are given reals, \(\nu\) is a fixed complex number, \(Gu(x, t)\) is a linear integral operator, and \(\Delta\) is the Laplace operator on \(S^n\). It is shown that the set of parameters \((\nu, b)\) for which the above problem admits a unique solution is a measurable set of full measure in \(\mathbb{C} \times \mathbb{R}^+\).

§1

In [4, 5] it was discovered that, for some partial differential equations, the set of periods for which a periodic solution is unique may have an unexpectedly complicated structure. In this paper, we study this issue for a class of linear equations on the multidimensional sphere.

We consider the problem of periodic solutions for the nonlocal Schrödinger type equation

\[(1) \quad \left(\frac{1}{i} \frac{\partial}{\partial t} - a\Delta - \lambda\right)u(x, t) = \nu G(u - f)\]

with the \(t\)-periodicity condition

\[(2) \quad u|_{t=0} = u|_{t=b}.\]

Here \(u(x, t)\) is a complex function on \(S^n \times [0, b]\), where \(S^n\) is the multidimensional sphere, \(n \geq 2\); \(a \neq 0, \lambda\), and \(\nu\) are given complex numbers; \(f(x, t)\) is a given function.

The change of variables \(t = b\tau\) reduces our problem to a problem with a fixed period, but with a new equation in which the coefficient of the \(\tau\)-derivative is equal to \(\frac{1}{b}\),

\[\left(\frac{1}{i} \frac{\partial}{b \partial \tau} - a\Delta - \lambda\right)u(x, b\tau) = \nu G(u(x, b\tau) - f(x, b\tau)).\]

§2

Thus, problem (1), (2) turns into a problem on periodic solutions of the equation

\[(3) \quad (L - \lambda)u \equiv \left(\frac{1}{i} \frac{\partial}{b \partial \tau} - a\Delta - \lambda\right)u(x, t) = \nu G(u - f)\]

with the fixed periodicity condition

\[(4) \quad u|_{\tau=0} = u|_{\tau=1}.\]

2000 Mathematics Subject Classification. Primary 35K20.
Key words and phrases. Schrödinger-type equation, periodicity condition.
Here $G_u(x, t) = \int_{S^n} g(x, y)u(y, t)dy$ (dy is the Lebesgue–Hausdorff measure on the sphere $S^n$) is an integral operator on the space $L_2(S^n \times [0, 1])$ with smooth kernel $g(x, y)$ defined on $S^n \times S^n$. The differential operation $\frac{1}{i} \frac{\partial}{\partial t} - a\Delta$ is assumed to be defined on the functions $u(x, t) \in C^\infty(S^n \times [0, 1])$ such that $u|_{t=0} = u|_{t=1}$. Let $L$ denote the closure of this operation $\frac{1}{i} \frac{\partial}{\partial t} - a\Delta$ in $\mathcal{H} = L_2(S^n \times [0, 1])$. So, an element $u \in \mathcal{H}$ belongs to the domain $\mathcal{D}(L)$ of $L = \frac{1}{i} \frac{\partial}{\partial t} - a\Delta$ if and only if there is a sequence $\{u_j\} \subset C^\infty(S^n \times [0, 1])$, $u_j|_{t=0} = u|_{t=1}$, such that $\lim u_j = u$, $\lim Lu_j = Lu$ in $\mathcal{H}$.

It is well known that the eigenvalues of the Laplace operator $\Delta$ on the sphere $S^n$ are of the form $-k(k+n-1)$, $k \in \mathbb{Z}$, $k \geq 0$, and that $\Delta$ admits the corresponding orthonormal basis of eigenfunctions $\psi_k(x) \in C^\infty(S^n)$ (see, e.g., [3]).

