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SPECTRAL SUBSPACES OF L^p FOR $p < 1$

A. B. ALEKSANDROV

Abstract. Let Ω be an open subset of $\mathbb{R}^n$. Denote by $L^p_\Omega(\mathbb{R}^n)$ the closure in $L^p(\mathbb{R}^n)$ of the set of all functions $\varepsilon \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ whose Fourier transform has compact support contained in Ω. The subspaces of the form $L^p_\Omega(\mathbb{R}^n)$ are called the *spectral* subspaces of $L^p(\mathbb{R}^n)$. It is easily seen that each spectral subspace is translation invariant; i.e., $f(x+a) \in L^p_\Omega(\mathbb{R}^n)$ for all $f \in L^p_\Omega(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$. Sufficient conditions are given for the coincidence of $L^p_\Omega(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$. In particular, an example of a set Ω is constructed such that the above spaces coincide for sufficiently small p but not for all $p \in (0,1)$. Moreover, the boundedness of the functional $f \mapsto (\mathcal{F}f)(a)$ with $a \in \Omega$, which is defined initially for sufficiently "good" functions in $L^p_\Omega(\mathbb{R}^n)$, is investigated. In particular, estimates of the norm of this functional are obtained. Also, similar questions are considered for spectral subspaces of $L^p(G)$, where G is a locally compact Abelian group.

§1. Introduction

The translation invariant subspaces of $L^2(\mathbb{R}^n)$ can be described easily with the help of the Fourier transformation $\mathcal{F}$. In this paper, by a subspace we mean a closed subspace. Let X be a translation invariant subspace of $L^2(\mathbb{R}^n)$. Then it can be represented in the form $X = X^2_E(\mathbb{R}^n) \stackrel{\text{def}}{=} (\mathcal{F})^{-1}(\mathbb{1}_E \cdot L^2(\mathbb{R}^n))$, where E is a measurable subset of $\mathbb{R}^n$. Here and in what follows, $\mathbb{1}_E$ denotes the characteristic function of the set E. Clearly, $X^2_{E_1}(\mathbb{R}^n) = X^2_{E_2}(\mathbb{R}^n)$ if and only if E_1 coincides with E_2 up to a set of zero measure. It is easily seen that for every open set $\Omega \subset \mathbb{R}^n$ the space $X^2_\Omega(\mathbb{R}^n)$ coincides with the spectral space $L^2_\Omega(\mathbb{R}^n)$, which was defined above in the abstract. Thus, the space $X^2_E(\mathbb{R}^n)$ is spectral if and only if the set E is open up to a set of zero measure. Moreover, $X^2_E(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ if and only if the Lebesgue measure of the set $\mathbb{R}^n \setminus E$ equals zero. In particular, $L^2_\Omega(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ if and only if the Lebesgue measure of the set $\mathbb{R}^n \setminus \Omega$ equals zero. Note also that, obviously, $L^1_\Omega(\mathbb{R}^n) = L^1(\mathbb{R}^n)$ if and only if $\Omega = \mathbb{R}^n$. It is easy to show that if $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for some $p \in [1, +\infty)$, then Ω is dense in $\mathbb{R}^n$.

For $p < 1$ the situation is quite different. §5 is devoted to this phenomenon. There exist very rarefied open sets Ω satisfying $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. For example, Theorem 9.1 in the author's paper [1] (see also Theorem 5.1 below) implies that for $p < 1$ we have $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ whenever Ω is centrally symmetric and contains arbitrarily large balls. On the other hand, we shall see that there is an open set Ω with countable complement $\mathbb{R}^n \setminus \Omega$ satisfying $L^p_\Omega(\mathbb{R}^n) \neq L^p(\mathbb{R}^n)$ for $p < 1$. This is the case for $\Omega = \mathbb{R}^n \setminus \mathbb{Z}^n$, but arbitrarily small perturbations of the complement of Ω change the situation radically (see Theorem 6.13).

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Note that the space $L^p_{(0,+\infty)}(\mathbb{R})$ can be identified naturally with the classical Hardy space $H^p = H^p(\mathbb{C}_+)$, which consists of functions holomorphic in the upper half-plane $\mathbb{C}_+$. Similarly, the space $L^p_{(0,+\infty)^n}(\mathbb{R}^n)$ can be identified with the Hardy class $H^p((\mathbb{C}_+)^n)$. In fact, the Cartesian product $(0,+\infty)^n$ can be replaced with a nondegenerate (i.e., nonempty and containing no straight lines) open convex cone Ω with vertex at the origin. In this case, the space $L^p_\Omega(\mathbb{R}^n)$ identifies naturally with the Hardy class H^p in the tube domain $T_{\Omega^*} \stackrel{\text{def}}{=} \{z \in \mathbb{C}^n : \operatorname{Im} z \in \Omega^*\}$, where Ω^* is the interior of the cone adjoint to Ω. Concerning the Hardy classes in tube domains, we refer the reader to Chapter 3 of the Stein and Weiss monograph [29].

It is well known (see [12] and [7]) that the space $L^p_\Omega(\mathbb{R}^n)$, where Ω is a nondegenerate open convex cone, can be viewed as a space of distributions, and $\mathcal{F}f \in C(\mathbb{R}^n)$ for every $f \in L^p_\Omega(\mathbb{R}^n)$ provided $p \le 1$. This implies that the functional $f \mapsto (\mathcal{F}f)(a)$ is continuous on $L^p_\Omega(\mathbb{R}^n)$ for $p \le 1$. We denote by $\Phi^\Omega_p(a)$ the norm of this functional, i.e., $\Phi^\Omega_p(a) = \sup\{|(\mathcal{F}f)(a)| : f \in L^p_\Omega(\mathbb{R}^n), \|f\|_{L^p} \le 1\}$.

In §2 we define the functional Φ^Ω_p for every open $\Omega \subset \mathbb{R}^n$ and obtain lower estimates for it (Theorem 2.11). Sometimes, this estimate is right in the sense that the same estimate from above is true with another constant. In particular, it is right for all open convex sets Ω; see §3. Note also that if a convex set Ω contains no straight lines, then $\Phi^\Omega_p(a) < +\infty$ for all $a \in \Omega$ and $p \le 1$. But if a convex set Ω contains a straight line, then there are no continuous linear functionals on $L^p_\Omega(\mathbb{R}^n)$ for $p < 1$ (Theorem 3.9), so that $\Phi^\Omega_p(a) = +\infty$ for all $a \in \Omega$ and $p < 1$.

We shall see in §8 that Theorem 2.11 yields a correct lower estimate of Φ^Ω_p also for some nonconvex sets. In particular, so it is for the set $\Omega = \left\{\xi \in \mathbb{R}^n : \prod_{j=1}^n |\xi_j| < 1\right\}$. The principal method for obtaining an upper estimate for $\Phi^\Omega_p(a)$ is based mainly on Theorems 8.1 and 8.2. To obtain an upper estimate for $\Phi^\Omega_p(0)$ (since $\Phi^\Omega_p(a) = \Phi^{\Omega-a}_p(0)$, we may assume that $a = 0$), it suffices to construct a lattice Λ in $\mathbb{R}^n$ whose intersection with Ω is small in a sense (in particular, the lattices fit whose intersection with Ω consists only of the origin). Each such lattice Λ yields the upper estimate $\Phi^\Omega_p(0) \le \operatorname{const}(\det\Lambda)^{\frac{1}{p}-1}$. Thus, possibly, this matter is related to the geometry of numbers. In particular, for $\Omega = \left\{\xi \in \mathbb{R}^n : \prod_{j=1}^n |\xi_j| < 1\right\}$ we use "purely algebraic" methods.

In §7 we introduce the notion of a meager subspace. A translation invariant subspace X of $L^p(\mathbb{R}^n)$ is said to be *meager* if $X + Y$ is not dense in $L^p(\mathbb{R}^n)$ for any proper translation invariant subspace Y of $L^p(\mathbb{R}^n)$. We prove that for $p < 1$ the space $L^p_\Omega(\mathbb{R}^n)$ is meager in the case where Ω is a strip (the definition of a strip is given in §7). We shall see that the properties of the space $L^p_\Omega(\mathbb{R}^n)$ may change drastically when the strip Ω is "distorted". In particular, there exist "distorted" strips Ω such that $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

Note that the Lebesgue measure of a strip is infinite if $n \ge 2$. On the other hand, there exist open sets Ω of finite Lebesgue measure such that the space $L^p_\Omega(\mathbb{R}^n)$ is not meager.

The notion of a spectral subspace can also be introduced for locally compact Abelian groups. Let G be a locally compact Abelian group, and let Γ be the group of characters of G. As before, with every open subset Ω of Γ we can associate the spectral subspace $L^p_\Omega(G)$. Though this paper is mainly devoted to the case where $G = \mathbb{R}^n$, sometimes it is convenient and natural to go beyond the framework of $\mathbb{R}^n$. In particular, for the investigation of spectral subspaces of $L^p(\mathbb{R}^n)$, the treatment of spectral subspaces of $L^p(\mathbb{T}^n)$ can be useful, where $\mathbb{T}^n$ denotes the n-dimensional torus, i.e.,

$\mathbb{T}^n = \{z \in \mathbb{C}^n : |z_j| = 1 \quad \text{for } j = 1, 2 \ldots, n\}$ with coordinatewise multiplication. Sometimes it is convenient to identify this group with the additive group $\mathbb{R}^n/\mathbb{Z}^n$. The group of characters of $\mathbb{T}^n$ (and of $\mathbb{R}^n/\mathbb{Z}^n$) is identified with $\mathbb{Z}^n$.

It is well known that if the group G is compact and $p \in [1, +\infty)$, then every translation invariant subspace of $L^p(G)$ is spectral. For the one-dimensional torus $\mathbb{T}$ this follows from the classical properties of the Fejér kernel and the Cesàro means. In the general case, the Fejér kernel can be replaced with an arbitrary approximate identity consisting of trigonometric polynomials on G (see [26, Theorem 2.6.8]).

De Leeuw [8] observed that for $p < 1$ the intersection $L^p_{\mathbb{Z}_+}(\mathbb{T}) \cap L^p_{\mathbb{Z}_-}(\mathbb{T})$ of two spectral subspaces, where $\mathbb{Z}_+ \stackrel{\text{def}}{=} \{n \in \mathbb{Z} : n \geq 0\}$ and $\mathbb{Z}_- \stackrel{\text{def}}{=} \{n \in \mathbb{Z} : n < 0\}$, is a translation invariant subspace that is not spectral, because it contains nonzero functions but no nonzero trigonometric polynomials. Similar arguments show also that $L^p_{(0,+\infty)}(\mathbb{R}) \cap L^p_{(-\infty,0)}(\mathbb{R})$ is an example of a nonspectral translation invariant subspace of $L^p(\mathbb{R})$. Both examples are closely related to the Hardy classes in the disk and in the half-plane. The theory of the so-called real Hardy classes also yields examples of nonspectral translation invariant subspaces, but in this case sometimes we deal with spaces of vector-valued functions, in particular, with $\mathbb{R}^{n+1}$-valued functions.[1] For example, an important role is played by the space $\mathcal{H}^p_1(\mathbb{R}^n)$, which can be defined as the closure in the space $L^p(\mathbb{R}^n, \mathbb{R}^{n+1})$ of the set of all functions $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{n+1}) \in L^1(\mathbb{R}^n, \mathbb{R}^{n+1}) \cap L^p(\mathbb{R}^n, \mathbb{R}^{n+1})$ such that $R_j \varphi_{n+1} = \varphi_j$ for $j = 1, 2, \ldots, n$, where R_j denotes the Riesz transformation (see [28]). The space $\mathcal{H}^p_1(\mathbb{R}^n)$ is closely related to the real Hardy space $\mathcal{H}^p(\mathbb{R}^n)$. In particular, it is well known that for $p > \frac{n-1}{n}$ the space $\mathcal{H}^p(\mathbb{R}^n)$ is isomorphic naturally to the space $\mathcal{H}^p_1(\mathbb{R}^n)$.

Wolff [33] proved that this is not true for $n = 2$ and sufficiently small p; moreover, he established that $\mathcal{H}^p_1(\mathbb{R}^2) = L^p(\mathbb{R}^2, \mathbb{R}^3)$ for sufficiently small p. The methods of Wolff's paper also show that there exists a number $p(n) \in [0, \frac{n-1}{n}]$ such that $\mathcal{H}^p_1(\mathbb{R}^n) = L^p(\mathbb{R}^n, \mathbb{R}^{n+1})$ if and only if $0 < p < p(n)$. In the paper [4] it was proved that $p(n) \geq \frac{n-1}{n+1}$. Clearly $p(1) = 0$. For $n \geq 2$ the sharp value of the constant $p(n)$ is unknown. In [4], a space of scalar functions $Y^p(\mathbb{R}^n)$ was introduced for which the relation $Y^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ is equivalent to the inequality $p < p(n)$ (see [4, Theorem 8.9]). Thus, for $n \geq 2$ we have $Y^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for all sufficiently small p but not for all $p \in (0, 1)$.

In §10 we shall see that a similar phenomenon may occur for spectral spaces even in the one-dimensional case. We shall prove that there exists a number $p_0 \in (0, 1)$ and an open subset Ω of $\mathbb{R}$ such that $L^p_\Omega(\mathbb{R}) = L^p(\mathbb{R})$ for $p < p_0$ and $L^p_\Omega(\mathbb{R}) \neq L^p(\mathbb{R})$ for $p \geq p_0$.

Nearly always, we consider only the case of $p < 1$, and usually this is done without explicit mention. Thus, in what follows, if nothing is said about p, then it is assumed that $0 < p < 1$.

Note also that in Wolff's paper [33] it was shown that the condition

$$\int_{\mathbb{R}^n} (|1 + f(x)|^p - 1)\, dx \geq 0 \tag{1.1}$$

plays an important role for the study of translation and dilation invariant subspaces of $L^p(\mathbb{R}^n)$. Condition (1.1) turns into the condition

$$\int_{\mathbb{R}^n} (\log |1 + f(x)|)\, dx \geq 0 \tag{1.2}$$

as $p \to 0$. In particular, if X is a proper translation and dilation invariant subspace of $L^p(\mathbb{R}^n)$, then (1.1) is fulfilled for all functions $f \in X$ (see [33] and [4]).

[1] Rigorously speaking, we should introduce the notion of a spectral subspace for spaces of $\mathbb{R}^{n+1}$-valued functions, but here we shall not do this.

It can be proved that for each open subset Ω of $\mathbb{R}^n$ there exists a unique number $r(\Omega) \in [0,1]$ with the following two properties:

1) condition (1.1) is fulfilled for all $f \in L^1_\Omega(\mathbb{R}^n)$ and with $p \in (r(\Omega), 1)$;

2) for every $p \in (0, r(\Omega))$ there exists $f \in L^1_\Omega(\mathbb{R}^n)$ for which (1.1) fails.

It is easy to verify that if $0 < r(\Omega) < 1$, then condition (1.1) is fulfilled also for $p = r(\Omega)$, whenever $f \in L^1_\Omega(\mathbb{R}^n)$. It is not difficult to prove that $r(\Omega) = 1$ if $\Omega \cap (-\Omega) \neq \varnothing$. Moreover, $r(\Omega) = 0$ if and only if condition (1.2) is fulfilled for all $f \in L^1_\Omega(\mathbb{R}^n)$.

The results of §9 show that $r(\Omega) = 0$ if and only if the convex hull of Ω does not contain the origin. To state the corresponding result for locally compact Abelian groups, it suffices to replace the words "the convex hull of Ω" with the words "the semigroup generated by Ω".

More precisely, let G and Γ denote the same as before, and let Ω be an open subset of Γ. Again, we can introduce the number $r_G(\Omega) \in [0,1]$. We shall prove that $r_G(\Omega) = 0$ if and only if the semigroup generated by Ω contains the zero of the group Γ (Theorem 9.17).

Suppose that the semigroup generated by an open subset Ω of Γ contains the zero of Γ. Then there exists a positive integer N such that $\Omega + \Omega + \cdots + \Omega \ni 0$ (N terms). In this case $r(\Omega)$ is positive by Theorem 9.17. We shall prove (Theorem 9.14) that the number $r(\Omega)$ admits a lower estimate that depends only on N; moreover, the lower bound for $r(\Omega)$ is attained when $G = \mathbb{T}^{N-1}$ and $\Omega = \{e_1, e_2, \dots, e_{N-1}, -\sum_{k=1}^{N-1} e_k\} \subset \mathbb{Z}^{N-1}$, where $\{e_k\}_{k=1}^{N-1}$ is the standard basis in $\mathbb{R}^{N-1}$.

We conclude this section with a few words about notation. As to the locally compact Abelian groups, we use the same notation and agreements as in [26]. In particular, in the abstract situation, the groups G and Γ are written additively. Next, for $x \in G$ and $\gamma \in \Gamma$ we write (x, γ) instead of $\gamma(x)$. We denote by $\boldsymbol{m}_G$ the Haar measure on the locally compact Abelian group G. If G is compact, we always assume that $\boldsymbol{m}_G$ is a probability measure. For $G = \mathbb{R}^n$, we normalize the Haar measure so as to make it coincide with the Lebesgue measure in $\mathbb{R}^n$, and we write $\int_{\mathbb{R}^n} f(x)\,dx$ instead of $\int_{\mathbb{R}^n} f(x)\,d\boldsymbol{m}_{\mathbb{R}^n}(x)$. Moreover, if A is a measurable subset of $\mathbb{R}^n$, then the symbol $|A|$ (or $|A|_n$) will denote the Lebesgue measure of A. We denote by $\boldsymbol{m}$ the normalized Lebesgue measure on the one-dimensional torus $\mathbb{T} \overset{\text{def}}{=} \{z \in \mathbb{C} : |z| = 1\}$, i.e., $\boldsymbol{m} \overset{\text{def}}{=} \boldsymbol{m}_{\mathbb{T}}$. We put $\boldsymbol{m}_n \overset{\text{def}}{=} \boldsymbol{m}_{\mathbb{T}^n}$.

§2. Preliminary remarks on spectral subspaces

We use the standard notation $S'(\mathbb{R}^n)$ for the space of tempered distributions and $\mathcal{S}(\mathbb{R}^n)$ for the corresponding space of test functions. Recall that the space $\mathcal{S}(\mathbb{R}^n)$ consists of all functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\sup\{|x^\beta D^\alpha \varphi(x)| : x \in \mathbb{R}^n\} < +\infty$ for all multi-indices $\alpha, \beta \in \mathbb{Z}^n_+$.

Let $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ denote the Fourier transformation, initially defined on $\mathcal{S}(\mathbb{R}^n)$ by the formula

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i (x,\xi)}\,dx.$$

All that we need concerning the Fourier transformation on $\mathbb{R}^n$ can be found in [29]. Let $a \in \mathbb{R}^n$. The obvious identity $\mathcal{F}(e^{2\pi i (x,a)}\varphi) = (\mathcal{F}\varphi)(\xi - a)$ implies

$$e^{2\pi i (x,a)} L^p_\Omega(\mathbb{R}^n) = L^p_{a+\Omega}(\mathbb{R}^n). \tag{2.1}$$

Let L be an invertible linear transformation of $\mathbb{R}^n$. Then $\mathcal{F}(\varphi \circ L) = |\det L|^{-1} (\mathcal{F}\varphi) \circ (L^*)^{-1}$. Consequently,

$$f \in L^p_{L^*(\Omega)}(\mathbb{R}^n) \Longleftrightarrow f \circ (L^{-1}) \in L^p_\Omega(\mathbb{R}^n). \tag{2.2}$$

2.1. The first several theorems of this section are remarks related, in one way or another, to the definition of $L^p_\Omega(\mathbb{R}^n)$. Let Ω be an open subset of $\mathbb{R}^n$, and let $\mathcal{S}_\Omega(\mathbb{R}^n)$ denote the set of all functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$ whose partial derivatives $D^\alpha(\mathcal{F}\varphi)$, $\alpha \in \mathbb{Z}^n_+$, are identically equal to zero on $\mathbb{R}^n \setminus \Omega$. Let $\mathcal{S}^0_\Omega(\mathbb{R}^n)$ be the set of all functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp} \mathcal{F}\varphi$ is a compact subset of Ω. Clearly, $\mathcal{S}^0_\Omega(\mathbb{R}^n) \subset \mathcal{S}_\Omega(\mathbb{R}^n)$ and $\mathcal{S}_{\mathbb{R}^n}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$. Put $\mathcal{S}^0(\mathbb{R}^n) \stackrel{\text{def}}{=} \mathcal{S}^0_{\mathbb{R}^n}(\mathbb{R}^n)$.

Theorem 2.1. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$, $p \in (0, +\infty)$. Suppose that the support of the distribution u is compact and $\mathcal{F}^{-1}u \in L^p(\mathbb{R}^n)$. Then $\mathcal{F}^{-1}u \in L^p_\Omega(\mathbb{R}^n)$ for every open set Ω satisfying* $\operatorname{supp} u \subset \Omega \subset \mathbb{R}^n$.

Proof. We take a compactly supported function $\Phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \Phi(x)\,dx = 1$ and put $\Phi_\varepsilon(x) \stackrel{\text{def}}{=} \varepsilon^{-n}\Phi(\varepsilon^{-1}x)$, where $\varepsilon \in (0, +\infty)$. The identity

$$\mathcal{F}^{-1}(\Phi_\varepsilon * u) = (\mathcal{F}^{-1}\Phi)(\varepsilon x)(\mathcal{F}^{-1}u)(x)$$

implies that $\lim_{\varepsilon\to 0} \mathcal{F}^{-1}(\Phi_\varepsilon * u) = \mathcal{F}^{-1}u$ in the space $L^p(\mathbb{R}^n)$, by the Lebesgue dominated convergence theorem. It remains to observe that

$$\mathcal{F}^{-1}(\Phi_\varepsilon * u) \in \mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$$

for all $\varepsilon > 0$, and that the support of the function $\Phi_\varepsilon * u$ is compact for all $\varepsilon > 0$ and is contained in Ω for sufficiently small $\varepsilon > 0$. □

Corollary 2.2. *If Ω is an open subset of $\mathbb{R}^n$, then the set $\mathcal{S}^0_\Omega(\mathbb{R}^n)$ is dense in $L^p_\Omega(\mathbb{R}^n)$.*

Proof. It suffices to verify that the closure in $L^p(\mathbb{R}^n)$ of the set $\mathcal{S}^0_\Omega(\mathbb{R}^n)$ contains each function $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ whose Fourier transform has compact support contained in Ω. Put $u = \mathcal{F}f$. Then $\lim_{\varepsilon\to 0} \mathcal{F}^{-1}(\Phi_\varepsilon * u) = f$ in $L^p(\mathbb{R}^n)$ (see the proof of Theorem 2.1). □

Remark 2.3. It can be proved that if $p < 1$ and the assumptions of Theorem 2.1 are fulfilled, then the distribution u is a continuous function. Hence, in the proof of Theorem 2.1 for $p < 1$, it was not necessary to approximate the function $\mathcal{F}^{-1}u$ by "good" functions, because $\mathcal{F}^{-1}u \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

This statement will not be used in the sequel, so we omit the (rather easy) proof.

Theorem 2.4. *Let Ω be an open subset of $\mathbb{R}^n$. Then $\mathcal{S}_\Omega(\mathbb{R}^n) \subset L^p_\Omega(\mathbb{R}^n)$.*

Proof. We need to check that $\operatorname{clos}_{L^p}(\mathcal{S}^0_\Omega(\mathbb{R}^n)) \supset \mathcal{S}_\Omega(\mathbb{R}^n)$. For this it suffices to verify that the closure of $\mathcal{S}^0_\Omega(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^n)$ coincides with $\mathcal{S}_\Omega(\mathbb{R}^n)$. Certainly, the latter statement is well known; for example, it can be proved with the help of the Whitney partition of unity (see §1.3 in Chapter 6 of the monograph [28]). □

2.2. The Hardy space H^p as a spectral subspace. Recall that the Hardy space H^p in the upper half-plane $\mathbb{C}_+ \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is defined as the set of all holomorphic functions $f : \mathbb{C}_+ \to \mathbb{C}$ such that

$$\|f\|^p_{H^p} \stackrel{\text{def}}{=} \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p\,dx < +\infty.$$

Each function f in H^p has angular boundary values f^* almost everywhere on $\mathbb{R}$, and $\|f\|_{H^p} = \|f^*\|_{L^p(\mathbb{R})}$. Let $\varphi \in \mathcal{S}_{(0,+\infty)}(\mathbb{R})$. Put

$$F(z) = \int_{\mathbb{R}} (\mathcal{F}\varphi)(\xi) e^{2\pi i z\xi}\,d\xi = \int_0^{+\infty} (\mathcal{F}\varphi)(\xi) e^{2\pi i z\xi}\,d\xi$$

for $z \in \mathbb{C}_+$. Clearly, $F \in H^p$ for all p, the function F is smooth up to the boundary, and $F^* = \varphi$. It is well known that the set of all such functions F is dense in H^p for all

$p \in (0, +\infty)$.[2] This allows us to identify the Hardy class H^p with the subspace $L^p_{\mathbb{R}_+}(\mathbb{R})$ of $L^p(\mathbb{R})$, where $\mathbb{R}_+ \stackrel{\text{def}}{=} (0, +\infty)$.

2.3. The functional $f \mapsto (\mathcal{F}f)(a)$ on the space $L^p_\Omega(\mathbb{R}^n)$. Note that for $p \geq 1$, the space $L^p(\mathbb{R}^n)$ embeds into the space of tempered distributions in a natural way, and the corresponding embedding is continuous.

Let $p < 1$. We say that the space $L^p_\Omega(\mathbb{R}^n)$ is a space of distributions if the identity embedding $\mathcal{S}_\Omega(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ extends up to a continuous operator $J^p_\Omega : L^p_\Omega(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. The author does not know whether the extended operator J^p_Ω is injective.

Note the space $L^p(\mathbb{R}^n) = L^p_{\mathbb{R}^n}(\mathbb{R}^n)$ is not a space of distributions. Indeed, if $\varphi \in \mathcal{S}(\mathbb{R}^n) = \mathcal{S}_{\mathbb{R}^n}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$, then the sequence $\{k^n\varphi(kx)\}_{k=1}^\infty$ converges to 0 in $L^p(\mathbb{R}^n)$ for $p < 1$, whereas it converges to the δ-function in $\mathcal{S}'(\mathbb{R}^n)$.

