NORMALITY IN GROUP RINGS

V. A. BOVDI and S. SICILIANO

Dedicated to Professor P. M. Gudivok on the occasion of his 70th birthday

ABSTRACT. Let $KG$ be the group ring of a group $G$ over a commutative ring $K$ with unity. The rings $KG$ are described for which $xx^\sigma = x^\sigma x$ for all $x = \sum_{g \in G} \alpha_g g \in KG$, where $x \mapsto x^\sigma = \sum_{g \in G} \alpha_g f(g)\sigma(g)$ is an involution of $KG$; here $f : G \to U(K)$ is a homomorphism and $\sigma$ is an antiautomorphism of order two of $G$.

Let $R$ be a ring with unity. We denote by $U(R)$ the group of units of $R$. A (bijective) map $\sigma : R \to R$ is called an involution if for all $a, b \in R$ we have $(a + b)^\sigma = a^\sigma + b^\sigma$, $(ab)^\sigma = b^\sigma \cdot a^\sigma$ and $a^\sigma^2 = a$. Let $KG$ be the group ring of a group $G$ over a commutative ring $K$ with unity, let $\sigma$ be an antiautomorphism of order two of $G$, and let $f : G \to U(K)$ be a homomorphism from $G$ onto $U(K)$. For an element $x = \sum_{g \in G} \alpha_g g \in KG$, we define $x^\sigma = \sum_{g \in G} \alpha_g f(g)\sigma(g) \in KG$. Clearly, $x \mapsto x^\sigma$ is an involution of $KG$ if and only if $g\sigma(g) \in \text{Ker } f = \{ h \in G \mid f(h) = 1 \}$ for all $g \in G$.

The ring $KG$ is said to be $\sigma$-normal if

$$xx^\sigma = x^\sigma x$$

for each $x \in KG$. The properties of the classical involution $x \mapsto x^*$ (where $* : g \mapsto g^{-1}$ for $g \in G$) and the properties of normal group rings (i.e., $xx^* = x^*x$ for each $x \in KG$) have been used actively for the investigation of the group of units $U(KG)$ of the group ring $KG$ (see [1, 2]). Moreover, they also have important applications in topology (see [7, 8]). Our aim is to describe the structure of the $\sigma$-normal group ring $KG$ for an arbitrary order 2 antiautomorphism $\sigma$ of the group $G$. Note that descriptions of the classical normal group rings and the twisted group rings were obtained in [1, 3] and [4, 5], respectively.

The notation used throughout the paper is essentially standard. $C_n$ denotes the cyclic group of order $n$; $\zeta(G)$ and $C_G(H)$ are the center of the group $G$ and the centralizer of $H$ in $G$, respectively; $(g, h) = g^{-1}h^{-1}gh = g^{-1}g^h$ ($g, h \in G$); $\gamma_i(G)$ is the $i$th term of the lower central series of $G$, i.e., $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for $i \geq 1$; $\Phi(G)$ denotes the Frattini subgroup of $G$. We say that $G = A \ast B$ is a central product of its subgroups $A$ and $B$ if $A$ and $B$ commute elementwise and, taken together, they generate $G$, provided that $A \cap B$ is a subgroup of $\zeta(G)$.

A non-Abelian 2-generated nilpotent group $G = \langle a, b \rangle$ with an antiautomorphism $\sigma$ of order 2 is called a $\sigma$-group if $G'$ has order 2, $\sigma(a) = a(a, b)$, and $\sigma(b) = b(a, b)$.

Our main result reads as follows.

Theorem. Let $KG$ be the noncommutative group ring of a group $G$ over a commutative ring $K$ and $f : G \to U(K)$ a homomorphism. Assume that $\sigma$ is an antiautomorphism of order two of $G$ such that $x \mapsto x^\sigma$ is an involution of $KG$. Put $\mathfrak{R}(G) = \{ g \in G \mid \sigma(g) = g \}$.

2000 Mathematics Subject Classification. Primary 16S34.

Key words and phrases. Group ring, normality.

This research was supported by OTKA no. T 037202 and no. T 038059.