**Lemma 1.** The functions $e_{km}(x, t) = e^{i2\pi mt}\psi_k(x)$, $k, m \in \mathbb{Z}$, $k \geq 0$, are eigenfunctions of the operator $L$ in the space $\mathcal{H} = L_2(S^n \times [0, 1])$ that correspond to the eigenvalues

$$\lambda_{km} = \frac{2m\pi}{b} + ak(k + n - 1) = \frac{2m\pi}{b} + \lambda_k.$$  

These functions form an orthonormal basis in $\mathcal{H}$. The domain of $L$ is given by the formula

$$\mathcal{D}(L) = \left\{ u = \sum u_{km}e_{km} \mid \sum |\lambda_{km}u_{km}|^2 < \infty, \sum |u_{km}|^2 < \infty \right\}.$$  

The spectrum $\sigma(L)$ is the closure of the set $\{\lambda_{km}\}$.

**Lemma 2.**

$$||G||^2 \leq M_0^2 = \int_{S^n} \int_{S^n} |g(x, y)|^2 \, dxdy.$$  

Proof. We have

$$|G_u(x, t)|^2 = \left| \int_{S^n} g(x, y)u(y, t) \, dy \right|^2 \leq \int_{S^n} |g(x, y)|^2 \, d y \int_{S^n} |u(y, t)|^2 \, dy,$$

$$||G_u(x, t)||^2 = \int_0^1 \int_{S^n} |G_u(x, t)|^2 \, dx \, dt$$

$$\leq \int_0^1 \int_{S^n} \left( \int_{S^n} |g(x, y)|^2 \, dy \int_{S^n} |u(y, t)|^2 \, dy \right) \, dx \, dt,$$

$$||G_u(x, t)||^2 \leq \int_{S^n} \int_{S^n} |g(x, y)|^2 \, dxdy \int_0^1 \int_{S^n} |u(y, t)|^2 \, dy \, dt = M_0^2 ||u||^2,$$

$$||G|| \leq M_0.$$  

The lemma is proved.  

We note that the Laplace operator is formally selfadjoint relative to the scalar product $(u, v) = \int_{S^n} u(x)v(x)\, dx$ on the space $C^\infty(S^n)$. The product $\Delta_x \circ G = \Delta_x G$ coincides with the integral operator with the kernel $\Delta_x g(x, y)$. We put $M = \max\{||\Delta_x G||, ||G||\}$.

**Lemma 3.** Let $v = Gu = \sum v_{km}e_{km}$; then

$$|v_{km}|^2 \leq \frac{4M^2}{(k(k+n-1)+1)^2} ||u||^2.$$  

Also, for $k \neq 0$ we have

$$|v_{km}|^2 \leq \frac{4|\alpha_{km}|^2}{(k(k+n-1)+1)^2},$$

where $\alpha_{km} = (\Delta_x Gu, e_{km})$, and $\sum |\alpha_{km}|^2 \leq M^2 ||u||^2$. 
Proof. The Parseval identity $\sum |v_{km}|^2 = \|Gu\|^2$ yields

$$|v_{km}|^2 \leq \|G\|^2|\|u\|^2 \leq 4M^2|\|u\|^2.$$  

Since the Laplace operator is selfadjoint, for $k \neq 0$ we have

$$\alpha_{km} = \langle \Delta_x Gu, e_{km} \rangle = \langle Gu, \Delta_x e_{km}(x, t) \rangle = \langle Gu, -k(k+n-1)e_{km}(x, t) \rangle,$$

$$\alpha_{km} = -k(k+n-1)\langle Gu, e_{km}(x, t) \rangle = -k(k+n-1)v_{km}.$$  

It follows that

$$|v_{km}|^2 = \frac{|\alpha_{km}|^2}{(k(k+n-1))^2} \leq \frac{4|\alpha_{km}|^2}{(k(k+n-1)+1)^2}.$$  