On the other hand, it is well known that $L^p_{\mathbb{R}_+}(\mathbb{R})$ is a space of distributions, and $\mathcal{F}f$ is continuous for every $f \in L^p_{\mathbb{R}_+}(\mathbb{R})$ $(0 < p < 1)$ (see [12]). A similar statement is true for the spaces $L^p_\Omega(\mathbb{R}^n)$, where Ω is a nondegenerate open convex cone. In particular, in this case the functional $f \mapsto (\mathcal{F}f)(a)$ is continuous on $L^p_\Omega(\mathbb{R}^n)$ for every $a \in \Omega$ provided $p < 1$.

To investigate this phenomenon in the general setting, we put

$$\Phi^\Omega_p(a) \stackrel{\text{def}}{=} \sup\{|(\mathcal{F}f)(a)| : f \in \mathcal{S}_\Omega(\mathbb{R}^n), \|f\|_{L^p} \leq 1\}.$$

Clearly, $\Phi^\Omega_p(a) > 0$ for all $a \in \Omega$. If $\Phi^\Omega_p(a) < +\infty$, then the functional $f \mapsto (\mathcal{F}f)(a)$, $f \in \mathcal{S}_\Omega(\mathbb{R}^n)$, extends up to a continuous functional $\mathcal{G}^\Omega_a : L^p_\Omega(\mathbb{R}^n) \to \mathbb{C}$, and $\|\mathcal{G}^\Omega_a\| = \Phi^\Omega_p(a)$. But if $\Phi^\Omega_p(a) = +\infty$, then such an extension does not exist.

By using formula (2.2), it is easy to obtain the relation

$$\Phi^{L(\Omega)}_p(La) = |\det L|^{\frac{1}{p}-1}\Phi^\Omega_p(a) \tag{2.3}$$

whenever L is an invertible linear transformation of $\mathbb{R}^n$. Observe also that $\Phi^\Omega_p(a) = \Phi^{\Omega-a}_p(0)$.

Theorem 2.5. *Suppose Ω is an open subset of $\mathbb{R}^n$ and $a \in \Omega$. Then*:

a) $L^p_{\Omega\setminus\{a\}}(\mathbb{R}^n) = \operatorname{Ker}\mathcal{G}^\Omega_a$ *provided* $\Phi^\Omega_p(a) < +\infty$;

b) $L^p_{\Omega\setminus\{a\}}(\mathbb{R}^n) = L^p_\Omega(\mathbb{R}^n)$ *provided* $\Phi^\Omega_p(a) = +\infty$.

It is not difficult to see that Theorem 2.5 can be restated as follows:

$$L^p_{\Omega\setminus\{a\}}(\mathbb{R}^n) \supset \{\varphi \in \mathcal{S}_\Omega(\mathbb{R}^n) : (\mathcal{F}\varphi)(a) = 0\}. \tag{2.4}$$

In other words, either

$$L^p_{\Omega\setminus\{a\}}(\mathbb{R}^n) \cap \mathcal{S}_\Omega(\mathbb{R}^n) = \{\varphi \in \mathcal{S}_\Omega(\mathbb{R}^n) : (\mathcal{F}\varphi)(a) = 0\}, \tag{2.5}$$

or

$$L^p_{\Omega\setminus\{a\}}(\mathbb{R}^n) \supset \mathcal{S}_\Omega(\mathbb{R}^n). \tag{2.6}$$

It is known and easy to show that for $p > 1$, the inclusion (2.6) is always true. For $p = 1$, we always have (2.5). In fact, this means that a singleton is a set of synthesis (see, e.g., [19, Chapter 5]). A standard proof of this fact (see [19] and Lemma 2.6 below) allows us to prove the inclusion (2.4) for $p \in (\frac{n}{n+1}, 1)$. The case where $p \leq \frac{n}{n+1}$ requires a little additional effort.

[2] If $p > 1$, then this statement is obvious, because the Riesz projection is continuous in the space $L^p(\mathbb{R})$ for $p \in (1, +\infty)$. For $p = 1$, the corresponding statement for the multidimensional real Hardy spaces can be found in §3.3.3 of Chapter VII in Stein's monograph [28]. The same proof works for the classical Hardy spaces; moreover, the results of [12] make it possible to carry the proof over to the case of $p < 1$.

Lemma 2.6. *Let $\varphi, f \in \mathcal{S}(\mathbb{R}^n)$. Suppose $(\mathcal{F}f)(\xi) = \mathcal{O}(|\xi|^N)$ as $\xi \to 0$. Then*

$$\|\varphi(\varepsilon x) * f\|_{L^p(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{N-n/p})$$

as $\varepsilon \to 0$.

Proof. Passage to the Fourier transform shows that the limit $\lim_{\varepsilon\to 0} \varepsilon^{-n-N} f(\varepsilon^{-1}x) * \varphi$ exists in $\mathcal{S}(\mathbb{R}^n)$. Hence, the limit $\lim_{\varepsilon\to 0} \varepsilon^{-n-N}\|f(\varepsilon^{-1}x) * \varphi\|_{L^p(\mathbb{R}^n)}$ exists and is finite. Now the required result follows from the identity

$$\|\varphi(\varepsilon x) * f\|_{L^p(\mathbb{R}^n)} = \varepsilon^{-n-n/p}\|f(\varepsilon^{-1}x) * \varphi\|_{L^p(\mathbb{R}^n)}.$$

□

Let $h \in \mathbb{R}^n$. We put $\Delta_h f \stackrel{\text{def}}{=} f(x+h) - f(x)$, where f is a function defined on $\mathbb{R}^n$. Observe that $(\mathcal{F}\Delta_h^N f)(\xi) = \mathcal{O}(|\xi|^N)$ as $\xi \to 0$ for every function $f \in \mathcal{S}(\mathbb{R}^n)$, every vector $h \in \mathbb{R}^n$, and every positive integer N.

Lemma 2.7. *Suppose $0 \in \Omega$, where Ω is an open subset of $\mathbb{R}^n$. Then the closure in $\mathcal{S}(\mathbb{R}^n)$ of the linear hull of the set of all functions representable as $\Delta_h f$, where $h \in \mathbb{R}^n$ and $f \in \mathcal{S}_\Omega(\mathbb{R}^n)$, coincides with the set of all $f \in \mathcal{S}_\Omega(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} f(x)\,dx = 0$.*

Proof. Let $g \in \mathcal{S}_\Omega(\mathbb{R}^n)$ be such that $(\mathcal{F}g)(0) = 0$. We prove that g belongs to the closure in $\mathcal{S}(\mathbb{R}^n)$ of the linear hull of the family $\{\Delta_h f\}_{h,f}$. As was already noted in the proof of Theorem 2.4, the set $\mathcal{S}_\Omega^0(\mathbb{R}^n)$ is dense in $\mathcal{S}_\Omega(\mathbb{R}^n)$. Therefore, we may assume that $g \in \mathcal{S}_\Omega^0(\mathbb{R}^n)$. Moreover, we may assume that $g \in \mathcal{S}_B^0(\mathbb{R}^n)$, where $B \subset G$ is a ball either centered at the origin or not containing the origin. Suppose $B \not\ni 0$. Then there exists a vector $h \in \mathbb{R}^n$ such that $0 < (\xi, h) < 1$ for all $\xi \in B$. It is clear that there exists a function $f \in \mathcal{S}(\mathbb{R}^n)$ such that $(e^{2\pi i(\xi,h)} - 1)(\mathcal{F}f)(\xi) = (\mathcal{F}g)(\xi)$, whence $g = \Delta_h f$, where $f \in \mathcal{S}_B^0(\mathbb{R}^n) \subset \mathcal{S}_\Omega(\mathbb{R}^n)$. Now, let B be a ball centered at the origin. We can write $\mathcal{F}g(\xi) = \sum_{j=1}^n \xi_j u_j(\xi)$, where $u_j \in C^\infty(\mathbb{R}^n)$ is an infinitely differentiable function whose support is a compact subset of B, $1 \le j \le n$. Clearly, for every j there exists a function $f_j \in \mathcal{S}(\mathbb{R}^n)$ such that $(e^{\frac{\pi i}{r}\xi_j} - 1)(\mathcal{F}f_j)(\xi) = \xi_j u_j(\xi)$, where r is the radius of B. Then $g = \sum_{j=1}^n \Delta_{(2r)^{-1}e_j} f_j$, where $\{e_1, e_2, \dots, e_n\}$ is the standard basis in the space $\mathbb{R}^n$, and $f_j \in \mathcal{S}_B^0(\mathbb{R}^n) \subset \mathcal{S}_\Omega(\mathbb{R}^n)$. □

Lemma 2.8. *Suppose $0 \in \Omega$, where Ω is an open subset of $\mathbb{R}^n$. Then for every vector $h \in \mathbb{R}^n$ and every integer $N \ge 2$, the closure in $L^p(\mathbb{R}^n)$ of the set $\Delta_h^N(\mathcal{S}_\Omega(\mathbb{R}^n))$ includes the set $\Delta_h(\mathcal{S}_\Omega(\mathbb{R}^n))$.*

Proof. Observe that

$$\Delta_h f - \frac{1}{2}\Delta_{2h} f = -\frac{1}{2}\Delta_h^2 f.$$

By induction,

$$\Delta_h f - \frac{1}{2^s}\Delta_{2^s h} f = -\sum_{j=0}^{s-1} \frac{1}{2^{j+1}}\Delta_{2^j h}^2 f.$$

It is easily seen that $\Delta_{kh}(\mathcal{S}_\Omega(\mathbb{R}^n)) \subset \Delta_h(\mathcal{S}_\Omega(\mathbb{R}^n))$ for every integer k. Consequently,

$$\Delta_h f - \frac{1}{2^s}\Delta_{2^s h} f \in \Delta_h^2(\mathcal{S}_\Omega(\mathbb{R}^n))$$

for every $f \in \mathcal{S}_\Omega(\mathbb{R}^n)$, whence

$$\Delta_h^{N-1} f - \frac{1}{2^s}\Delta_h^{N-2}\Delta_{2^s h} f \in \Delta_h^N(\mathcal{S}_\Omega(\mathbb{R}^n)).$$

Letting $s \to +\infty$, we obtain

$$\Delta_h^{N-1} f \in \operatorname{clos}_{L^p}(\Delta_h^N(\mathcal{S}_\Omega(\mathbb{R}^n))),$$

whence

$$\operatorname{clos}_{L^p}(\Delta_h^{N-1}(\mathcal{S}_\Omega(\mathbb{R}^n))) \subset \operatorname{clos}_{L^p}(\Delta_h^N(\mathcal{S}_\Omega(\mathbb{R}^n))).$$

Now Lemma 2.8 is obvious. □

Proof of Theorem 2.5. We prove (2.4). It can be assumed that $a = 0$. Let $f \in \mathcal{S}_\Omega(\mathbb{R}^n)$, $\mathcal{F}f(0) = 0$. Take a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\mathcal{F}\varphi = 1$ in a neighborhood of the origin. The Fourier transforms of the functions $\varepsilon^n\varphi(\varepsilon x) * f$ and f coincide in a neighborhood of the origin. Hence, $f - \varepsilon^n\varphi(\varepsilon x) * f \in \mathcal{S}_{\Omega\setminus\{0\}}(\mathbb{R}^n)$. Thus, by Lemma 2.6, if $\mathcal{F}f(\xi) = \mathcal{O}(|\xi|^N)$ as $\xi \to 0$, where $N > n - n/p$, then $f \in L^p_{\Omega\setminus\{a\}}(\mathbb{R}^n)$. Note that the condition $\mathcal{F}f(0) = 0$ implies $\mathcal{F}f(\xi) = \mathcal{O}(|\xi|)$ as $\xi \to 0$. This allows us to complete the proof for $p > \frac{n}{n+1}$. If $p \le \frac{n}{n+1}$, we should invoke Lemmas 2.7 and 2.8. □

Let $f \in L^p_\Omega(\mathbb{R}^n)$, where Ω is an open subset of $\mathbb{R}^n$. Assume that $\Phi_p^\Omega(a) < +\infty$, where $a \in \Omega$. Then we can define the Fourier transform of f at the point a by putting $(\mathcal{F}f)(a) \stackrel{\text{def}}{=} \mathcal{G}_a^\Omega(f)$. The local boundedness of the function Φ_p^Ω implies that the Fourier transform of any $f \in L^p_\Omega(\mathbb{R})$ is continuous. More precisely, the following obvious assertion is true. We state it without proof.

Theorem 2.9. *Let $f \in L^p_\Omega(\mathbb{R}^n)$. Suppose that $\sup\{\Phi_p^\Omega(\xi) : \xi \in U \cap \Omega\} < +\infty$, where U is an open subset of $\mathbb{R}^n$. Then*:

a) *the function $\mathcal{F}f$ is continuous on U if $U \subset \Omega$;*

b) *$\lim_a(\mathcal{F}f)|_{U\cap\Omega} = 0$ if $a \in U$ and a is a boundary point of Ω.*

In §6 we shall see that, in general, the condition $\Phi_p^\Omega(a) < +\infty$ does not imply that Φ_p^Ω is bonded near the point a.

Theorem 2.10. *Let Ω_1 and Ω_2 be open subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. Let $a_1 \in \Omega_1$, $a_2 \in \Omega_2$. Put $\Omega = \Omega_1 \times \Omega_2$, $a = (a_1, a_2)$. Then $\Phi_p^\Omega(a) = \Phi_p^{\Omega_1}(a_1)\Phi_p^{\Omega_2}(a_2)$.*

Proof. Note that if $f \in L^p_{\Omega_1}(\mathbb{R}^{n_1})$ and $g \in L^p_{\Omega_2}(\mathbb{R}^{n_2})$, then $f(x)g(y) \in L^p_\Omega(\mathbb{R}^{n_1+n_2})$ and $\|f(x)g(y)\|_{L^p(\mathbb{R}^{n_1+n_2})} = \|f\|_{L^p(\mathbb{R}^{n_1})}\|g\|_{L^p(\mathbb{R}^{n_2})}$. It follows that $\Phi_p^\Omega(a) \ge \Phi_p^{\Omega_1}(a_1)\Phi_p^{\Omega_2}(a_2)$. We prove the reverse inequality. We may assume that $a_1 = 0$ and $a_2 = 0$. Let $f = f(x, y) \in \mathcal{S}^0_\Omega(\mathbb{R}^{n_1+n_2})$, where $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$. It is easily seen that for each $x \in \mathbb{R}^{n_1}$, the function $y \mapsto f(x, y)$ belongs to the space $\mathcal{S}^0_{\Omega_2}(\mathbb{R}^{n_2})$. Hence,

$$\Big|\int_{\mathbb{R}^{n_2}} f(x,y)\,dy\Big|^p \le (\Phi_p^{\Omega_2}(0))^p \int_{\mathbb{R}^{n_2}} |f(x,y)|^p\,dy \tag{2.7}$$

for each $x \in \mathbb{R}^{n_1}$. Note that the function $\int_{\mathbb{R}^{n_2}} f(x, y)\,dy$ belongs to $\mathcal{S}^0_{\Omega_1}(\mathbb{R}^{n_1})$. Consequently,

$$\Big|\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} f(x,y)\,dy\,dx\Big|^p \le (\Phi_p^{\Omega_1}(0))^p \int_{\mathbb{R}^{n_1}} \Big|\int_{\mathbb{R}^{n_2}} f(x,y)\,dy\Big|^p\,dx. \tag{2.8}$$

Inequalities (2.7) and (2.8) imply that $\Phi_p^\Omega(0) \le \Phi_p^{\Omega_1}(0)\Phi_p^{\Omega_2}(0)$. □

2.4. A lower estimate for the functional Φ_p^Ω. Let $a \in \Omega$, where Ω is an open subset of $\mathbb{R}^n$. Put

$$V_\Omega(a) \stackrel{\text{def}}{=} \sup\{|K| : K \subset \Omega, 2a - K = K, K \text{ is convex}\},$$

$$\widetilde{V}_\Omega(a) \stackrel{\text{def}}{=} \sup\{|K| : K \subset \Omega, 2a - K = K, K \text{ is an ellipsoid}\}.$$

Clearly, $\widetilde{V}_\Omega(a) \le V_\Omega(a)$. Moreover, the John theorem [18] yields $V_\Omega(a) \le n^{\frac{n}{2}}\widetilde{V}_\Omega(a)$.

Theorem 2.11. *Let Ω be an open subset of $\mathbb{R}^n$. Then $\Phi_p^\Omega(a) \ge C(n,p)(V_\Omega(a))^{\frac{1}{p}-1}$ for all $a \in \Omega$.*

Proof. We may assume that $a = 0$. Put $B_n(p) \stackrel{\text{def}}{=} \Phi_p^U(0)$, where U is the open unit ball in $\mathbb{R}^n$, $U \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^n : |\xi| < 1\}$. It is easily seen that $B_n(p) < +\infty$. The results of §4 (see Theorem 4.1) imply that $B_n(p) \le |U|^{\frac{1}{p}-1}$. Applying formula (2.3), we obtain

$$\Phi_p^\Omega(0) \ge \Phi_p^{L(U)}(0) = B_n(p)|\det L|^{\frac{1}{p}-1} = B_n(p)|U|^{1-\frac{1}{p}}|L(U)|^{\frac{1}{p}-1}, \tag{2.9}$$

where L is a nondegenerate linear transformation of $\mathbb{R}^n$ satisfying $L(U) \subset \Omega$. Passing in (2.9) to the supremum over all such transformations L, we obtain $\Phi_p^\Omega(0) \ge B_n(p)|U|^{1-\frac{1}{p}}(\widetilde{V}_\Omega(0))^{\frac{1}{p}-1}$. □

We shall see that this simple lower estimate is rather often sharp in the sense that a similar upper estimate $\Phi_p^\Omega(a) \le C_\Omega(n,p)(V_\Omega(a))^{\frac{1}{p}-1}$ is valid with another constant. In §3 we prove that this is the case for any convex set Ω. Other examples of such sets Ω will be presented in §8.

Nevertheless, the upper estimate $\Phi_p^\Omega(a) \le C_\Omega(n,p)(V_\Omega(a))^{\frac{1}{p}-1}$ fails in general. For example, in the one-dimensional case we have $V_\Omega(a) < +\infty$ for all $a \in \Omega$ if $\Omega \ne \mathbb{R}$. At the same time it is easy to construct examples of sets $\Omega \ne \mathbb{R}$ such that $\Phi_p^\Omega(a) = +\infty$ for all $a \in \Omega$. Other examples of this kind are given at the end of §6; see also §10, where it is shown that, in general, the inequality $\Phi_p^\Omega(a) < +\infty$ does not imply that $\Phi_q^\Omega(a) < +\infty$ for $q < p$.

2.5. Let X be a translation invariant subspace of $L^p(\mathbb{R}^n)$. Put

$$\mathcal{U}_X \stackrel{\text{def}}{=} \bigcup_{\varphi \in X \cap \mathcal{S}^0(\mathbb{R}^n)} \{\mathcal{F}\varphi \ne 0\}.$$

It is clear that $\mathcal{U}_X$ is an open subset of $\mathbb{R}^n$.

Let ω be a nonnegative measurable function on $\mathbb{R}^n$. We denote by $L^1(\mathbb{R}^n, \omega)$ the space L^1 with the weight ω.

Lemma 2.12. *Let $f, g \in L^1(\mathbb{R}^n, (1+|x|)^N)$, where $N > n(p^{-1}-1)$. Suppose that $f \in X$, where X is a translation invariant subspace of $L^p(\mathbb{R}^n)$. Then $f * g \in X$.*

Proof. By the Hölder inequality, we have $L^1(\mathbb{R}^n, (1+|x|)^N) \subset L^p(\mathbb{R}^n)$, and the corresponding embedding is continuous. It is easily seen and well known that the space $L^1(\mathbb{R}^n, (1+|x|)^N)$ is stable under convolution. Hence, $f * g \in L^1(\mathbb{R}^n, (1+|x|)^N)$. We prove that $f * g \in X$. Put $f_t(x) \stackrel{\text{def}}{=} f(x-t)$. Clearly, $f_t \in X$ for all $t \in \mathbb{R}^n$. We can view the convolution $f * g$ as the Bochner integral of an $L^1(\mathbb{R}^n, (1+|x|)^N)$-valued function:

$$f * g = \int_{\mathbb{R}^n} g(t) f_t \, dt.$$

Hence, the function $f * g$ belongs to the closure in $L^1(\mathbb{R}^n, (1+|x|)^N)$ (and so in $L^p(\mathbb{R}^n)$) of the linear hull of the family $\{f_t\}_{t \in \mathbb{R}^n}$. □

Corollary 2.13. *Let $0 < p < +\infty$. Suppose that $f \in X \cap \mathcal{S}(\mathbb{R}^n)$, where X is a translation invariant subspace of $L^p(\mathbb{R}^n)$. Then $f * \varphi \in X$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. It suffices to consider the case where $p < 1$. In this case the result follows from Lemma 2.12 because $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n, (1+|x|)^N)$. □

Theorem 2.14. *Let $0 < p < +\infty$. Let X be a translation invariant subspace of $L^p(\mathbb{R}^n)$. Then, for every open set $\Omega \subset \mathbb{R}^n$, the inclusion $L^p_\Omega(\mathbb{R}^n) \subset X$ is true if and only if $\Omega \subset \mathcal{U}_X$.*

Proof. Let $L^p_\Omega(\mathbb{R}^n) \subset X$. We prove that $\Omega \subset \mathcal{U}_X$. For an arbitrary point $a \in \Omega$, there exists a function $\varphi \in \mathcal{S}^0_\Omega(\mathbb{R}^n)$ such that $\varphi(a) = 1$. Hence, $a \in \mathcal{U}_X$.

Now, let $\Omega \subset \mathcal{U}_X$. We prove that $L^p_\Omega(\mathbb{R}^n) \subset X$. It suffices to check that $\mathcal{S}^0_\Omega(\mathbb{R}^n) \subset X$. Let $\varphi \in \mathcal{S}^0_\Omega(\mathbb{R}^n)$. Since $\Omega \subset \mathcal{U}_X$ and the support of $\mathcal{F}\varphi$ is compact, there exists a finite family $\{\psi_j\}_{j\in J}$ in $X \cap \mathcal{S}^0(\mathbb{R}^n)$ such that

$$\operatorname{supp} \mathcal{F}\varphi \subset \bigcup_{j\in J} \{\mathcal{F}\psi_j \neq 0\}.$$

It is easily seen that there is a function $h \in \mathcal{S}^0(\mathbb{R}^n)$ such that $\mathcal{F}\varphi = (\sum_{j\in J} |\mathcal{F}\psi_j|^2)\mathcal{F}h$. Hence, $\varphi = \sum_{j\in J} \psi_j * \widetilde{\psi}_j * h$, where $\widetilde{\psi}(x) \overset{\text{def}}{=} \overline{\psi(-x)}$. Thus, $\varphi \in X$ by Corollary 2.13. $\square$

Let Ω be an open subset of $\mathbb{R}^n$. Put $[\,\Omega]_p \overset{\text{def}}{=} \mathcal{U}_{L^p_\Omega(\mathbb{R}^n)}$, where $p \in (0, +\infty)$. Clearly, the set $[\,\Omega]_p$ is open and $[\,\Omega]_p \supset \Omega$. The set $[\,\Omega]_p$ is called the *L^p-completion* of Ω. It follows easily from Theorem 2.14 that $[\mathcal{U}_X]_p = \mathcal{U}_X$. In particular, $[\,[\,\Omega]_p]_p = [\,\Omega]_p$. Note that $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ if and only if $[\,\Omega]_p = \mathbb{R}^n$. Let L be an invertible affine transformation of $\mathbb{R}^n$. Then properties (2.1) and (2.2) imply

$$[L(\Omega)]_p = L([\,\Omega]_p)\,. \tag{2.10}$$

An open subset $\Omega \subset \mathbb{R}^n$ is said to be *L^p-complete* if $[\,\Omega]_p = \Omega$. It is clear that every open set is L^1-complete. It is easily seen that $[\,\Omega]_p$ is contained in the closure $\overline{\Omega}$ of the set Ω for all $p \in (1, +\infty)$. In particular, an open set Ω is L^p-complete if it coincides with the interior of its closure provided $p \in (1, +\infty)$. Hence, every open convex set is L^p-complete for all $p \in [1, +\infty)$. In the next section we shall see that this is also true for $p \in (0, 1)$.

Theorem 2.15. *If the space $L^p_\Omega(\mathbb{R}^n)$ is a space of distributions, then $[\Omega]_p \subset \overline{\Omega}$.*

Proof. Suppose the contrary. Then there exists a nonempty open set $U \subset [\Omega]_p$ such that $U \cap \Omega = \varnothing$. Let $\varphi \in \mathcal{S}_U(\mathbb{R}^n)$, $\varphi \neq 0$. There exists a sequence $\{\varphi_j\}$ in $\mathcal{S}_\Omega(\mathbb{R}^n)$ that converges to φ in L^p. Clearly, this sequence cannot tend to φ in the space $\mathcal{S}'(\mathbb{R}^n)$, and we arrive at a contradiction. $\square$

Note also that the definition of $\Phi^\Omega_p(a)$ readily implies the following statement.

Theorem 2.16. *If $a \in [\Omega]_p \setminus \Omega$, then $\Phi^{[\Omega]_p}_p(a) = +\infty$.*

Theorem 2.17. *Let $0 < p < +\infty$. Let $\{\Omega_\alpha\}_{\alpha\in A}$ be an arbitrary family of open subsets in $\mathbb{R}^n$. Put $\Omega = \bigcup_{\alpha\in A} \Omega_\alpha$. Then the space $L^p_\Omega(\mathbb{R}^n)$ is the closure in $L^p(\mathbb{R}^n)$ of the linear hull of the family $\{L^p_{\Omega_\alpha}(\mathbb{R}^n)\}_{\alpha\in A}$.*

Proof. Let X denote the closure in $L^p(\mathbb{R}^n)$ of the linear hull of the family of subspaces $\{L^p_{\Omega_\alpha}(\mathbb{R}^n)\}_{\alpha\in A}$. The inclusion $X \subset L^p_\Omega(\mathbb{R}^n)$ is obvious. We prove that $L^p_\Omega(\mathbb{R}^n) \subset X$. For this, note that $\mathcal{U}_X \supset \mathcal{U}_{L^p_{\Omega_\alpha}(\mathbb{R}^n)} \supset \Omega_\alpha$. Consequently, $\mathcal{U}_X \supset \Omega$, whence $X \supset L^p_\Omega(\mathbb{R}^n)$. $\square$

Remark 2.18. It is easily seen that $L^p_{\Omega_1\cap\Omega_2}(\mathbb{R}^n) \subset L^p_{\Omega_1}(\mathbb{R}^n) \cap L^p_{\Omega_2}(\mathbb{R}^n)$. But in general, $L^p_{\Omega_1\cap\Omega_2}(\mathbb{R}^n) \neq L^p_{\Omega_1}(\mathbb{R}^n) \cap L^p_{\Omega_2}(\mathbb{R}^n)$; see also Remark 3.10 below.

