

ON THE NUMBER OF CLOSED BRAIDS
OBTAINED AS A RESULT
OF SINGLE STABILIZATIONS AND DESTABILIZATIONS
OF A CLOSED BRAID

A. V. MALYUTIN

ABSTRACT. Sufficient conditions for a closed n -braid $\widehat{\beta}$ to have infinite sets $\mathfrak{D}(\widehat{\beta})$ and $\mathfrak{S}(\widehat{\beta})$ are given, where $\mathfrak{D}(\widehat{\beta})$ denotes the set of all closed $(n-1)$ -braids that are obtained from $\widehat{\beta}$ via Markov destabilization, while $\mathfrak{S}(\widehat{\beta})$ denotes the set of all closed $(n+1)$ -braids that are obtained from $\widehat{\beta}$ via Markov stabilization. New integer-valued conjugacy invariants for the braid group are introduced.

INTRODUCTION

0.1. Braids. The *Artin braid group* B_n of index n is defined by the presentation

$$B_n := \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geq 2; \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle.$$

The elements of the groups B_1, B_2, B_3, \dots are called *braids*. The generators σ_i are *Artin's generators*. By a braid group we usually mean a braid group with a fixed system of Artin's generators.

In what follows, by a *closed braid of index n* (a *closed n -braid*) we mean a conjugacy class in B_n . The conjugacy class of a braid β (i.e., the corresponding closed braid) is denoted by $\widehat{\beta}$.

0.2. Representation of links by braids. There exists a standard procedure that maps a braid β to an oriented link type $L(\beta)$ in \mathbb{R}^3 . This procedure is described, e.g., in [3]. We say that β *represents* $L(\beta)$. Since conjugate braids represent one and the same link type, each closed braid represents a certain link type ($L(\widehat{\beta}) := L(\beta)$). By the Alexander theorem [1], for each link type L there exists a closed braid $\widehat{\beta}$ such that $L = L(\widehat{\beta})$. By the Markov theorem [4], two links $L(\alpha)$ and $L(\beta)$ coincide if and only if the braids α and β are related by a sequence of conjugations, stabilizations, and destabilizations. (Definitions of the latter two operations are given in Section 0.4.)

0.3. Agreement. It is convenient to regard braid groups with smaller indices as subgroups of braid groups with greater indices: if $m < n$, then B_m is embedded in B_n by the homomorphism that takes the generator $\sigma_i \in B_m$ to the generator $\sigma_i \in B_n$ for each $i \in \{1, \dots, m-1\}$. Given a braid $\beta \in B_m$, we denote the image of this braid under the canonical embedding $B_m \rightarrow B_n$ by the same letter β .

2000 *Mathematics Subject Classification.* Primary 20F36, 57M25.

Key words and phrases. Braid group, Markov destabilization, Markov stabilization, conjugacy invariant, link theory.

0.4. Stabilization and destabilization of braids. Let β be a braid in B_n . We say that the braid $\beta\sigma_n$ in B_{n+1} (see Agreement 0.3) is obtained from the braid $\beta \in B_n$ via *positive stabilization*, while $\beta\sigma_n^{-1}$ is obtained from β via *negative stabilization*. We also say that β is obtained from $\beta\sigma_n$ via *positive destabilization*, and from $\beta\sigma_n^{-1}$ via *negative destabilization*.

0.5. Stabilization and destabilization of closed braids. We say that a closed $(n+1)$ -braid $\widehat{\beta}$ is obtained from a closed n -braid $\widehat{\alpha}$ via positive (respectively, negative) *stabilization* if there are braids $\gamma \in \widehat{\alpha}$ and $\gamma' \in \widehat{\beta}$ such that γ' is obtained from γ via positive (respectively, negative) stabilization. We also say that $\widehat{\alpha}$ is obtained from $\widehat{\beta}$ via positive (respectively, negative) *destabilization*.

Remark. The Markov theorem implies that two closed braids $\widehat{\alpha}$ and $\widehat{\beta}$ represent one and the same link type if and only if $\widehat{\alpha}$ and $\widehat{\beta}$ are related by a sequence of stabilizations and destabilizations.

Notation. For a closed n -braid $\widehat{\beta}$, we denote by

$$\mathfrak{D}(\widehat{\beta})$$

the set of all closed $(n-1)$ -braids that are obtained from $\widehat{\beta}$ via destabilization, and we denote by

$$\mathfrak{S}(\widehat{\beta})$$

the set of all closed $(n+1)$ -braids that are obtained from $\widehat{\beta}$ via stabilization. Thus we have

$$\widehat{\alpha} \in \mathfrak{D}(\widehat{\beta}) \iff \widehat{\beta} \in \mathfrak{S}(\widehat{\alpha}).$$

0.6. In this paper, we deal with the cardinality of the sets $\mathfrak{S}(\widehat{\beta})$ and $\mathfrak{D}(\widehat{\beta})$ for a closed braid $\widehat{\beta}$. Concerning the set $\mathfrak{S}(\widehat{\beta})$, we prove the following theorem.

Theorem 1. *Suppose that a braid β of index $n \geq 3$ represents a knot (i.e., a one-component link). Then the set $\mathfrak{S}(\widehat{\beta})$ is infinite.*

The following conjecture seems to be plausible.