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The group ring $KG$ is $\sigma$-normal if and only if $f : G \to \{\pm 1\}$, $G$, $K$, and $\sigma$ satisfy one of the following conditions:

(i) $G$ has an Abelian subgroup $H$ of index 2 such that $G = \langle H, b \rangle$, $f(b) = -1$, $f(h) = 1$, $\sigma(b) = b$, and $\sigma(h) = b^{-1}hb = hbb^{-1}$ for all $h \in H$;

(ii) $G = H \times \mathfrak{C}$ is a central product of a $\sigma$-group $H = \langle a, b \rangle$ and an Abelian group $\mathfrak{C}$ such that $G' = \langle c \mid c^2 = 1 \rangle$ and $H \subset \text{Ker}(f)$. Moreover, either $\sigma(d) = d$ for all $d \in \mathfrak{C}$, $\mathfrak{C} \subset \text{Ker}(f)$, $\mathfrak{K}(G) = \zeta(G)$, and

$$G/\mathfrak{K}(G) = \langle a\zeta(G), b\zeta(G) \rangle \cong C_2 \times C_2,$$

or $\mathfrak{K}(G)$ is of index 2 in $\zeta(G)$ and

$$G/\mathfrak{K}(G) = \langle g\mathfrak{K}(G), h\mathfrak{K}(G), d\mathfrak{K}(G) \rangle \cong C_2 \times C_2 \times C_2,$$

where $d \in \mathfrak{C}$, $\sigma(d) = dc$, and $f(d) = -1$;

(iii) $\text{char}(K) = 2$, $G = SY\mathfrak{C}$ is a central product of $S = \prod_{i=1}^n H_i$ and an Abelian group $\mathfrak{C}$ such that $H_i = \langle a_i, b_i \rangle$ is a $\sigma$-group and $G = \text{Ker}(f)$. Moreover, $G' = \langle c \mid c^2 = 1 \rangle$, $n \geq 2$, where $n$ is not necessarily a finite number, $\sigma(a_i) = a_ic$, $\sigma(b_i) = b_ic$ for all $i = 1, 2, \ldots$, and $\exp(G/\mathfrak{K}(G)) = 2$.

Furthermore, if $n$ is finite, then either $\sigma(d) = d$ for all $d \in \mathfrak{C}$ and

$$G/\mathfrak{K}(G) = \prod_{i=1}^n \langle a_i\zeta(G), b_i\zeta(G) \rangle \cong \prod_{i=1}^{2n} C_2,$$

or $\mathfrak{K}(G)$ is of index 2 in $\zeta(G)$ and

$$G/\mathfrak{K}(G) = \prod_{i=1}^n \langle a_i\mathfrak{K}(G), b_i\mathfrak{K}(G), d\mathfrak{K}(G) \rangle \times \langle d\mathfrak{K}(G) \rangle \cong \prod_{i=1}^{2n+1} C_2,$$

where $d \in \mathfrak{C}$ and $\sigma(d) = dc$.

Note that, in parts (i) and (ii) of the theorem, the group $\mathfrak{C}$ may be equal to 1.

To make the statements less cumbersome, in what follows we shall often talk of $\sigma$-normal group rings $KG$ without specifying the homomorphism $f : G \to U(K)$ and the antiautomorphism $\sigma$ of order two of $G$. In order to prove the main theorem, we need some preliminary lemmas.

**Lemma 1.** Let $U(R)$ be the group of units of the ring $R$, and let $x \mapsto x^\circ$ be an involution of $R$. Suppose that $xx^\circ = x^\circ x$ for all $x \in R$. If $a \in U(R)$, then $a^\circ = ta$, at $= ta$, and $t^\circ = t^{-1}$, where $t \in U(R)$.

**Proof.** Clearly $a^\circ = at$ for some $t \in U(R)$, and $a^\circ a = aa^\circ$ implies that $ata = a^2t$ and $at = ta$. Now $a = a^\circ = (at)^\circ = t^\circ at = t^\circ ta$, whence $t^\circ = t^{-1}$. $\square$

**Lemma 2.** Let $K$ be a commutative ring, let $H = \langle a, b \rangle$ be a non-Abelian 2-generated subgroup of a group $G$, and let $f : G \to U(K)$ be a homomorphism. If the group ring $KG$ is $\sigma$-normal, then $f : H \to \{\pm 1\}$ and one of the following conditions is fulfilled:

(i) $f(a) = 1$, $f(b) = -1$, $\sigma(a) = (a, b)a$, $\sigma(b) = b$, $(b^2, a) = 1$, $(ab)^2 = (ba)^2$, $(a, b, a) = 1$, and $((a, b), b) = (a, b)^{-2}$;

