

MISHCHENKO AND QUILLEN FORMULAS FOR ORIENTED COHOMOLOGY PRETHEORIES

E. SHINDER

ABSTRACT. Two formulas mentioned in the title are generalized to arbitrary oriented cohomology pretheories on algebraic varieties.

INTRODUCTION

We generalize some well-known formulas and relations for complex cobordism to oriented cohomology pretheories on algebraic varieties. The definition and properties of oriented pretheories were given in [Pa].

Given an oriented cohomology pretheory, Mishchenko's theorem expresses the coefficients of the related invariant differential form in terms of classes of projective spaces. For the complex cobordism theory, this theorem was stated and proved in Mishchenko's Appendix to [N]. In the first part of the present paper, we generalize this theorem to an arbitrary oriented cohomology pretheory (see Theorem 1). The proof is based on the complex cobordism theorem mentioned above. For the first time, Mishchenko's theorem was proved in this generality by I. A. Panin in a slightly different way; we present it here for completeness.

Quillen's formula expresses the action of the direct image of a projection for projectivized bundles in terms of the formal group law and the invariant differential form that are related to the cohomology theory in question. This formula was stated without proof for the complex cobordism theory in [Qu1, p. 1294]. In the second part of the present paper, we generalize Quillen's formula to the case of an arbitrary theory. The proof is based on the Mishchenko theorem treated in the first part of the paper, rather than on the formula in [Qu1]. In this generality, Quillen's formula is a new result (see Theorem 3). The proofs of the two main theorems are based on Theorem 2 describing the direct image of an embedding of projectivized bundles.

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In this paper, we use the following notation:

- X is a smooth algebraic variety over an algebraically closed field k ;
- $pt = \text{Spec}(k)$;
 \mathbf{P}^n is the n -dimensional projective space over a field k ;
- $\mathbf{1}_X^n$ is the trivial bundle of rank n over X ;
 E^\vee is the dual bundle for a bundle E ;
 $\mathbf{P}(E)$ is the projectivization of a bundle E ;
 $\mathcal{O}_E(-1)$ and $\mathcal{O}_E(1)$ are the tautological and the canonical bundle over $\mathbf{P}(E)$, respectively;

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- A is an oriented cohomology pretheory [Pa];
 MU is the complex cobordism theory;
- $\mathcal{L}(D)$ is the line bundle corresponding to a divisor $D \subset X$ [Har];
- $c_0^A(E), c_1^A(E), \dots, c_n^A(E)$ are the Chern classes of a bundle E of rank n ;
 $c_{\max}^A(E) = c_n^A(E)$ is the highest Chern class;
 $c_t^A(E) = c_0^A(E) + c_1^A(E)t + \dots + c_n^A(E)t^n$ is the Chern polynomial;
 $\xi_1, \xi_2, \dots, \xi_n$ are the Chern roots, i.e., the formal variables for which $(1 + \xi_1 t)(1 + \xi_2 t) \cdots (1 + \xi_n t) = c_t(E)$;
- $F_A(t_1, t_2)$ is the formal group law related to a pretheory A ;
 $\omega_A(t)$ is the invariant differential form for a formal group law $F_A(t_1, t_2)$;
- $[\mathbf{P}^n]_A = p_A(1) \in A(pt)$ for a projection $p : \mathbf{P}^n \rightarrow X$;
 $[\mathbf{CP}^n] = [\mathbf{P}^n]_{MU} \in MU(pt)$.

§1. THE GENERALIZED MISHCHENKO THEOREM

The following theorem was proved in [N].

Theorem (Mischenko).

$$(1) \quad \omega_{MU}(t) = \sum_{n=0}^{\infty} [\mathbf{CP}^n] t^n dt.$$

We use this fact to obtain the next statement.