Conjecture. *Let $\beta \in B_n$. Then the set $\mathfrak{S}(\widehat{\beta})$ is finite if and only if β lies in the center of B_n .*

We easily see that the set $\mathfrak{S}(\widehat{\beta})$ is two-element whenever β lies in the center of B_n . Consequently, if the above conjecture is true, then for any closed braid $\widehat{\beta}$ the set $\mathfrak{S}(\widehat{\beta})$ is either infinite or two-element.

The question on the cardinality of $\mathfrak{D}(\widehat{\beta})$ is more difficult. Even finding out whether $\mathfrak{D}(\widehat{\beta})$ is empty for a given β is a hard problem in the general case.

It turns out that there exist closed braids with infinite $\mathfrak{D}(\widehat{\beta})$, i.e., closed braids that can be destabilized in an infinite number of ways. Furthermore, any link type L is represented by infinitely many closed braids $\widehat{\beta}$ with infinite $\mathfrak{D}(\widehat{\beta})$. The latter fact easily follows from our Theorem 3. Note that while the existence of closed braids $\widehat{\beta}$ with infinite sets $\mathfrak{S}(\widehat{\beta})$ is natural in a certain sense, the existence of closed braids with infinite $\mathfrak{D}(\widehat{\beta})$ is “unexpected” at first glance. Indeed, destabilization reduces the index of a closed braid; hence it seems that this operation must “simplify” any braid.

To formulate our results on the cardinality of $\mathfrak{D}(\widehat{\beta})$, we need the notion of *double destabilization*.

0.7. Double stabilization and double destabilization. Let β be a braid of index $n \geq 2$. We consider the following two braids in B_{n+2} :

$$\mathfrak{s}_{2+}(\beta) := \beta \sigma_{n-1}^{-2} \sigma_n \sigma_{n+1} \sigma_{n-1} \sigma_n \quad \text{and} \quad \mathfrak{s}_{2-}(\beta) := \beta \sigma_{n-1}^2 \sigma_n^{-1} \sigma_{n+1}^{-1} \sigma_{n-1}^{-1} \sigma_n^{-1}.$$

(Here we use our Agreement 0.3.)

We say that the braid $\mathfrak{s}_{2+}(\beta)$ is obtained from the braid $\beta \in B_n$ via *positive double stabilization*, and $\mathfrak{s}_{2-}(\beta)$ is obtained from β via *negative double stabilization*. We also say that β is obtained from $\mathfrak{s}_{2+}(\beta)$ via *positive double destabilization*, and from $\mathfrak{s}_{2-}(\beta)$ via *negative double destabilization*. If a braid $\alpha \in B_{n+2}$ is obtained from a braid $\beta \in B_n$ via double stabilization (either positive or negative), then we say that α *admits double destabilization* or is *doubly destabilizable*. Clearly, the result of double destabilization for a doubly destabilizable braid is uniquely determined.

Accordingly, we say that a closed $(n + 2)$ -braid $\widehat{\beta}$ is obtained from a closed n -braid $\widehat{\alpha}$ via *double stabilization* (and $\widehat{\alpha}$ is obtained from $\widehat{\beta}$ via *double destabilization*) if there are braids $\gamma \in \widehat{\alpha}$ and $\gamma' \in \widehat{\beta}$ such that γ' is obtained from γ via double stabilization. A closed braid $\widehat{\beta}$ *admits double destabilization* or is *doubly destabilizable* if it contains a doubly destabilizable braid $\gamma \in \widehat{\beta}$.

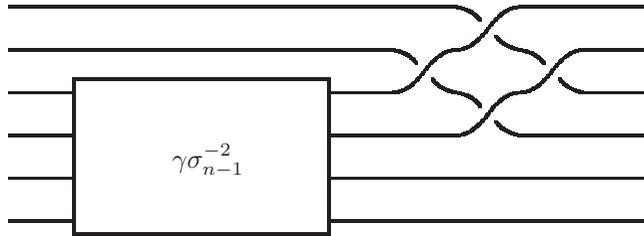


FIGURE 1. The template of (positively) doubly destabilizable braids.

The operation of double stabilization of a closed braid can be decomposed into two successive stabilizations. In other words, all closed braids obtained from a closed braid $\widehat{\beta}$ via double stabilization lie in the set $\mathfrak{S}(\mathfrak{S}(\widehat{\beta}))$, while all closed braids obtained from $\widehat{\beta}$ via double destabilization lie in the set $\mathfrak{D}(\mathfrak{D}(\widehat{\beta}))$. In particular, two closed braids related by double stabilizations and double destabilizations represent one and the same link type.

0.8. First of all, we mention the following result. (We place this result here for the convenience of comprehension. The theorem is neither proved nor used in the proofs below.)

Theorem 2. *If a closed braid $\widehat{\beta}$ has infinite set $\mathfrak{D}(\widehat{\beta})$, then $\widehat{\beta}$ is doubly destabilizable.*

Double destabilizability of $\widehat{\beta}$ is a necessary but not a sufficient condition for the infiniteness of $\mathfrak{D}(\widehat{\beta})$. The techniques of this paper provide powerful sufficient conditions for the infiniteness of $\mathfrak{D}(\widehat{\beta})$ for a given $\widehat{\beta}$. In particular, using these techniques we prove the following result.