For example, for $n = 1$, $\Omega_1 = \mathbb{R}_+ \overset{\text{def}}{=} (0, +\infty)$, and $\Omega_2 = \mathbb{R}_- \overset{\text{def}}{=} (-\infty, 0)$, we have $\{0\} = L^p_{\Omega_1\cap\Omega_2}(\mathbb{R}) \neq L^p_{\Omega_1}(\mathbb{R}) \cap L^p_{\Omega_2}(\mathbb{R})$ provided $p < 1$. This is related to the fact that for $p < 1$, the space $L^p_{\mathbb{R}_+}(\mathbb{R})$ contains nonzero real functions h. Every h of this sort also belongs to the space $L^p_{\mathbb{R}_-}(\mathbb{R})$. Moreover, it is not difficult to construct an open subset Ω of $\mathbb{R}^n$ such that $\Omega \cap (-\Omega) = \varnothing$ and $L^p_\Omega(\mathbb{R}^n) = L^p_{-\Omega}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for all $p < 1$.

Theorem 2.19. *Let $0 < p < +\infty$. Let $\{\Omega_\alpha\}_{\alpha\in A}$ be an arbitrary family of L^p-complete open subsets of $\mathbb{R}^n$. Then the interior of the set $\bigcap_{\alpha\in A}\Omega_\alpha$ is also L^p-complete.*

Proof. Let Ω denote the interior of $\bigcap_{\alpha\in A}\Omega_\alpha$. We prove that $[\,\Omega]_p = \Omega$. Clearly, $[\,\Omega]_p \subset \Omega_\alpha$ for all $\alpha \in A$. Consequently, $\Omega \subset [\,\Omega]_p \subset \bigcap_{\alpha\in A}\Omega_\alpha$, whence $[\,\Omega]_p = \Omega$ because $[\,\Omega]_p$ is open. □

Remark 2.20. The union of two L^p-complete sets may fail to be L^p-complete. Let $\Omega_1 = (0,+\infty)^n$, $\Omega_2 = (-\infty,0)^n$, and let $\Omega = \Omega_1 \cup \Omega_2$. The sets Ω_1 and Ω_2 are L^p-complete because they are convex, see Theorem 3.4 below, but $[\,\Omega]_p = \mathbb{R}^n$, because Ω is symmetric with respect to the origin and contains arbitrarily large balls; see Theorem 9.1 of the paper [1] or Theorem 5.1 below.

§3. Convex sets Ω

First we consider the case where $\Omega = \mathbb{R}^n_+ \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : x_1 > 0\}$.

Let $x \in \mathbb{R}^n$. Put $x' \stackrel{\text{def}}{=} (x_2, x_3, \dots, x_n)$. Then $x = (x_1, x')$. Thus, we identify the half-space $\mathbb{R}^n_+$ with the set $\mathbb{R}_+ \times \mathbb{R}^{n-1}$.

Theorem 3.1. *Let $f \in L^p(\mathbb{R}^n)$, where $0 < p < +\infty$. Then $f \in L^p_{\mathbb{R}^n_+}(\mathbb{R}^n)$ if and only if $f(\,\cdot\,, x') \in L^p_{\mathbb{R}_+}(\mathbb{R})$ for almost all $x' \in \mathbb{R}^{n-1}$.*

We need the following lemma.

Lemma 3.2. *Let $(S, \mathfrak{A}, \mu)$ and $(T, \mathfrak{B}, \nu)$ be two measure spaces. Suppose that μ and ν are σ-finite. Let X be a subspace of $L^p(S,\mu)$. Denote by $[X, L^p(T,\nu)]$ the set of all functions $f \in L^p(S\times T, \mu\otimes\nu)$ such that $f(\,\cdot\,, t) \in X$ for almost all $t \in T$. Then $[X, L^p(T,\nu)]$ is a subspace of $L^p(S\times T, \mu\otimes\nu)$, and the linear hull of all functions of the form $u(s)v(t)$, where $u \in X$, $v \in L^p(T,\nu)$, is dense in this subspace.*

Proof. The linearity of the set $[X, L^p(T,\nu)]$ is obvious. We omit an elementary proof of the fact that this set is closed. Taking an arbitrary function $f \in [X, L^p(T,\nu)]$, we prove that f can be approximated by linear combinations of functions of the form $u(s)v(t)$, where $u \in X$, $v \in L^p(T,\nu)$. Observe that the linear combinations of functions of the form $u(s)v(t)$, where $u \in L^p(S,\mu)$ and $v \in L^p(T,\nu)$, are dense in the space $L^p(S\times T, \mu\otimes\nu)$. Hence, for every $\varepsilon > 0$ there exist two finite sequences $\{u_k\}_{k=1}^M$ and $\{v_k\}_{k=1}^M$ such that $u_k \in L^p(S,\mu)$, $v_k \in L^p(T,\nu)$, and

$$\Big\| f(s,t) - \sum_{k=1}^M u_k(s)v_k(t) \Big\|_{L^p} < \varepsilon. \tag{3.1}$$

We may assume that each function v_k takes only finitely many values. Then the function $\sum_{k=1}^M u_k(s)v_k(t)$ can be represented in the form

$$\sum_{k=1}^M u_k(s)v_k(t) = \sum_{j=1}^N g_j(s)\mathbb{1}_{B_j}(t),$$

where $\{g_j\}_{j=1}^N$ is a finite sequence in the space $L^p(S,\mu)$, and $\{B_j\}_{j=1}^N$ is a finite sequence of pairwise disjoint measurable subsets of T with $0 < \nu(B_j) < +\infty$ for all j. Now inequality (3.1) can be rewritten as follows:

$$\begin{aligned} &\int_{T\setminus\bigcup_{j=1}^N B_j} \left(\int_S |f(s,t)|^p \, d\mu(s) \right) d\nu(t) \\ &\quad + \sum_{j=1}^N \int_{B_j} \left(\int_S |f(s,t) - g_j(s)|^p \, d\mu(s) \right) d\nu(t) < \varepsilon^p. \end{aligned} \tag{3.2}$$

It is clear that for every j, $1 \le j \le N$, there exists a point $t_j \in B_j$ such that $f(\,\cdot\,, t_j) \in X$ and

$$\nu(B_j) \int_S |f(s,t_j) - g_j(s)|^p \, d\mu(s) \le \int_{B_j} \left(\int_S |f(s,t) - g_j(s)|^p \, d\mu(s) \right) d\nu(t). \tag{3.3}$$

Put $h(s,t) \stackrel{\text{def}}{=} \sum_{j=1}^N f(s,t_j) \mathbb{1}_{B_j}(t)$. It remains to note that inequalities (3.2) and (3.3) imply $\|f - h\|_{L^p}^p \le \max(2, 2^p)\, \varepsilon^p$. □

Proof of Theorem 3.1. To simplify notation, we restrict ourselves to the case where $n = 2$. Suppose that the support of the Fourier transform of a function $\varphi \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ is a compact subset of $\mathbb{R}^2_+$. We may assume that $\operatorname{supp} \mathcal{F}\varphi \subset K_1 \times K_2$, where K_1 and K_2 are compact subsets of $\mathbb{R}_+$ and $\mathbb{R}$, respectively. Applying the inverse Fourier transformation, we see that

$$\varphi(x_1, x_2) = \int_{\mathbb{R}} u_{x_2}(\xi_1) e^{2\pi i x_1 \xi_1} \, d\xi_1,$$

where

$$u_{x_2}(\xi_1) \stackrel{\text{def}}{=} \int_{\mathbb{R}} (\mathcal{F}\varphi)(\xi_1, \xi_2) e^{2\pi i x_2 \xi_2} \, d\xi_2.$$

Clearly, the function u_{x_2} is continuous, and $\operatorname{supp} u_{x_2} \subset K_2$. Hence, $\varphi(\,\cdot\,, x_2) \in L^p_{\mathbb{R}_+}(\mathbb{R})$ if $\varphi(\,\cdot\,, x_2) \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, which is fulfilled for almost all $x_2 \in \mathbb{R}$. Thus, we have proved that $L^p_{\mathbb{R}^2_+}(\mathbb{R}^2) \subset [L^p_{\mathbb{R}_+}(\mathbb{R}), L^p(\mathbb{R})]$ (here we use the notation of Lemma 3.2). Now we prove that $[L^p_{\mathbb{R}_+}(\mathbb{R}), L^p(\mathbb{R})] \subset L^p_{\mathbb{R}^2_+}(\mathbb{R}^2)$. By Lemma 3.2, it suffices to check that $\varphi(x_1)\psi(x_2) \in L^p_{\mathbb{R}_+}(\mathbb{R}^2)$ for all $\varphi \in L^p_{\mathbb{R}_+}(\mathbb{R})$ and $\psi \in L^p(\mathbb{R})$. Obviously, we may assume in addition that $\varphi, \psi \in L^1(\mathbb{R})$, $\operatorname{supp} \mathcal{F}\varphi$ is a compact subset of $(0, +\infty)$, and $\operatorname{supp} \mathcal{F}\psi$ is a compact subset of $\mathbb{R}$. Then, clearly, $\varphi(x_1)\psi(x_2) \in L^p_{\mathbb{R}_+}(\mathbb{R}^2)$. □

Corollary 3.3. *Every open half-space of $\mathbb{R}^n$ is L^p-complete for all $p \in (0, +\infty)$.*

Proof. Relation (2.10) allows us to assume that $\Omega = \mathbb{R}^n_+$. Note that $a + \mathbb{R}^n_+ \subset \mathbb{R}^n_+$ for all $a \in \mathbb{R}^n_+$. Applying (2.10) once again, we see that $a + [\mathbb{R}^n_+]_p = [a + \mathbb{R}^n_+]_p \subset [\mathbb{R}^n_+]_p$ for every $a \in \mathbb{R}^n_+$. Hence, $[\mathbb{R}^n_+]_p + \mathbb{R}^n_+ \subset [\mathbb{R}^n_+]_p$; i.e., $a + \mathbb{R}^n_+ \subset [\mathbb{R}^n_+]_p$ for all $a \in [\mathbb{R}^n_+]_p$. Arguing in exactly the same way, we deduce that $[\mathbb{R}^n_+]_p + [\mathbb{R}^n_+]_p \subset [\mathbb{R}^n_+]_p$. Now it is clear that either $[\mathbb{R}^n_+]_p = \mathbb{R}^n_+$ or $[\mathbb{R}^n_+]_p = \mathbb{R}^n$, but the latter cannot occur because $L^p_{\mathbb{R}^n_+}(\mathbb{R}^n) \ne L^p(\mathbb{R}^n)$ by Theorem 3.1. □

Theorem 3.4. *Every open convex subset of $\mathbb{R}^n$ is L^p-complete for all $p \in (0, +\infty)$.*

Proof. Let Ω be an open convex subset of $\mathbb{R}^n$. The set $\mathbb{R}^n$ is L^p-complete for all $p \in (0, +\infty)$. Suppose $\Omega \ne \mathbb{R}^n$. Then Ω is the intersection of all open half-spaces containing Ω. It remains to use Corollary 3.3 and Theorem 2.19. □

Theorem 3.1 can be generalized: the space $L^p_{\mathbb{R}_+}(\mathbb{R})$ can be replaced with an arbitrary spectral space $L^p_\Omega(\mathbb{R})$ (and then, of course, the set $\mathbb{R}^n_+$ should be replaced with the set $\Omega \times \mathbb{R}^{n-1}$). A more or less final generalization can be stated conveniently for the case of locally compact Abelian groups.

Theorem 3.5. *Let $0 < p < +\infty$. Let G_1 and G_2 be locally compact Abelian groups, Γ_1 and Γ_2 the dual groups. Let Ω be an open subset of Γ_2. Then the space $L^p_{\Gamma_1 \times \Omega}(G_1 \times G_2)$ coincides with the set of all functions $f \in L^p(G_1 \times G_2)$ such that $f(x, \,\cdot\,) \in L^p_\Omega(G_2)$ for almost all $x \in G_1$.*

We omit the proof of this theorem because it is similar to that of Theorem 3.1.

The results of the paper [12] imply that $\Phi_p^{\mathbb{R}_+}(t) = \|\mathcal{G}_t^{\mathbb{R}_+}\|_{(L^p_{\mathbb{R}_+}(\mathbb{R}))^*} \le C(p)t^{\frac{1}{p}-1}$ for all $t>0$ (see §2, Subsection 2.3).[3] From a homogeneity argument and formula (2.3), we obtain $\Phi_p^{\mathbb{R}_+}(t) = t^{\frac{1}{p}-1}\Phi_p^{\mathbb{R}_+}(1) \stackrel{\text{def}}{=} \upsilon_p t^{\frac{1}{p}-1}$ for all $t>0$.

We define an operator A_t by the formula $(A_t f)(x') = \mathcal{G}_t^{\mathbb{R}_+}(f(\,\cdot\,,x'))$.

Lemma 3.6. *Let $t>0$. Then $(A_t f)(x') \in L^p(\mathbb{R}^{n-1})$ for every function $f \in L^p_{\mathbb{R}^n_+}(\mathbb{R}^n)$. The operator A_t acts continuously from $L^p_{\mathbb{R}^n_+}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^{n-1})$, and $\|A_t\| \le \upsilon_p\, t^{\frac{1}{p}-1}$.*

Proof. If $f \in \mathcal{S}^0_{\mathbb{R}^n_+}(\mathbb{R}^n)$, then $|\mathcal{G}_t^{\mathbb{R}_+}(f(\,\cdot\,,x'))|^p \le \upsilon_p^p\, t^{1-p}\int_{\mathbb{R}}|f(s,x')|^p\,ds$ for all $x' \in \mathbb{R}^{n-1}$. Integrating this inequality in x', we get $\|A_t f\|^p_{L^p} \le \upsilon_p^p\, t^{1-p}\|f\|^p_{L^p}$. A standard limit passage completes the proof. □

Corollary 3.7. *Let Ω be an open subset of $\mathbb{R}^n_+$. Put $\Omega_t \stackrel{\text{def}}{=} \{x' \in \mathbb{R}^{n-1} : (t,x') \in \Omega\}$, where $t>0$. Then $A_t(L^p_\Omega(\mathbb{R}^n)) \subset L^p_{\Omega_t}(\mathbb{R}^{n-1})$.*

Proof. We must prove that $A_t f \in L^p_{\Omega_t}(\mathbb{R}^{n-1})$ for every $f \in L^p_\Omega(\mathbb{R}^n)$. It suffices to consider the case where $f \in \mathcal{S}^0_\Omega(\mathbb{R}^n)$, and then the required inclusion is evident. □

Theorem 3.8. *Let Ω be an open convex subset of $\mathbb{R}^n$. Then*

$$C_1(n,p)(V_\Omega(a))^{\frac{1}{p}-1} \le \Phi_p^\Omega(a) \le C_2(n,p)(V_\Omega(a))^{\frac{1}{p}-1}$$

for all $a \in \Omega$.

Proof. The lower estimate is a special case of Theorem 2.11. Next, the theorem is obvious for $\Omega = \mathbb{R}^n$, because in this case $\Phi_p^\Omega(a) = V_\Omega(a) = +\infty$ for all $a \in \mathbb{R}^n$. We prove the upper estimate in the case where $\Omega \ne \mathbb{R}^n$. We use induction on n. Let $n=1$. Then it suffices to consider the following two cases: $\Omega = (0,+\infty)$ and $\Omega = (0,1)$. If $\Omega = (0,+\infty)$, then, as was noted before Lemma 3.6, $\Phi_p^{\mathbb{R}_+}(a) = \upsilon_p\, a^{\frac{1}{p}-1} = 2^{1-\frac{1}{p}}\upsilon_p\,(V_{\mathbb{R}_+}(a))^{\frac{1}{p}-1}$. Now, let $\Omega = (0,1)$, $a \in (0,1)$. Observe that $\Phi_p^{(0,1)}(a) = \Phi_p^{(0,1)}(1-a)$; therefore, we may assume that $a \in (0,\frac{1}{2}]$. Clearly, $\Phi_p^{(0,1)}(a) \le \Phi_p^{\mathbb{R}_+}(a) = \upsilon_p\, a^{\frac{1}{p}-1} = 2^{1-\frac{1}{p}}\upsilon_p\,(V_{(0,1)}(a))^{\frac{1}{p}-1}$ because $a \in (0,\frac{1}{2}]$.

Assuming that the claim is proved in the $(n-1)$-dimensional case, we prove it in the n-dimensional case. Let $t \stackrel{\text{def}}{=} \operatorname{dist}(a,\partial\Omega) = |a-b|$, where $b \in \partial\Omega$. We may assume that $b=0$ and $a=(t,0,0,\dots,0)$. Then $\Omega \subset \mathbb{R}^n_+$. Note that $V_\Omega(a) = |\Omega \cap (2a-\Omega)|$ because Ω is convex. Clearly, the set $\Omega\cap(2a-\Omega)$ contains the set $X \stackrel{\text{def}}{=} \bigcup_{s\in(0,1]}(\{s\,t\}\times(s\,\Omega_t \cap (-s\,\Omega_t)))$. Hence, $V_\Omega(a) \ge |X|_n = \frac{1}{n}t|\Omega_t \cap (-\Omega_t)|_{n-1}$. Applying Lemma 3.6 and Corollary 3.7, we get

$$\begin{aligned}\Phi_p^\Omega(a) &\le \upsilon_p\, t^{\frac{1}{p}-1}\Phi_p^{\Omega_t}(0)\\ &\le C_2(n-1,p)\upsilon_p\, t^{\frac{1}{p}-1}|\Omega_t \cap (-\Omega_t)|_{n-1}^{\frac{1}{p}-1} \le n^{\frac{1}{p}-1}C_2(n-1,p)\upsilon_p\, V_\Omega(a).\end{aligned}$$ □

[3] One can also refer to the known description of the continuous functionals on the Hardy space H^p for $p<1$ (see [9]). However, it should be noted that the paper [9] deals with the Hardy space $H^p(\mathbb{D})$ in the unit disk $\mathbb{D}$. The case of the Hardy class H^p in the half-plane can be treated in a similar way, see also [14], where the so-called real Hardy classes were considered (in the one-dimensional case, these are the spaces $H^p + \overline{H^p}$ in our notation). Then the estimate we need reduces to the elementary inequality $\|e^{2\pi i t x}\|_{\Lambda_{1/p-1}(\mathbb{R})} \le Ct^{\frac{1}{p}-1}$, where $\Lambda_s(\mathbb{R})$ denotes the Besov class $B^s_{\infty,\infty}(\mathbb{R})$ (see [31]).

Theorem 3.9. *Let Ω be an open convex subset of $\mathbb{R}^n$. Then the following two statements are true.*

a) *If Ω contains a straight line, then $\Phi_p^\Omega(a) = +\infty$ for all $a \in \Omega$, and the space of continuous linear functionals on $L_\Omega^p(\mathbb{R}^n)$ consists of the zero functional only.*

b) *If Ω contains no straight lines, then $\Phi_p^\Omega(a) < +\infty$ for all $a \in \Omega$, and the continuous linear functionals on $L_\Omega^p(\mathbb{R}^n)$ separate the points of this space.*

Proof. First, we prove statement a). Suppose Ω contains a straight line. Without loss of generality we may assume that this straight line is parallel to the vector $e_1 \stackrel{\text{def}}{=} (1, 0, \dots, 0)$. Then $\Omega = \mathbb{R} \times \Omega'$, where Ω' is an open convex subset of $\mathbb{R}^{n-1}$. Let H be a continuous linear functional on the space $L_\Omega^p(\mathbb{R}^n)$. Fixing an arbitrary function $\psi \in L_{\Omega'}^p(\mathbb{R}^{n-1})$, we define a functional $L = L_\psi$ on the space $L^p(\mathbb{R})$ by the formula $L(f) = H(f(x_1)\psi(x'))$. It is easily seen that L is a continuous linear functional on $L^p(\mathbb{R})$. Consequently, $L = 0$. Thus, we have proved that $H(f(x_1)\psi(x')) = 0$ for all $f \in L^1(\mathbb{R})$ and all $\psi \in L_{\Omega'}^p(\mathbb{R}^{n-1})$. Hence, $H = 0$ by Theorem 3.5 and Lemma 3.2.

Now, suppose that the set Ω contains no straight lines. We use induction on the dimension to prove that $\Phi_p^\Omega(a) < +\infty$ for all $a \in \Omega$ and that the continuous linear functionals $\{\mathcal{G}_a^\Omega\}_{a\in\Omega}$ separate the points of the space $L_\Omega^p(\mathbb{R}^n)$. For $n = 1$ this is obvious. Assuming that the claim is true in the $(n-1)$-dimensional case, we prove it in the n-dimensional case. We shall employ the notation of the proof of Theorem 3.8. In that proof the inequality $\Phi_p^\Omega(a) \le C_2(n-1,p)\upsilon_p\, t^{\frac{1}{p}-1}\Phi_p^{\Omega_t}(0)$ was established, whence $\Phi_p^\Omega(a) < +\infty$, because $\Phi_p^{\Omega_t}(0) < +\infty$ by the inductive hypothesis. Assume that $\mathcal{G}_a^\Omega f = 0$ for all $a \in \Omega$. Then, by the inductive hypothesis, $A_t f = 0$ for all $t > 0$, whence $f = 0$. □

Remark 3.10. Let

$$\Omega_j = \{x \in \mathbb{R}^n : x_j > 0\}, \quad \Omega = \bigcap_{j=1}^n \Omega_j = (0, +\infty)^n.$$

It is easily seen that

$$L_\Omega^p(\mathbb{R}^n) = \bigcap_{j=1}^n L_{\Omega_j}^p(\mathbb{R}^n)$$

for all $p \in [1, +\infty)$. In particular, $f \in L_\Omega^p(\mathbb{R}^n)$ if and only if the function $t \mapsto f(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n)$ belongs to the Hardy class H^p for all j and almost all $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \mathbb{R}^{n-1}$. The following example shows that this is not true for $p < 1$ and $n \ge 2$.

Example 3.11. Suppose $f \in L_{\mathbb{R}_+}^p(\mathbb{R})$ and $g \in L_{\mathbb{R}_+}^p(\mathbb{R}) \cap L_{\mathbb{R}_-}^p(\mathbb{R})$. Then

$$f(x_1)g(x_1 - x_2) \in L_{\mathbb{R}\times(0,+\infty)}^p(\mathbb{R}^2) \cap L_{(0,+\infty)\times\mathbb{R}}^p(\mathbb{R}^2), \tag{3.4}$$

but $f(x_1)g(x_1 - x_2) \notin L_{\mathbb{R}_+\times\mathbb{R}_+}^p(\mathbb{R}^2)$ if none of the functions f or g is equal to zero almost everywhere on $\mathbb{R}$.

Proof. The inclusion (3.4) follows from the Fubini theorem and the fact that $g(a - x) \in L_{\mathbb{R}_+}^p(\mathbb{R})$ for all $a \in \mathbb{R}$. There exists a function $F \in H^p$ such that $F^* = f$ almost everywhere on $\mathbb{R}$, where $F^*(x) = \lim_{t\to 0+} F(x + ti)$. Suppose that $f(x_1)g(x_1 - x_2) \in L_{\mathbb{R}_+\times\mathbb{R}_+}^p(\mathbb{R}^2)$. Then there exists a holomorphic function $\Psi : \mathbb{C}_+ \times \mathbb{C}_+ \to \mathbb{C}$ in the Hardy space $H^p(\mathbb{C}_+ \times \mathbb{C}_+)$ such that $\Psi^*(x_1, x_2) = f(x_1)g(x_1 - x_2)$ for almost all $x \in \mathbb{R}^2$, where $\Psi^*(x_1, x_2) = \lim_{t\to 0+} \Psi(x_1 + ti, x_2 + ti)$.

With each point $x = (x_1, x_2) \in \mathbb{R}^2$ we can associate the slice function $\Psi_x : \mathbb{C}_+ \to \mathbb{C}$, $\Psi_x(\lambda) \stackrel{\text{def}}{=} \Psi(x_1 + \lambda, x_2 + \lambda)$. It is easily seen that $\Psi_x \in H^p$ for almost all $x \in \mathbb{R}^2$. Clearly,

for almost all $x \in \mathbb{R}^2$ we have

$$\Psi_x^*(t) = \Psi^*(x_1 + t, x_2 + t) = F^*(x_1 + t)g(x_1 - x_2)$$

for almost all $t \in \mathbb{R}$. Hence, for almost all $x \in \mathbb{R}^2$ we have

$$\Psi_x(\lambda) = F(x_1 + \lambda)g(x_1 - x_2) \tag{3.5}$$

for all $\lambda \in \mathbb{C}_+$. If f is a nonzero function, then there exists a number $\lambda_0 \in \mathbb{C}$ such that $F(\lambda_0 + t) \neq 0$ for all $t \in \mathbb{R}$. Substituting $\lambda = \lambda_0$ in (3.5), we see that the function $g(x_1 - x_2)$ coincides almost everywhere on $\mathbb{R}^2$ with a continuous function defined on $\mathbb{R}^2$. Hence, we may assume that the function g is defined and continuous everywhere on $\mathbb{R}$. Since $g \in L^p_{\mathbb{R}_+}(\mathbb{R}) \cap L^p_{\mathbb{R}_-}(\mathbb{R})$, there exists a holomorphic function $G : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ belonging to the Hardy class H^p both in the upper half-plane and in the lower half-plane and such that $g(x) = \lim_{t\to 0+} G(x + ti) = \lim_{t\to 0-} G(x + ti)$ for almost all $x \in \mathbb{R}$. Since g is continuous, the function G extends up to an entire function. It remains to note that only the zero entire function G can satisfy the condition $\sup_{t\in\mathbb{R}} \int_{\mathbb{R}} |G(x + ti)|^p \, dx < +\infty$. □

§4. Open sets Ω of finite Lebesgue measure

Let $-\infty < a < b < +\infty$. It is not difficult to verify that the space $L^p_{(a,b)}(\mathbb{R})$ coincides with the set of all functions of the form $F|_{\mathbb{R}}$, where F is an entire function such that $e^{-2\pi iaz}F(z)|_{\mathbb{C}_+} \in H^p$ and $e^{2\pi ibz}\widetilde{F}(z)|_{\mathbb{C}_+} \in H^p$; here $\widetilde{F}(z) \stackrel{\text{def}}{=} \overline{F(\overline{z})}$. In particular, the space $L^p_{(0,b)}(\mathbb{R})$ is the so-called model space K^p_S corresponding to the inner function $S(z) = e^{2\pi ibz}$. There are many papers devoted to model spaces, which for $p = 2$ play an important role in operator theory (see [21]).