(ii) $f(a) = -1$, $f(b) = 1$, $\sigma(a) = a$, $\sigma(b) = (a, b)b$, $(b^2, a) = 1$, $(ab)^2 = (ba)^2$, $(a, b, b) = 1$, and $((a, b), a) = (a, b)^{-2}$;

(iii) $f(a) = f(b) = -1$, $\sigma(a) = a$, $\sigma(b) = b$, $(a^2, b) = (a, b^2) = 1$, $(ab)^2 = (ba)^2$, and $((a, b), ab) = 1$;

(iv) $f(a) = f(b) = 1$, $\sigma(a) = (a, b)a$, $\sigma(b) = (a, b)b$, and $\langle a, b \rangle$ is nilpotent of class 2 and such that $\gamma_2(\langle a, b \rangle)$ is of order 2.
Proof. Let $KG$ be a $\sigma$-normal ring. For any noncommutative $a, b \in G$ we can put $\sigma(a) = at$ and $\sigma(b) = bs$, where $s, t \in G$. By Lemma 1, $at = ta, bs = sb, \sigma(t) = t^{-1}$, and $\sigma(s) = s^{-1}$. Set $x = a + b \in KG$. Clearly, $x^\sigma = f(a)\sigma(a) + f(b)\sigma(b)$, and by (1) we have

$$(2) \quad f(b)a\sigma(b) + f(a)b\sigma(a) = f(a)\sigma(a)b + f(b)\sigma(b)a.$$ 

If $a\sigma(b) = b\sigma(a) = \sigma(a)b = \sigma(b)a$, then we get $s = t$ and $ab = ba$, a contradiction. Observe that if three of the elements $\{a\sigma(b), b\sigma(a), \sigma(a)b, \sigma(b)a\}$ coincide, then $s = t$ and $ab = ba$, a contradiction. We consider the following cases.

1. $a\sigma(b) = b\sigma(a)$. By (2), it follows that

$$(3) \quad f(a) + f(b) = 0, \quad a\sigma(b) = b\sigma(a), \quad \sigma(a)b = b\sigma(a).$$

2. $a\sigma(b) = \sigma(a)b$. This yields $ab = atb$, so that $s = t \in \zeta(H)$, and (2) ensures

$$(4) \quad f(a) = f(b), \quad \sigma(a) = at, \quad \sigma(b) = bt, \quad t \in \zeta(H).$$

3. $a\sigma(b) = \sigma(b)a$. Since $b\sigma(b) = \sigma(b)b$, we get $\sigma(b) \in \zeta(H)$, a contradiction. Now put $x = a(1 + b)$. Then $x^\sigma = (1 + f(b)\sigma(b))f(a)\sigma(a)$ and, by (1),

$$(5) \quad f(ab)a\sigma(ab) + f(a)b\sigma(ab) = f(a)\sigma(a)ab + f(ab)\sigma(ab)a.$$ 

We shall treat the following cases separately.

1. $a\sigma(ab) = ab\sigma(a)$. Formula (5) implies that

$$(6) \quad f(b) = -1, \quad \sigma(b) = ab^{-1}, \quad (\sigma(a)a) \cdot b = b \cdot (\sigma(a)a).$$

2. $a\sigma(ab) = \sigma(a)ab$ and $ab\sigma(a) = \sigma(ab)a$. By (1) we have $ab = \sigma(b)a$ and $\sigma(b) = aba^{-1}$. Since $a\sigma(ab) = \sigma(ab)a$, we get $aba^{-1}at = atb$ and $(b, t) = 1$. Recall that $\sigma(b) = bs = sb$. So, by (5),

$$(7) \quad f(b) = 1, \quad \sigma(b) = aba^{-1}, \quad t \in \zeta(H), \quad s = (a^{-1}b^{-1}) = (b, a^{-1}).$$

3. $a\sigma(b)a = \sigma(b)a\sigma(a)$. Then $a\sigma(b) = \sigma(b)a$ and $\sigma(b) \in \zeta(H)$, a contradiction. Assume that (3) and (6) are true. Then $f(b) = -1, f(a) = 1, \sigma(b) = b, \sigma(a) = b^{-1}ab = bab^{-1}$, whence $(b^2, a) = 1$. Since $\sigma(a) = a^{-1}b^{-1}ab = (bab^{-1}a^{-1})a$, we get $a^{-1}b^{-1}ab = bab^{-1}a^{-1}$ and $(ab)^2 = (ba)^2$. Obviously,$$

b^{-1}(a, b)b = b^{-1}(bab^{-1}a^{-1})b = ab^{-1}a^{-1}b = aba^{-1}b^{-1} = (a, b)^{-1},

so that $((a, b), b) = (a, b)^{-2}$, and statement (i) of our lemma follows.