Theorem 1 (generalized Mischenko theorem). *Let A be an oriented cohomology pretheory. Then*

$$(2) \quad \omega_A(t) = \sum_{n=0}^{\infty} [\mathbf{P}^n]_A t^n dt.$$

It is well known that the formal group law F_{MU} is the universal commutative one-dimensional formal group law, and the ring $MU(pt)$ coincides with the Lazard ring; see [Qu1]. Consequently, for every oriented cohomology pretheory A , there exists a unique ring homomorphism

$$l_A : MU(pt) \rightarrow A(pt)$$

such that

$$(l_A)_* F_{MU}(t_1, t_2) = F_A(t_1, t_2),$$

or equivalently,

$$l_A(a_{ij}^{MU}) = a_{ij}^A,$$

where the a_{ij}^A are the coefficients of the formal group law $F_A(t_1, t_2)$,

$$F_A(t_1, t_2) = t_1 + t_2 + \sum_{i,j \geq 1} a_{ij}^A t_1^i t_2^j.$$

It will be proved that

$$(l_A)([\mathbf{CP}^n]) = [\mathbf{P}^n]_A$$

for all $n \geq 0$.

First, we prove two lemmas.

Lemma 1.1. *Let $\Delta : \mathbf{P}^n \rightarrow \mathbf{P}^n \times \mathbf{P}^n$ be the diagonal embedding, and let $p_i : \mathbf{P}^n \times \mathbf{P}^n \rightarrow \mathbf{P}^n, i = 1, 2$, be projections. Put $\zeta_i = c_1(p_i^* \mathcal{O}(1))$. Then, in the ring $A(\mathbf{P}^n \times \mathbf{P}^n) = A(pt)[\zeta_1, \zeta_2]/(\zeta_1^{n+1}, \zeta_2^{n+1})$, we have*

$$(3) \quad \Delta_A(1) = \zeta_1^n + \zeta_2^n + \sum_{i,j \geq 1} b_{ij}^A \zeta_1^i \zeta_2^j,$$

where the $b_{ij}^A \in A(pt)$ are polynomials in the variables a_{ij}^A with integral coefficients independent of A .

Proof. In the ring $A(\mathbf{P}^n \times \mathbf{P}^n)$, we have the relation

$$(4) \quad \Delta_A(1) = c_n^A(p_1^* \mathcal{O}(1) \otimes p_2^* Q),$$

where $\mathcal{O}(1)$ is the canonical bundle over \mathbf{P}^n and Q is the canonical quotient bundle over \mathbf{P}^n occurring in the exact sequence

$$(5) \quad 0 \rightarrow \mathcal{O}(-1) \rightarrow V^{n+1} \rightarrow Q \rightarrow 0$$

(V^{n+1} is the trivial $(n + 1)$ -dimensional bundle over \mathbf{P}^n).

Formula (4) is a special case of Theorem 2 (see below) for $X = \mathbf{P}^n, E = \mathcal{O}(-1), F = V^{n+1}$, and $M = Q$.

We express the Chern class $c_n^A(p_1^* \mathcal{O}(1) \otimes p_2^* Q)$ on the right-hand side of (4) in terms of ζ_1 and ζ_2 . For this, we use the formula that expresses the highest Chern class of the tensor product of two bundles in terms of the Chern classes of the factors:

$$(6) \quad c_n(p_1^* \mathcal{O}(1) \otimes p_2^* Q) = \prod_{j=1}^n F_A(\zeta_1, \xi_j),$$

where the ξ_j are the Chern roots of the bundle $p_2^* Q$, i.e., the formal variables for which

$$c_t(p_2^* Q) = 1 + c_1(p_2^* Q)t + \dots + c_n(p_2^* Q)t^n = (1 + t\xi_1) \cdots (1 + t\xi_n).$$

We rearrange $F_A(\zeta_1, \xi_j)$ in (6) to obtain

$$\begin{aligned} &c_n(p_1^* \mathcal{O}(1) \otimes p_2^* Q) \\ &= \left(\zeta_1 + \xi_1 + \sum_{i,j \geq 1} a_{ij}^A \zeta_1^i \xi_1^j \right) \left(\zeta_1 + \xi_2 + \sum_{i,j \geq 1} a_{ij}^A \zeta_1^i \xi_2^j \right) \cdots \left(\zeta_1 + \xi_n + \sum_{i,j \geq 1} a_{ij}^A \zeta_1^i \xi_n^j \right). \end{aligned}$$

Removing parentheses, we arrive at the formula

$$(7) \quad c_n(p_1^* \mathcal{O}(1) \otimes p_2^* Q) = \zeta_1^n + c_n(p_2^* Q) + \sum_{i,j \geq 1} b_{ij}^A \zeta_1^i c_j(p_2^* Q),$$

where the b_{ij}^A satisfy the requirements of the lemma.