In this section we shall show that some elementary properties of the model space corresponding to the inner function $S(z) = e^{2\pi ibz}$ depend only on the fact that $K^p_S = L^p_\Omega(\mathbb{R})$, where Ω is an open set of finite Lebesgue measure.

Theorem 4.1. *Let Ω be an open subset of $\mathbb{R}^n$ with $|\Omega| < +\infty$. Suppose that $0 < p \le 2$. Then $L^p_\Omega(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ and $\|f\|_{L^q} \le |\Omega|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^p}$ for all $f \in L^p_\Omega(\mathbb{R}^n)$ and all $q \in [p, +\infty]$.*

Proof. It suffices to obtain the required estimate for $f \in \mathcal{S}_\Omega(\mathbb{R}^n)$. Moreover, it suffices to consider the case where $q = +\infty$. We have

$$\|f\|_{L^\infty} \le \|\mathcal{F}f\|_{L^1} \le |\Omega|^{\frac{1}{2}}\|\mathcal{F}f\|_{L^2} = |\Omega|^{\frac{1}{2}}\|f\|_{L^2}.$$

Consequently,

$$\|f\|_{L^2} \le \|f\|^{\frac{p}{2}}_{L^p}\|f\|^{1-\frac{p}{2}}_{L^\infty} \le |\Omega|^{\frac{1}{2}-\frac{p}{4}}\|f\|^{\frac{p}{2}}_{L^p}\|f\|^{1-\frac{p}{2}}_{L^2},$$

whence

$$\|f\|_{L^2} \le |\Omega|^{\frac{1}{p}-\frac{1}{2}}\|f\|_{L^p}.$$

Now it is clear that $\|f\|_{L^\infty} \le |\Omega|^{\frac{1}{p}}\|f\|_{L^p}$. □

Corollary 4.2. *Let Ω be the same as in Theorem 4.1. Let $a \in \mathbb{R}^n$. Then:*
a) *the functional $f \mapsto f(a)$ is continuous on $L^p_\Omega(\mathbb{R}^n)$ provided $0 < p \le 2$;*
b) *the functional $f \mapsto (\mathcal{F}f)(a)$ is continuous on $L^p_\Omega(\mathbb{R}^n)$ provided $0 < p \le 1$.*

Proof. To prove a), we observe that the identical embedding $L^p_\Omega(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ is continuous, and thus, so is the embedding $L^p_\Omega(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Statement b) follows from the continuity of the identical embedding $L^p_\Omega(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. □

Corollary 4.3. *Let Ω be the same as in Theorem 4.1. Then the continuous linear functionals separate the points of $L^p_\Omega(\mathbb{R}^n)$.*

Corollary 4.4. *Let Ω be the same as in Theorem 4.1. Then $[\Omega]_p = \Omega$ for $p \le 1$.*

Remark 4.5. The condition $p \le 2$ is essential in Theorem 4.1.

Proof. Let $p > 2$. It is well known[4] that there exists a nonzero function $f \in L^p(\mathbb{R}^n)$ such that $|\operatorname{supp} \mathcal{F}f| = 0$. Let Ω be an open set of finite Lebesgue measure containing the set $\bigcup_{k=0}^{\infty} 2^k \operatorname{supp} \mathcal{F}f$. It is easily seen that $f(2^k x) \in L^p_\Omega(\mathbb{R}^n)$ for all $k \in \mathbb{Z}_+$. It remains to observe that $\lim_{k\to+\infty} \frac{\|f(2^k x)\|_{L^q}}{\|f(2^k x)\|_{L^p}} = +\infty$ provided $q > p$. Hence, $L^p_\Omega(\mathbb{R}^n) \not\subset L^q(\mathbb{R}^n)$ for $q > p$. □

Remark 4.6. It is not difficult to prove that the inclusion $L^p_\Omega(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$, $q > p$, remains valid for $p > 2$ provided Ω is bounded, and $\|f\|_{L^q} \le C(n,p,q)(\operatorname{diam}\Omega)^{\frac{n}{p}-\frac{n}{q}}\|f\|_{L^p}$.

Remark 4.7. It can be seen from the proof of Theorem 4.1 that the argument is also applicable to functions f taking values in a Hilbert space. In particular, this leads to the following result. Suppose that $\Big(\sum_{n=1}^{\infty}|f_n|^2\Big)^{\frac12} \in L^p(\mathbb{R}^n)$, where $f_n \in L^p_\Omega(\mathbb{R}^n)$. Then

$$\Big\|\Big(\sum_{n=1}^{\infty}|f_n|^2\Big)^{\frac12}\Big\|_{L^q} \le |\Omega|^{\frac1p-\frac1q}\Big\|\Big(\sum_{n=1}^{\infty}|f_n|^2\Big)^{\frac12}\Big\|_{L^p} \quad (0<p\le 2,\ \ p\le q\le +\infty)$$

and

$$\Big\|\Big(\sum_{n=1}^{\infty}|f_n|^2\Big)^{\frac12}\Big\|_{L^q} \le C(n,p,q)(\operatorname{diam}\Omega)^{\frac{n}{p}-\frac{n}{q}}\Big\|\Big(\sum_{n=1}^{\infty}|f_n|^2\Big)^{\frac12}\Big\|_{L^p} \quad (2<p\le q\le +\infty).$$

Example 4.8. Let $\Omega = \bigcup_{n=1}^{\infty}(2^n, 2^n+1)$. Then $L^p_\Omega(\mathbb{R}) \subset L^q(\mathbb{R})$ for all $q \in [p,+\infty)$.

Proof. The required inclusion can be reduced to Remark 4.7 if we use the following statement.

A function $f \in L^p(\mathbb{R})$ belongs to $L^p_\Omega(\mathbb{R})$ if and only if $f = \sum_{n=1}^{\infty} f_n(x)e^{2^{n+1}\pi i x}$, where $f_n \in L^p_{(0,1)}(\mathbb{R})$ and $(\sum_{n=1}^{\infty}|f_n|^2)^{\frac12} \in L^p(\mathbb{R})$.

For $p > 1$, this follows from the well-known Littlewood–Paley inequality:

$$C_1(p)\|f\|_{L^p} \le \Big\|\Big(\sum_{n\in\mathbb{Z}}|f*\chi_n|^2\Big)^{\frac12}\Big\|_{L^p} \le C_2(p)\|f\|_{L^p}$$

for every $f \in \mathcal{S}_{\mathbb{R}_+}(\mathbb{R})$; here $\chi_n = \mathcal{F}^{-1}\mathbb{1}_{[2^n,2^{n+1})}$. A similar inequality is true for $p \le 1$ if we replace the functions $\mathbb{1}_{[2^n,2^{n+1})}$ with the corresponding smooth functions φ_n (see, e.g., [31]). We only need to note that the functions φ_n can be chosen in a such way that $\varphi_n = 1$ on the interval $(2^n, 3\cdot 2^{n-1})$. □

Theorem 4.1 can easily be extended to the case of locally compact Abelian groups.

Theorem 4.9. *Let G be a locally compact Abelian group. Let Ω be an open subset of the dual group Γ with $\boldsymbol{m}_\Gamma(\Omega) < +\infty$. Suppose that $0 < p \le 2$. Then $L^p_\Omega(G) \subset L^q(G)$ and $\|f\|_{L^q} \le (\boldsymbol{m}_\Gamma(\Omega))^{\frac1p-\frac1q}\|f\|_{L^p}$ for all $f \in L^p_\Omega(G)$ and $q \in [p,+\infty]$.*

§5. Sufficient conditions for the identity $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$

Let G be a locally compact Abelian group, Γ the dual group. An open subset Ω of Γ is said to be *sufficient* if for every compact subset K of Γ there exists an element $\gamma \in \Gamma$ such that $\gamma + K \subset \Omega$ and $-\gamma + K \subset \Omega$. The following theorem is essentially a special

[4] In particular, the Ivashev-Musatov theorem (see, e.g., [17]) shows that there exists a probability measure μ on $\mathbb{R}^n$ such that $\operatorname{supp}\mu$ is a compact set of Lebesgue measure zero and $|(\mathcal{F}^{-1}\mu)(x)| \le C\prod_{j=1}^{n}(1+|x_j|)^{-\frac12}$ for all $x \in \mathbb{R}^n$.

case of Theorem 9.1 in the author's paper [1].[5] For the reader's convenience, here we present the proof of this theorem in the case where $G = \mathbb{R}^n$.

Theorem 5.1. *Let Ω be a sufficient open subset of the group Γ. Then $L^p_\Omega(G) = L^p(G)$.*

Proof. Let $G = \mathbb{R}^n$. Then the group Γ is identified with $\mathbb{R}^n$ in a natural way. Let $\varphi \in \mathcal{S}^0(\mathbb{R}^n)$. There exists a vector $a \in \mathbb{R}^n$ such that $-a + \operatorname{supp} \mathcal{F}\varphi \subset \Omega$ and $a + \operatorname{supp} \mathcal{F}\varphi \subset \Omega$. Put $f_t \stackrel{\text{def}}{=} \frac{1}{2}(e^{2\pi i((x,a)+t)} + e^{-2\pi i((x,a)+t)})\varphi$. It is clear that $f_t \in L^p_\Omega(\mathbb{R}^n)$ for all $t \in [0,1)$. By the Fubini theorem,

$$\int_0^1 \|\varphi - f_t\|^p_{L^p}\, dt = \eta_p^p \|\varphi\|^p_{L^p},$$

where $\eta_p^p \stackrel{\text{def}}{=} \int_0^1 (1 - \frac{1}{2}(e^{2\pi i t} + e^{-2\pi i t}))^p\, dt < \big(\int_0^1 (1 - \frac{1}{2}(e^{2\pi i t} + e^{-2\pi i t}))\, dt\big)^p = 1$. Consequently, there exists $t_0 \in [0,1)$ such that $\|\varphi - f_{t_0}\|^p_{L^p} \le \eta_p^p \|\varphi\|^p_{L^p}$. Thus, we have proved that $\operatorname{dist}_{L^p}(\varphi, L^p_\Omega(\mathbb{R}^n)) \stackrel{\text{def}}{=} \inf\{\|\varphi - f\|_{L^p} : f \in L^p_\Omega(\mathbb{R}^n)\} \le \eta_p \|\varphi\|_{L^p}$ for all $\varphi \in \mathcal{S}^0(\mathbb{R}^n)$. Hence, this inequality is true for all $\varphi \in L^p(\mathbb{R}^n)$, because the set $\mathcal{S}^0(\mathbb{R}^n)$ is dense in the space $L^p(\mathbb{R}^n)$. Now it is clear that $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. □

Corollary 5.2. *Assume that the upper density of a subset A of $\mathbb{Z}$ is equal to 1, i.e.,*

$$\limsup_{k \to +\infty} \frac{\operatorname{card}(A \cap [-k,k])}{2k+1} = 1.$$

Then $L^p_A(\mathbb{T}) = L^p(\mathbb{T})$.

This corollary was proved in [1, Corollary 8.5] in an even stronger form. Of course, a similar statement can be proved in exactly the same way for subsets of the group $\mathbb{Z}^n$. An analog of this statement for the group $\mathbb{R}$ (again we restrict ourselves to the one-dimensional case only) looks like this.

Corollary 5.3. *Let $\Omega \subset \mathbb{R}$. Suppose that*

$$\liminf_{t \to +\infty} \frac{\operatorname{card}((\mathbb{R} \setminus \Omega) \cap [-t,t])}{2t} = 0.$$

Then $L^p_\Omega(\mathbb{R}) = L^p(\mathbb{R})$.

It is of interest to note that in this statement the number of elements in the set $(\mathbb{R} \setminus \Omega) \cap [-t,t]$ cannot be replaced with the Lebesgue measure of this set. In the next section we shall see that $L^p_{\mathbb{R}\setminus\mathbb{Z}}(\mathbb{R}) \ne L^p(\mathbb{R})$.

Remark 5.4. If we replace the set $\mathcal{S}^0(\mathbb{R}^n)$ with the set $\{f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) : \operatorname{supp} \mathcal{F}f$ is compact$\}$ in the above proof of Theorem 5.1, then this proof extends almost literally to the case of locally compact Abelian groups.

Example 5.5. Let Ω be a nonempty open convex cone in $\mathbb{R}^n$. Then Theorem 5.1 implies that $L^p_{(-\Omega)\cup\Omega}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. In fact, Theorem 9.1 of [1] implies even the following stronger statement: $L^p_{-\Omega}(\mathbb{R}^n) + L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

It is not difficult to understand that the cone can be replaced with a much smaller set. The following example allows us to illustrate this in the two-dimensional case.

Example 5.6. Let $h : (\sigma, +\infty) \to \mathbb{R}$ be a continuous function with $\lim_{t\to+\infty} h(t) = +\infty$. Define the set $\Omega \stackrel{\text{def}}{=} \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 > \sigma, |\xi_2| < h(\xi_1)\}$. Then by Theorem 5.1 we have $L^p_{\Omega\cup(-\Omega)}(\mathbb{R}^2) = L^p(\mathbb{R}^2)$; moreover, as in the preceding example, the stronger property $L^p_\Omega(\mathbb{R}^2) + L^p_{(-\Omega)}(\mathbb{R}^2) = L^p(\mathbb{R}^2)$ is true.

[5] Rigorously speaking, the definition of the space $L^p_\Omega(G)$ in [1] differs from that used in the present paper, but practically this does not affect the proof.

Theorem 5.7. *Let Π and Ω be open subsets in $\mathbb{R}^n$ with $\Omega \subset \Pi$. Suppose that for every compact subset K of Π there exists a vector $a \in \mathbb{R}^n$ such that $-a+K \subset \Omega$ and $a+K \subset \Omega$. Then $L^p_\Pi(\mathbb{R}^n) = L^p_\Omega(\mathbb{R}^n)$.*

Proof. We can repeat the proof of Theorem 5.1 almost word for word, except that the space $\mathcal{S}^0(\mathbb{R}^n)$ should be replaced with the space $\mathcal{S}^0_\Pi(\mathbb{R}^n)$. □

Corollary 5.8. *Let Π be an open subset of $\mathbb{R}^n$. Suppose that Π is periodic; i.e., $a+\Pi = \Pi$ for some nonzero vector $a \in \mathbb{R}^n$. Then $L^p_{\Pi\setminus K}(\mathbb{R}^n) = L^p_\Pi(\mathbb{R}^n)$ for every compact subset K of $\mathbb{R}^n$.*

Corollary 5.9. *Let U be an open subset in $\mathbb{R}^n$, and let $a \in \mathbb{R}^n$. Put $\Pi = \bigcup_{k\in\mathbb{Z}}(ka+U)$ and $\Omega = \bigcup_{k\in\mathbb{Z}\setminus\{0\}}(ka+U)$. Then $L^p_\Pi(\mathbb{R}^n) = L^p_\Omega(\mathbb{R}^n)$.*

Remark 5.10. Theorem 5.7 can be extended to the case of groups.

Example 5.11. Let $\Pi = \{(\xi_1,\xi_2) : \xi_2 > 0\}$. Let h be the same as in Example 5.6. Define the set $\Omega = \{(\xi_1,\xi_2) : |\xi_1| > \sigma, 0 < \xi_2 < h(|\xi_1|)\}$. Then $L^p_\Pi(\mathbb{R}^n) = L^p_\Omega(\mathbb{R}^n)$.

Theorem 5.12. *Let Π and Ω be open subsets in $\mathbb{R}^n$. Suppose that for every point $x \in \Pi \setminus \Omega$ there exists a neighborhood U of x and a nonzero vector $a \in \mathbb{R}^n$ such that $U + ka \subset \Omega$ for all $k \in \mathbb{Z}$, $k \neq 0$. Then $L^p_\Pi(\mathbb{R}^n) \subset L^p_\Omega(\mathbb{R}^n)$.*

Proof. A subset K of Π is said to be *small* if there exists a (not necessarily nonzero) vector $a \in \mathbb{R}^n$ such that $K + ka \subset \Omega$ for all $k \in \mathbb{Z}$, $k \neq 0$. Note that for every point $x \in \Pi$ there exists a small neighborhood of x.

Let $\varphi \in \mathcal{S}^0_\Pi(\mathbb{R}^n)$. We prove that $\varphi \in L^p_\Omega(\mathbb{R}^n)$. The support of the function $\mathcal{F}\varphi$ can be covered by finitely many small neighborhoods. A smooth partition of unity subordinate to this covering allows us to restrict ourselves to the case where there exists a small neighborhood U of the set $\operatorname{supp}\mathcal{F}\varphi$. If $U \subset \Omega$, then there is nothing to prove. Otherwise, there exists a nonzero vector $a \in \mathbb{R}^n$ such that $ka + U \subset \Omega$ for all $k \in \mathbb{Z}$, $k \neq 0$. Then the inclusion $\varphi \in L^p_\Omega(\mathbb{R}^n)$ follows from Corollary 5.9. □

§6. The space $L^p_\Omega(\mathbb{R}^n)$ for $\Omega = \mathbb{R}^n \setminus \mathbb{Z}^n$

We define an operator $\mathcal{E} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n/\mathbb{Z}^n)$ by the formula $\mathcal{E}f = \sum_{k\in\mathbb{Z}^n} f(x+k)$. The proof of the following theorem shows that the above multiple series converges absolutely for almost all $x \in \mathbb{R}^n$.

Theorem 6.1. *Let $0 < p \le 1$. Then the operator $\mathcal{E}$ acts continuously from the space $L^p(\mathbb{R}^n)$ to the space $L^p(\mathbb{R}^n/\mathbb{Z}^n)$, $\|\mathcal{E}\|_{L^p\to L^p} = 1$. Moreover, $\mathcal{E}(L^p(\mathbb{R}^n)) = L^p(\mathbb{R}^n/\mathbb{Z}^n)$.*

Proof. Let $f \in L^p(\mathbb{R}^n)$. Clearly,

$$\|\mathcal{E}f\|^p_{L^p(\mathbb{R}^n/\mathbb{Z}^n)} = \|\mathbb{1}_{[0,1)^n}\mathcal{E}f\|^p_{L^p(\mathbb{R}^n)} \le \sum_{k\in\mathbb{Z}^n} \|\mathbb{1}_{[0,1)^n} f(x+k)\|^p_{L^p(\mathbb{R}^n)} = \|f\|^p_{L^p(\mathbb{R}^n)}.$$

Moreover, $\mathcal{E}(\mathbb{1}_{[0,1)^n} g) = g$ for every $g \in L^p(\mathbb{R}^n/\mathbb{Z}^n)$. □

Theorem 6.2. *Let $0 < p \le 1$. Then the kernel of the operator $\mathcal{E} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n/\mathbb{Z}^n)$ coincides with $L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$. In particular, $L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n) \neq L^p(\mathbb{R}^n)$.*

Proof. Let $\varphi \in \mathcal{S}_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$. Then, by the Poisson summation formula (see, e.g., [29, §2 of Chapter 7]), we have

$$\sum_{k\in\mathbb{Z}^n} \varphi(x+k) = \sum_{k\in\mathbb{Z}^n} (\mathcal{F}\varphi)(k) e^{2\pi i(x,k)} = 0.$$

Hence, $L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n) \subset \ker\mathcal{E}$.

We prove that $\ker\mathcal{E}\subset L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$. Let $f\in\ker\mathcal{E}$, and let a function $g\in L^p(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ be such that $\|f-g\|_{L^p}<\varepsilon$ and the support of g is compact. Note that $g-\mathbb{1}_{[0,1)^n}\mathcal{E}g\in\ker\mathcal{E}$ and

$$\|f-(g-\mathbb{1}_{[0,1)^n}\mathcal{E}g)\|^p_{L^p}=\|(f-g)-\mathbb{1}_{[0,1)^n}\mathcal{E}(f-g)\|^p_{L^p}\le 2\|f-g\|^p_{L^p}<2\varepsilon^p.$$

Thus, we may assume that $g\in\ker\mathcal{E}$. Taking a function $\Phi\in\mathcal{S}^0(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\Phi(x)\,dx=1$, we put $\Phi_\varepsilon(x)\overset{\text{def}}{=}\varepsilon^{-n}\Phi(\varepsilon^{-1}x)$, where $\varepsilon>0$. Clearly, $\lim_{\varepsilon\to0}\Phi_\varepsilon*g=g$ in $L^p(\mathbb{R}^n)$ and $\Phi_\varepsilon*g\in\ker\mathcal{E}$ for all $\varepsilon>0$. Hence, the function f can be approximated in $L^p(\mathbb{R}^n)$ as closely as we wish by a function $h\in\mathcal{S}^0(\mathbb{R}^n)\cap\ker\mathcal{E}$. Thus, we need to prove that $\mathcal{S}^0(\mathbb{R}^n)\cap\ker\mathcal{E}\subset L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$. Let $f\in\mathcal{S}^0(\mathbb{R}^n)\cap\ker\mathcal{E}$. We check the inclusion $f\in L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$. The Poisson summation formula shows that $\mathcal{F}f(k)=0$ for all $k\in\mathbb{Z}^n$. Clearly, it suffices to consider the case where $\operatorname{diam}(\operatorname{supp}\mathcal{F}f)<1$. Then the intersection $\mathbb{Z}^n\cap\operatorname{supp}\mathcal{F}f$ consists of at most one element. If it is empty, then $f\in\mathcal{S}^0_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)\subset L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$. If this intersection is nonempty, then we can find a neighborhood U of the set $\operatorname{supp}\mathcal{F}f$ such that $\operatorname{diam}U<1$. Let $U\cap\mathbb{Z}^n=\{a\}$. Note that $f\in\mathcal{S}^0_U(\mathbb{R}^n)\subset\mathcal{S}_U(\mathbb{R}^n)$ and $(\mathcal{F}f)(a)=0$. By Theorem 2.5, for every $\varepsilon>0$ there exists a function $g\in\mathcal{S}_{U\setminus\{a\}}(\mathbb{R}^n)\subset\mathcal{S}_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)\subset L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$ such that $\|f-g\|_{L^p}<\varepsilon$. □

Corollary 6.3. *Let $\Sigma=\mathbb{R}^n\setminus[0,1)^n$. Then the operator $f\mapsto f|_\Sigma$ is an isomorphism from $L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$ onto $L^p(\Sigma)$.*

Proof. Theorem 6.2 implies that the operator $V:L^p(\Sigma)\to L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$,

$$(Vg)(x)\overset{\text{def}}{=}\begin{cases}-\sum\limits_{k\in\mathbb{Z}^n\setminus\{0\}}g(x+k) & \text{if } x\in[0,1)^n,\\ g(x) & \text{if } x\notin[0,1)^n,\end{cases}$$

is inverse to the restriction operator $f\mapsto f|_\Sigma$. □

Remark 6.4. It is clear that the space $L^p_{\mathbb{R}\setminus\mathbb{Z}}(\mathbb{R})$ coincides with the closure in $L^p(\mathbb{R})$ of the linear hull of the family of subspaces $\{e^{2\pi inx}L^p_{(0,1)}(\mathbb{R})\}_{n\in\mathbb{Z}}$. Thus, the space $L^p_{\mathbb{R}\setminus\mathbb{Z}}(\mathbb{R})$ can be defined in terms of the model space K^p_S mentioned at the beginning of §4.

A measurable set $\Sigma\subset\mathbb{R}^n$ is called a *uniqueness set* for a subspace X of $L^p(\mathbb{R}^n)$ if for $f\in X$ we have $f=0$ almost everywhere on $\mathbb{R}$ whenever $f=0$ almost everywhere on Σ.

Theorem 6.5. *Let E be a measurable subset of $\mathbb{R}^n$. The following statements are equivalent.*

1) *$\mathbb{R}^n\setminus E$ is a uniqueness set for the space $L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$.*
2) *$|E\cap(k+E)|=0$ for all $k\in\mathbb{Z}^n\setminus\{0\}$.*
3) *$\mathcal{E}\mathbb{1}_E\le1$ almost everywhere on $\mathbb{R}^n$.*

Proof. The equivalence 2) $\iff$ 3) is obvious. We prove that 2) $\implies$ 1). Assume that a function $f\in L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$ vanishes on the set $\mathbb{R}^n\setminus E$. Condition 2) implies that the functions in the family $\{f(x+k)\}_{k\in\mathbb{Z}^n}$ are supported on pairwise disjoint sets. Hence, if $\mathcal{E}f=0$, then $f=0$.

It remains to prove that 1) $\implies$ 2). Let $k\in\mathbb{Z}^n\setminus\{0\}$. Put $h=\mathbb{1}_E\mathbb{1}_{E+k}$. The function $f=h(x)-h(x+k)$ belongs to the space $L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$ and equals 0 almost everywhere outside of E. Consequently, $f=0$, whence $h=0$. □

A measurable set $\Sigma\subset\mathbb{R}^n$ is said to be *interpolating* for a subspace X of $L^p(\mathbb{R}^n)$ if for every $g\in L^p(\mathbb{R}^n)$ there exists $f\in X$ such that $f|_\Sigma=g|_\Sigma$ almost everywhere.

Theorem 6.6. *Let E be a measurable subset of $\mathbb{R}^n$. The following statements are equivalent.*

1) *$\mathbb{R}^n \setminus E$ is an interpolating set for the space $L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$.*
2) *The family $\{k+E\}_{k\in\mathbb{Z}^n}$ covers $\mathbb{R}^n$ up to a set of measure 0.*
3) *$\mathcal{E}\mathbb{1}_E \geq 1$ almost everywhere on $\mathbb{R}^n$.*

Proof. We only prove that 1) $\Longleftrightarrow$ 2), because the equivalence of 2) and 3) is obvious. To prove that 1) $\Longrightarrow$ 2), suppose that condition 2) is not fulfilled. Then $|A| > 0$, where $A = \mathbb{R}^n \setminus \bigcup_{k\in\mathbb{Z}^n}(E+k)$. The set A must be interpolating for the space $L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$, because $A \subset \mathbb{R}^n \setminus E$. But on the other hand, since $A + k = A$ for all $k \in \mathbb{Z}^n$, we see that A cannot be interpolating, because $\sum_{k\in\mathbb{Z}^n} f(x+k) = 0$ almost everywhere on A for every $f \in L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$, a contradiction.