By the Parseval identity, we have $\sum |\alpha_{km}|^2 = \|\Delta_x Gu\|^2 \leq M^2|\|u\|^2|$, whence

$$|v_{km}|^2 \leq \frac{4M^2|\|u\|^2}{(k(k+n-1)+1)^2}.$$  

The lemma is proved. \qed

We assume that $a$ and $\lambda$ are real numbers. Then, by Lemma 1 the spectrum $\sigma(L)$ lies on the real axis. Most typical and interesting is the case where the number $ab/(2\pi)$ is irrational. The H. Weyl theorem (see, e.g., [11]) says that, in this case, the set of the numbers $\lambda_{km}$ is everywhere dense on $\mathbb{R}$ and $\sigma(L) = \mathbb{R}$. Now, suppose that $\lambda \neq \lambda_{km}$ for all $k, m \in \mathbb{Z}, k \geq 0$. Then the inverse operator $(L - \lambda)^{-1}$ is well defined, but unbounded. The expression for this inverse operator involves small denominators:

$$(L - \lambda)^{-1}v(x, t) = \sum_{k, m \in \mathbb{Z}, k \geq 0} \frac{v_{km}}{\lambda_{km} - \lambda}e_{km},$$

where the $v_{km}$ are the Fourier coefficients of the series

$$v(x, t) = \sum_{k, m \in \mathbb{Z}, k \geq 0} v_{km}e_{km}.$$  

For positive numbers $\sigma$ and $C$, let $A_{\sigma}(C)$ denote the set of all positive $b$ such that

$$|\lambda_{km} - \lambda| \geq \frac{C}{(k+1)^{1+\sigma}}$$

for all $m, k \in \mathbb{Z}, k \geq 0$.

This definition shows that the sets $A_{\sigma}(C)$ extend as $C$ reduces and as $\sigma$ grows. Therefore, in what follows, to prove that such a set or its part is nonempty, we require that $C$ be sufficiently small and $\sigma$ sufficiently large. Let $A_\sigma$ denote the union of the sets $A_{\sigma}(C)$ over all $C > 0$.

If inequality (8) is fulfilled for some $b$ and all $m, k$, then it is fulfilled for $m = 0$; this provides a condition necessary for the nonemptiness of $A_{\sigma}(C)$:

$$C \leq (k+1)^{1+\sigma}|ak(k+n-1) - \lambda|, \quad \forall k \geq 0.$$  

We put $d = \min_{k \in \mathbb{Z}, k \geq 0}(k+1)^{1+\sigma}|ak(k+n-1) - \lambda| > 0$.

Theorem 1. The sets $A_{\sigma}(C)$ and $A_{\sigma}$ are Borel. The set $A_{\sigma}$ has full measure, i.e., its complement to the half-line $\mathbb{R}^+$ is of zero measure.

Proof. Obviously, the sets $A_{\sigma}(C)$ are closed in $\mathbb{R}^+$. The set $A_{\sigma} = \bigcup_{r=1}^{\infty} A_{\sigma}(1/r)$ is Borel, being a countable union of closed sets. We show that $A_{\sigma}$ has full measure in $\mathbb{R}^+$. Suppose $b, l > 0$ and $C \leq \frac{d}{2}$; we consider the complement $(0, l) \setminus A_{\sigma}(C)$. This set consists of all positive numbers $b$ for which there exist $m$ and $k$ such that

$$|\lambda_{km} - \lambda| < \frac{C}{(k+1)^{1+\sigma}}.$$
Solving this inequality for $b$, we see that, for $m$, $k$ fixed, the numbers $b$ form an interval $I_{k,m} = (\alpha_k, m\beta_k)$, where $m = 1, 2, 3, \ldots$,

$$\alpha_k = \frac{2\pi}{|ak(n-1) - \lambda| + \frac{C}{(k+1)^{1+\sigma}}}$$

$$\beta_k = \frac{2\pi}{|ak(n-1) - \lambda| - \frac{C}{(k+1)^{1+\sigma}}}.$$  

The length of $I_{k,m}$ is $m\delta_k$, with

$$\delta_k = \frac{4\pi C(k+1)^{-1-\sigma}}{|ak(n-1) - \lambda|^2 - C^2(k+1)^{-2-2\sigma}}.$$  

Since $C \leq \frac{d}{4}$ by assumption, we have

$$\delta_k \leq \frac{16\pi C}{3(k+1)^{1+\sigma}|ak(n-1) - \lambda|^2}.$$  

For $k$ fixed and $m$ varying, there are only finitely many intervals $I_{k,m}$ that intersect the given segment $(0, l)$.