To prove formula (3), we calculate $c_j(p_2^* Q) = p_2^* c_j(Q)$. Application of the Cartan formula to the exact sequence (5) yields

$$c_t(\mathcal{O}(-1))c_t(Q) = 1.$$

Since $c_t(\mathcal{O}(-1)) = 1 - \zeta t$, we have

$$c_t(Q) = \frac{1}{1 - \zeta t} = 1 + \zeta t + \dots + \zeta^n t^n.$$

In other words, $c_j(Q) = \zeta^j$. Substituting these expressions in (7), we obtain (3). □

Lemma 1.2. *For every $n > 0$, the following expansion is valid in $A(pt)$:*

$$(8) \quad [\mathbf{P}^n]_A = \sum_{j=0}^{n-1} \alpha_j^A [\mathbf{P}^j]_A,$$

where the $\alpha_j^A \in A(pt)$ are polynomials in the variables α_{ij}^A with integral coefficients independent of A .

Proof. Consider the projection $p_1 : \mathbf{P}^n \times \mathbf{P}^n \rightarrow \mathbf{P}^n$ and its direct image

$$(p_1)_A : A(\mathbf{P}^n \times \mathbf{P}^n) \rightarrow A(\mathbf{P}^n).$$

We calculate the action of $(p_1)_A$ on the additive generator $\zeta_1^i \zeta_2^j$ of the $A(pt)$ -module $A(\mathbf{P}^n \times \mathbf{P}^n)$. We have

$$(p_1)_A(\zeta_1^i \zeta_2^j) = (p_1)_A((p_1)^A(\zeta^i) \zeta_2^j) = \zeta^i (p_1)_A(\zeta_2^j) = \zeta^i [\mathbf{P}^{n-j}]_A.$$

In particular,

$$(p_1)_A(\zeta_1^n) = \zeta^n [\mathbf{P}^n]_A, \quad (p_1)_A(\zeta_2^n) = 1.$$

Now, we apply the operator $(p_1)_A$ to both sides of formula (3). Using Lemma 1.1 and the relation $(p_1)_A \Delta_A = id$, we obtain

$$1 = \zeta^n [\mathbf{P}^n]_A + 1 + \sum_{i,j \geq 1} b_{ij}^A \zeta^i [\mathbf{P}^{n-j}]_A.$$

Since the elements $1, \zeta, \dots, \zeta^n$ are linearly independent over $A(pt)$, the coefficient at ζ^n on the right-hand side must be equal to zero. Hence, we obtain the required expansion (8). The properties of the coefficients α_i^A follow from the corresponding properties of the b_{ij}^A . The lemma is proved. \square

Corollary. For the coefficients α_j^A occurring in Lemma 2, we have

$$l_A(\alpha_j^{MU}) = \alpha_j^A.$$

Proposition. $(l_A)([\mathbf{CP}^n]) = [\mathbf{P}^n]_A$.

Proof. We use induction on n . If $n = 0$, the statement is trivial because $[\mathbf{P}^0]_A = 1$. For $n > 0$, we employ Lemma 2 two times: for the theory MU and for the theory A . We obtain

$$l_A([\mathbf{CP}^n]) = l_A\left(\sum_{j=0}^{n-1} \alpha_j^{MU} [\mathbf{CP}^j]\right) = \sum_{j=0}^{n-1} \alpha_j^A [\mathbf{P}^j]_A = [\mathbf{P}^n]_A.$$

The proposition is proved. \square

Proof of Theorem 1. Since the homomorphism l_A takes the coefficients of F_{MU} to those of F_A , we see that this homomorphism takes the coefficients of ω_{MU} to those of ω_A . Using (1) and the above proposition, we obtain (2). The theorem is proved. \square