Now we prove that 2) $\Longrightarrow$ 1). Suppose condition 2) is fulfilled. We may assume that $\{k+E\}_{k\in\mathbb{Z}^n}$ is a covering of $\mathbb{R}^n$ (in the literal sense). Then $\mathbb{R}^n \setminus E \subset \bigcup_{k\in\mathbb{Z}^n\setminus\{0\}}(k+E)$. Hence, there exists a measurable partition $\{e_k\}_{k\in\mathbb{Z}^n\setminus\{0\}}$ of the set $\mathbb{R}^n \setminus E$ such that $e_k \subset k+E$ for all $k \in \mathbb{Z}^n$, $k \neq 0$. Let $g \in L^p(\mathbb{R}^n)$. Put $f = \sum_{k\in\mathbb{Z}^n\setminus\{0\}}(g_k(x) - g_k(x-k))$, where $g_k = g\mathbb{1}_{e_k}$. Note that the series converges unconditionally in $L^p(\mathbb{R}^n)$, because

$$\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \|g_k(x) - g_k(x-k)\|^p_{L^p} \leq 2 \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \|g_k(x)\|^p_{L^p} \leq 2\|g\|^p_{L^p}.$$

Moreover, $f \in L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$ because $g_k(x) - g_k(x-k) \in L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$ for all k. It remains to note that $f|_{\mathbb{R}^n\setminus E} = g|_{\mathbb{R}^n\setminus E}$. □

Theorem 6.7. *Let Λ be a subset of $\mathbb{Z}^n$. Then*

$$L^p_{\mathbb{R}^n\setminus\Lambda}(\mathbb{R}^n) = \Big\{ f \in L^p(\mathbb{R}^n) : \mathcal{E}f \in L^p_{\mathbb{Z}^n\setminus\Lambda}(\mathbb{R}^n/\mathbb{Z}^n) \Big\}$$

for all $p \in (0,1]$.

Proof. It suffices to check the inclusion

$$\Big\{ f \in L^p(\mathbb{R}^n) : \mathcal{E}f \in L^p_{\mathbb{Z}^n\setminus\Lambda}(\mathbb{R}^n/\mathbb{Z}^n) \Big\} \subset L^p_{\mathbb{R}^n\setminus\Lambda}(\mathbb{R}^n), \tag{6.1}$$

because the reverse inclusion can be obtained with the help of the Poisson summation formula in exactly the same way as in the case of $\Lambda = \mathbb{Z}^n$ (see the proof of Theorem 6.2). First, we prove (6.1) for $\Lambda = \mathbb{Z}^n \setminus \{0\}$. For this, we must verify that if $f \in L^p(\mathbb{R}^n)$ and $\mathcal{E}f = \text{const}$, then $f \in L^p_{\mathbb{R}^n\setminus(\mathbb{Z}^n\setminus\{0\})}(\mathbb{R}^n)$. We may assume that $\mathcal{E}f = 1$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $(\mathcal{F}\varphi)(0) = 1$ and $\operatorname{diam}(\operatorname{supp}\mathcal{F}\varphi) < 1$. Then $\varphi \in \mathcal{S}^0_{\mathbb{R}^n\setminus(\mathbb{Z}^n\setminus\{0\})}(\mathbb{R}^n) \subset L^p_{\mathbb{R}^n\setminus(\mathbb{Z}^n\setminus\{0\})}(\mathbb{R}^n)$. Moreover, $f - \varphi \in L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n) \subset L^p_{\mathbb{R}^n\setminus(\mathbb{Z}^n\setminus\{0\})}(\mathbb{R}^n)$ by Theorem 6.2. So, we have proved (6.1) for $\Lambda = \mathbb{Z}^n \setminus \{0\}$. By (2.1), the case where $\Lambda = \mathbb{Z}^n \setminus \{k\}$ with $k \in \mathbb{Z}^n$ reduces to the case where $\Lambda = \mathbb{Z}^n \setminus \{0\}$. Now we prove the inclusion (6.1) for an arbitrary set $\Lambda \subset \mathbb{Z}^n$.

Let $g \in L^p_{\mathbb{Z}^n\setminus\Lambda}(\mathbb{R}^n/\mathbb{Z}^n)$, and let $f \in L^p(\mathbb{R}^n)$ be such that $\mathcal{E}f = g$. We need to prove that $f \in L^p_{\mathbb{R}^n\setminus\Lambda}(\mathbb{R}^n)$. Note that $f - g\mathbb{1}_{[0,1)^n} \in L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n) \subset L^p_{\mathbb{R}^n\setminus\Lambda}(\mathbb{R}^n)$. It remains to show that $g\mathbb{1}_{[0,1)^n} \in L^p_{\mathbb{R}^n\setminus\Lambda}(\mathbb{R}^n)$. For this, it suffices to consider the case where $g = e^{2\pi i(x,k)}$ with $k \in \mathbb{Z}^n \setminus \Lambda$. It remains to note that this case reduces to the inclusion (6.1) for $\Lambda = \mathbb{Z}^n \setminus \{k\}$, which has already been proved. □

Theorem 6.8. *Let Λ be a closed subset of $\mathbb{R}^n$. Suppose that the set $\Lambda \setminus \mathbb{Z}^n$ is bounded. Then $L^p_{\mathbb{R}^n\setminus\Lambda}(\mathbb{R}^n) = L^p_{\mathbb{R}^n\setminus(\Lambda\cap\mathbb{Z}^n)}(\mathbb{R}^n)$.*

Proof. Put $\Omega = \mathbb{R}^n \setminus \Lambda$, $\Pi = \mathbb{R}^n \setminus (\Lambda \cap \mathbb{Z}^n)$. Then the sets Ω and Π satisfy the assumptions of Theorem 5.12. □

Remark 6.9. The condition of boundedness imposed on the set $\Lambda \setminus \mathbb{Z}^n$ in Theorem 6.8 can be replaced with a much weaker condition. For example, it suffices to assume that the projection of this set onto a straight line is bounded.

We may apply Theorem 3.5 to $G_1 = \mathbb{R}^m$, $G_2 = \mathbb{R}^n$, $\Omega = \mathbb{R}^n \setminus \mathbb{Z}^n$. Then, in the case where $m = n = 1$, we get the following statement.

Theorem 6.10. *Let* $\Omega = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_2 \notin \mathbb{Z}\}$. *Then*

$$L^p_\Omega(\mathbb{R}^2) = \Big\{ f \in L^p(\mathbb{R}^2) : \sum_{k\in\mathbb{Z}} f(x_1, x_2 + k) = 0 \text{ for almost all } \ x \in \mathbb{R}^2 \Big\}$$

for $0 < p \le 1$.

Now we state a generalization of this theorem to the case of locally compact Abelian groups.

Theorem 6.11. *Let* G *be a locally compact Abelian group,* Γ *the dual group. Let* H *be a closed subgroup of* Γ*, and let* Λ *be the annihilator of* H*. Then* $L^p_{\Gamma\setminus\Lambda}(G) \ne L^p(G)$ *if and only if* H *is a discrete group. If* H *is discrete, then*

$$L^p_{\Gamma\setminus\Lambda}(G) = \Big\{ f \in L^p(G) : \sum_{h\in H} f(x + h) = 0 \text{ for almost all } x \in G \Big\}.$$

We omit the proof of this theorem.

Remark 6.12. Note that

$$L^1_{\Gamma\setminus\Lambda}(G) = \Big\{ f \in L^1(G) : \int_H f(x + h)\, d\boldsymbol{m}_H(h) = 0 \text{ for a.e. } x \in G \Big\} \ne L^1(G)$$

for any closed subgroup H of G (see [26, Theorem 2.7.4]).

Now, once again, we return to the space $\mathbb{R}^n$ to find out what happens to the space $L^p_{\mathbb{R}^n\setminus\mathbb{Z}^n}(\mathbb{R}^n)$ under a perturbation of the set $\mathbb{R}^n \setminus \mathbb{Z}^n$. We state the corresponding result in the one-dimensional case.

Theorem 6.13. *Let* $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ *be a two-sided infinitesimal sequence of nonzero real numbers. Denote by* Λ *the set of all values of the sequence* $\{k + \varepsilon_k\}_{k\in\mathbb{Z}}$. *Then* $L^p_{\mathbb{R}\setminus\Lambda}(\mathbb{R}) = L^p(\mathbb{R})$.

Proof. Note that for every $\varepsilon \in (0, 1/2)$ the inclusion $(k + \varepsilon, k + 1 - \varepsilon) \subset \mathbb{R} \setminus \Lambda$ is fulfilled for all k whose absolute values are sufficiently large. Consequently, $L^p_{\mathbb{R}\setminus\Lambda}(\mathbb{R}) \supset L^p_{\mathbb{R}\setminus\mathbb{Z}}(\mathbb{R})$ by Theorem 5.12, whence we get

$$L^p_{\mathbb{R}\setminus\Lambda}(\mathbb{R}) \supset \operatorname{clos}_{L^p}(L^p_{\mathbb{R}\setminus\Lambda}(\mathbb{R}) + L^p_{\mathbb{R}\setminus\mathbb{Z}}(\mathbb{R})) = L^p_{\mathbb{R}\setminus(\Lambda\cap\mathbb{Z})}(\mathbb{R}) = L^p(\mathbb{R}^n),$$

because $\Lambda \cap \mathbb{Z}$ is finite. □

Of course, a multidimensional analog of Theorem 6.13 is true, but in the multidimensional case a stronger result is valid: we may eliminate each point λ of Λ, where Λ is a perturbation of the lattice $\mathbb{Z}^n$, together with an appropriate hyperplane passing through λ. The following theorem illustrates this in the two-dimensional case.

Theorem 6.14. *Let* Λ *be the same as in Theorem* 6.13*, and let* Ω *denote the set* $\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_2 \notin \Lambda\}$. *Then* $L^p_\Omega(\mathbb{R}^2) = L^p(\mathbb{R}^2)$.

Proof. It suffices to apply Theorems 6.13 and 3.5. □

Remark 6.15. Theorem 6.13 shows that a small perturbation of the set of integers can affect its properties greatly. Other examples that exhibit such oddity and instability of the subset $\mathbb{Z}$ of $\mathbb{R}$ were known before (see, e.g., [6, 23] and [24]).

Theorem 6.16. *There exists a strictly increasing two-sided sequence in* $\mathbb{R}$ *such that* $\lim_{|n|\to+\infty}(\lambda_{n+1}-\lambda_n)=0$, $\lim_{|n|\to+\infty}|\lambda_n|=+\infty$ *and* $L^p_\Omega(\mathbb{R})=L^p(\mathbb{R})$, *where* $\Omega=\{x\in\mathbb{R}: x\neq\lambda_n\}$.

Proof. Let η be a positive even function on $\mathbb{R}$ tending to zero at infinity. Put $\Lambda_j \stackrel{\text{def}}{=} \{2^{-j}k+\eta(2^{-j}k): k\in\mathbb{Z}\}$, where $j\in\mathbb{N}$. Then $\Lambda_j\subset\Lambda_{j+1}$ for all $j\in\mathbb{N}$. Take a sequence $\{f_j\}_{j\in\mathbb{N}}$ dense in $L^p(\mathbb{R})$. By Theorem 6.13, we have $L^p_{\mathbb{R}\setminus\Lambda_j}(\mathbb{R})=L^p(\mathbb{R})$ for all $j\in\mathbb{N}$; therefore, there exists a sequence $\{\varphi_j\}_{j\in\mathbb{Z}}$ in $\mathcal{S}^0_{\mathbb{R}\setminus\Lambda_j}(\mathbb{R})$ such that $\|f_j-\varphi_j\|_{L^p}<2^{-j}$. We take a monotone increasing unbounded sequence of positive numbers $\{a_j\}_{j=1}^\infty$ such that $\operatorname{supp}\mathcal{F}\varphi_k\subset[-a_j,a_j]$ for all $k<j$, and put $\Lambda\stackrel{\text{def}}{=}\mathbb{R}\setminus\big(\bigcup_{j=1}^\infty([-a_j,a_j]\setminus\Lambda_j)\big)$. Clearly, Λ is the set of all values of a strictly monotone increasing sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$ such that $\lim_{|n|\to+\infty}(\lambda_{n+1}-\lambda_n)=0$ and $\lim_{|n|\to+\infty}|\lambda_n|=+\infty$. The identity $L^p_{\mathbb{R}\setminus\Lambda}(\mathbb{R})=L^p(\mathbb{R})$ follows from the fact that the sequence $\{\varphi_j\}_{j=1}^\infty$ is dense in $L^p(\mathbb{R})$ and $\Lambda\cap\operatorname{supp}\mathcal{F}\varphi_k=\varnothing$ for all $k\in\mathbb{N}$. □

The results of this section allow us to construct examples of open sets $\Omega\subset\mathbb{R}^n$ with bad behavior of the function Φ^Ω_p.

Example 6.17. Let $a\in\Omega=(-1,+\infty)\cup\bigcup_{k=1}^\infty(-k-1,-k)$. Then $\Phi^\Omega_p(a)<+\infty$ if and only if a is a nonnegative integer.

Proof. By Theorem 6.7, $f\in L^p_\Omega(\mathbb{R})$ if and only if $\mathcal{E}f\in L^p_{\mathbb{Z}_+}(\mathbb{R}/\mathbb{Z})$, where $\mathbb{Z}_+$ is the set of all nonnegative integers. The space $L^p_{\mathbb{Z}_+}(\mathbb{R}/\mathbb{Z})$ is identified in a natural way with the Hardy space $H^p(\mathbb{D})$ in the disk $\mathbb{D}$. Hence, for every $k\in\mathbb{Z}_+$ the functional $f\mapsto(\mathcal{F}f)(k)$ is continuous on $L^p_{\mathbb{Z}_+}(\mathbb{R}/\mathbb{Z})$. This and the Poisson summation formula imply that $\Phi^\Omega_p(k)<+\infty$ for all $k\in\mathbb{Z}_+$. Now we prove that $\Phi^\Omega_p(a)=+\infty$ if $a\in\mathbb{R}\setminus\mathbb{Z}$. For this, it suffices to observe that if Ψ is a continuous functional on $L^p_\Omega(\mathbb{R})$, then $\Psi(f(x))=\Psi(f(x+1))$ for all $f\in L^p_\Omega(\mathbb{R})$, because $f(x)-f(x+1)\in L^p_{\mathbb{R}\setminus\mathbb{Z}}(\mathbb{R})$ by Theorem 6.2 and $\Psi|_{L^p_{\mathbb{R}\setminus\mathbb{Z}}(\mathbb{R})}\equiv 0$ by Corollary 6.3. □

Similarly, we can obtain the following statement.

Example 6.18. Let $\Omega=(\mathbb{R}\setminus\mathbb{Z})\cup A$, where A is a finite set of integers. Then for $a\in\Omega$, we have $\Phi^\Omega_p(a)<+\infty$ if and only if $a\in A$.

§7. Strips and meager invariant subspaces

7.1. Strips. The set $\Xi_n\stackrel{\text{def}}{=}\{x\in\mathbb{R}^n: 0<x_1<1\}$ is called the *standard strip* in the space $\mathbb{R}^n$. Let L be a nondegenerate affine transformation of $\mathbb{R}^n$. The set of the form $L(\Xi_n)$ is called a *strip* in the space $\mathbb{R}^n$.

Theorem 7.1. *Let* Ω *be a finite union of strips in* $\mathbb{R}^n$. *Then* $[\Omega]_p=\Omega$.

Note that in the one-dimensional case Theorem 7.1 is a special case of Corollary 4.4. There are some differences in the multidimensional case. First, in the one-dimensional case the functionals separate the points of $L^p_\Omega(\mathbb{R})$, whereas it is easily seen that the dual of $L^p_\Omega(\mathbb{R}^n)$ is trivial in the multidimensional case. Second, as is clear from Example 5.6, the situation can change radically even in the two-dimensional case if we deform the strip by making its width tend to infinity, though arbitrarily slowly.

To prove Theorem 7.1, we need a lemma. We start with some notation. Let $M_p(\mathbb{R}^n)$ denote the set of all discrete measures on $\mathbb{R}^n$ such that $\|\mu\|_p^p\stackrel{\text{def}}{=}\sum_{x\in\mathbb{R}^n}|\mu(\{x\})|^p<+\infty$. Note that if $f\in L^p(\mathbb{R}^n)$, then $\mu*f\in L^p(\mathbb{R}^n)$ and $\|\mu*f\|_{L^p}\le\|\mu\|_p\|f\|_{L^p}$ for every $\mu\in M_p(\mathbb{R}^n)$. Thus, each measure $\mu\in M_p(\mathbb{R}^n)$ gives rise to an operator

$R_\mu : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $R_\mu f \stackrel{\text{def}}{=} \mu * f$, that commutes with the translations, i.e., the operators of the form R_{δ_a}, where δ_a denotes the δ-measure at $a \in \mathbb{R}^n$. It is easy to show that $\|R_\mu\| = \|\mu\|_p$. Moreover, it is known that the operators $\{R_\mu\}_{\mu \in M_p(\mathbb{R}^n)}$ exhaust all operators from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ that commute with the translations (see [22]).

It is clear that if X is a translation invariant subspace of $L^p(\mathbb{R}^n)$, then $R_\mu X \subset X$ for every $\mu \in M_p(\mathbb{R}^n)$.

Lemma 7.2. *Let I and J be disjoint closed bounded intervals on the real line. Then there exists a measure $\mu \in M_p(\mathbb{R})$ such that $\mathcal{F}\mu = 0$ on I and $\mathcal{F}\mu = 1$ on J.*

Proof. Take a periodic infinitely differentiable function f such that $f = 0$ on I and $f = 1$ on J. It remains to put $\mu \stackrel{\text{def}}{=} \mathcal{F}^{-1} f$. □

Corollary 7.3. *Let I and J be disjoint closed bounded intervals on the real line. Then there exists a measure $\mu \in M_p(\mathbb{R}^n)$ such that $(\mathcal{F}\mu)(x) = 0$ if $x_1 \in I$, and $(\mathcal{F}\mu)(x) = 1$ if $x_1 \in J$.*

Proof of Theorem 7.1. Let $\Omega = \bigcup_{j=1}^N \Omega_j$, where Ω_j is a strip for every j. We use induction on N. For $N = 0$ the claim is obvious. Let $\left[\bigcup_{j=1}^{N-1} \Omega_j\right]_p = \bigcup_{j=1}^{N-1} \Omega_j \stackrel{\text{def}}{=} \Pi$. We prove that $[\Omega]_p = \Omega$. Suppose this is not true. Then there exists an open ball B in $\mathbb{R}^n$ and a nonzero function $\varphi \in \mathcal{S}_B^0(\mathbb{R}^n)$ such that $\overline{B} \cap \overline{\Omega} = \varnothing$ and $\varphi \in L^p_\Omega(\mathbb{R}^n)$. We could assume from the outset that Ω_N is the standard strip. Denote by J the projection of $\overline{B}$ onto the first coordinate axis. By Corollary 7.3, there exists a measure $\mu \in M_p(\mathbb{R}^n)$ such that $(\mathcal{F}\mu)(x) = 0$ if $x_1 \in [0,1]$, and $(\mathcal{F}\mu)(x) = 1$ if $x_1 \in J$. Clearly, $\varphi * \mu \in L^p_\Pi(\mathbb{R}^n)$ and $\varphi * \mu = \varphi$. Hence, $[\Pi]_p \supset \{\operatorname{supp} \mathcal{F}\varphi \neq 0\}$, a contradiction. □

7.2. Meager invariant subspaces. A translation invariant subspace $X \subset L^p(G)$ is said to be *meager* if $X + Y$ is not dense in $L^p(G)$ for every proper translation invariant subspace $Y \subset L^p(G)$.

Theorem 7.4. *Let $\{X_j\}_{j \in J}$ be a finite family of meager invariant subspaces of $L^p(G)$. Then $\operatorname{clos}_{L^p}(\sum_{j \in J} X_j)$ is a meager invariant subspace of $L^p(G)$.*

Proof. This can be proved easily by induction on the number of elements in the set J. □

Theorem 7.5. *Let Ω be a strip in $\mathbb{R}^n$. Then $L^p_\Omega(\mathbb{R}^n)$ is a meager invariant subspace.*

Proof. We may assume that Ω is the standard strip in $\mathbb{R}^n$. Let Y be a proper invariant subspace of $L^p(\mathbb{R}^n)$. Suppose that $Y + L^p_\Omega(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. We prove that $Y \supset L^p_\Pi(\mathbb{R}^n)$, where Π is a strip in $\mathbb{R}^n$ such that $\overline{\Pi} \cap \overline{\Omega} = \varnothing$.

Let $f \in L^p_\Pi(\mathbb{R}^n)$. For every $\varepsilon > 0$ there exists $g \in L^p_\Omega(\mathbb{R}^n)$ and $h \in Y$ such that

$$\|f - g - h\|^p_{L^p} \le \varepsilon^p.$$

By Corollary 7.3, there exists a measure $\mu \in M_p(\mathbb{R}^n)$ such that $f * \mu = f$ and $g * \mu = 0$. Then

$$\|f - h * \mu\|^p_{L^p} \le \varepsilon^p \|\mu\|^p_p.$$

Hence, $f \in Y$. It is clear that the union of all strips Π in $\mathbb{R}^n$ such that $\overline{\Pi} \cap \overline{\Omega} = \varnothing$ is $\mathbb{R}^n \setminus \overline{\Omega}$. Thus, by Theorem 2.17, we have $Y \supset L^p_{\mathbb{R}^n \setminus \overline{\Omega}}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, where the latter relation follows from Theorem 5.1, and we arrive at a contradiction. □

Corollary 7.6. *Let Ω be an open subset in $\mathbb{R}^n$, Π a finite union of strips in $\mathbb{R}^n$. If $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, then $L^p_{\Omega \setminus \overline{\Pi}}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.*

Proof. Consider a finite union of strips $\widetilde{\Pi}$ such that $\widetilde{\Pi} \supset \overline{\Pi}$. Assume that $L^p_{\Omega\setminus\overline{\Pi}}(\mathbb{R}^n) \neq L^p(\mathbb{R}^n)$. Then $L^p_\Omega(\mathbb{R}^n) \subset \operatorname{clos}_{L^p}(L^p_{\Omega\setminus\overline{\Pi}}(\mathbb{R}^n) + L^p_{\widetilde{\Pi}}(\mathbb{R}^n)) \neq L^p(\mathbb{R}^n)$ because the space $L^p_{\widetilde{\Pi}}(\mathbb{R}^n)$ is meager, a contradiction. □

Corollary 7.7. *If $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, then $L^p_{\Omega\setminus K}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for every compact set $K \subset \mathbb{R}^n$.*

Theorem 7.8. *Let Ω be an open subset of $\mathbb{R}^n$. Suppose that $L^p_\Omega(\mathbb{R}^n)$ is a meager invariant subspace. Then $(\mathbb{R}^n\setminus\Omega) + L(\mathbb{Z}^n) = \mathbb{R}^n$ for every nondegenerate linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$.*

Proof. It suffices to consider the case where L is the identity operator. Assume that $(\mathbb{R}^n\setminus\Omega) + \mathbb{Z}^n \neq \mathbb{R}^n$. Then $a \notin (\mathbb{R}^n\setminus\Omega) + \mathbb{Z}^n$ for some $a \in \mathbb{R}^n$. Let $\mathcal{O}$ denote the complement of $a + \mathbb{Z}^n$. Then $\mathcal{O} \cup \Omega = \mathbb{R}^n$. Hence, $L^p_{\mathcal{O}}(\mathbb{R}^n) + L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, a contradiction. □

Corollary 7.9. *Let Ω be a subset of $\mathbb{R}^n$. If $\Omega \supset \mathbb{Z}^n$, then $L^p_\Omega(\mathbb{R}^n)$ is not meager.*

Corollary 7.10. *There exists an open set $\Omega \subset \mathbb{R}^n$ such that $|\Omega| < +\infty$ and $L^p_\Omega(\mathbb{R}^n)$ is not meager.*

Corollary 7.11. *If Ω is a dense open subset of $\mathbb{R}^n$, then the invariant subspace $L^p_\Omega(\mathbb{R}^n)$ is not meager.*

Proof. By the Baire category theorem, there exists a vector $a \in \mathbb{R}^n$ such that $(a + \Omega) \supset \mathbb{Z}^n$, and we may refer to Corollary 7.9. □

Theorem 7.12. *Let Ω be an open subset of $\mathbb{R}^n$. Assume that $L^p_\Omega(\mathbb{R}^n)$ is a meager invariant subspace. Then there exists a number $R > 0$ such that $\mathbb{R}^n\setminus\Omega$ intersects each ball of radius R.*

Proof. Assume that there are no such numbers R. Then the set $L^p_\Omega(\mathbb{R}^n) + L^p_{-\Omega}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ by Theorem 5.1, and we get a contradiction. □

Example 7.13. Let Y be the set of all functions in $L^p(\mathbb{R})$ such that $f = 0$ almost everywhere on $(0, \infty)$. Then $Y + L^p_{(-1,1)}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

Proof. Consider the operator $T : L^0(\mathbb{R}) \to L^0(0, +\infty)$, $Tf \stackrel{\text{def}}{=} f|_{(0,+\infty)}$. It suffices to prove that $T(L^p_{(-1,1)}(\mathbb{R}))$ is dense in $L^p(0, +\infty)$. By a duality argument, $T(L^2_{(-1,1)}(\mathbb{R}))$ is dense in $L^2(0, +\infty)$. We fix a positive integer n such that $\frac{2}{n} < p$ and note that if $f_j \in L^2_{(-\frac{1}{n},\frac{1}{n})}(\mathbb{R})$ for $j = 1, 2, \ldots, n$, then $\prod_{j=1}^n f_j \in L^p_{(-1,1)}(\mathbb{R})$. Thus, the result reduces to the case of $p = 2$. □

This example shows that should we not demand the invariance of Y in the definition of a meager subspace, the space $L^p_{(-1,1)}(\mathbb{R})$ would not be meager.

7.3. Curved strips. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $\Pi^\varepsilon_f \stackrel{\text{def}}{=} \{(\xi_1, \xi_2) \in \mathbb{R}^2 : f(\xi_1) - \varepsilon < \xi_2 < f(\xi_1) + \varepsilon\}$. If f is an affine function, then the set Π^ε_f is a strip in $\mathbb{R}^2$. In the general case the set Π^ε_f can be viewed as a curved strip in $\mathbb{R}^2$.