Such intervals arise for the values of $m = 1, 2, \ldots$ satisfying $m\alpha_k < l$, i.e.,

$$0 < m < \frac{l}{2\pi} |ak(n-1) - \lambda| + C(k+1)^{-1-\sigma}.$$  

Since $C(k+1)^{-1-\sigma} \leq \frac{1}{4}|ak(n-1) - \lambda|$, we can write simpler restrictions on $m$:

$$0 < m < \frac{l}{2\pi} \frac{3}{2} |ak(n-1) - \lambda| < \frac{l}{\pi} |ak(n-1) - \lambda|.$$  

The measure of the intervals indicated (for $k$ fixed) is dominated by $\delta_k \tilde{S}_k$, where $\tilde{S}_k = \tilde{S}_k(l)$ is the sum of all integers $m$ satisfying $\delta_k$. Summing the arithmetic progression, we obtain

$$\tilde{S}_k \leq \frac{l}{2\pi^2} |ak(n-1) - \lambda| \{ l|ak(n-1) - \lambda| + \pi \}.$$  

Passing to the union of the intervals in question over $k$ and $m$ and using (11), we see that

$$\mu((0, l) \setminus A_\sigma(C)) \leq \sum_{k=0}^\infty \delta_k \tilde{S}_k \leq CS(l),$$  

where

$$S = S(l) = \sum_{k=0}^\infty \frac{8l\{ l|ak(n-1) - \lambda| + \pi \}}{3\pi(k+1)^{1+\sigma}|ak(n-1) - \lambda|}.$$  

Observe that the quantity

$$\frac{l|ak(n-1) - \lambda| + \pi}{\pi|ak(n-1) - \lambda|}$$

is dominated by a constant $D$; therefore,

$$S(l) \leq \frac{8}{3} \frac{1}{D} \sum_{k=0}^\infty \frac{1}{(k+1)^{1+\sigma}} < \infty.$$  

We have

$$\mu((0, l) \setminus A_\sigma) \leq \mu((0, l) \setminus A_\sigma(C)) \leq CS(l)$$

for all $C > 0$. It follows that $\mu((0, l) \setminus A_\sigma) = 0$ for all $l > 0$. Thus, $\mu((0, \infty) \setminus A_\sigma) = 0$ and $A_\sigma$ has full measure. The theorem is proved.
Theorem 2. Suppose $g(x, y)$ is a function defined on $S^n \times S^n$ and such that the function $\Delta x g(x, y)$ is continuous on $S^n \times S^n$. Let $0 < \sigma < 1$, and let $b \in A_{\sigma}(C)$. Then the inverse operator $(L - \lambda)^{-1}$ is well defined, and the operator $(L - \lambda)^{-1} \circ G$ is compact.

Proof. Since $b \in A_{\sigma}(C)$, we have $\lambda_{km} \neq \lambda$ for all $k, m \in \mathbb{Z}$, $k \geq 0$, so that $(L - \lambda)^{-1}$ is well defined and looks like the expression in (1). Observe that $\lim_{k \to \infty} \frac{(k + 1)^{2+2\sigma}}{(k(k + n - 1) + 1)^2} = 0$ as $k \to \infty$. Therefore, given $\varepsilon > 0$, we can find an integer $k_0 > 0$ such that

$$\frac{(k + 1)^{2+2\sigma}}{(k(k + n - 1) + 1)^2} < \left( \frac{\varepsilon C}{2M} \right)^2$$

for all $k > k_0$.