If (3) and (7) are fulfilled, then $f(b) = 1, f(a) = -1, \sigma(a) = a, \sigma(b) = a^{-1}ba$, and $(a^2, b) = 1$. Since $\sigma(b) = b(b^{-1}a^{-1}ba) = (aba^{-1}b^{-1})b, we obtain s = b^{-1}a^{-1}ba = aba^{-1}b^{-1}b^{-1}$ and $(ab)^2 = (ba)^2$. Therefore, $\sigma(a) = a, \sigma(b) = a^{-1}ba, (a^2, b) = 1, and we arrive at statement (iii).

Assume (4) and (6). Then $f(a) = f(b) = -1, \sigma(a) = a, \sigma(b) = b, and (2a, b) = 1$. Moreover, $f(ab) = 1 and \sigma(ab) = ba = a^{-1}(ab)a$. We put $x = b(1 + a)$. Clearly, $x^\sigma = a^{-1}b)$, and (1) implies $b^2a + b^2a^2 = ab^2a + ab^2$, whence $(b^2, a) = 1$. Thus, statement (iii) of our lemma is fulfilled.

Finally, if (4) and (7) are true, then $f(a) = f(b) = 1 and (b, a^{-1}) \in \zeta(H)$. Using the identity $(a^\beta, \gamma) = (a, \gamma)(a, \gamma)(\beta, \gamma), where \alpha, \beta, \gamma \in G, we see that 1 = (a^{-1}a, b) = (a^{-1}b)(a, b), whence s = (b, a^{-1}) = (a, b) \in \zeta(H) and \sigma(a) = (a, b)a, \sigma(b) = (a, b)b$. Since $a = \sigma^2(a) and (a, b) \in \zeta(H), we have (a, b)^2 = 1, which yields statement (iv). The proof is complete. 

□

Lemma 3. Let $KG$ be a $\sigma$-normal group ring of a non-Abelian group $G$. Then $H = \langle w \in G \mid \sigma(w) \neq w \rangle$ is a normal subgroup in $G$. If $H$ is Abelian, then $G$ satisfies statement (i) of the theorem.
Lemma 4. Let $KG$ be a $\sigma$-normal group ring, let $W = \{w \in G \mid \sigma(w) \neq w\}$, and let $a, b \in W$ be such that $(a, b) \neq 1$. Put $\mathcal{R} = \{g \in G \mid \sigma(g) = g\}$ and $\mathcal{C} = C_G(\langle a, b \rangle)$. Then

$$
\langle a, b \rangle \text{ is a } \sigma\text{-group, } \Phi(\langle a, b \rangle) = \zeta(\langle a, b \rangle) = \{g \in \langle a, b \rangle \mid \sigma(g) = g\}, \text{ and }
$$

$$
\sigma(g) = \begin{cases} 
g & \text{if } g \in \zeta(\langle a, b \rangle), 
g(a, b) & \text{if } g \not\in \zeta(\langle a, b \rangle). 
\end{cases}
$$

Moreover, $G = \langle a, b \rangle \mathcal{Y} \mathcal{C}$, and either $\sigma(c) = (a, b)c$, or $\sigma(c) = c$, where $c \in \mathcal{C}$. Also, the following is true:

(i) if $\mathcal{C}$ is Abelian, then $G$ satisfies statement (ii) of the theorem;
(ii) if $\mathcal{C}$ is not Abelian, then $\text{char}(K) = 2$.

Proof. Let $a, b \in W$ satisfy $(a, b) \neq 1$. By Lemma 2, $f(a) = f(b) = 1$, $\langle a, b \rangle$ is nilpotent of class 2 and such that $|\gamma_2(\langle a, b \rangle)| = 2$, and $\sigma(a) = b^{-1}ab$, $\sigma(b) = a^{-1}ba$. Thus $\langle a, b \rangle$ is a $\sigma$-group. Any element $g \in \langle a, b \rangle$ can be written as $g = a^ib^j(a, b)^k$, where $i, j, k \in \mathbb{N}$. 