Theorem 7.14. *Let $f \in C^1(\mathbb{R})$. Suppose that f is even, $\lim_\infty f' = 0$, and $f(\mathbb{R}) = \mathbb{R}$. Then $L^p_{\Pi^\varepsilon_f}(\mathbb{R}^2) = L^p(\mathbb{R}^2)$ for all $\varepsilon > 0$.*

Proof. First, we prove that for every $a \in \mathbb{R}$ there exists a sufficient open subset Ω of $\mathbb{R}$ such that $\Omega \times (a - \varepsilon/2, a + \varepsilon/2) \subset \Pi_f^\varepsilon$. Let a sequence $\{\tau_k\}_{k=1}^\infty$ of positive numbers be such that $\lim_{k\to+\infty} \tau_k = +\infty$ and $f(\tau_k) = a$ for all $k \geq 1$. We put $h(t) \stackrel{\text{def}}{=} \sup_{x\geq t} |f'(x)|$. Clearly, $\lim_{+\infty} h = 0$. Consider the set $\Omega = \Sigma \cup (-\Sigma)$, where $\Sigma = \bigcup_{k=1}^\infty (\tau_k, \tau_k + \frac{\varepsilon}{2h(\tau_k)})$. It is easily seen that $\Omega \times (a - \varepsilon/2, a + \varepsilon/2) \subset \Pi_f^\varepsilon$. Note that $L^p_{\Omega\times(a-\varepsilon/2,a+\varepsilon/2)}(\mathbb{R}^2) = L^p_{\mathbb{R}\times(a-\varepsilon/2,a+\varepsilon/2)}(\mathbb{R}^2)$ by Theorem 5.7. Consequently, $L^p_{\Pi_f^\varepsilon}(\mathbb{R}^2) \supset L^p_{\mathbb{R}\times(a-\varepsilon/2,a+\varepsilon/2)}(\mathbb{R}^2)$ for every $a \in \mathbb{R}$, whence $[\Pi_f^\varepsilon]_p = \mathbb{R}^2$. □

Corollary 7.15. *Let f be the same as in Theorem 7.14. Then there exists a continuous function $g : \mathbb{R} \to (0, +\infty)$ such that $\lim_\infty g = 0$ and the L^p-completion of the set $\{(\xi_1, \xi_2) \in \mathbb{R}^2 : f(\xi_1) < \xi_2 < f(\xi_1) + g(\xi_1)\}$ coincides with $\mathbb{R}^2$.*

Proof. Let $\{f_k\}_{k=1}^\infty$ be a sequence dense in $L^p(\mathbb{R}^2)$. We put $\Omega_k \stackrel{\text{def}}{=} \{(\xi_1, \xi_2) \in \mathbb{R}^2 : f(\xi_1) < \xi_2 < f(\xi_1) + 1/k\}$, $\Omega_k^{(j)} \stackrel{\text{def}}{=} \{(\xi_1, \xi_2) \in \Omega_k, |\xi_1| < j\}$. Theorem 7.14 shows that $L^p_{\Omega_k}(\mathbb{R}^2) = L^p(\mathbb{R}^2)$, whence $\lim_{j\to+\infty} \operatorname{dist}_{L^p}(f_k, L^p_{\Omega_k^{(j)}}(\mathbb{R}^2)) = 0$ for all positive integers k. Hence, for every k there exists a number j_k such that $\operatorname{dist}_{L^p}(f_k, L^p_{\Omega_k^{(j_k)}}(\mathbb{R}^2)) < 1/k$. We may assume that the sequence $\{j_k\}_{k=1}^\infty$ is strictly monotone increasing to infinity. Now it is clear that for the role of g we may take any even continuous function g monotone decreasing on $(0, +\infty)$ and such that $g(j_k) = 1/k$. □

Remark 7.16. From Theorem 4.1, it follows that $\int_\mathbb{R} g(t)\, dt = +\infty$.

Remark 7.17. The condition $\lim_\infty f' = 0$ in Theorem 7.14 can be relaxed: it can be replaced with the condition $\lim_{t\to+\infty} \max_{0\leq s\leq 1} |f(t + s) - f(t)| = 0$. This allows us to lift the differentiability condition on f.

Remark 7.18. Theorem 7.14 remains true if we replace the set Π_f^ε with the ε-neighborhood of the graph of f.

Now we state a multidimensional version of Theorem 7.14. Let $\mathcal{U}_\varepsilon[f]$ denote the ε-neighborhood of the graph $\Pi_f = \{(t, f(t)) \in \mathbb{R}^n : t \in \mathbb{R}\}$ of the function $f : \mathbb{R} \to \mathbb{R}^{n-1}$.

Theorem 7.19. *Let $f : \mathbb{R} \to \mathbb{R}^{n-1}$ be a differentiable function. Suppose that f is even, $\lim_\infty f' = 0$, and $f(\mathbb{R})$ is dense in $\mathbb{R}^{n-1}$. Then $L^p_{\mathcal{U}_\varepsilon[f]}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $\varepsilon > 0$.*

Proof. The proof of this theorem is similar to that of Theorem 7.14, and we omit it. □

Remark 7.17 also concerns Theorem 7.19.

Corollary 7.20. *Let f be the same as in Theorem 7.19. Then there exists a continuous function $\rho : \mathbb{R} \to (0, +\infty)$ such that $\lim_\infty \rho = 0$ and $L^p_{\mathcal{U}_\rho[f]}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, where $\mathcal{U}_\rho[f] = \bigcup_{t\in\mathbb{R}} B_{(t,f(t))}(\rho(t))$.*

Proof. The proof of this corollary is similar to that of Corollary 7.15, and we omit it. □

7.4. The case of the two-dimensional torus. It is not difficult to prove that if f satisfies the assumptions of Theorem 7.14, then $L^p_A(\mathbb{T}^2) = L^p(\mathbb{T}^2)$ for $A = \{(k, [f(k)]) : k \in \mathbb{Z}\}$, where $[x]$ denotes the integral part of x. The intersection of A with every vertical "straight line in $\mathbb{Z}^2$" consists of one element, whereas the intersections of A with the horizontal "straight lines" are infinite sets. We shall construct a set $A \subset \mathbb{Z}^2$ such that $L^p_A(\mathbb{T}^2) = L^p(\mathbb{T}^2)$ and the intersection of A with "a straight line in $\mathbb{Z}^2$" can consist of more than one element only for lines whose slope is sufficiently close to a number given in advance (see Theorem 7.21). Let k and l be two different points in $\mathbb{R}^2$. We denote by $\alpha(k, l)$ the slope of the straight line passing through k and l; $\alpha(k, l) \stackrel{\text{def}}{=} \infty$ if this line is vertical.

Theorem 7.21. *For every real α_0 and every positive ε there exists a set $A \subset \mathbb{Z}^2$ such that $L^p_A(\mathbb{T}^2) = L^p(\mathbb{T}^2)$ for all $p \in (0,1)$, and $|\alpha(k,l) - \alpha_0| < \varepsilon$ for every two different points k and l in A.*

We start with a lemma.

Lemma 7.22. *Let G be a locally compact Abelian group, and let ζ and ξ be two elements of the dual group Γ. If X denotes the linear hull of the family of characters*[6] *$\{\xi + k\zeta\}_{0<|k|\le N}$, then $\operatorname{dist}_{L^p}(\xi, X) \le C(p)N^{1-\frac{1}{p}}$.*

Proof. We may assume that $\xi = 0$. There exists a trigonometric polynomial $f = \sum_{k=-N}^{N} \widehat{f}(k)z^k$ such that $\widehat{f}(0) = 1$ and $\|f\|_{L^p} \le C(p)N^{1-\frac{1}{p}}$ (see, e.g., [1, Chapter 4, §2]). We have

$$\int_{\mathbb{T}}\int_G \Big|\sum_{k=-N}^{N} \widehat{f}(k)z^k(x,\zeta)^k\Big|^p \, d\boldsymbol{m}_G(x)\, d\boldsymbol{m}(z)$$
$$= \int_G\int_{\mathbb{T}} \Big|\sum_{k=-N}^{N} \widehat{f}(k)z^k(x,\zeta)^k\Big|^p \, d\boldsymbol{m}(z)\, d\boldsymbol{m}_G(x) \le C(p)N^{p-1},$$

whence $\Big\|\sum_{k=-N}^{N} \widehat{f}(k)z^k(\,\cdot\,,\zeta)^k\Big\|^p_{L^p(G)} \le C(p)N^{p-1}$ for some $z \in \mathbb{T}$. □

Proof of Theorem 7.21. We may assume that α_0 is a rational number. Let $\alpha_0 = \frac{n_1}{n_2}$, where $n_1, n_2 \in \mathbb{Z}$, $n_2 \ne 0$. Put $n \overset{\text{def}}{=} (n_1, n_2)$, $\chi \overset{\text{def}}{=} z_1^{n_1} z_2^{n_2}$. We can enumerate the elements of $\mathbb{Z}^2$ by a sequence $\{k^{(j)}\}_{j=1}^{\infty}$ in such a way that each element of $\mathbb{Z}^2$ occurs in this sequence infinitely many times. Let $\{N_j\}_{j=1}^{\infty}$ be a sequence of positive integers to be chosen later. Put

$$A_j \overset{\text{def}}{=} \{k^{(j)} + sN_j n : s \in \mathbb{Z},\ 0 < |s| \le j\}, \quad A \overset{\text{def}}{=} \bigcup_{j=1}^{\infty} A_j.$$

Lemma 7.22 implies that the space $L^p_A(\mathbb{T}^2)$ contains all characters of the group $\mathbb{T}^2$, i.e., $L^p_A(\mathbb{T}^2) = L^p(\mathbb{T}^2)$. The sequence $\{N_j\}_{j=1}^{\infty}$ will be constructed by induction. If at the jth step for the role of N_j we choose a sufficiently large positive integer, then the inequality $|\alpha(k,l) - \alpha_0| < \varepsilon$ will be fulfilled for every two different points k and l such that $k, l \in \bigcup_{r=1}^{j} A_r$. □

Corollary 7.23. *There exists a set $T \subset \mathbb{Z}$ and a strictly monotone increasing function $f : T \to \mathbb{Z}$ such that $L^p_A(\mathbb{T}^2) = L^p(\mathbb{T}^2)$ for all $p \in (0,1)$, where $A = \{(k, f(k)) : k \in T\}$.*

Proof. It suffices to apply Theorem 7.21 with $\alpha_0 = \varepsilon = 1$. □

Remark 7.24. From Theorem 7.21 it is clear that we can require that, moreover,

$$\lim_{\substack{k,l\in A, k\ne l\\ \max(|k|,|l|)\to\infty}} \alpha(k,l) = \alpha_0.$$

[6]Recall that, in accordance with our notation, $(x, \xi + k\zeta) = (x,\xi)(x,\zeta)^k$ for all $x \in G$.

§8. Estimates of the functional Φ_p^Ω

The function Φ_p^Ω was defined in §2 in the case where Ω is an open subset of $\mathbb{R}^n$. Now we define it in a more general situation. Let Ω be an open subset of the locally compact Abelian group Γ that is the set of all characters of a locally compact Abelian group G. We denote by $\mathcal{L}_\Omega^p(G)$ the set of all functions $f \in L^1(G) \cap L^p(G)$ such that $\operatorname{supp} \mathcal{F}f$ is a compact subset of Ω and put

$$\Phi_p^\Omega(a) \stackrel{\text{def}}{=} \Phi_p^\Omega(a,G) \stackrel{\text{def}}{=} \sup\{|(\mathcal{F}f)(a)| : f \in \mathcal{L}_\Omega^p(G),\ \|f\|_{L^p} \le 1\}.$$

It should be noted that the quantity $\Phi_p^\Omega(a)$ (like the p-norm in $L^p(G)$) depends also on the choice of Haar measure on G. Recall that if the group G is compact, then we always choose the Haar measure in such a way that $\boldsymbol{m}_G(G) = 1$. Clearly, $\Phi_p^\Omega(a) = \Phi_p^{\Omega-a}(0)$. If $\Phi_p^\Omega(a) < +\infty$, then the functional $f \mapsto (\mathcal{F}f)(a)$, $f \in \mathcal{L}_\Omega^p(G)$, extends up to a continuous functional $\mathcal{G}_a^\Omega : L_\Omega^p(G) \to \mathbb{C}$, and $\|\mathcal{G}_a^\Omega\| = \Phi_p^\Omega(a)$. But if $\Phi_p^\Omega(a) = +\infty$, then such an extension does not exist. It is easily seen that in the case where Ω is an open subset of $\mathbb{R}^n$ this definition of Φ_p^Ω is equivalent to the definition in §2. Note also that if the group G is compact, then $\Phi_p^\Omega(a) = 1$ for all $p \in [1,+\infty)$, where Ω is an arbitrary subset of the dual group Γ, $a \in \Omega$.

Let A be an arbitrary subset of $\mathbb{R}^n$. A point $a \in A$ is said to be *exposed* if there exists a hyperplane H in $\mathbb{R}^n$ such that $a \in H$ and $A \setminus \{a\}$ lies in one of the open half-spaces bounded by the hyperplane H.

Theorem 8.1. *Let a be an exposed point of a set $A \subset \mathbb{Z}^n \subset \mathbb{R}^n$. Then $|(\mathcal{F}f)(a)| \le \|f\|_{L^p}$ for every $f \in \mathcal{L}_A^p(\mathbb{T}^n)$. In other words, $\Phi_p^A(a,\mathbb{T}^n) = 1$.*

Proof. We may assume that $a = 0$. The case where A is contained in a closed half-space P bounded by the hyperplane ∂P such that $\partial P \cap \mathbb{Z}^n = \{0\}$, is well known (see, e.g., [26, Theorem 8.4.1] and Example 8.1.7). The general case reduces to this special one, because it suffices to obtain the required result for the finite sets A. □

Theorem 8.2. *Suppose Ω is an open subset of $\mathbb{R}^n$, $\Omega \ni a$, and $A = \mathbb{Z}^n \cap L^{-1}(-a+\Omega)$, where $L : \mathbb{R}^n \to \mathbb{R}^n$ is a nondegenerate linear operator. Then*

$$\Phi_p^\Omega(a,\mathbb{R}^n) \le |\det L|^{\frac{1}{p}-1} \Phi_p^A(0,\mathbb{R}^n/\mathbb{Z}^n).$$

Proof. We may assume that $a = 0$. Formula (2.3) allows us to restrict ourselves to the case where L is the identity operator. Let $\mathcal{E} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n/\mathbb{Z}^n)$ denote the operator defined in §6. The Poisson summation formula shows that $\mathcal{E}(L_\Omega^p(\mathbb{R}^n)) \subset L_A^p(\mathbb{R}^n/\mathbb{T}^n)$ and $(\mathcal{F}f)(0) = (\mathcal{F}(\mathcal{E}f))(0)$ for every $f \in \mathcal{S}_\Omega(\mathbb{R}^n)$, where $\mathcal{F}$ denotes the Fourier transformation on the group $\mathbb{R}^n$ on the left-hand side of the latter identity, and the Fourier transformation on the group $\mathbb{R}^n/\mathbb{Z}^n$ on the right-hand side of the latter identity. It remains to apply Theorem 6.1. □

Let P be the strip in $\mathbb{R}^n$ bounded by hyperplanes H_1 and H_2. Let a belong to the exterior of P. Put

$$\lambda(a,P) \stackrel{\text{def}}{=} \frac{\operatorname{dist}(a,H_1) + \operatorname{dist}(a,H_2)}{\min\big(\operatorname{dist}(a,H_1), \operatorname{dist}(a,H_2)\big)}.$$

Theorem 8.3. *Suppose Ω is an open subset of $\mathbb{R}^n$, P is a strip, and $a \in \Omega \setminus \overline{P}$. Then $\Phi_p^\Omega(a) \le C(p)\big(\lambda(a,P)\big)^{\frac{1}{p}-1} \Phi_p^{\Omega\setminus\overline{P}}(a)$.*

The proof is based on the following quantitative version of Lemma 7.2.

Lemma 8.4. *Let $0 < \alpha < \beta < +\infty$. Then there exists a measure $\mu \in M_p(\mathbb{R})$ such that $\mathcal{F}\mu = 1$ in a neighborhood of zero, $\mathcal{F}\mu = 0$ in a neighborhood of $[\alpha, \beta]$, and $\|\mu\|_p^p \le (C(p))^p\, \alpha^{p-1}(\alpha+\beta)^{1-p}$.*

Proof. Let $\varphi \in C^\infty(\mathbb{R})$ be an even function such that $\operatorname{supp}\varphi \subset (-1,1)$ and $\varphi = 1$ in a neighborhood of zero. The Poisson summation formula easily implies the identity

$$\sum_{k\in\mathbb{Z}} \varphi\Big(\frac{x}{\alpha} + \frac{(\alpha+\beta)k}{\alpha}\Big) = \frac{\alpha}{\alpha+\beta}\sum_{k\in\mathbb{Z}} e^{\frac{2\pi i k x}{\alpha+\beta}}(\mathcal{F}\varphi)\Big(\frac{\alpha k}{\alpha+\beta}\Big) = \mathcal{F}\mu,$$

where

$$\mu = \frac{\alpha}{\alpha+\beta}\sum_{k\in\mathbb{Z}} (\mathcal{F}\varphi)\Big(\frac{\alpha k}{\alpha+\beta}\Big)\delta_{\frac{k}{\alpha+\beta}}.$$

Here δ_a denotes the δ-measure at a point $a \in \mathbb{R}$.

Applying the elementary inequality

$$t\sum_{k\in\mathbb{Z}} (1+t|k|)^{-q} \le C(q),$$

which is valid for all $t \in (0,1]$ and all $q > 1$, we get

$$\|\mu\|_p^p = \frac{\alpha^p}{(\alpha+\beta)^p}\sum_{k\in\mathbb{Z}}\Big|(\mathcal{F}\varphi)\Big(\frac{\alpha k}{\alpha+\beta}\Big)\Big|^p \le C(\varphi,p)\frac{(\alpha+\beta)^{1-p}}{\alpha^{1-p}}. \qquad \square$$

Corollary 8.5. *Let P be a strip in $\mathbb{R}^n$, and let $a \in \mathbb{R}^n \setminus \overline{P}$. There exists a measure $\mu \in M_p(\mathbb{R}^n)$ such that $\mathcal{F}\mu = 1$ in a neighborhood of a, $\mathcal{F}\mu = 0$ in a neighborhood of the closure of P, and $\|\mu\|_p \le C(p)\big(\lambda(a,P)\big)^{\frac{1}{p}-1}$.*

Proof of Theorem 8.3. Let μ be a measure, the existence of which is stated in Corollary 8.5. Take a function $f \in \mathcal{S}^0_\Omega(\mathbb{R}^n)$. It is clear that $f * \mu \in \mathcal{S}^0_{\Omega\setminus\overline{P}}(\mathbb{R}^n)$. Consequently,

$$\begin{aligned}(\mathcal{F}(f*\mu))(a) &\le \Phi_p^{\Omega\setminus\overline{P}}(a)\|f*\mu\|_{L^p}\\ &\le \Phi_p^{\Omega\setminus\overline{P}}(a)\|f\|_{L^p}\|\mu\|_p \le C(p)\big(\lambda(a,P)\big)^{\frac{1}{p}-1}\Phi_p^{\Omega\setminus\overline{P}}(a)\|f\|_{L^p},\end{aligned}$$

whence $(\mathcal{F}f)(a) \le C(p)\big(\lambda(a,P)\big)^{\frac{1}{p}-1}\Phi_p^{\Omega\setminus\overline{P}}(a)\|f\|_{L^p}$. $\square$

The next statement follows from Theorem 8.3 by induction.

Theorem 8.6. *Suppose Ω is an open subset of $\mathbb{R}^n$, $\{P_j\}_{j=1}^N$ is a finite family of strips in $\mathbb{R}^n$, and $a \in \Omega_0 \overset{\text{def}}{=} \Omega \setminus \bigcup_{j=1}^N \overline{P}_j$. Then*

$$\Phi_p^\Omega(a) \le C(p)^N\Big(\prod_{j=1}^N \lambda(a,P_j)\Big)^{\frac{1}{p}-1}\Phi_p^{\Omega_0}(a).$$

8.1. Examples. Let $\Omega_\alpha = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : |\xi_1|^\alpha|\xi_2| < 1\}$.

Theorem 8.7. *Let $a \in \Omega_\alpha$, where $\alpha \in (0,1]$. Then*

$$\begin{aligned}&C_1(p,\alpha)\big(|a_2|^{(1-\frac{1}{\alpha})}(1-|a_1|^\alpha|a_2|)\big)^{\frac{1}{p}-1}\\ &\quad\le \Phi_p^{\Omega_\alpha}(a_1,a_2) \le C_2(p,\alpha)\big(|a_2|^{(1-\frac{1}{\alpha})}(1-|a_1|^\alpha|a_2|)\big)^{\frac{1}{p}-1},\end{aligned}$$

where $0^{(1-\frac{1}{\alpha})} \overset{\text{def}}{=} +\infty$ for $\alpha < 1$ and $0^{(1-\frac{1}{\alpha})} \overset{\text{def}}{=} 1$ for $\alpha = 1$.

Proof. Let $(a_1, a_2) \in \Omega_\alpha$. We may assume that $a_1, a_2 \geq 0$. Moreover, we assume that $a_1 a_2 > 0$, leaving the case of $a_1 a_2 = 0$ to the reader. First, we prove the upper estimate. Consider the linear operator $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula

$$L(k_1, k_2) = \Big(k_1 - \sqrt{2}\, k_2, 9\Big(\frac{2}{a_2}\Big)^{\frac{1}{\alpha}-1}(k_1 + \sqrt{2}\, k_2)\Big).$$

Let A be the same as in Theorem 8.2 with $\Omega = \{(\xi_1, \xi_2) \in \Omega_\alpha : |\xi_1| > \frac{a_1}{2}, |\xi_2| > \frac{a_2}{2}\}$. We prove that $A = \{(0, 0)\}$. Let $(k_1, k_2) \in \mathbb{Z}^2$. Then $(k_1, k_2) \in A$ if and only if $|x|^\alpha |y| < 1$, $2|x| > a_1$ and $2|y| > a_2$, where $x = k_1 - \sqrt{2}\, k_2 + a_1$ and $y = 9\big(\frac{2}{a_2}\big)^{\frac{1}{\alpha}-1}(k_1 + \sqrt{2}\, k_2) + a_2$. Note that the inequalities $2|x| > a_1$ and $2|y| > a_2$ imply that $|k_1 - \sqrt{2}\, k_2| < 3|x|$ and $9\big(\frac{2}{a_2}\big)^{\frac{1}{\alpha}-1}|k_1 + \sqrt{2}\, k_2| < 3|y|$. Moreover, from the inequalities $|x|^\alpha |y| < 1$ and $2|y| > a_2$ it follows that $|x| < \big(\frac{2}{a_2}\big)^{\frac{1}{\alpha}}$. Consequently,

$$9\Big(\frac{2}{a_2}\Big)^{\frac{1}{\alpha}-1}|k_1^2 - 2k_2^2| < 9|xy| \leq 9|x|^{1-\alpha} < 9\Big(\frac{2}{a_2}\Big)^{\frac{1}{\alpha}-1},$$

whence $k_1 = k_2 = 0$. Now, using Theorems 8.2 and 8.6, we see that

$$\Phi_p^{\Omega_\alpha}(a_1, a_2) \leq C(p, \alpha) a_2^{(1-\frac{1}{\alpha})(\frac{1}{p}-1)} \leq 2^{1-\frac{1}{p}} C(p, \alpha)\big(a_2^{(1-\frac{1}{\alpha})}(1 - a_1^\alpha a_2)\big)^{\frac{1}{p}-1}$$

if $2a_1^\alpha a_2 \leq 1$.

To obtain the required inequality in the case where $2a_1^\alpha a_2 \geq 1$, consider the operator $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula

$$L(k_1, k_2) = \Big(2a_2^{-\frac{1}{\alpha}} k_1, (a_1^{-\alpha} - a_2)k_2\Big).$$

Let A be the same as in Theorem 8.2 with $\Omega = \Omega_\alpha$. We prove that $(0, 0)$ is an exposed point of A. Let $(k_1, k_2) \in \mathbb{Z}^2$. Then $(k_1, k_2) \in A$ if and only if $|x|^\alpha |y| < 1$, where $x = a_1 + 2a_2^{-\frac{1}{\alpha}} k_1$, $y = a_2 + (a_1^{-\alpha} - a_2)k_2$. Observe that if $k_2 \geq 1$, then $y \geq a_1^{-\alpha}$. Moreover, $|x| \geq a_1$. Thus, $k_2 \leq 0$ if $(k_1, k_2) \in A$, because $|x|^\alpha |y| < 1$. If $k_1 \neq 0$ and $k_2 = 0$, then $|x| \geq a_2^{-\frac{1}{\alpha}}$ and $y = a_2$; so $|x|^\alpha |y| \geq 1$. Hence, $(0, 0)$ is an exposed point of A. By Theorems 8.1 and 8.2, we have

$$\Phi_p^{\Omega_\alpha}(a_1, a_2) \leq \big(2a_2^{-\frac{1}{\alpha}} a_1^{-\alpha}(1 - a_1^\alpha a_2)\big)^{\frac{1}{p}-1} \leq \big(4a_2^{1-\frac{1}{\alpha}}(1 - a_1^\alpha a_2)\big)^{\frac{1}{p}-1}$$

if $2a_1^\alpha a_2 \geq 1$.

Now we pass to the lower estimate. The proof is based on Theorem 2.11. First, we consider the case where $2a_1^\alpha a_2 \geq 1$. Let P be the parallelogram centered at the point (a_1, a_2) and such that one of its sides is the intersection of the straight line $y = -\alpha a_1^{-\alpha-1}(x - a_1) + 2a_2 - a_1^{-\alpha}$ with the rectangle $\{0 \leq x \leq 2a_1, 0 \leq y \leq 2a_2\}$. Then the opposite side lies on the straight line $y = -\alpha a_1^{-\alpha-1}(x - a_1) + a_1^{-\alpha}$, which is tangent to the graph of the convex function $x \mapsto x^{-\alpha}$, $x > 0$. This tangent line and so the entire open parallelogram lies under the graph of this function. Hence, $P \subset \Omega_\alpha$ and $\Phi_p^{\Omega_\alpha}(a_1, a_2) \geq C(p)|P|^{\frac{1}{p}-1}$. It is not difficult to verify that $|P| \geq C(\alpha) a_2^{1-\frac{1}{\alpha}}$ (we omit the computations). Now, let $2a_1^\alpha a_2 \leq 1$. In this case, for the role of the parallelogram P we take the rectangle centered at the point (a_1, a_2) whose sides are parallel to the coordinate axes and one of the vertices is at the point $((\frac{3}{2}a_2)^{-\frac{1}{\alpha}}, \frac{3}{2}a_2)$. Then $\Phi_p^{\Omega_\alpha}(a_1, a_2) \geq C(p)|P|^{\frac{1}{p}-1}$, where

$$|P| = 2a_2^{1-\frac{1}{\alpha}}((2/3)^{\frac{1}{\alpha}} - a_1 a_2^{\frac{1}{\alpha}}) \geq C(\alpha) a_2^{1-\frac{1}{\alpha}}(1 - a_1^\alpha a_2),$$

because $2a_1^\alpha a_2 \leq 1$. $\square$

Corollary 8.8. *Let $a \in \Omega_\alpha$, where $\alpha \in (1, +\infty)$. Then*

$$C_1(p,\alpha)\big(|a_1|^{(1-\alpha)}(1-|a_1|^\alpha|a_2|)\big)^{\frac{1}{p}-1} \\ \le \Phi_p^{\Omega_\alpha}(a_1,a_2) \le C_2(p,\alpha)\big(|a_1|^{(1-\alpha)}(1-|a_1|^\alpha|a_2|)\big)^{\frac{1}{p}-1},$$

where $0^{(1-\alpha)} \stackrel{\text{def}}{=} +\infty$ for $\alpha > 1$.