We write

$$(L - \lambda)^{-1} v(x, t) = Q_{k_0}^1 v + Q_{k_0}^2 v, \quad v = Gu,$$

where

$$Q_{k_0}^1 v = \sum_{0 \leq k \leq k_0} \frac{v_{km}}{\lambda_{km} - \lambda} e_{km}, \quad Q_{k_0}^2 v = \sum_{k > k_0} \frac{v_{km}}{\lambda_{km} - \lambda} e_{km}.$$ 

For the operator $Q_{k_0}^1$ we have

$$\|Q_{k_0}^1 v\|^2 = \sum_{0 \leq k \leq k_0} \frac{|v_{km}|^2}{|\lambda_{km} - \lambda|^2}.$$

Observe that if $0 < k \leq k_0$, then

$$\lim_{m \to \infty} \frac{1}{|2m^2 + ak(k + n - 1) - \lambda|^2} = 0$$

as $|m| \to \infty$. Therefore, the quantity $\frac{1}{|2m^2 + ak(k + n - 1) - \lambda|^2}$ is dominated by a constant $C(k_0)$. Then

$$\|Q_{k_0}^1 v\|^2 \leq \sum_{0 \leq k \leq k_0} |v_{km}|^2 C(k_0) \leq C(k_0) \|v\|^2,$$

which means that $Q_{k_0}^1$ is a bounded operator.

Consider the operator $Q_{k_0}^2 \circ G$. By (3) and Lemma (8), we have

$$\|Q_{k_0}^2 v\|^2 = \|Q_{k_0}^2 \circ Gu\|^2 = \sum_{k > k_0} \frac{|v_{km}|^2}{|\lambda_{km} - \lambda|^2} \leq \sum_{k > k_0} \frac{4|\alpha_{km}|^2}{(k(k + n - 1) + 1)^2} \left( \frac{1}{G} \right)^2 (k + 1)^{2+2\sigma} \leq \left( \frac{1}{G} \right)^2 \left( \frac{\varepsilon C}{2M} \right)^2 \sum_{k > k_0} 4|\alpha_{km}|^2 \leq \varepsilon^2 \|u\|^2.$$

Consequently, $\|Q_{k_0}^2 \circ G\| \leq \varepsilon$.

Since $G$ is compact and $Q_{k_0}^2$ is bounded, $Q_{k_0}^1 \circ G$ is compact. Next, we have

$$\|(L - \lambda)^{-1} \circ G - Q_{k_0}^1 \circ G\| = \|Q_{k_0}^2 \circ G\| < \varepsilon.$$

Thus, the operator $(L - \lambda)^{-1} \circ G$ is the limit of a sequence of compact operators. Therefore, it is compact itself. The theorem is proved. \qed

We denote $K = K_b = (L - \lambda)^{-1} \circ G$.

Theorem 3. Suppose $b \in A_{\sigma}(C)$. Then problem (1), (2) admits a unique periodic solution with period $b$ for all $\nu \in C$ except, possibly, an at most countable discrete set of values of $\nu$. 

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Lemma 5. The operator-valued function for all $\sigma 
subseteq K - \frac{1}{\nu}$.

Since $K$ is a compact operator, its spectrum $\sigma(K)$ is at most countable, and the limit point of $\sigma(K)$ (if any) can only be zero. Therefore, the set $S = \{\nu \neq 0 \mid \frac{1}{\nu} \in \sigma(K)\}$ is at most countable and discrete, and if $\nu \neq 0$ and $\nu \not\in S$, then the operator $(K - \frac{1}{\nu})$ is invertible, i.e., equation (\ref{eq:inverse}) is uniquely solvable. The theorem is proved. \hfill $\square$

We pass to the question of the solvability of problem (\ref{eq:1}), (\ref{eq:2}) for fixed $\nu$. We need to study the structure of the set $E \subset \mathbb{C} \times \mathbb{R}^+$ that consists of all pairs $(\nu, b)$ with $\nu \neq 0$ and $\frac{1}{\nu} \not\in \sigma(K_b)$, where $K_b = (L - \sigma)^{-1} \circ G$.

Theorem 4. $E$ is a measurable set of full measure in $\mathbb{C} \times \mathbb{R}^+$.

For the proof, we need several auxiliary statements.