Proof. Set $W = \{w \in G \mid \sigma(w) \neq w\}$. Let $g \notin W$ be such that $g^2 \notin \zeta(G)$. Then $(g^2, h) \neq 1$ and $(g, h) \neq 1$, respectively, for some $h \in G$.

We consider the following cases.

1. $\text{char}(K) \neq 2$. Since $\sigma(g^2) = g^2$, we can use Lemma 2 for the group $\langle g^2, h \rangle$ to show that $-1 = f(g^2) = (\pm 1)^2 = 1$, a contradiction.

2. $\text{char}(K) = 2$. Using Lemma 2 for $(g, h)$, we get $(g^2, h) = 1$, again a contradiction.

Thus, $g^2 \in \zeta(G)$ for any $g \notin W$. Now, if $w \in W$, $g \in G \setminus W$, and $g^{-1}wg \notin W$, then $\sigma(g^{-1}wg) = g^{-1}wg$ and

$$
g^{-1}wg = \sigma(g^{-1}wg) = g\sigma(w)g^{-2}g = g^{-1}\sigma(w)g
$$

so that $\sigma(w) = w$, a contradiction. Therefore, $g^{-1}wg \in W$ and the subgroup $H = \langle W \rangle$ is normal in $G$.

Suppose that $H = \langle W \rangle$ is Abelian. If $a \in W$ and $c \in C_G(W) \setminus H$, then $ca \notin H$. Therefore, $ca = \sigma(ca) = \sigma(a)c$, whence $\sigma(a) = a$, a contradiction. This shows that $C_G(W) = H$ and for each $b \notin H$ there exists $w \in W$ such that $(b, w) \neq 1$.

We claim that if $b_1, b_2 \in G \setminus H$, then $b_1b_2 \in H$. The following cases will be treated separately:

1. $\text{char}(K) \neq 2$ and $b_1b_2 \in G \setminus H$. For each $b_i$ we choose $w_i \in W$ such that $(b_i, w_i) \neq 1$. By (i) or (ii) of Lemma 2, in $(w_i, b_i)$ we have $f(b_i) = -1$, so that $f(b_1b_2) = 1$ and there exists $w \in W$ for which $(b_1b_2, w) \neq 1$. Since $\sigma(b_1b_2) = b_1b_2$, by (i) or (ii) of Lemma 2 we get $f(b_1b_2) = -1$, a contradiction.

2. $\text{char}(K) = 2$ and $b_1b_2 \in G \setminus H$. Obviously, $b_1b_2 = \sigma(b_1b_2) = b_2b_1$, whence $(b_1, b_2) = 1$. Now, there is $w \in W$ with $(w, b_1) \neq 1$, and by Lemma 2 we get $\sigma(w) = b_1^{-1}wb_1 = b_1wb_1^{-1}$. Furthermore, $b_1b_2w \in G \setminus H$ and

$$
b_1b_2w = \sigma(b_1b_2w) = \sigma(w)b_1b_2 = b_1wb_2,
$$

implying $(b_2, w) = 1$. Now $(b_1b_2w) = (b_1, w) \neq 1$ and $b_2w \in G \setminus H$; applying Lemma 2 in $(b_1, b_2w)$, we obtain $b_2w = \sigma(b_2w) = \sigma(w)b_2$ and $\sigma(w) = w$, a contradiction.

We have proved that $b_1b_2 \in H$ for every $b_1, b_2 \in G \setminus H$. Hence, $G = \langle H, b \mid b \notin H, b^2 \in H \rangle$, $f(b) = -1$, and $f(h) = 1$ for all $h \in H$.

Finally, let $w \in W$ be such that $(b, w) = 1$. Since $b \notin H = C_G(W)$, there exists $w_1 \in W$ with $w_1 \neq b^{-1}w_1b = \sigma(w_1)$. Clearly, we have $(ww_1, b) \neq 1$; using Lemma 2 for $\langle ww_1, b \rangle$, we obtain

$$
\sigma(w)\sigma(w_1) = \sigma(w_1w) = b^{-1}w_1wb = \sigma(w_1)w,
$$

whence $\sigma(w) = w$, a contradiction. Thus, $b^{-1}hb = \sigma(h)$ for all $h \in H$. \qed
Since \( \sigma(g) = g \), we conclude that \( i \) and \( j \) are even. Now by [6, Theorems 10.4.1 and 10.4.3] we obtain

\[
\Phi(\langle a, b \rangle) = \zeta(\langle a, b \rangle) = \{ g \in \langle a, b \rangle \mid \sigma(g) = g \}.
\]