§2. GENERALIZED QUILLEN FORMULA

Theorem 2. Let X be a smooth variety, and let

$$(9) \quad 0 \rightarrow E \rightarrow F \rightarrow M \rightarrow 0$$

be an exact sequence of vector bundles over X . Consider the commutative diagram

$$\begin{array}{ccc} \mathbf{P}(E) & \xrightarrow{i} & \mathbf{P}(F) \\ \downarrow p & \swarrow q & \\ X & & \end{array}$$

Then for the direct image i_A of the morphism $i : \mathbf{P}(E) \rightarrow \mathbf{P}(F)$ we have

$$(10) \quad i_A(1) = c_{\max}(\mathcal{O}_F(1) \otimes q^*M),$$

$$(11) \quad i_A(h(\zeta_E)) = h(\zeta_F) c_{\max}(\mathcal{O}_F(1) \otimes q^*M),$$

where $h(t)$ is an arbitrary polynomial with coefficients in $A(X)$, $\zeta_E = c_1(\mathcal{O}_E(1)) \in A(\mathbf{P}(E))$, and $\zeta_F = c_1(\mathcal{O}_F(1)) \in A(\mathbf{P}(F))$.

Proof. First, we note that relation (11) follows from (10) and the projection formula. To prove formula (10), we apply the splitting principle, by which we may assume that the sequence (9) is split, $F = M \oplus E$, and that $M = \bigoplus_{i=1}^m L_i$, where the L_i , $i = 0, \dots, m$, are line bundles. We use induction on m .

Base of induction: $m = 1$. In this case, $i : P(E) \hookrightarrow P(F)$ is an embedding of a divisor, so that $i_A(1) = c(\mathcal{L})$, where $\mathcal{L} = L(\mathbf{P}(E))$. We prove that $\mathcal{L} = \mathcal{O}(1) \otimes q^*M$. First, we consider the case where $M = \mathbf{1}$. Then $F = \mathbf{1} \oplus E$ and $\mathcal{L} = \mathcal{O}_{\mathbf{1} \oplus E}(1)$, because the latter has a natural section t (the first coordinate in $\mathbf{1} \oplus E$) whose zeros coincide with $\mathbf{P}(E)$. For an arbitrary M , we have

$$\begin{aligned} \mathbf{P}(M \oplus E) &= \mathbf{P}((M \oplus E) \otimes M^\vee) = \mathbf{P}(\mathbf{1} \oplus (E \otimes M^\vee)), \\ \mathcal{L} &= \mathcal{O}_{\mathbf{1} \oplus (E \otimes M^\vee)}(1) = \mathcal{O}_{M \oplus E}(1) \otimes q^*M. \end{aligned}$$

Induction step. Suppose $M_k = \bigoplus_{i=1}^k L_i$ and $F_k = E \oplus M_k$. We have the commutative diagram

$$\begin{array}{ccccc} \mathbf{P}(E) & \xrightarrow{i} & \mathbf{P}(F_k) & \xrightarrow{j} & \mathbf{P}(F_{k+1}) \\ & \searrow p & \downarrow q & \swarrow q' & \\ & & X & & \end{array}$$

Let i' be the composition of i and j . We must prove that

$$i'_A(1) = c_{k+1}(\mathcal{O}_{F_{k+1}}(1) \otimes q'^*M_{k+1}).$$

The exact sequence

$$0 \rightarrow E \rightarrow F_k \rightarrow M_k \rightarrow 0$$

and the inductive hypothesis imply that $i_A(1) = c_k(\mathcal{O}_{F_k}(1) \otimes q^*M_k)$, and the sequence

$$0 \rightarrow F_k \rightarrow F_{k+1} \rightarrow L_k \rightarrow 0$$

yields $j_A(1) = c_1(\mathcal{O}_{F_{k+1}}(1) \otimes q'^*L_k)$. Now, we calculate $i'_A(1)$:

$$\begin{aligned} i'_A(1) &= j_A(i_A(1)) = j_A(c_k(\mathcal{O}_{F_k}(1) \otimes q^*M_k)) \\ &= j_A(j^A(c_k(\mathcal{O}_{F_{k+1}}(1) \otimes q'^*M_k))) = j_A(1)c_k(\mathcal{O}_{F_{k+1}}(1) \otimes q'^*M_k) \\ &= c_1(\mathcal{O}_{F_{k+1}}(1) \otimes q'^*L_{k+1})c_k(\mathcal{O}_{F_{k+1}}(1) \otimes q'^*M_k) \\ &= c_{k+1}(\mathcal{O}_{F_{k+1}}(1) \otimes q'^*M_{k+1}). \end{aligned}$$