Theorem 3. *Suppose that a closed braid $\widehat{\beta}$ of odd index represents a knot and admits double destabilization. Then the set $\mathfrak{D}(\widehat{\beta})$ is infinite.*

Remarks. 1. Theorems 2 and 3 imply that a one-component closed braid $\widehat{\beta}$ of odd index admits double destabilization if and only if the set $\mathfrak{D}(\widehat{\beta})$ is infinite.

2. It can be shown that if a one-component doubly destabilizable braid $\widehat{\beta}$ has even index, then the set $\mathfrak{D}(\widehat{\beta})$ can be finite as well as infinite.

0.9. Examples of closed braids with infinite $\mathfrak{D}(\widehat{\beta})$. Theorem 3 provides numerous examples of closed braids with infinite set $\mathfrak{D}(\widehat{\beta})$. Indeed, take a one-component braid β of odd index; then the braid $\mathfrak{s}_{2+}(\beta) \stackrel{\text{def}}{=} \beta\sigma_{n-1}^{-2}\sigma_n\sigma_{n+1}\sigma_{n-1}\sigma_n$ admits double destabilization, has odd index, and represents the same link type as β (hence, $\mathfrak{s}_{2+}(\beta)$ is also a one-component braid); then Theorem 3 implies that $\mathfrak{D}(\widehat{\mathfrak{s}_{2+}(\beta)})$ is infinite.

One of the simplest examples of closed braids with infinite $\mathfrak{D}(\widehat{\beta})$ is the conjugacy class of the 5-braid

$$b := \sigma_1\sigma_2^{-1}\sigma_3\sigma_4\sigma_2\sigma_3.$$

We observe that b represents the unknot, and that b is obtained from the 3-braid $\sigma_1\sigma_2$ via positive double stabilization. Following the pattern of the proof of Theorem 3, we see that the set $\mathfrak{D}(\widehat{b})$ contains conjugacy classes of all braids of the form $\sigma_1\sigma_2^{z+1}\sigma_3\sigma_2^{-z}$, where $z \in \mathbb{Z}$ (cf. Proposition 2.1). We also see that the set of indicated conjugacy classes is infinite (cf. Proposition 2.2).

0.10. \mathcal{X} -invariants. In Section 1 below, we introduce a series of integer-valued conjugacy invariants in the braid group. We call them “ \mathcal{X} -invariants”. The \mathcal{X} -invariants are involved in the proofs of Theorems 1 and 3. The invariants are of independent interest.

§1. \mathcal{X} -INVARIANTS

In this section, we describe a series of integer-valued conjugacy invariants in the braid group.

1.1. The symmetric group. Let S_n denote the group of all permutations of the set $\{1, \dots, n\}$, i.e., the *symmetric group of degree n* . We denote by \mathcal{S} the following standard homomorphism from B_n to S_n :

$$\mathcal{S} : B_n \rightarrow S_n, \quad \sigma_i \mapsto (i, i + 1).$$

We recall that a braid $\beta \in B_n$ represents a knot if and only if $\mathcal{S}(\beta)$ is a cycle of length n . We consider the *left* action of the group S_n on the set $\{1, \dots, n\}$. If i is an integer and β is a braid, then we denote by $\beta(i)$ the image of i under $\mathcal{S}(\beta)$:

$$\beta(i) := \mathcal{S}(\beta)(i).$$

As in the case of braid groups, we assume that the group S_m is canonically embedded in S_n whenever $m < n$. It is also convenient to assume that S_n acts on \mathbb{N} . (If $j > n$, then j is fixed under S_n .)

1.2. Definition: intersection indices of strings. We define a set of mappings

$$X_{i,j} : B_n \rightarrow \mathbb{Z}, \quad \text{where } i, j \in \mathbb{N},$$

by the following rules:

$$(1) \quad X_{i,j}(\sigma_k^p) := \begin{cases} p & \text{if } \{i, j\} = \{k, k + 1\}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(2) \quad X_{i,j}(\alpha\beta) = X_{\beta(i),\beta(j)}(\alpha) + X_{i,j}(\beta).$$

The correctness of this definition (i.e., the existence and uniqueness of a set of mappings satisfying the rules (1) and (2)) is proved by a routine verification. [The verification consists of the following two parts. First, we check that there exists a unique set of \mathbb{Z} -valued mappings (say, $X'_{i,j}$) from the set of words in Artin’s generators and their inverses that satisfy the rules (1) and (2). Secondly, we check that each of these $X'_{i,j}$ takes equal values on words representing one and the same braid. To prove the latter fact, it suffices

to show that $X_{i,j}(W') = X_{i,j}(W)$ whenever W' is obtained from W via a single application of one of the following elementary relations: $\sigma_k\sigma_k^{-1} = e$, $\sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}$, and $\sigma_k\sigma_l = \sigma_l\sigma_k$ for $|k - l| \geq 2$.]

The mapping $X_{i,j}$ is called the *intersection index of the strings i and j* .