Suppose \( c \in W \) and \( (a, c) \neq 1 \). Again by Lemma 2, \( \langle a, c \rangle \) is nilpotent of class 2 and \( \sigma(a) = c^{-1}ac = b^{-1}ab \), so that \( (a, b) = (a, c) \). Now, let \( c, d \in W \) be such that \( (c, d) \neq 1 \) and \( \langle c, d \rangle \in \mathfrak{C} \). Obviously, \( (ac, b) = (a, b) \neq 1 \) and \( (ac, d) = (c, d) \neq 1 \). By Lemma 2,

\[
\sigma(ac) = b^{-1}(ac)b = d^{-1}(ac)d \quad \text{and} \quad (a, b) = (c, d),
\]

which shows that \( H' \) has order two and is central in \( G \).

Let \( g \in G \setminus \mathfrak{C} \cdot \langle a, b \rangle \). If \( (a, g) \neq 1 \), then, using Lemma 2 for \( \langle a, g \rangle \), we get \( \sigma(a) = g^{-1}ag = b^{-1}ab \) and \( (a, g) = (a, b) \). Similarly, if \( (b, g) \neq 1 \), then \( (b, g) = (a, b) \).

The following cases are possible:

1. \( (g, a) = 1 \) and \( (g, b) \neq 1 \). Then we have \( (ga, b) = (ga, a) = 1 \), which implies that \( g = (ga) \cdot a^{-1} \in \mathfrak{C} \cdot \langle a, b \rangle \).

2. \( (g, a) \neq 1 \) and \( (g, b) = 1 \). Then we have \( (gb, a) = (gb, b) = 1 \), which implies that \( g = (gb) \cdot b^{-1} \in \mathfrak{C} \cdot \langle a, b \rangle \).

3. \( (g, a) \neq 1 \) and \( (g, b) \neq 1 \). Then we have \( (gab, b) = (gab, a) = 1 \), which implies that \( g = (gab) \cdot (ab)^{-1} \in \mathfrak{C} \cdot \langle a, b \rangle \).

Since each of these cases leads to a contradiction, we have \( G = \mathfrak{C} Y \langle a, b \rangle \).

Let \( d \in \mathfrak{C} \setminus H \). Since \( \sigma(ad) = ad \), we get \( ad = \sigma(ad) = \sigma(a)d \), whence \( \sigma(a) = a \), a contradiction. Since \( G = \mathfrak{C} \cdot \langle a, b \rangle \), it follows that \( G = H = \langle W \rangle \). If \( d \in \zeta(G) \cap W \), then \( \sigma(ad) = ad \) and \( (ad, b) = (a, b) \neq 1 \); using Lemma 2 for \( \langle ad, b \rangle \), we obtain

\[
-1 = f(ad) = f(a)f(d) = f(d).
\]

Now, we let \( \zeta(G) \cap W = \emptyset \) and put \( x = ac + b \), where \( c \in \mathfrak{C} \). Then there exists \( d \in G \) such that \( (c, d) \neq 1 \), and Lemma 2 implies that \( f(g) = 1 \) for all \( g \in G \). Thus,

\[
x^\sigma = (a\sigma(c) + b)(a, b), \quad \text{and by (1) we have} \quad (\sigma(c) - c)(1 - (a, b)) = 0.
\]

It follows that either \( \sigma(c) = c \), or \( \text{char}(K) = 2 \) and \( \sigma(c) = (a, b)c \). Therefore, if \( \mathfrak{C} \) is Abelian, we obtain statement (ii) of the theorem.

Finally, assume that \( \text{char}(K) \neq 2 \). Suppose there exist \( c, d \in \mathfrak{C} \) such that \( (c, d) \neq 1 \). If \( \sigma(c) = c \), then \( f(c) = 1 \) by what has already been proved, but, by Lemma 2 in \( \langle c, d \rangle \), we have \( f(c) = -1 \), a contradiction. Therefore, \( c \in W \) and similarly \( d \in W \). We put \( x = ac + d \). Clearly, \( x^\sigma = ac + d(a, b) \) and \( (a, b) = 1 \) by (1), a contradiction. Thus, if \( \mathfrak{C} \) is not Abelian, then \( \text{char}(K) = 2 \), and the proof is complete.