Lemma 4. For any $\varepsilon > 0$, there exists an integer $k_0$ such that $\|K_b - \widetilde{K}_b\| < \varepsilon$ for all $b \in A_{\nu}(\frac{1}{\nu})$, $0 < \sigma < 1$, where $r = 1, 2, \ldots$,

$$K_b u = (L_b - \lambda)^{-1} v = \sum_{\alpha \leq k \leq k_0} \frac{v_{\lambda k m}}{\lambda k m(\lambda) - \lambda} e_k, \quad K_b u = \sum_{\alpha \leq k \leq k_0} \frac{v_{\lambda k m}}{\lambda k m(\lambda) - \lambda} e_k.$$

Proof. Observe that for any $\varepsilon > 0$ there is an integer $k_0$ such that

$$\left(\frac{(k + 1)^{2 + 2\sigma}}{k(k + n - 1 + 1)^{2}} \right) \leq \left(\frac{\varepsilon}{2rM}\right)^2$$

for all $k \geq k_0$, $0 < \sigma < 1$. We have

$$(K_b - \tilde{K}_b) u = K_{k_0} b u = \sum_{k > k_0} \frac{v_{\lambda k m}}{\lambda k m(\lambda) - \lambda} e_k, \quad \|K_{k_0} b u\|^2 = \sum_{k > k_0 \lambda n m(\lambda) - \lambda} e_k^2 \leq \sum_{k > k_0} 4r^2 \sigma_{\lambda k m}(k + 1)^{2 + 2\sigma} (k + n - 1 + 1)^{2} \leq r^2 \left(\frac{\varepsilon}{2rM}\right)^2 4 \sum_{k > k_0} |\lambda n m|^2 \leq r^2 \left(\frac{\varepsilon}{2rM}\right)^2 4M^2 \|u\|^2 = \varepsilon^2 \|u\|^2.$$

Thus, $\|K_b - \tilde{K}_b\| = \|K_{k_0} b\| < \varepsilon$, as required. \hfill $\square$

Lemma 5. The operator-valued function $b \mapsto K_b$ is continuous for $b \in A_{\nu}(\frac{1}{\nu})$.

Proof. Suppose $b, b + \Delta b \in A_{\nu}(\frac{1}{\nu})$ and $\varepsilon > 0$. By Lemma 4, for some $k_0$ (independent of $b$ and $b + \Delta b$), we have $\|K_b - \tilde{K}_b\| = \|K_{k_0} b\| < \varepsilon$ and $\|K_{b + \Delta b} - \tilde{K}_b\| = \|K_{k_0}(b + \Delta b)\| < \varepsilon$. Next,

$$K_{b + \Delta b} - K_b = (\tilde{K}_{b + \Delta b} + K_{k_0}(b + \Delta b)) - (\tilde{K}_b + K_{k_0} b),$$

whence we obtain

$$\|K_{b + \Delta b} - K_b\| \leq \|\tilde{K}_{b + \Delta b} - \tilde{K}_b\| + \|K_{k_0}(b + \Delta b)\| + \|K_{k_0} b\|.$$
Consider the operators \( \tilde{K}_{b+\Delta b} \) and \( \tilde{K}_b \). We have

\[
(K_{b+\Delta b} - \tilde{K}_b)u = \sum_{0 \leq k \leq k_0} \left( \frac{1}{\lambda_{km}(b+\Delta b) - \lambda} - \frac{1}{\lambda_{km}(b) - \lambda} \right) v_{km} e_{km},
\]

(14)

\[
\|K_{b+\Delta b} - \tilde{K}_b\|_{b}^2 = \frac{|\Delta b|^2}{|b(b+\Delta b)|^2} \sum_{0 \leq k \leq k_0} \frac{|v_{km}|^2}{|\lambda_{km}(b+\Delta b) - \lambda|^2} \frac{4m^2\pi^2}{|\lambda_{km}(b) - \lambda|^2}.
\]

If \( b + \Delta b \in A_{\sigma}(b) \), \( 0 \leq k \leq k_0 \), and \( 0 < \sigma < 1 \), then

\[
\frac{|v_{km}|^2}{|\lambda_{km}(b+\Delta b) - \lambda|^2} \leq |v_{km}|^2 r^2(k+1)^{2+2\sigma} \leq r^2(k+1)^4 |v_{km}|^2.
\]