The theorem is proved. □

Notation. For every vector bundle E of rank n over a smooth variety X , we consider the formal power series $f_E(t) \in A(X)[[t]]$ defined by the following formula:

$$f_E(t) = \prod_{i=1}^n (t - \lambda_i),$$

where the λ_i , $i = 1, \dots, n$, are the Chern roots of the bundle E .

For example, we have $f_E(t) = t^n$ for $E = \mathbf{1}^n$.

Lemma 2.1. Consider the projection $\pi : X \times \mathbf{P}^\infty \rightarrow X$ and identify $A(X \times \mathbf{P}^\infty)$ with $A(X)[[t]]$, $t = c_1(\mathcal{O}(1))$. Then in the ring $A(X \times \mathbf{P}^\infty)$ we have

$$c_n(\mathcal{O}(1) \otimes \pi^*E) = f_{E^\vee}(t).$$

Proof. By the splitting principle, we may assume that

$$E = \bigoplus_{i=1}^n L_i.$$

Then

$$\begin{aligned} c_n(\mathcal{O}(1) \otimes \pi^* E) &= c_n\left(\mathcal{O}(1) \otimes \pi^* \bigoplus_{i=1}^n L_i\right) = c_n\left(\bigoplus_{i=1}^n (\mathcal{O}(1) \otimes \pi^* L_i)\right) \\ &= \prod_{i=1}^n c_1(\mathcal{O}(1) \otimes \pi^* L_i) = \prod_{i=1}^n F_A(t, \lambda_i) = f_{E^\vee}(t). \quad \square \end{aligned}$$

Lemma 2.2. *Let*

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

be an exact sequence of bundles over X . Then

$$f_E(t) = f_{E'}(t)f_{E''}(t).$$

Proof. It suffices to apply the Cartan formula to the Chern class of Lemma 2.1. \square

Lemma 2.3. *The power series $f_E(t)$ is invertible in $A(X)((t))$.*

Proof. By the splitting principle and the definition of $f_E(t)$, it suffices to prove the statement for a line bundle. In this case, the element $f_E(t) = t -_F \lambda = F(t, I\lambda) = t - \lambda + \sum_{i,j \geq 1} a_{ij} t^i \lambda^j = -\lambda + t(1 + \nu + t(\dots))$ is invertible, being the sum of a nilpotent and an invertible element. \square

Theorem 3 (Quillen's formula). *Let E be a bundle of rank n over a smooth variety X . Let $\mathbf{P}(E)$ be its projectivization and $p : \mathbf{P}(E) \rightarrow X$ the projection. We denote by $\omega(t)$ the invariant differential form corresponding to $F_A(t_1, t_2)$. Then for the direct image p_A we have*

$$(12) \quad p_A(\zeta_E^k) = \operatorname{Res}_{t=0} \frac{t^k \omega(t)}{f_{E^\vee}(t)},$$

where $\zeta_E = c_1(\mathcal{O}_E(1)) \in A(\mathbf{P}(E))$.

Lemma 2.4. *Formula (12) is valid for a trivial bundle E .*

Proof. It is well known that the direct image of the projection $p : X \times \mathbf{P}^{n-1} \rightarrow pt$ acts by the formula

$$p_A(\zeta^k) = [\mathbf{P}^{n-k-1}].$$

Rearranging the right-hand side of the above relation, we obtain

$$\operatorname{Res}_{t=0} \frac{t^k \omega(t)}{f_{E^\vee}(t)} = \operatorname{Res}_{t=0} \frac{t^k \omega(t)}{t^n} = \operatorname{Res}_{t=0} (t^{k-n} \omega(t)).$$