Now we are in a position to prove our main theorem.

\[\text{Proof of the “if” part of the theorem. Set } W = \{ w \in G \mid \sigma(w) \neq w \} \text{ and } H = \langle W \rangle. \]

If \( H \) is Abelian, then, by Lemma 3, statement (i) of the theorem is valid for \( G \).

Suppose that \( H \) is non-Abelian and that \( a, b \in W \) satisfy \( (a, b) \neq 1 \). By Lemma 4, \( G = \langle a, b \rangle Y \mathfrak{C} = \langle W \rangle \), where \( \mathfrak{C} = C_G(\langle a, b \rangle) \). If \( \mathfrak{C} \) is Abelian, then statement (ii) of our theorem is valid for \( G \) by Lemma 4.

Let \( c, d \in C_G(\langle a, b \rangle) \) be such that \( (c, d) \neq 1 \) (i.e., \( \mathfrak{C} \) is non-Abelian). By Lemma 4, we have \( \text{char}(K) = 2 \). If \( c, d \in W \), then, by Lemma 4,

\[
G = C_G(\langle a, b \rangle) \cdot \langle a, b \rangle = C_G(\langle c, d \rangle) \cdot \langle c, d \rangle.
\]

Obviously, \( C_G(\langle a, b \rangle) \cap \langle a, b \rangle \subseteq \zeta(G) \). Therefore, \( G \) contains the subgroup \( H_2 = \langle a, b \rangle Y \langle c, d \rangle \), which cannot be a direct product because \( G' \) has order 2.

Since \( G' \subseteq \mathfrak{R}(G) \), we see that \( G/\mathfrak{R}(G) \) is an elementary Abelian 2-group. Let \( \tau : G \to G/\mathfrak{R}(G) = \bigtimes_{i \geq 1} \langle a_i \mid a_i^2 = 1 \rangle \) be such that \( \tau^{-1}(a_1) = a \) and \( \tau^{-1}(a_2) = b \). We put \( \zeta_i = \tau^{-1}(a_i) \) for all \( i \geq 3 \) and \( \mathfrak{B} = \{ a_i \mid i \geq 3 \} \).
Suppose that for some \( s \geq 3 \) we have \((\bar{\alpha}, \bar{\alpha}_i) = 1\) for all \( i \geq 3 \). Such an element is unique, because if \( \bar{\alpha}_i \neq \bar{\alpha}_s \) commutes with all \( \bar{\alpha}_s \), then \( \sigma(\bar{\alpha}_i \bar{\alpha}_s) = \bar{\alpha}_i \bar{\alpha}_s \), whence \( a_s a_t = 1 \), a contradiction. Put \( \mathfrak{B} = \mathfrak{B} \setminus \{a_s, b_0 = a_s, b_1 = a_1, b_2 = a_2\} \). Note that if such an element \( a_s \) does not exist, then we put \( b_0 = 1 \).

Choose \( a_i \in \mathfrak{B} \). There is \( a_j \in \mathfrak{B} \) such that \((\bar{\alpha}_j, \bar{\alpha}_i) \neq 1\), and we consider the following cases.

1. \( \bar{\alpha}_i, \bar{\alpha}_j \in W \). Put \( b_3 = a_i, b_4 = a_j \) and \( \mathfrak{B} = \mathfrak{B} \setminus \{a_i, a_j\} \).
2. \( \bar{\alpha}_i \in W \) and \( \bar{\alpha}_j \notin W \). Clearly, \( \langle \bar{\alpha}_1, \bar{\alpha}_2 \rangle Y \langle \bar{\alpha}_3, \bar{\alpha}_4 \rangle \cong \langle \bar{\alpha}_1 \bar{\alpha}_2, \bar{\alpha}_3 \bar{\alpha}_4 \rangle \) and \( \bar{\alpha}_1 \bar{\alpha}_2, \bar{\alpha}_3 \bar{\alpha}_4 \in W \). Put \( b_1 = \tau(\bar{\alpha}_1 \bar{\alpha}_2), b_2 = a_2, b_3 = a_4 = \tau(\bar{\alpha}_2 \bar{\alpha}_4) \), and \( \mathfrak{B} = \mathfrak{B} \cup \{a_1, a_4\} \setminus \{b_1, b_2, b_3, b_4\} \).
3. \( \bar{\alpha}_i, \bar{\alpha}_j \notin W \). Obviously, we have \( \bar{\alpha}_i \bar{\alpha}_j \in W \), so that this case reduces to the preceding one.