The relation \( \lim_{m \to \infty} \frac{4m^2\pi^2}{|\lambda_{km}(b) - \lambda|^2} \) is dominated by a constant \( C(k_0) \) depending on \( k_0 \). Therefore,

\[
\frac{|\Delta b|^2}{|b(b+\Delta b)|^2} \sum_{0 \leq k \leq k_0} \frac{|v_{km}|^2}{|\lambda_{km}(b+\Delta b) - \lambda|^2} \frac{4m^2\pi^2}{|\lambda_{km}(b) - \lambda|^2}
\]

\[
\leq \frac{|\Delta b|^2}{|b(b+\Delta b)|^2} \sum_{0 \leq k \leq k_0} r^2(k+1)^4 C(k_0) |v_{km}|^2
\]

\[
\leq \frac{|\Delta b|^2}{|b(b+\Delta b)|^2} r^2(k+1)^4 C(k_0) \sum_{0 \leq k \leq k_0} |v_{km}|^2.
\]

Since

\[
\sum_{0 \leq k \leq k_0} |v_{km}|^2 \leq \|v\|^2 \leq M^2 \|u\|^2,
\]

we arrive at the estimate

\[
\|K_{b+\Delta b} - \tilde{K}_b\|_{b}^2 \leq \frac{|\Delta b|^2}{|b(b+\Delta b)|^2} M^2 r^2(k+1)^4 C(k_0).
\]

We choose \( \Delta b \) so as to satisfy the condition

\[
\frac{|\Delta b|^2}{|b(b+\Delta b)|^2} M^2 r^2(k+1)^4 C(k_0) < \varepsilon.
\]

Then \( \|K_{b+\Delta b} - \tilde{K}_b\| < 3\varepsilon \). This shows that the operator-valued function \( b \mapsto K_b \) is continuous on \( A_{\sigma}(\frac{1}{4}) \). The lemma is proved. \( \square \)

**Lemma 6.** The spectrum \( \sigma(K) \) of the compact operator \( K \) depends continuously on \( K \) in the space \( \text{Comp}(\mathcal{H}) \) of compact operators on \( \mathcal{H} \), in the sense that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all compact (and even bounded) operators \( B \) with \( \|B - K\| < \delta \) we have

\[
\sigma(B) \subset \sigma(K) + V_{\varepsilon}(0), \quad \sigma(K) \subset \sigma(B) + V_{\varepsilon}(0).
\]

Here \( V_{\varepsilon}(0) = \{ \lambda \in \mathbb{C} \mid |\lambda| < \varepsilon \} \) is the \( \varepsilon \)-neighborhood of the point \( 0 \) in \( \mathbb{C} \).

**Proof.** Let \( K \) be a compact operator; we fix \( \varepsilon > 0 \). The structure of the spectrum of a compact operator shows that there exists \( \varepsilon_1 < \varepsilon / 2 \) such that \( \varepsilon_1 \neq |\lambda| \) for all \( \lambda \in \sigma(K) \). Let \( S = \{ \lambda_1, \ldots, \lambda_k \} \) be the set of all spectrum points \( \lambda \) with \( |\lambda| > \varepsilon_1 \), and let \( V = \bigcup_{\lambda \in S(0)} V_{\epsilon_1}(\lambda) \). Then \( V \) is a neighborhood of \( \sigma(K) \), and \( V \subset \sigma(K) + V_{\varepsilon}(0) \).

By the well-known property of spectra (see, e.g., [2], Theorem 10.20), there exists \( \delta > 0 \)
such that $\sigma(B) \subset V$ for any bounded operator $B$ with $\|B - K\| < \delta$. Moreover (see, e.g., [2] p. 293, Exercise 20), the number $\delta > 0$ can be chosen so that $\sigma(B) \cap V_\varepsilon(\lambda) \neq \emptyset$ for all $\lambda \in S \cup \{0\}$. Then for all bounded operators $B$ with $\|B - K\| < \delta$ the required inclusions $\sigma(K) \subset \sigma(B) + V_{2\varepsilon}(0) \subset \sigma(B) + V_\varepsilon(0)$ and $\sigma(B) \subset V \subset \sigma(K) + V_\varepsilon(0)$ are fulfilled. The lemma is proved. \hfill $\square$

It is easy to deduce the following statement from Lemma 5.