Remarks. 1. The intersection index $X_{i,j} : B_n \rightarrow \mathbb{Z}$ is not a trivial mapping if and only if $i \neq j$ and $i, j \in \{1, \dots, n\}$. Obviously, we have $X_{i,j} \equiv X_{j,i}$. We remark that on B_n there are $n(n - 1)/2$ pairwise distinct nontrivial intersection indices.

2. If $n > 2$, each nontrivial index $X_{i,j}$ on B_n is not a homeomorphism. However, the sum of all intersection indices on B_n is a homeomorphism, which is equal to the doubled sum of powers of the Artin generators in a word representing a braid.

1.3. Definition: \mathcal{X} -invariants. Let $k \in \mathbb{Z}$. We define the mapping

$$\mathcal{X}_k : B_n \rightarrow \mathbb{Z}$$

by the following rule:

$$\mathcal{X}_k(\beta) := \sum_{\{(i,j) \mid \beta^k(i)=j\}} X_{i,j}(\beta).$$

Remark. If $\beta^k(i) = j$ and $\beta^k(j) = i$, then the sum $\mathcal{X}_k(\beta)$ contains both of the (equal) numbers $X_{i,j}(\beta)$ and $X_{j,i}(\beta)$ as summands.

1.4. Proposition. For each $k \in \mathbb{Z}$, the mapping $\mathcal{X}_k : B_n \rightarrow \mathbb{Z}$ is a conjugacy invariant.

Proof. For a permutation S and an integer k , we define:

$$\mathcal{R}_k(S) := \{(i, j) \mid S^k(i) = j\} \subset \mathbb{N} \times \mathbb{N}.$$

We consider the induced action of the symmetric group on the set $\mathbb{N} \times \mathbb{N}$:

$$S(i, j) = (S(i), S(j)).$$

For any two permutations S and T , we have

$$(*) \quad \mathcal{R}_k(TST^{-1}) = T(\mathcal{R}_k(S)).$$

(To check this, we observe that the condition $(i, j) \in \mathcal{R}_k(TST^{-1})$ is equivalent to the condition $TS^kT^{-1}(i) = j$. In turn, the condition $TS^kT^{-1}(i) = j$ is equivalent to the condition $S^kT^{-1}(i) = T^{-1}(j)$. The latter condition means that $(T^{-1}(i), T^{-1}(j)) \in \mathcal{R}_k(S)$, i.e., $(i, j) \in T(\mathcal{R}_k(S))$.)

If $S = T$, then equality (*) takes the following form:

$$(**) \quad S(\mathcal{R}_k(S)) = \mathcal{R}_k(S).$$

For a braid β , we will denote by $\mathcal{R}_k(\beta)$ the set $\mathcal{R}_k(\mathcal{S}(\beta))$:

$$\mathcal{R}_k(\beta) := \mathcal{R}_k(\mathcal{S}(\beta)).$$

With this notation, the definition of \mathcal{X}_k takes the following form:

$$\mathcal{X}_k(\beta) = \sum_{(i,j) \in \mathcal{R}_k(\beta)} X_{i,j}(\beta).$$

Now, we pass to the main part of our proof. It is necessary (and it is sufficient) to prove that for any two braids α and β we have

$$\mathcal{X}_k(\alpha\beta\alpha^{-1}) = \mathcal{X}_k(\beta).$$

We put:

$$A := \mathcal{X}_k(\alpha\beta\alpha^{-1}) \stackrel{\text{def}}{=} \sum_{(i,j) \in \mathcal{R}_k(\alpha\beta\alpha^{-1})} X_{i,j}(\alpha\beta\alpha^{-1}).$$

By equality (*), we have

$$A = \sum_{(i,j) \in \alpha(\mathcal{R}_k(\beta))} X_{i,j}(\alpha\beta\alpha^{-1}).$$

Now, formula (2) of Definition 1.2 implies the following equality:

$$A = \sum_{(i,j) \in \beta\alpha^{-1}\alpha(\mathcal{R}_k(\beta))} X_{i,j}(\alpha) + \sum_{(i,j) \in \alpha^{-1}\alpha(\mathcal{R}_k(\beta))} X_{i,j}(\beta) + \sum_{(i,j) \in \alpha(\mathcal{R}_k(\beta))} X_{i,j}(\alpha^{-1}).$$

We observe that $\alpha^{-1}\alpha\mathcal{R}_k(\beta) = \mathcal{R}_k(\beta)$ and that equality (**) implies $\beta(\mathcal{R}_k(\beta)) = \mathcal{R}_k(\beta)$. We have thus obtained

$$A = \sum_{(i,j) \in \mathcal{R}_k(\beta)} X_{i,j}(\alpha) + \sum_{(i,j) \in \mathcal{R}_k(\beta)} X_{i,j}(\beta) + \sum_{(i,j) \in \alpha(\mathcal{R}_k(\beta))} X_{i,j}(\alpha^{-1}).$$

Once more using Definition 1.2, we get

$$\sum_{(i,j) \in \alpha(\mathcal{R}_k(\beta))} X_{i,j}(\alpha^{-1}) + \sum_{(i,j) \in \mathcal{R}_k(\beta)} X_{i,j}(\alpha) = \sum_{(i,j) \in \mathcal{R}_k(\beta)} X_{i,j}(\alpha^{-1}\alpha) = 0,$$

whence it follows that, indeed,

$$A = \sum_{(i,j) \in \mathcal{R}_k(\beta)} X_{i,j}(\beta) \stackrel{\text{def}}{=} \mathcal{X}_k(\beta). \quad \square$$

§2. PROOFS OF AUXILIARY PROPOSITIONS

In this section, we prove two propositions. In Section 3, we use them to prove Theorems 1 and 3.