Furthermore, if \( C_G(\{\bar{\alpha}_1, \bar{\alpha}_2\}) \) contains a noncommuting pair of elements, then this pair can be chosen in \( W \). By continuing this process, we can conclude that \( G \) contains a subgroup \( \mathfrak{M} = A_1 Y A_2 Y \cdots \) that is a central product, where each \( A_i = \langle g_i, h_i \rangle \) is a \( \sigma \)-group and \( C_G(\mathfrak{M}) \) is Abelian. Applying Lemma 4, we arrive at statement (iii) of the theorem, and the proof is complete.

**Proof of the “only if” part of the theorem.** (i) We can write any \( x \in KG \) as \( x = x_1 + x_2 b \), where \( x_i \in KH \). Clearly, \( x^\sigma = x_1^\sigma + f(b)\sigma(b)x_2^\sigma = x_1^\sigma - x_2 b^2 \) and

\[
xx^\sigma = x_1 x_1^\sigma - x_2 x_2^\sigma b^2 = x_1 x_1 - x_2 x_2 b^2 = x^\sigma x,
\]

so that \( KG \) is a \( \sigma \)-normal ring.

(ii) Any \( x \in KH \) can be written as \( x = x_0 + x_1 g + x_2 h + x_3 gh \), where \( x_i \in K \langle g^2, h^2, c \rangle \) and \( c = (g, h) \). Clearly, \( x^\sigma = x_0 + (x_1 g + x_2 h + x_3 gh)c \) and \( xx^\sigma = x^\sigma x \), so that \( KH \) is \( \sigma \)-normal. Suppose that \( \sigma(d) = dc \), with \( c = (a, b) \). Any \( x \in KG \) can be written as \( x = (w_0 + u_1) + (w_2 + u_3)d \), where \( u_1 = \alpha_1 a + \alpha_2 b + \alpha_3 ab \), \( u_3 = \beta_1 a + \beta_2 b + \beta_3 ab \), and \( \alpha_i, \beta_i, w_0, w_2 \in K \mathfrak{R} \). Then \( x^\sigma = (w_0 + u_1 c) - (w_2 + u_3 c) dc \) and \( xx^\sigma - x^\sigma x = (u_3 u_1 - u_1 u_3)(1 + c)d \). Since \( ab - ba = ba(c - 1) \) and \( c^2 = 1 \), it follows that \( xx^\sigma - x^\sigma x = 0 \).

Thus, \( KG \) is \( \sigma \)-normal. In the case where \( \sigma(d) = d \), the proof is similar.

(iii) Put \( G_n = A_1 Y \cdots Y A_n \), where \( A_i = \langle a_i, b_i | c = (a_i, b_i) \rangle \) is a \( \sigma \)-subgroup. We use induction on \( n \). Any \( x \in KG_n \) can be written as \( x = x_0 + x_1 a_n + x_2 b_n + x_3 a_n b_n \), where \( x_i \in K \langle G_{n-1}, a_n^2, b_n^2 \rangle \). Obviously, \( x^\sigma = x_0^\sigma + (x_1^\sigma a_n + x_2^\sigma b_n + x_3^\sigma a_n b_n)c \). Since \( KG_{n-1} \) is \( \sigma \)-normal, we get \( x_i x_i^\sigma = x_i x_i^\sigma \) and \( x_i^\sigma (1 + c) = x_i (1 + c) \). The formula

\[
(x_i + x_j)(x_i + x_j)^\sigma = (x_i + x_j)^\sigma (x_i + x_j)
\]

shows that

\[
x_i x_j^\sigma + x_j x_i^\sigma = x_i x_j + x_j x_i.
\]

Proceeding as in the preceding case, we conclude that

\[
xx^\sigma = x^\sigma x,
\]

and the proof is complete.

**References**


Institute of Mathematics, University of Debrecen, P.O. Box 12, H-4010 Debrecen, Hungary
E-mail address: vbovdi@math.klte.hu
Current address: Institute of Mathematics and Informatics, College of Nyíregyháza, Sóstói út 31/b, H-4410 Nyíregyháza, Hungary

Dipartimento di Matematica “E. De Giorgi”, Università degli Studi di Lecce, Via Provinciale Lecce-Arnesano, 73100-LECCE, Italy
E-mail address: salvatore.siciliano@unile.it

Received 31/AUG/2006
Originally published in English