Proof. The result reduces to Theorem 8.7 if we use the inequalities $1-|a_1|^\alpha|a_2| \le \alpha(1-|a_1||a_2|^{1/\alpha}) \le \alpha(1-|a_1|^\alpha|a_2|)$ and observe that $(a_1,a_2) \in \Omega_\alpha$ if and only if $(a_2,a_1) \in \Omega_{1/\alpha}$. □

Theorem 8.9. *Let $\Omega_\alpha^+ = \{(\xi_1,\xi_2) \in \Omega_\alpha : \xi_2 > 0\}$, where $0 < \alpha < +\infty$. Then*

$$C_1(p,\alpha)\big(a_2^{(1-\frac{1}{\alpha})}(1-|a_1|^\alpha a_2)\big)^{\frac{1}{p}-1} \\ \le \Phi_p^{\Omega_\alpha^+}(a_1,a_2) \le C_2(p,\alpha)\big(a_2^{(1-\frac{1}{\alpha})}(1-|a_1|^\alpha a_2)\big)^{\frac{1}{p}-1}$$

for all $(a_1,a_2) \in \Omega_\alpha^+$.

Proof. To get the lower estimate, we can repeat the corresponding part of the proof of Theorem 8.7 word by word, because all parallelograms P in that proof are contained in Ω_α^+. We pass to the upper estimate. The case where $0 < \alpha \le 1$ follows from Theorem 8.7. Now, let $\alpha > 1$. If $2|a_1|^\alpha a_2 \ge 1$, then the upper estimate reduces to Corollary 8.8 because in this case $|a_1|^{1-\alpha} \le (2a_2)^{1-\frac{1}{\alpha}}$. Let $2|a_1|^\alpha a_2 \le 1$. Consider the linear mapping $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(k_1,k_2) = (2a_2^{-\frac{1}{\alpha}}k_1, a_2k_2)$. Let A be the same as in Theorem 8.2 with $\Omega = \Omega_\alpha^+$. It is easily seen that $(0,0)$ is an exposed point of A. Hence, $\Phi_p^{\Omega_\alpha^+}(a_1,a_2) \le \big(2a_2^{1-\frac{1}{\alpha}}\big)^{\frac{1}{p}-1}$ as required, because $2|a_1|^\alpha a_2 \le 1$. □

Similarly, we can prove the following statement.

Theorem 8.10. *Let $\Omega_\alpha^{++} = \{(\xi_1,\xi_2) \in \Omega_\alpha : \xi_1 > 0, \xi_2 > 0\}$, where $0 < \alpha < +\infty$. Then*

$$C_1(p,\alpha)\big(a_1a_2(1-a_1^\alpha a_2)\big)^{\frac{1}{p}-1} \le \Phi_p^{\Omega_\alpha^{++}}(a_1,a_2) \le C_2(p,\alpha)\big(a_1a_2(1-a_1^\alpha a_2)\big)^{\frac{1}{p}-1}$$

for all $(a_1,a_2) \in \Omega_\alpha^{++}$.

Theorem 8.11. *Let $0 < \alpha \le 1$. Then*:

a) *$L^p_{\Omega_\alpha^+}(\mathbb{R}^2)$ is not a space of distributions for $p < 1-\alpha$;*

b) *$L^p_{\Omega_\alpha}(\mathbb{R}^2)$ is a space of tempered distributions for $p > 1-\alpha$; moreover, $\mathcal{F}(L^p_{\Omega_\alpha}(\mathbb{R}^2)) \subset L^1_{\mathrm{loc}}(\mathbb{R}^2)$ for $1-\alpha < p \le 1$.*

Proof. Let $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ satisfy $\operatorname{supp}\mathcal{F}\varphi \subset (-1,1)$, $\operatorname{supp}\mathcal{F}\psi \subset (0,1)$, $(\mathcal{F}\varphi)(0) = 1$, and $\psi(0) = 1$. Clearly, $\varphi(t^{-1}x_1)\psi(t^\alpha x_2) \in L^p_{\Omega_\alpha^+}(\mathbb{R}^2)$ for all $t \in (0,1]$. Note that $\lim_{t\to 0+} t^{-1}\varphi(t^{-1}x_1)\psi(t^\alpha x_2) = \delta_0(x_1)$ in the space of tempered distributions. On the other hand, it is easily seen that $\lim_{t\to 0+} t^{-1}\varphi(t^{-1}x_1)\psi(t^\alpha x_2) = 0$ in $L^p(\mathbb{R}^2)$ for $p < 1-\alpha$. Hence, $L^p_{\Omega_\alpha^+}(\mathbb{R}^2)$ is not a space of distributions for $p < 1-\alpha$.

To prove statement b), it suffices to note that $|\xi_2|^{(1-\frac{1}{\alpha})(\frac{1}{p}-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^2) \cap \mathcal{S}'(\mathbb{R}^n)$ for $p > 1-\alpha$, and to use Theorem 8.7. □

Now we consider a multidimensional analog of the set Ω_1.

Theorem 8.12. *Let $\Omega_{[n]} \stackrel{\text{def}}{=} \Big\{\xi \in \mathbb{R}^n : \prod_{j=1}^n |\xi_j| < 1\Big\}$. Then for all $a \in \Omega_{[n]}$,*

$$C_1(n,p)\Big(1-\prod_{j=1}^n |a_j|\Big)^{1-\frac{1}{p}} \le \Phi_p^\Omega(a) \le C_2(n,p)\Big(1-\prod_{j=1}^n |a_j|\Big)^{1-\frac{1}{p}}.$$

Proof. We start with the upper estimate. As in the proof of Theorem 8.7, we assume that $a_j>0$ for all $j=1,2,\dots,n$. We take a polynomial $q(x)=x^n+\sum_{j=0}^{n-1}c_jx^j$ with integral coefficients that is irreducible over the field of rational numbers and has n real zeros $\lambda_1,\lambda_2,\dots,\lambda_n$. Consider the linear mapping $L:\mathbb{R}^n\to\mathbb{R}^n$ determined by the matrix $\{3\lambda_j^{k-1}\}_{1\le j,k\le n}$. Let A be the same as in Theorem 8.2 with $\Omega=\{\xi\in\Omega_{[n]}:|\xi_j|>\frac{a_j}{2},j=1,2,\dots,n\}$. We prove that $A=\{0\}$. Let $\tau\in\mathbb{Z}^n$. Then $\tau\in A$ if and only if $\prod_{j=1}^n|x_j|<1$ and $2|x_j|>a_j$ for all $j,j=1,2,\dots,n$, where $x_j=a_j+3\sum_{k=1}^n\lambda_j^{k-1}\tau_k$. The inequality $2|x_j|>a_j$ implies $\big|\sum_{k=1}^n\lambda_j^{k-1}\tau_k\big|<|x_j|$. Hence, $\Big|\prod_{j=1}^n\big(\sum_{k=1}^n\lambda_j^{k-1}\tau_k\big)\Big|<1$. Note that $H_\tau(\lambda_1,\lambda_2,\dots,\lambda_n)\overset{\text{def}}{=}\prod_{j=1}^n\big(\sum_{k=1}^n\lambda_j^{k-1}\tau_k\big)$ is a symmetric polynomial in $\lambda_1,\lambda_2,\dots,\lambda_n$ with integral coefficients. As is well known, this implies that H_τ is a polynomial with integral coefficients in the elementary symmetric polynomials $\sigma_1,\sigma_2,\dots,\sigma_n$. Note that $\sigma_j(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{Z}$ by the Viète theorem. Hence, $H_\tau(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{Z}$ for all $\tau\in\mathbb{Z}^n$. Thus, the inequality $\Big|\prod_{j=1}^n\big(\sum_{k=1}^n\lambda_j^{k-1}\tau_k\big)\Big|<1$ implies $\prod_{j=1}^n\big(\sum_{k=1}^n\lambda_j^{k-1}\tau_k\big)=0$. It follows that $\tau=0$, because the polynomial q is irreducible over the field of rational numbers. Now Theorems 8.2 and 8.6 imply that

$$\Phi_p^{\Omega_{[n]}}(a)\le C(n)^{\frac1p-1}\le C(n,p)\Big(1-\prod_{j=1}^n a_j\Big)^{1-\frac1p}$$

if $2\prod_{j=1}^n a_j\le 1$.

To get the required inequality in the case where $2\prod_{j=1}^n a_j\ge 1$, we consider the linear operator $L:\mathbb{R}^n\to\mathbb{R}^n$ given by the formula

$$L(\tau_1,\tau_2,\dots,\tau_3)=\Big(\Big(-a_1+\prod_{j=2}^n a_j^{-1}\Big)\tau_1,3a_2\tau_2,\dots,3a_n\tau_n\Big).$$

Let A be the same as in Theorem 8.2 with $\Omega=\Omega_{[n]}$. We prove that 0 is an exposed point of A. Let $\tau\in\mathbb{Z}^n$. Then $\tau\in A$ if and only if $\prod_{j=1}^n|x_j|<1$, where $x_1=a_1+\big(-a_1+\prod_{j=2}^n a_j^{-1}\big)\tau_1$ and $x_j=a_j+3a_j\tau_j$ for $j=2,\dots,n$. Note that if $\tau_1\ge1$, then $\prod_{j=1}^n|x_j|\ge1$ because $x_1\ge\prod_{j=2}^n a_j^{-1}$ and $x_j\ge a_j$ for $j=2,\dots,n$. Thus, $\tau_1\le0$ for all $\tau\in A$. Suppose that τ is not an exposed point. Then there exists a nonzero vector $\tau\in A$ such that $\tau_1=0$. Clearly, $|x_j|\ge a_j$ for all j. The fact that τ is nonzero and $\tau_1=0$ implies that $|x_j|\ge 2a_j$ for some $j\ge2$, whence $\prod_{j=1}^n|x_j|\ge2\prod_{j=1}^n a_j\ge1$, and we get a contradiction. Hence, 0 is an exposed point of A. By Theorems 8.1 and 8.2, we have

$$\Phi_p^{\Omega_{[n]}}(a)\le\Big(3^{n-1}\Big(1-\prod_{j=1}^n a_j\Big)\Big)^{\frac1p-1}$$

if $2\prod_{j=1}^n a_j\ge1$.

Now we pass to the lower estimate. Formula (2.3) allows us to assume that, moreover, $a_1=a_2=\dots=a_n\overset{\text{def}}{=}\alpha\in(0,1)$. First we treat the case where $\alpha\ge2/3$. Note that a is the center of the cylinder

$$W\overset{\text{def}}{=}\Big\{\xi\in\mathbb{R}^n:\sum_{k=1}^n\Big(\xi_k-n^{-1}\sum_{j=1}^n\xi_j\Big)^2=(2\alpha-1)^2,\ 2\alpha-1<n^{-1}\sum_{k=1}^n\xi_k<1\Big\},$$

contained in the set $\{\xi\in\Omega_{[n]}:\xi_j>0,\ j=1,2,\dots,n\}$. Hence,

$$\Phi_p^{\Omega_{[n]}}(a)\ge C_1(n,p)|W|^{\frac1p-1}\ge C_2(n,p)((2\alpha-1)^{n-1}(1-\alpha))^{\frac1p-1}\ge C_3(n,p)(1-\alpha^n)^{\frac1p-1}.$$

Now, let $\alpha \le 2/3$. Then a is the center of the cube

$$W = \{\xi \in \mathbb{R}^n : \alpha - 1/3 < \xi_j < \alpha + 1/3,\ j = 1, 2, \dots, n\},$$

and $W \subset \Omega_{[n]}$. Now it is clear that $\Phi_p^{\Omega_{[n]}}(a) \ge C_1(n,p) \ge C_2(n,p)(1-\alpha^n)^{\frac{1}{p}-1}$ if $\alpha \le 2/3$. $\square$

§9. The exponent $r(\Omega)$

Let Ω be a subset of $\mathbb{Z} \setminus \{0\}$. Note that the function $p \mapsto \Phi_p^{\Omega\cup\{0\}}(0,\mathbb{T})$ is monotone decreasing and lower semicontinuous. Moreover, $\Phi_p^{\Omega\cup\{0\}}(0,\mathbb{T}) = 1$ for $p \ge 1$.

Hence, there exists a unique number $r(\Omega) \in [0,1]$ such that $\Phi_p^{\Omega\cup\{0\}}(0,\mathbb{T}) > 1$ if and only if $0 < p < r(\Omega)$. Observe also that the condition $\Phi_p^{\Omega\cup\{0\}}(0,\mathbb{T}) > 1$ is equivalent to the inequality

$$\inf\Big\{\int_{\mathbb{T}} (|1+f|^p - 1)\, d\boldsymbol{m} : f \in \mathcal{L}^1_\Omega(\mathbb{T})\Big\} < 0.$$

Put $r(\Omega) \stackrel{\text{def}}{=} 1$ if $\Omega \ni 0$. It is clear that if $\Omega_1 \subset \Omega_2 \subset \mathbb{Z}$, then $r(\Omega_1) \le r(\Omega_2)$.

Lemma 9.1. *Let $p \in (0,1]$. Then*

$$|1+z|^p \ge 1 + \frac{p}{2}(z+\overline{z}) - \frac{p(2-p)}{8}(z^2+\overline{z}^2) + \frac{p^2}{4}|z|^2 - 2|z|^3$$

for all $z \in \mathbb{C}$.

Proof. First, assume that $|z| < 1$. Put

$$h(z) \stackrel{\text{def}}{=} (1+z)^{\frac{p}{2}} - 1 - \frac{p}{2}z + \frac{p(2-p)}{8}z^2 = \sum_{k=3}^{\infty} \binom{p/2}{k} z^k.$$

Clearly,

$$|h(z)| \le -|z|^3 \sum_{k=3}^{\infty} (-1)^k \binom{p/2}{k} = |z|^3 \sum_{k=0}^{2} (-1)^k \binom{p/2}{k} = \frac{(2-p)(4-p)}{8}|z|^3.$$

Hence, using the obvious inequality

$$\Big|1 + \frac{p}{2}z - \frac{p(2-p)}{8}z^2\Big| \le \frac{4+3p}{4},$$

we get

$$\begin{aligned}
|1+z|^p &= \Big|1 + \frac{p}{2}z - \frac{p(2-p)}{8}z^2 + h(z)\Big|^2 \\
&= 1 + \frac{p}{2}(z+\overline{z}) - \frac{p(2-p)}{8}(z^2+\overline{z}^2) + \frac{p^2}{4}|z|^2 \\
&\quad - \frac{p^2(2-p)}{16}(z^2\overline{z} + \overline{z}^2 z) + \frac{p^2(2-p)^2}{64}|z|^4 \\
&\quad + 2\operatorname{Re}\Big(\Big(1 + \frac{p}{2}z - \frac{p(2-p)}{8}z^2\Big)\overline{h(z)}\Big) + |h(z)|^2 \\
&\ge 1 + \frac{p}{2}(z+\overline{z}) - \frac{p(2-p)}{8}(z^2+\overline{z}^2) \\
&\quad + \frac{p^2}{4}|z|^2 - \frac{p^2(2-p)}{8}|z|^3 - \frac{(4+3p)(2-p)(4-p)}{16}|z|^3 \\
&\ge 1 + \frac{p}{2}(z+\overline{z}) - \frac{p(2-p)}{8}(z^2+\overline{z}^2) + \frac{p^2}{4}|z|^2 - 2|z|^3.
\end{aligned}$$

Now we consider the case where $|z| > 1$. It suffices to check that in this case the right-hand side of the required inequality is negative. Indeed,

$$\begin{aligned}1 &+ \frac{p}{2}(z+\overline{z}) - \frac{p(2-p)}{8}(z^2+\overline{z}^{\,2}) + \frac{p^2}{4}|z|^2 - 2|z|^3 \\ &= -\frac{1}{2}\left(p(2-p)\Big(\frac{z+\overline{z}}{2}\Big)^2 - 2p\Big(\frac{z+\overline{z}}{2}\Big) + 4|z|^3 - p|z|^2 - 2\right) < 0,\end{aligned}$$

because

$$p(2-p)x^2 - 2px + r > 0$$

for all $x \in \mathbb{R}$ if $r > 1$. □

Theorem 9.2. *Let $\Omega \subset \mathbb{Z}$. Suppose that $\Omega \cap (-\Omega) \neq \varnothing$. Then $r(\Omega) = 1$. If Ω is finite and $\Omega \cap (-\Omega) = \varnothing$, then $r(\Omega) < 1$.*

Proof. The first statement follows from the fact that $\|1 + (z^{-n} + z^n)/2\|_{L^p} < 1$ for all $p < 1$ and all $n \in \mathbb{Z} \setminus \{0\}$.

We prove the second statement. Let $f \in \mathcal{L}^1_\Omega(\mathbb{T})$. Note that $\int_\mathbb{T} f\,d\boldsymbol{m} = \int_\mathbb{T} \overline{f}\,d\boldsymbol{m} = 0$ because $0 \notin \Omega$. Moreover, $\int_\mathbb{T} f^2\,d\boldsymbol{m} = \int_\mathbb{T} \overline{f}^{\,2}\,d\boldsymbol{m} = 0$ because $\Omega \cap (-\Omega) = \varnothing$. Assume that $\|1 + f\|_{L^p} < 1$. Then, by Lemma 9.1,

$$\begin{aligned}1 > \int_\mathbb{T} |1+f|^p\,d\boldsymbol{m} &\geq 1 + \frac{p^2}{4}\int_\mathbb{T} |f|^2\,d\boldsymbol{m} - 2\int_\mathbb{T} |f|^3\,d\boldsymbol{m} \\ &\geq 1 + \Big(\frac{p^2}{4} - 2\|f\|_{L^\infty}\Big)\|f\|^2_{L^2}.\end{aligned} \tag{9.1}$$

Hence, $\|f\|_{L^\infty} \geq \frac{p^2}{8}$. Assume that $r(\Omega) = 1$. Then we can construct sequences $\{p_n\}$ in $(1/2, 1)$ and $\{f_n\}$ in $\mathcal{L}^1_\Omega(\mathbb{T})$ such that $\lim_{n\to\infty} p_n = 1$ and $\|1 + f_n\|_{L^{p_n}} < 1$ for all n. Note that $\|f_n\|_{L^\infty} \geq \frac{p^2}{8} \geq \frac{1}{32}$ and $\|f_n\|^{p_n}_{L^{\frac{1}{2}}} \leq \|f_n\|^{p_n}_{L^{p_n}} \leq 1 + \|1 + f_n\|^{p_n}_{L^{p_n}} < 2$ for all n. Thus, the sequence $\{f_n\}$ is bounded in the finite-dimensional space $\mathcal{L}^1_\Omega(\mathbb{T})$, so that we can extract a converging subsequence $\{f_{n_k}\}$. Then for $f = \lim_{k\to\infty} f_{n_k}$ we have $\int_\mathbb{T} |1+f|\,d\boldsymbol{m} \leq 1$, whence $\operatorname{Im} f = 0$, i.e., $f = 0$, because $\Omega \cap (-\Omega) = \varnothing$. But on the other hand, $\|f\|_{L^\infty} = \lim_{k\to\infty} \|f_{n_k}\|_{L^\infty} \geq \frac{1}{32}$, and we get a contradiction. □

Remark 9.3. The above proof of the second part of Theorem 9.2 is similar to the proof of Theorem 2.7 in [2].

Theorem 9.4. *Let $\Omega \subset \mathbb{Z}$. Then $r(\Omega) = 0$ if and only if $\Omega \subset \mathbb{N}$ or $\Omega \subset \mathbb{Z}_-$.*

Proof. The condition $r(\Omega) = 0$ is fulfilled if and only if $\int_\mathbb{T}(|1 + f|^p - 1)\,d\boldsymbol{m} \geq 0$ for all $p > 0$ and all $f \in \mathcal{L}^1_\Omega(\mathbb{T})$. The latter condition can be rewritten also as follows:

$$\int_\mathbb{T} \log|1+f|\,d\boldsymbol{m} \geq 0 \tag{9.2}$$

for all $f \in \mathcal{L}^1_\Omega(\mathbb{T})$.

Let $\Omega \subset \mathbb{N}$. Then inequality (9.2) is well known. The case where $\Omega \subset \mathbb{Z}_-$ reduces easily to the case where $\Omega \subset \mathbb{N}$.

Now, suppose that (9.2) is true. Then $\int_\mathbb{T} f^n\,d\boldsymbol{m} = 0$ for all $n \geq 1$ and $f \in \mathcal{L}^1_\Omega(\mathbb{T})$ by [3, Lemma 2.1]. It remains to note that if $m, n \in \mathbb{N}$, then $\int_\mathbb{T}(z^{-n} + z^m)^{m+n}\,d\boldsymbol{m} = \frac{(m+n)!}{m!n!} \neq 0$. □

Remark 9.5. Theorem 9.4 can be reformulated as follows. The identity $r(\Omega) = 0$ is true if and only if the semigroup generated by Ω does not contain the zero element. We shall show that in this formulation the result extends to the setting of locally compact Abelian groups.

9.1. Compact Abelian groups. Everything said above about the exponent $r(\Omega)$ can easily be extended to the case of compact Abelian groups. Let G be a compact Abelian group, Γ the dual group. Let Ω be a subset of Γ. We denote by $r_G(\Omega)$ the infimum of the set of all $p \in (0,1]$ such that $\int_G (|1+f|^p - 1)\, d\boldsymbol{m}_G \geq 0$ for all $f \in \mathcal{L}^1_\Omega(G)$. If $0 \in \Omega$, then there is no such p, and we put $r_G(\Omega) \stackrel{\text{def}}{=} 1$. Now, let $0 \notin \Omega$. Then the infimum is attained provided $r_G(\Omega) > 0$. In this case, $r_G(\Omega)$ is the smallest exponent p with $\Phi_p^{\Omega\cup\{0\}}(0,G) = 1$. Moreover, the fact that $r_G(\Omega) = 0$ means that $\Phi_p^{\Omega\cup\{0\}}(0,G) = 1$ for all $p > 0$. The latter is true if and only if $\int_G \log|1+f|\, d\boldsymbol{m}_G \geq 0$ for all $f \in \mathcal{L}^1_\Omega(G)$.

Theorem 9.2 extends almost word by word to the case of a compact Abelian group.

Theorem 9.6. *Let G be a compact Abelian group. Let Ω be a subset of the dual group Γ. Suppose that $\Omega \cap (-\Omega) \neq \varnothing$. Then $r_G(\Omega) = 1$. If Ω is finite and $\Omega \cap (-\Omega) = \varnothing$, then $r_G(\Omega) < 1$.*

Now we state a generalization of Theorem 9.4.

Theorem 9.7. *Let G be a compact Abelian group, and let Ω be a subset of the dual group Γ. Then $r_G(\Omega) = 0$ if and only if the semigroup generated by Ω does not contain the zero element of Γ.*

Proof. Assume that the semigroup generated by Ω contains the zero element. This means that there exists a finite sequence $\{\lambda_k\}_{k=1}^N$ in Ω such that $\sum_{k=1}^N \lambda_k = 0$. Put $f(x) = \Big(\sum_{k=1}^N (x,\lambda_k)\Big)^N$, $x \in G$. Then f is the sum of N^N terms of the form $\prod_{j=1}^N (x,\lambda_{k_j}) = \Big(x, \sum_{j=1}^N \lambda_{k_j}\Big)$, and each of these terms is a character of the group G. Hence, $\int_G f^N\, d\boldsymbol{m}_G \neq 0$ because at least one ($k_j = j$ for $j = 1,2,\dots,N$) of these terms is identically equal to 1. Thus, by [3, Lemma 2.1], we have $r_G(\Omega) > 0$ (see the proof of Theorem 9.4).

Now we prove the converse statement, although probably it is well known. So, suppose that the semigroup generated by Ω does not contain the zero element. We prove that $r_G(\Omega) = 0$. Let P denote the semigroup generated by Ω. Clearly, $P \cap (-P) = \varnothing$; otherwise P would have contained the zero element. We denote by A the uniform algebra generated by the family of characters $\{(x,\lambda)\}_{\lambda \in P\cup\{0\}}$. The functional $f \stackrel{\varphi}{\mapsto} \int_G f\, d\boldsymbol{m}_G$ is multiplicative on A, because $P \cap (-P) = \varnothing$. Note that the algebra A is invariant under translations; i.e., $f \in A \Longrightarrow f(x+h) \in A$ for all $h \in G$. We need to prove that $\boldsymbol{m}_G$ is a Jensen measure for the functional φ; see [13] for the definition and elementary properties of Jensen measures. By Theorem 2.2.4 in the monograph [13], there exists a probability measure μ on G that is a Jensen measure for φ. The functional φ is invariant under translations. Hence, all translations of this measure are also Jensen measures for φ. Averaging all these translations over the Haar measure $\boldsymbol{m}_G$, we see that the Haar measure $\boldsymbol{m}_G$ is itself a Jensen measure for φ. □

Let Ω be a nonempty subset of the group Γ. Put $\Omega_1 \stackrel{\text{def}}{=} \Omega + \Omega$, $\Omega_2 \stackrel{\text{def}}{=} \Omega_1 + \Omega$, $\dots$, $\Omega_{k+1} \stackrel{\text{def}}{=} \Omega_k + \Omega$, $\dots$. From Theorem 9.7, it follows that if $0 \in \Omega_k$ for some k, then $r_G(\Omega) > 0$. Our next aim is to obtain the following quantitative refinement of this result: if $0 \in \Omega_k$, then $r_G(\Omega) \geq p_k$. Theorem 9.6 shows that we can take 1 for the role of p_1.

Let $G = \mathbb{T}^n$. Then we can identify the dual group Γ with $\mathbb{Z}^n$ in a natural way. Let $\{e_k\}_{k=1}^n$ be the standard basis in $\mathbb{R}^n$. Put $\Lambda_{[n]} \stackrel{\text{def}}{=} \{e_1, e_2, \dots, e_n, -\sum_{k=1}^n e_k\}$. Then $\Lambda_{[n]}$ is an $(n+1)$-element subset of $\mathbb{Z}^n$. Note that $0 \in (\Lambda_{[n]})_n$. Put $\tau_n \stackrel{\text{def}}{=} r_{\mathbb{T}^n}(\Lambda_{[n]})$. Clearly, $\tau_1 = 1$. We shall see that τ_k can be taken for the role of p_k.