Proposition 2.1. *Let β be a braid in B_n . Let $t \in \mathbb{Z}$, and let β_t denote the braid $\beta\sigma_{n-1}^t\sigma_n\sigma_{n-1}^{-t}$ in B_{n+1} :*

$$\beta_t := \beta\sigma_{n-1}^t\sigma_n\sigma_{n-1}^{-t} \in B_{n+1}.$$

Then the closed braid $\widehat{\beta}_t$ can be obtained from $\widehat{\beta}$ via stabilization. At the same time, $\widehat{\beta}_t$ can be obtained via destabilization from the closed braid $\widehat{\mathfrak{s}_{2+}(\beta)}$. (We recall that the braid $\mathfrak{s}_{2+}(\beta)$ is obtained from β via positive double destabilization.) In other words, for each $t \in \mathbb{Z}$ the closed braid $\widehat{\beta}_t$ lies in the set $\mathfrak{S}(\widehat{\beta}) \cap \mathfrak{D}(\widehat{\mathfrak{s}_{2+}(\beta)})$. Therefore, if the set $\{\widehat{\beta}_t\}_{t \in \mathbb{Z}}$ is infinite, then the sets $\mathfrak{S}(\widehat{\beta})$ and $\mathfrak{D}(\widehat{\mathfrak{s}_{2+}(\beta)})$ are also infinite.

Proof. We observe that $\widehat{\beta}_t$ contains the braid

$$\gamma' := \sigma_{n-1}^{-t}\beta_t\sigma_{n-1}^t = \sigma_{n-1}^{-t}\beta\sigma_{n-1}^t\sigma_n \in B_{n+1},$$

while $\widehat{\beta}$ contains the braid

$$\gamma := \sigma_{n-1}^{-t}\beta\sigma_{n-1}^t \in B_n.$$

Clearly, γ' is obtained from γ via stabilization. By definition, this means that

$$\widehat{\beta}_t \in \mathfrak{S}(\widehat{\beta}).$$

Now, we show that $\widehat{\beta}_t \in \mathfrak{D}(\widehat{\mathfrak{s}_{2+}(\beta)})$. We recall that

$$\mathfrak{s}_{2+}(\beta) \stackrel{\text{def}}{=} \beta\sigma_{n-1}^{-2}\sigma_n\sigma_{n+1}\sigma_{n-1}\sigma_n.$$

We consider the following braid α_t , which is conjugate to $\mathfrak{s}_{2+}(\beta)$:

$$\alpha_t := \sigma_{n+1}^{-t-1}\mathfrak{s}_{2+}(\beta)\sigma_{n+1}^{t+1} = \sigma_{n+1}^{-t-1}\beta\sigma_{n-1}^{-2}\sigma_n\sigma_{n+1}\sigma_{n-1}\sigma_n\sigma_{n+1}^{t+1}.$$

It is easy to see that in the braid group we have

$$\sigma_i^s\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}^s.$$

Using this relation and the fact that the braid $\beta \in B_{n+2}$ commutes with σ_{n+1} (because β is represented by a word in the generators $\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}$), we obtain the following chain of equalities:

$$\begin{aligned}
 \alpha_t &= \sigma_{n+1}^{-t-1} \beta \sigma_{n-1}^{-2} \sigma_n \sigma_{n+1} \sigma_{n-1} \sigma_n \sigma_{n+1}^{t+1} \\
 &= \beta \sigma_{n-1}^{-2} \sigma_{n+1}^{-t-1} \sigma_n \sigma_{n+1} \sigma_{n-1} \sigma_n \sigma_{n+1}^{t+1} \\
 &= \beta \sigma_{n-1}^{-2} \sigma_n \sigma_{n+1} \sigma_n^{-t-1} \sigma_{n-1} \sigma_n \sigma_{n+1}^{t+1} \\
 &= \beta \sigma_{n-1}^{-2} \sigma_n \sigma_{n+1} \sigma_{n-1} \sigma_n \sigma_{n-1}^{-t-1} \sigma_{n+1}^{t+1} \\
 &= \beta \sigma_{n-1}^{-2} \sigma_n \sigma_{n-1} \sigma_{n+1} \sigma_n \sigma_{n+1}^{t+1} \sigma_{n-1}^{-t-1} \\
 &= \beta \sigma_{n-1}^{-2} \sigma_n \sigma_{n-1} \sigma_n^{t+1} \sigma_{n+1} \sigma_n \sigma_{n-1}^{-t-1} \\
 &= \beta \sigma_{n-1}^{t-1} \sigma_n \sigma_{n-1} \sigma_{n+1} \sigma_n \sigma_{n-1}^{-t-1}.
 \end{aligned}$$