**Corollary 1.** The function $\rho(\lambda, K) = \text{dist}(\lambda, \sigma(K))$ is continuous on $\mathbb{C} \times \text{Comp}(H)$.

**Proof.** Suppose $\lambda \in \mathbb{C}$, $K \in \text{Comp}(H)$, and $\varepsilon > 0$. By Lemma 6 there exists $\delta > 0$ such that for any operator $H$ lying in the $\delta$-neighborhood of $K$, $\|H - K\| < \delta$, the inclusions (5) are fulfilled; these inclusions directly imply the estimate $|\rho(\lambda, K) - \rho(\lambda, H)| < \varepsilon$. Then for all $\mu \in \mathbb{C}$ with $|\mu - \lambda| < \varepsilon$ and all $H$ with $\|H - K\| < \delta$ we have

$$|\rho(\mu, K) - \rho(\lambda, H)| \leq |\rho(\mu, K) - \rho(\lambda, K)| + |\rho(\lambda, K) - \rho(\lambda, H)| < |\mu - \lambda| + \varepsilon < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the function $\rho(\lambda, K)$ is continuous. The proposition is proved. \hfill $\square$

Combining Proposition 5 and Lemma 5, we obtain the following fact.

**Corollary 1.** The function $\rho(\lambda, b) = \text{dist}(\lambda, \sigma(K_b))$ is continuous on $(\lambda, b) \in \mathbb{C} \times A_\sigma(\frac{1}{\nu})$.

Now we are ready to prove Theorem 4.

**Proof of Theorem 4.** By Corollary 1 the function $\rho(1/\nu, b)$ is continuous with respect to the variables $(\nu, b) \in (\mathbb{C} \setminus \{0\}) \times A_\sigma(\frac{1}{\nu})$. Consequently, the set

$$B_\sigma = \left\{(\nu, b) \mid \rho(1/\nu, b) \neq 0, \ b \in A_\sigma(\frac{1}{\nu})\right\}$$

is measurable, and so is the set $B = \bigcup_\nu B_\sigma$. Clearly, $B \subset E$ and $E = B \cup B_0$, where $B_0 = E \setminus B$. Obviously, $B_0$ lies in the set $\mathbb{C} \times (\mathbb{R}^+ \setminus A_\sigma)$ of zero measure (recall that, by Theorem 1, $A_\sigma$ has full measure in $\mathbb{R}^+$). Since the Lebesgue measure is complete, $B_0$ is measurable. Thus, the set $E$ is measurable, being the union of two measurable sets. Next, by Theorem 4 for $b \in A_\sigma$ the section $E_\nu = \{\nu \in \mathbb{C} \mid (\nu, b) \in E\}$ has full measure, because its complement $\{1/\nu \mid \nu \in \sigma(K_b)\}$ is at most countable. Therefore, the set $E$ is of full plane Lebesgue measure. The theorem is proved. \hfill $\square$

The following important statement is a consequence of Theorem 4.

**Corollary 2.** For a.e. $\nu \in \mathbb{C}$, problem (1), (2) has a unique periodic solution with almost every period $b \in \mathbb{R}^+$.

**Proof.** Since the set $E$ is measurable and has full measure, for a.e. $\nu \in \mathbb{C}$ the section $E_\nu = \{b \in \mathbb{R}^+ \mid (\nu, b) \in E\} = \{b \in \mathbb{R}^+ \mid 1/\nu \notin \sigma(K_b)\}$ has full measure, and for such $b$'s, problem (1), (2) has a unique periodic solution with period $b$. The corollary is proved. \hfill $\square$

I am sincerely grateful to E. Yu. Panov for his attention to my work.

**References**


**Division of Mathematical Analysis, Novgorod State University, Bol’shaya St.-Peterburgskaya Ulitsa 41, 173003, Velikiy Novgorod, Russia**

_E-mail address:_ dangkhanhhoi@yahoo.com

Received 1/DEC/2005

Translated by A. PLOTKIN