Theorem 1 states that

$$\omega_A(t) = ([\mathbf{P}^0] + [\mathbf{P}^1]t + \dots + [\mathbf{P}^n]t^n + \dots) dt.$$

Therefore, the required residue is equal to $[\mathbf{P}^{n-k-1}]$, and the right-hand side in (12) coincides with the left-hand side, which proves the lemma. \square

Proof of Theorem 3. Using the Jouanolou trick [P1], we assume that X is affine. On an affine variety, we have a natural equivalence between the category of vector bundles and the category of projective modules. Since a projective module is a direct summand of a free module, we obtain the following exact sequence of vector bundles over X :

$$0 \rightarrow E \rightarrow \mathbf{1}^N \rightarrow M \rightarrow 0.$$

Consider the commutative diagram

$$\begin{array}{ccc} \mathbf{P}(E) & \xrightarrow{i} & \mathbf{P}(\mathbf{1}^N) \\ \downarrow p & \swarrow q & \\ X & & \end{array}$$

The direct image p_A of p can be calculated as the composition of the mappings i_A and q_A . By Theorem 2,

$$i_A(\zeta_E^k) = \zeta_{\mathbf{1}^N}^k c_m(\mathcal{O}_{\mathbf{1}^N}(1) \otimes q^* M).$$

Arguing as in the proof of Lemma 2.1, it is easy to check that

$$c_m(\mathcal{O}_{\mathbf{1}^N}(1) \otimes q^* M) = f_{M^\vee}(t)|_{t=\zeta}.$$

Lemmas 2.2 and 2.3 allow us to find $f_{M^\vee}(t)$:

$$f_{M^\vee}(t) = \frac{t^N}{f_{E^\vee}(t)}.$$

As a result, we see that $i_A(\zeta_E^k) = h(t)|_{t=\zeta_{\mathbf{1}^N}}$ with $h(t) = \frac{t^{N+k}}{f_{E^\vee}(t)} \in A(X)[[t]]$. Substituting this expression in the formula for q_A (see Lemma 2.4), we obtain

$$p_A(\zeta_E^k) = q_A(i_A(\zeta_E^k)) = q_A(h(\zeta_{\mathbf{1}^N})) = \operatorname{Res}_{t=0} \frac{h(t)\omega(t)}{t^N}.$$

After simplification, we arrive at the formula

$$p_A(\zeta_E^k) = \operatorname{Res}_{t=0} \frac{t^k \omega(t)}{f_{E^\vee}(t)}.$$

The theorem is proved. □

Corollary. *Under the assumptions and in the notation of Theorem 2, we have*

$$(13) \quad p_A(h(\zeta)) = \operatorname{Res}_{t=0} \frac{h(t)\omega(t)}{f_{E^\vee}(t)},$$

where $h(t)$ is an arbitrary polynomial with coefficients in $A(X)$.

Proof. Formula (13) follows from (12) and the theorem on projectivized bundles. □

REFERENCES

- [Har] R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math., No. 52, Springer-Verlag, New York–Heidelberg, 1977. MR0463157 (57:3116)
- [P1] I. Panin, *Push-forwards in oriented cohomology theories of algebraic varieties*. II, <http://www.math.uiuc.edu/K-theory/0619/2003>.
- [PS] I. Panin and A. Smirnov, *Push-forwards in oriented cohomology theories of algebraic varieties*, <http://www.math.uiuc.edu/K-theory/0459/2000>.
- [Pa] I. Panin, *Riemann–Roch theorems for oriented cohomology*, *Axiomatic, Enriched, and Motivic Homotopy Theory*, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 2004, pp. 261–333. MR2061857 (2005g:14025)

- [Qu1] D. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298. MR0253350 (40:6565)
- [N] S. P. Novikov, *Methods of algebraic topology from the point of view of cobordism theory*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), no. 4, 855–951; English transl. in USSR-Izv. **1** (1967). MR0221509 (36:4561)

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, UNIVERSITETSKIĬ PROSPECT 28, STARYĬ PETERHOF, ST. PETERSBURG 198904, RUSSIA
E-mail address: `shinder@list.ru`

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