Furthermore, we observe that the braid β_t is obtained from α_t via successive use of conjugation (in B_{n+2}), destabilization (from B_{n+2} to B_{n+1}), and conjugation (in B_{n+1}):

$$\begin{aligned}
 \alpha_t &= \beta \sigma_{n-1}^{t-1} \sigma_n \sigma_{n-1} \sigma_{n+1} \sigma_n \sigma_{n-1}^{-t-1} \\
 &\quad \downarrow \text{conjugation} \\
 \sigma_n \sigma_{n-1}^{-t-1} \beta \sigma_{n-1}^{t-1} \sigma_n \sigma_{n-1} \sigma_{n+1} & \\
 &\quad \downarrow \text{destabilization} \\
 \sigma_n \sigma_{n-1}^{-t-1} \beta \sigma_{n-1}^{t-1} \sigma_n \sigma_{n-1} & \\
 &\quad \downarrow \text{conjugation} \\
 \beta \sigma_{n-1}^{t-1} \sigma_n \sigma_{n-1} \sigma_n \sigma_{n-1}^{-t-1} &= \beta \sigma_{n-1}^{t-1} \sigma_{n-1} \sigma_n \sigma_{n-1} \sigma_{n-1}^{-t-1} = \beta \sigma_{n-1}^t \sigma_n \sigma_{n-1}^{-t} \stackrel{\text{def}}{=} \beta_t.
 \end{aligned}$$

Now we recall that $\alpha_t \in \widehat{\mathfrak{s}_{2+}(\beta)}$. Therefore, $\widehat{\beta}_t \in \mathfrak{D}(\widehat{\mathfrak{s}_{2+}(\beta)})$. \square

The formulation of the following proposition uses the notation $\beta(i) \stackrel{\text{def}}{=} \mathcal{S}(\beta)(i)$, which was introduced in Subsection 1.1.

Proposition 2.2. *Let β be a braid in B_n . Suppose that β represents a knot. Furthermore, suppose that for a certain $k \in \mathbb{Z}$ the following conditions are satisfied:*

$$\beta^k(n-1) = n \text{ and } \beta^k(n) \neq n-1.$$

Then the set $\{\widehat{\beta}_t\}_{t \in \mathbb{Z}}$, where β_t denotes the braid $\beta \sigma_{n-1}^t \sigma_n \sigma_{n-1}^{-t}$ in B_{n+1} , is infinite.

Remark. In Proposition 2.2, the assumption that β represents a knot is redundant, but it simplifies the proof a great deal.

Proof. Let r be the least nonnegative integer such that n divides $k-r$. In this proof, we show that the invariant \mathcal{X}_r takes arbitrarily large values on braids in the set $\{\beta_t\}_{t \in \mathbb{Z}}$. This immediately implies that the set $\{\widehat{\beta}_t\}_{t \in \mathbb{Z}}$ is infinite.

Claim 1. *For each $s \in \mathbb{Z}$, we have*

$$\beta_{2s}^r(n-1) = n \text{ and } \beta_{2s}^r(n+1) \neq n-1.$$

(Here, $\beta_{2s} \stackrel{\text{def}}{=} \beta \sigma_{n-1}^{2s} \sigma_n \sigma_{n-1}^{-2s} \in B_{n+1}$.)

Proof of Claim 1. Since β lies in B_n and represents a knot, it follows that β^n is a pure braid, whence $\beta^n(j) = j$ for each j . Then we have $\beta^r(j) = \beta^k(j)$ by the choice of r . In particular, we have

$$\beta^r(n-1) = \beta^k(n-1) = n \text{ and } \beta^r(n) = \beta^k(n) \neq n-1.$$

Since $\beta^r(n-1) = n \neq (n-1)$, it follows that $r \neq 0$, whence, by the choice of r , we have

$$0 < r < n.$$

Furthermore, since β represents a knot, it follows that the permutation $\mathcal{S}(\beta)$ is a cycle of length n . Now, consider the permutation $\mathcal{S}(\beta_{2s})$. We obviously have

$$\begin{aligned} \mathcal{S}(\beta_{2s}) &= \mathcal{S}(\beta) \circ \mathcal{S}(\sigma_{n-1}^{2s} \sigma_n \sigma_{n-1}^{-2s}) \\ &= \mathcal{S}(\beta) \circ (n-1, n)^{2s} \circ (n, n+1) \circ (n-1, n)^{2s} \\ &= \mathcal{S}(\beta) \circ (n, n+1). \end{aligned}$$

It follows that the permutation $\mathcal{S}(\beta_{2s})$ is a cycle of length $n+1$ and for each $1 \leq \ell \leq n$, we have $\beta_{2s}^\ell(n+1) = \beta^\ell(n)$. In particular, since $0 < r < n$, we have

$$\beta_{2s}^r(n+1) = \beta^r(n) \neq n-1.$$

We also easily see that for each $0 \leq \ell \leq r$ we have $\beta_{2s}^\ell(n-1) = \beta^\ell(n-1)$. In particular,

$$\beta_{2s}^r(n-1) = \beta^r(n-1) = n,$$

as required. □

By definition, we have

$$\beta_{2s} = \beta_0 \sigma_n^{-1} \sigma_{n-1}^{2s} \sigma_n \sigma_{n-1}^{-2s}.$$

Also, we observe that the permutation $\mathcal{S}(\sigma_n^{-1} \sigma_{n-1}^{2s} \sigma_n \sigma_{n-1}^{-2s})$ is trivial. By formula (2) of Definition 1.2, this yields

$$X_{i,j}(\beta_{2s}) = X_{i,j}(\beta_0) + X_{i,j}(\sigma_n^{-1} \sigma_{n-1}^{2s} \sigma_n \sigma_{n-1}^{-2s}).$$

This easily implies the following claim.

Claim 2. *For each $s \in \mathbb{Z}$, we have*

$$\begin{aligned} X_{n-1,n}(\beta_{2s}) &= X_{n-1,n}(\beta_0) - 2s, \\ X_{n-1,n+1}(\beta_{2s}) &= X_{n-1,n+1}(\beta_0) + 2s, \end{aligned}$$

and

$$X_{i,j}(\beta_{2s}) = X_{i,j}(\beta_0) \text{ if } \{i, j\} \notin \{\{n-1, n\}, \{n-1, n+1\}\}.$$

Furthermore, we estimate the difference

$$\mathcal{X}_r(\beta_{2s}) - \mathcal{X}_r(\beta_0).$$

We observe that the permutation $\mathcal{S}(\beta_{2s})$ does not depend on the choice of $s \in \mathbb{Z}$. Consequently, the following set R of ordered pairs of indices also does not depend on the choice of $s \in \mathbb{Z}$:

$$R := \{(i, j) \mid \beta_{2s}^r(i) = j\}.$$

In particular, we have

$$\{(i, j) \mid \beta_0^r(i) = j\} = R.$$

Thus, by the definitions of the invariant \mathcal{X}_r and of the set R , we have

$$\mathcal{X}_r(\beta_{2s}) = \sum_{(i,j) \in R} X_{i,j}(\beta_{2s}) \text{ and } \mathcal{X}_r(\beta_0) = \sum_{(i,j) \in R} X_{i,j}(\beta_0),$$

whence, by Claim 2, we obtain

$$\mathcal{X}_r(\beta_{2s}) - \mathcal{X}_r(\beta_0) = \sum_{(i,j) \in R \cap Q} (X_{i,j}(\beta_{2s}) - X_{i,j}(\beta_0)),$$

where

$$Q := \{(n-1, n), (n, n-1), (n-1, n+1), (n+1, n-1)\}.$$

By Claim 1, we have $\beta_{2s}^r(n-1) = n$, i.e.,

$$(n-1, n) \in R.$$

At the same time, the equality $\beta_{2s}^r(n-1) = n$ implies that $\beta_{2s}^r(n-1) \neq n+1$, i.e.,

$$(n-1, n+1) \notin R.$$

The second inequality $\beta_{2s}^r(n+1) \neq n-1$ of Claim 1 means that

$$(n+1, n-1) \notin R.$$

Let us remark that, using only the assumptions of Proposition 2.2, it is impossible to determine whether $(n, n-1) \in R$ or not. (We remark that $(n, n-1) \in R$ if and only if $2r = n+1$.)

We let $\lambda = 1$ if $(n, n-1) \in R$, and we let $\lambda = 0$ otherwise. Then, by the above, we have

$$\mathcal{X}_r(\beta_{2s}) - \mathcal{X}_r(\beta_0) = X_{n-1,n}(\beta_{2s}) - X_{n-1,n}(\beta_0) + \lambda(X_{n,n-1}(\beta_{2s}) - X_{n,n-1}(\beta_0)).$$

Combining this with the first two equalities of Claim 2, we obtain

$$\mathcal{X}_r(\beta_{2s}) - \mathcal{X}_r(\beta_0) = -2s + \lambda(-2s).$$

Thus, the number $\mathcal{X}_r(\beta_{2s}) - \mathcal{X}_r(\beta_0)$ is in the set $\{-2s, -4s\}$. In particular, the invariant \mathcal{X}_r takes arbitrarily large values on braids in the set $\{\beta_{2s}\}_{s \in \mathbb{Z}} \subset \{\beta_t\}_{t \in \mathbb{Z}}$. This means that the set $\{\widehat{\beta}_t\}_{t \in \mathbb{Z}}$ is infinite. □

§3. PROOFS OF THEOREMS 1 AND 3

Theorem 1. *Suppose that a braid β of index $n \geq 3$ represents a knot (i.e., a one-component link). Then the set $\mathfrak{S}(\widehat{\beta})$ is infinite.*

Proof. Since β represents a knot, it follows that the corresponding permutation $\mathcal{S}(\beta)$ in S_n is a cycle of length n . All such cycles in S_n are obviously conjugate. Furthermore, the homomorphism $\mathcal{S} : B_n \rightarrow S_n$ is an epimorphism. Consequently, there exists a braid $\gamma \in \widehat{\beta}$ such that

$$\mathcal{S}(\gamma) = (1, 2, \dots, n).$$

Then $\gamma(n-1) = n$ and $\gamma(n) = 1$. Proposition 2.2 implies (take $k = 1$) that the set $\{\widehat{\gamma}_t\}_{t \in \mathbb{Z}}$, where $\widehat{\gamma}_t$ is the braid $\gamma \sigma_{n-1}^t \sigma_n \sigma_{n-1}^{-t}$ in B_{n+1} , is infinite. By Proposition 2.1, this means that the set $\mathfrak{S}(\widehat{\gamma})$ is infinite. To complete the proof, we observe that $\widehat{\gamma} = \widehat{\beta}$ and so $\mathfrak{S}(\widehat{\gamma}) = \mathfrak{S}(\widehat{\beta})$. □

Theorem 3. *Suppose that a closed braid $\widehat{\beta}$ of odd index represents a knot and admits double destabilization. Then the set $\mathfrak{D}(\widehat{\beta})$ is infinite.*

Proof. We denote by m the index of $\widehat{\beta}$. By the definition of double destabilization, the closed braids of indices 1, 2, and 3 are not doubly destabilizable. Consequently, since m is odd by the assumption, we have $m \geq 5$.

1) Assume at first that $\widehat{\beta}$ contains a *positively* doubly destabilizable braid β' . Let $\alpha \in B_n$, where $n := m - 2$, be the result of positive double destabilization of β' , i.e., $\beta' = \mathfrak{s}_{2+}(\alpha)$. Then, by the definition of double destabilization, α represents the same

link type as $\widehat{\beta}$ does (i.e., $L(\alpha) = L(\widehat{\beta})$). This means that α represents a knot and that the corresponding permutation $\mathcal{S}(\alpha) \in S_n$ is a cycle of length n . Consequently, there exists an integer k such that $\alpha^k(n-1) = n$. Since the length of the cycle $\mathcal{S}(\alpha)$ is odd ($n = m - 2$), the equality $\alpha^k(n-1) = n$ implies the inequality $\alpha^k(n) \neq n-1$. Then, by Proposition 2.2, the set $\{\widehat{\alpha}_t\}_{t \in \mathbb{Z}}$, where α_t denotes the braid $\alpha \sigma_{n-1}^t \sigma_n \sigma_{n-1}^{-t}$ in B_{n+1} , is infinite, and Proposition 2.1 implies that the set $\mathfrak{D}(\widehat{\mathfrak{s}}_{2+}(\alpha))$ is also infinite. To complete the proof in this case, we observe that $\mathfrak{s}_{2+}(\alpha) = \beta'$ and so $\mathfrak{D}(\widehat{\mathfrak{s}}_{2+}(\alpha)) = \mathfrak{D}(\widehat{\beta})$.

2) If $\widehat{\beta}$ is not positively doubly destabilizable (hence, it is negatively doubly destabilizable), our assertion follows from the above by the “symmetry” argument. Namely: we recall that the braid group B_n has an automorphism T_n determined by the following condition: T_n takes the generator σ_i to its inverse σ_i^{-1} for each $i \in \{1, \dots, n-1\}$. (The braid $T_n(b)$ is the “mirror image” of b .) For a closed n -braid \widehat{b} , the image $T_n(\widehat{b})$ is also a closed braid, because every group automorphism maps a conjugacy class to a conjugacy class. It immediately follows from definitions that for each closed n -braid \widehat{b} we have the equality $\mathfrak{D}(T_n(\widehat{b})) = T_{n-1}(\mathfrak{D}(\widehat{b}))$, and \widehat{b} is negatively doubly destabilizable if and only if $T_n(\widehat{b})$ is positively doubly destabilizable.

Thus, negative double destabilizability of $\widehat{\beta}$ implies positive double destabilizability of $T_n(\widehat{\beta})$. Then, by the proof in case 1), the set $\mathfrak{D}(T_n(\widehat{\beta}))$ is infinite. Since $\mathfrak{D}(T_n(\widehat{\beta})) = T_{n-1}(\mathfrak{D}(\widehat{\beta}))$, it follows that the set $\mathfrak{D}(\widehat{\beta})$ is also infinite. \square

REFERENCES

- [1] J. W. Alexander, *A lemma on system of knotted curves*, Proc. Nat. Acad. Sci. U.S.A. **9** (1923), 93–95.
- [2] E. Artin, *Theorie der Zöpfe*, Abh. Math. Sem. Univ. Hamburg **4** (1925), 47–72.
- [3] J. S. Birman, *Braids, links, and mapping class groups*, Ann. of Math. Stud., No. 82, Princeton Univ. Press, Princeton, NJ, 1974. MR0375281 (51:11477)
- [4] A. A. Markov, *Über die freie Äquivalenz der geschlossenen Zöpfe*, Mat. Sb. (N.S.) **1** (43) (1936), no. 1, 73–78. (German)

ST. PETERSBURG BRANCH, STEKLOV MATHEMATICAL INSTITUTE, RUSSIAN ACADEMY OF SCIENCES,
 FONTANKA 27, ST. PETERSBURG 191023, RUSSIA
E-mail address: malyutin@pdmi.ras.ru

Received 12/JUN/2006

Translated by THE AUTHOR