Theorem 9.8. *The sequence $\{\tau_n\}_{n=1}^\infty$ is strictly monotone decreasing and tends to zero at infinity.*

To prove this theorem we need several lemmas.

Lemma 9.9. *Let G be a compact Abelian group, Ω a finite set of nonconstant characters of G. Suppose that $r_G(\Omega)>0$. Then there exists a nonzero function $f\in\mathcal{L}^1_\Omega(G)$ such that $\int_G|1+f|^p\,d\boldsymbol{m}_G=1$ for $p=r_G(\Omega)$.*

Proof. First, assume that $r_G(\Omega)=1$. Then $\Omega\cap(-\Omega)\neq\varnothing$ by Theorem 9.6. Let $\gamma\in\Omega\cap(-\Omega)$. If $\gamma=0$, then put $f=-2$. Otherwise, put $f(x)=\frac{(x,\gamma)+(x,-\gamma)}{2}=\mathrm{Re}(x,\gamma)$.

Now, let $r_G(\Omega)<1$. Then $\Omega\cap(-\Omega)=\varnothing$. This allows us to use an analog of inequality (9.1) for $f\in\mathcal{L}^1_\Omega(G)$, together with other arguments from the proof of Theorem 9.2. □

Let $z_1,z_2,\dots,z_n$ be the independent variables on the n-dimensional torus $\mathbb{T}^n$. We put

$$\sigma_p\overset{\text{def}}{=}\inf\left\{\left\|\sum_{j=1}^n a_jz_j\right\|_{L^p(\mathbb{T}^n)}:\sum_{j=1}^n|a_j|^2=1,\ n\in\mathbb{N}\right\}\quad\text{for }\ p\in(0,2)$$

and

$$\sigma_p\overset{\text{def}}{=}\inf\left\{\left\|\sum_{j=1}^n a_jz_j\right\|^{-1}_{L^p(\mathbb{T}^n)}:\sum_{j=1}^n|a_j|^2=1,\ n\in\mathbb{N}\right\}\quad\text{for }\ p\in(2,+\infty).$$

Clearly, $\sigma_p<1$ for all positive $p\neq2$. The well-known Khinchin inequality (see, e.g., [28]) asserts that $\sigma_p>0$ for all p. It is known that $\sigma_0\overset{\text{def}}{=}\lim_{p\to0+}\sigma_p>0$; see [32] and [10]. Moreover, in [10] the inequality $\sigma_0\ge\frac12$ was announced, which can even be improved by the methods of [11]. The case of negative p, $-1<p<0$, was considered in [15].

Lemma 9.10. *Let $f=a_0\prod_{j=1}^n z_j^{-1}+\sum_{j=1}^n a_jz_j$. Suppose that $\|1+f\|_{L^p}<1$, where $p\in(0,1)$. Then $\|f\|_{L^2(\mathbb{T}^n)}>\frac{p^2\sigma_3^3}{8}$ if $n\ge2$.*

Proof. Lemma 9.1 shows that $\frac{p^2}{4}\|f\|^2_{L^2(\mathbb{T}^n)}<2\|f\|^3_{L^3(\mathbb{T}^n)}$. It remains to observe that $\|f\|^3_{L^3(\mathbb{T}^n)}\le\sigma_3^{-3}\|f\|^3_{L^2(\mathbb{T}^n)}$, because

$$\left\|a_0\prod_{j=1}^n z_j^{-1}+\sum_{j=1}^n a_jz_j\right\|_{L^p(\mathbb{T}^n)}=\left\|a_0z_{n+1}+\sum_{j=1}^n a_jz_j\right\|_{L^p(\mathbb{T}^{n+1})}$$

for all $p\in(0,+\infty)$. □

Lemma 9.11. *Let $0<p\le2$. Then $\|1+az\|_{L^p(\mathbb{T})}\ge(1+\frac p2|a|^2)^{1/2}$ for all $a\in\mathbb{C}$.*

Proof. First, let $|a|\le1$. Then

$$\|1+az\|^p_{L^p}=\|(1+az)^{\frac p2}\|^2_{L^2}=\left\|1+\frac p2az+\cdots\right\|^2_{L^2}\ge1+\frac{p^2}{4}|a|^2\ge\Big(1+\frac p2|a|^2\Big)^{\frac p2}.$$

But if $|a|\ge1$, then

$$\|1+az\|^p_{L^p}=|a|^p\|1+a^{-1}z\|^p_{L^p}\ge|a|^p\Big(1+\frac p2|a|^{-2}\Big)^{\frac p2}=\Big(|a|^2+\frac p2\Big)^{\frac p2}\ge\Big(1+\frac p2|a|^2\Big)^{\frac p2}.$$

□

Lemma 9.12. *Let $f=\sum_{j=1}^n a_jz_j$. Then*

$$\|1+f\|_{L^p(\mathbb{T}^n)}\ge\Big(1+\frac{p\,\sigma_p^2}{2}\|f\|^2_{L^2(\mathbb{T}^n)}\Big)^{1/2}$$

for all $p\in(0,2)$.

Proof. We have

$$\begin{aligned}\|1+f\|_{L^p}^p &= \int_{\mathbb{T}^n}\int_{\mathbb{T}} |1+\zeta f(z_1,\dots,z_n)|^p\, d\boldsymbol{m}(\zeta)\, d\boldsymbol{m}_n(z_1,\dots,z_n)\\ &\ge \int_{\mathbb{T}^n}\Big(1+\frac{p}{2}|f|^2\Big)^{p/2}\, d\boldsymbol{m}_n\\ &= \Big\|1+\frac{p}{2}|f|^2\Big\|_{L^{p/2}}^{p/2} \ge \Big(\|1\|_{L^{p/2}}+\Big\|\frac{p}{2}|f|^2\Big\|_{L^{p/2}}\Big)^{p/2}\\ &= \Big(1+\frac{p}{2}\|f\|_{L^p}^2\Big)^{p/2} \ge \Big(1+\frac{p\,\sigma_p^2}{2}\|f\|_{L^2}^2\Big)^{p/2}. \end{aligned}$$

□

Proof of Theorem 9.8. Assume that the sequence $\{\tau_n\}$ is not strictly monotone decreasing. Then $\tau_{n-1}\le\tau_n$ for some $n\ge 2$. By Lemma 9.9, there exists a nonzero finite sequence $\{a_j\}_{j=0}^n$ of numbers such that

$$\int_{\mathbb{T}^n}\Big|1+a_0\prod_{j=1}^n z_j^{-1}+\sum_{j=1}^n a_j z_j\Big|^p\, d\boldsymbol{m}_n(z_1,\dots,z_n)=1,$$

where $p=\tau_n$. We may assume that the sequence $\{|a_j|\}_{j=0}^n$ increases. Lemma 9.12 shows that $a_0\ne 0$, and so $a_j\ne 0$ for all j. Note that every finite sequence $\{b_j\}_{j=-1}^{n-1}$ of numbers satisfies the inequality

$$\int_{\mathbb{T}^{n-1}}\Big|b_{-1}+b_0\prod_{j=1}^{n-1} z_j^{-1}+\sum_{j=1}^{n-1} b_j z_j\Big|^p\, d\boldsymbol{m}_{n-1}(z_1,\dots,z_{n-1})\ge |b_{-1}|^p$$

because $p=\tau_n\ge\tau_{n-1}$. Consequently, the inequality

$$\int_{\mathbb{T}^n}\Big|1+a_0\prod_{j=1}^n z_j^{-1}+\sum_{j=1}^n a_j z_j\Big|^p\, d\boldsymbol{m}_{n-1}(z_1,\dots,z_{n-1})\ge |1+a_n z_n|^p$$

is fulfilled for every $z_n\in\mathbb{T}$, whence

$$\int_{\mathbb{T}^n}\Big|1+a_0\prod_{j=1}^n z_j^{-1}+\sum_{j=1}^n a_j z_j\Big|^p\, d\boldsymbol{m}_n(z_1,\dots,z_n)\ge\int_{\mathbb{T}}|1+a_n z_n|^p\, d\boldsymbol{m}(z_n)>1,$$

because $a_n\ne 0$. The contradiction obtained proves the first statement of the theorem.

We pass to the proof of the second part. Now we put $p\overset{\text{def}}{=}\lim_{n\to\infty}\tau_n$. Assume that $p>0$. Fix a positive integer n. Since $\tau_n>p$, there exists a function of the form $f=a_0\prod_{j=1}^n z_j^{-1}+\sum_{j=1}^n a_j z_j$ such that $\|1+f\|_{L^p(\mathbb{T}^n)}<1$. Again, we assume that the sequence $\{|a_j|\}_{j=0}^n$ increases. Then $|a_0|^2\le An^{-1}$, where $A\overset{\text{def}}{=}\sum_{j=1}^n|a_j|^2$. Hence,

$$1>\|1+f\|_{L^p}^p\ge\Big\|1+\sum_{j=1}^n a_j z_j\Big\|_{L^p}^p-|a_0|^p\ge\Big(1+\frac{p\,\sigma_p^2}{2}A\Big)^{p/2}-(An^{-1})^{p/2}\tag{9.3}$$

by Lemma 9.12. Note that the inequality $\|1+f\|_{L^p}<1$ implies $\|f\|_{L^p}\le 2^{\,1/p}$, whence $A\le\|f\|_{L^2}^2\le\sigma_p^{-2}2^{\,2/p}$. Moreover, from Lemma 9.10 it follows that $A\ge\frac12\|f\|_{L^2}^2\ge\frac{p^4\sigma_3^6}{128}$. Thus, inequality (9.3) implies that

$$\Big(1+\frac{p^5\sigma_3^6\,\sigma_p^2}{256}\Big)^{p/2}\le\Big(1+\frac{p\,\sigma_p^2}{2}A\Big)^{p/2}<1+(An^{-1})^{p/2}\le 1+2\,\sigma_p^{-p}n^{\,-p/2},$$

and we get a contradiction if n is sufficiently large. □

9.2. Locally compact Abelian groups. Let G be a locally compact Abelian group, Γ the dual group. Let Ω be an open subset of Γ. We denote by $r_G(\Omega)$ the infimum of the set of all $p \in (0,1]$ such that $\int_G (|1+f|^p - 1)\, d\boldsymbol{m}_G \geq 0$ for all $f \in \mathcal{L}^1_\Omega(G)$. It is not difficult to verify that there are no such numbers p if $0 \in \Omega$. In this case we put $r_G(\Omega) \stackrel{\text{def}}{=} 1$. Now suppose $0 \notin G$. Then the infimum is attained if $r_G(\Omega) > 0$. Moreover, we have $r_G(\Omega) = 0$ if and only if $\int_G \log|1+f|\, d\boldsymbol{m}_G \geq 0$ for all $f \in \mathcal{L}^1_\Omega(G)$.

Remark 9.13. The following identity is true:

$$\inf\Big\{\int_G (|1+f|^p - 1)\, d\boldsymbol{m}_G : f \in \mathcal{L}^1_\Omega(G)\Big\} = \inf\Big\{\int_G (|1+f|^p - 1)\, d\boldsymbol{m}_G : f \in \mathcal{L}^p_\Omega(G)\Big\}.$$

Proof. It is easy to check that the function $f \mapsto \int_G (|1+f|^p - 1)\, d\boldsymbol{m}_G$ is continuous on $L^1(G)$ (and even on $L^1(G) + L^p(G)$). Thus, it suffices to prove that the closure of $\mathcal{L}^p_\Omega(G)$ in $L^1(G)$ contains $\mathcal{L}^1_\Omega(G)$. Let N be a positive integer such that $pN \geq 2$. With every compact neighborhood U of the zero in Γ, we associate the function $\varphi_U \stackrel{\text{def}}{=} \frac{1}{\boldsymbol{m}_\Gamma(U)} \mathcal{F}^{-1} \mathbb{1}_U$. It is clear that $\varphi_U \in L^2(G) \cap L^\infty(G)$, whence $\varphi_U^N \in L^p(G)$. Each function $f \in \mathcal{L}^1_\Omega(G)$ is the limit in $L^1(G)$ of the net $\{\varphi_U^N f\}_U$ as $U \to 0$. It remains to note that $\varphi_U^N f \in \mathcal{L}^p_\Omega(G)$ if the neighborhood U is sufficiently small. □

The next identity can be proved in a similar way:

$$\inf\Big\{\int_G (|1+f|^p - 1)\, d\boldsymbol{m}_G : f \in \mathcal{L}^1_\Omega(G)\Big\} = \inf\Big\{\int_G (|1+f|^p - 1)\, d\boldsymbol{m}_G : f \in \mathcal{L}^q_\Omega(G)\Big\},$$

where $q \in [p,1]$.

Theorem 9.14. *Let G be a locally compact Abelian group, Ω an open subset of the dual group Γ. Suppose that there exists a finite sequence $\{\gamma_k\}_{k=0}^n$ in Ω such that $\sum_{k=0}^n \gamma_k = 0$, where $n \geq 1$. Then $r_G(\Omega) \geq \tau_n$.*

We start with proving several auxiliary statements.

Lemma 9.15. *Let $\varphi \in L^1(G)$, where G is a locally compact Abelian group. Then for every positive ε there exists a compact subset E of G such that $\boldsymbol{m}_G(E) > 0$ and $\|\varphi * \mathbb{1}_E - (\int_G \varphi\, d\boldsymbol{m}_G)\mathbb{1}_E\|_{L^1} \leq \varepsilon \|\mathbb{1}_E\|_{L^1}$.*

Proof. It suffices to prove the required statement for a function $\varphi \in L^1(G)$ with compact support. Every compact subset of G is contained in an open compactly generated subgroup G_0 of G. Thus, we may assume that G is a compactly generated group, and then, by the well-known structure theorems (see, e.g., [20, Theorem 24]), G is isomorphic to a group of the form $\mathbb{R}^m \times \mathbb{Z}^n \times K$, where K is a compact Abelian group and m, n are nonnegative integers. Thus, the lemma reduces to the case where $G = \mathbb{R}^m \times \mathbb{Z}^n \times K$. We shall seek a set E of the form

$$E = E_A = \big\{(x,k) \in \mathbb{R}^{m+n} \times K : x \in \mathbb{R}^m \times \mathbb{Z}^n,\ |x_j| \leq A \text{ for } j = 1,2,\dots,m+n\big\}.$$

Let $\operatorname{supp}\varphi \subset E_{A_0}$. Note that $\varphi * \mathbb{1}_{E_A} = \int_G \varphi\, d\boldsymbol{m}_G$ everywhere on E_{A-A_0} for $A > A_0$. Moreover, $\operatorname{supp}(\varphi * \mathbb{1}_{E_A}) \subset E_{A+A_0}$. Consequently,

$$\Big\|\varphi * \mathbb{1}_{E_A} - \Big(\int_G \varphi\, d\boldsymbol{m}_G\Big)\mathbb{1}_{E_A}\Big\|_{L^1} \leq 2\,\|\varphi\|_{L^1}\, \boldsymbol{m}_G(E_{A+A_0} \setminus E_{A-A_0}) \leq \varepsilon \|\mathbb{1}_{E_A}\|_{L^1}$$

if the number A is sufficiently large, because $\boldsymbol{m}_G(E_A) \sim CA^{m+n}$ as $A \to +\infty$. □

Lemma 9.16. *Let G be a locally compact Abelian group, Γ the dual group. Let $h : [0,1] \to \mathbb{R}$ be a continuous function such that $h(t) \leq Ct$ and $h(1) < 0$. Then for every neighborhood U of zero in Γ there exists a function $f \in \mathcal{L}^1_U(G)$ such that $0 \leq f \leq 1$ and $\int_G (h \circ f)\, d\boldsymbol{m}_G < 0$.*

Proof. Fix a nonnegative function $\varphi \in \mathcal{L}^1_U(G)$ such that $\int_G \varphi \, d\boldsymbol{m}_G = 1$. We shall seek the required function f in the form $f = \varphi * \mathbb{1}_E$, where $\mathbb{1}_E$ is the characteristic function of a compact subset E of G. It is clear that $f \in \mathcal{L}^1_U(G)$, $0 \le f \le 1$, and $\int_G f \, d\boldsymbol{m}_G = \boldsymbol{m}_G(E)$. Since h is continuons, the inequality $h < 0$ is fulfilled in a neighborhood of the point 1. Hence, there exists $\tau \in (0,1)$ such that $h(t) \le -\tau$ for all $t \in [1-\tau, 1]$. Then we have

$$\begin{aligned}\int_G (h \circ f) \, d\boldsymbol{m}_G &\le C \int_{\{f \le 1-\tau\}} f \, d\boldsymbol{m}_G - \tau \boldsymbol{m}_G\{f > 1-\tau\} \\ &\le (C+\tau) \int_{\{f \le 1-\tau\}} f \, d\boldsymbol{m}_G - \tau \int_G f \, d\boldsymbol{m}_G \\ &\le (C+\tau)(1+\tau^{-1}) \|f - \mathbb{1}_E\|_{L^1} - \tau \|\mathbb{1}_E\|_{L^1},\end{aligned}$$

because the inequality $f(x) \le 1-\tau$ implies $f(x) \le (1+\tau^{-1})|f(x) - \mathbb{1}_E|$. It remains to apply Lemma 9.15. □

Proof of Theorem 9.14. There exists a neighborhood U of zero in Γ such that $\gamma_k + U \subset \Omega$ for $k = 0, 1, \dots, n$. We take a nonzero function $f \in \mathcal{L}^1_U(G)$. Let $p < \tau_n$. Then there exists a nonzero finite sequence $\{a_j\}_{j=0}^n$ of numbers such that

$$\int_{\mathbb{T}^n} \Big(\Big| 1 + a_0 \prod_{j=1}^n z_j^{-1} + \sum_{j=1}^n a_j z_j \Big|^p - 1 \Big) \, d\boldsymbol{m}_n(z_1, \dots, z_n) < 0.$$

Consider the function $h : [0,1] \to \mathbb{R}$ defined by the formula

$$h(t) = \int_{\mathbb{T}^n} \Big(\Big| 1 + t a_0 \prod_{j=1}^n z_j^{-1} + t \sum_{j=1}^n a_j z_j \Big|^p - 1 \Big) \, d\boldsymbol{m}_n(z_1, \dots, z_n).$$

Clearly, h is continuous. Note that $h(1) < 0$. Moreover,

$$\begin{aligned}h(t) &= \Big\| 1 + t a_0 \prod_{j=1}^n z_j^{-1} + t \sum_{j=1}^n a_j z_j \Big\|_{L^p}^p - 1 \le \Big\| 1 + t a_0 \prod_{j=1}^n z_j^{-1} + t \sum_{j=1}^n a_j z_j \Big\|_{L^2}^p - 1 \\ &= \Big(1 + t^2 \sum_{j=0}^n |a_j|^2 \Big)^{\frac{p}{2}} - 1 \le Ct^2 \le Ct.\end{aligned}$$

Let f be a real integrable function on G such that $0 \le f \le 1$ almost everywhere on G. Then

$$\begin{aligned}&\int_{\mathbb{T}^n} \int_G \Big(\Big| 1 + a_0 f(x)(x, \gamma_0) \prod_{j=1}^n z_j^{-1} + \sum_{j=1}^n a_j f(x)(x, \gamma_j) z_j \Big|^p - 1 \Big) \, d\boldsymbol{m}_G(x) \, d\boldsymbol{m}_n(z_1, \dots, z_n) \\ &\quad = \int_G h(f(x)) \, d\boldsymbol{m}_G(x).\end{aligned}$$

Hence, there exists a point $z = (z_1, \dots, z_n) \in \mathbb{T}^n$ such that

$$\begin{aligned}&\int_G \Big(\Big| 1 + a_0 f(x)(x, \gamma_0) \prod_{j=1}^n z_j^{-1} + \sum_{j=1}^n a_j f(x)(x, \gamma_j) z_j \Big|^p - 1 \Big) \, d\boldsymbol{m}_G(x) \\ &\quad = \int_G h(f(x)) \, d\boldsymbol{m}_G(x).\end{aligned}$$

Thus, the theorem reduces to Lemma 9.16. □

Theorem 9.17. *Let G be a locally compact Abelian group, Ω an open subset of the dual group Γ. Then $r_G(\Omega) = 0$ if and only if the semigroup generated by Ω does not contain the zero element in Γ.*

Proof. If the semigroup generated by Ω contains the zero element, then $r_G(\Omega) > 0$ by Theorem 9.14. Now assume that the semigroup generated by Ω does not contain the zero element. Note that for every nonzero function $f \in \mathcal{L}^1_\Omega(G)$ the semigroup generated by $\operatorname{supp} \mathcal{F}f$ is an open compactly generated subgroup of Γ. Thus, it suffices to consider the case where Γ is a compactly generated group. Hence, as in the proof of Lemma 9.15, we may restrict ourselves to the groups of the form $\mathbb{R}^m \times \mathbb{Z}^n \times K$, where K is a compact Abelian group and m, n are nonnegative integers; but now $\Gamma = \mathbb{R}^m \times \mathbb{Z}^n \times K$. First we consider the case where $K = \{0\}$, i.e., $\Gamma = \mathbb{R}^m \times \mathbb{Z}^n$. Then $G = \mathbb{R}^m \times (\mathbb{R}^n/\mathbb{Z}^n)$. We take a positive number t and define an operator $\mathcal{E}_t : L^1(\mathbb{R}^m \times (\mathbb{R}^n/\mathbb{Z}^n)) \to L^1((\mathbb{R}^m/t\mathbb{Z}^m) \times (\mathbb{R}^n/\mathbb{Z}^n))$ by putting $(\mathcal{E}_t f)(x, y) \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}^n} f(x + tk, y)$. In a natural way, we identify the group dual to $(\mathbb{R}^m/t\mathbb{Z}^m) \times (\mathbb{R}^n/\mathbb{Z}^n)$ with the group $t^{-1}\mathbb{Z}^m \times \mathbb{Z}^n$. It is easily seen that if $f \in \mathcal{L}^1_\Omega(G)$, then $\mathcal{E}_t f \in \mathcal{L}_{\Omega\cap(t^{-1}\mathbb{Z}^m\times\mathbb{Z}^n)}((\mathbb{R}^m/t\mathbb{Z}^m) \times (\mathbb{R}^n/\mathbb{Z}^n))$. Clearly, the semigroup generated by $\Omega \cap (t^{-1}\mathbb{Z}^m \times \mathbb{Z}^n)$ does not contain the zero element of the group $t^{-1}\mathbb{Z}^m \times \mathbb{Z}^n$. Hence, $\int_{[0,t)^m\times[0,1)^n} \log|1 + \mathcal{E}_t f(x, y)|\, dxdy \geq 0$ by Theorem 9.7, whence $\int_{[0,t)^m\times[0,1)^n} (|1 + \mathcal{E}_t f(x, y)|^p - 1)\, dxdy \geq 0$ for all $p \in (0, 1]$. It is not difficult to verify that

$$\begin{aligned}\int_{\mathbb{R}^m\times[0,1)^n} &(|1 + f(x, y)|^p - 1)\, dxdy \\ &= \lim_{t\to+\infty} \int_{[-t/2,t/2)^m\times[0,1)^n} (|1 + \mathcal{E}_t f(x, y)|^p - 1)\, dxdy\end{aligned} \tag{9.4}$$

for every $f \in L^1(\mathbb{R}^m \times (\mathbb{R}^n/\mathbb{Z}^n))$. This identity is obvious if the support of f is compact. The general case follows from the uniform continuity of the function $f \mapsto \int(|1+f|^p-1)\, d\mu$ on the bounded subsets of $L^1(\mu)$. By (9.4), we have $\int_{\mathbb{R}^m\times[0,1)^n}(|1+f(x, y)|^p-1)\, dxdy \geq 0$ for all $f \in \mathcal{L}^1_\Omega(G)$ and $p \in (0, 1]$, whence $\int_{\mathbb{R}^m\times[0,1)^n}(\log|1 + f(x, y)|)\, dxdy \geq 0$ for all $f \in \mathcal{L}^1_\Omega(G)$.

Now, let K be an arbitrary compact Abelian group. Then $\Gamma = \Gamma_0 \times K$, where $\Gamma_0 = \mathbb{R}^m \times \mathbb{Z}^n$, and $G = G_0 \times \Sigma$, where Σ is the group dual to K. We may assume that Ω is an open semigroup of $\Gamma_0 \times K$ not containing the zero element. We prove that $\Omega \cap (\{0\} \times K) = \varnothing$. Assume that $a \in \Omega \cap (\{0\} \times K)$. Then there exists a neighborhood U of zero in Γ such that $a + \overline{U} \subset \Omega$. Hence, $na + \overline{U} \subset \Omega$ for all positive integers n. Since K is compact, 0 is a limit point of the sequence $\{na\}_{n=1}^\infty$, whence $0 \in \overline{U} \subset \Omega$, which contradicts the condition $0 \notin \Omega$. Hence, $\Omega \cap (\{0\} \times K) = \varnothing$. Then the zero element of Γ does not belong to the open semigroup $\Omega + (\{0\} \times K)$. Thus, we may assume that $\Omega = \Omega + (\{0\} \times K)$. Therefore, $\Omega = \Omega_0 \times K$, where Ω_0 is an open semigroup of Γ_0, and Ω_0 does not contain the zero element of Γ_0. Let $f(x, y) \in \mathcal{L}^1_\Omega(G)$. Here $x \in G_0$ and $y \in \Sigma$. Note that Σ is a discrete group. Thus, $f(\cdot, y)$ belongs to $\mathcal{L}^1_{\Omega_0}(G_0)$ for all $y \in \Sigma$. Consequently, as was already established, $\int_{G_0} \log|1 + f(x, y)|\, d\boldsymbol{m}_{G_0}(x) \geq 0$ for all $y \in \Sigma$, whence we get $\int_G \log|1 + f(x, y)|\, d\boldsymbol{m}_{G_0}(x) d\boldsymbol{m}_\Sigma(y) \geq 0$. □

§10. Again about the identity $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. Dependence on p

10.1. The identity $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and n-sufficient sets. In §5 the notion of a sufficient set was introduced. Now we are going to generalize this notion, defining an n-sufficient set for each positive integer n. Let G be a locally compact Abelian group, Γ the dual group. An open subset Ω of Γ is said to be *n-sufficient* if for every compact subset K of Γ there exists a nonempty finite subset Λ of Γ satisfying the following three conditions:

a) $\operatorname{card}(\Lambda) \leq n + 1$; b) $\sum_{\lambda\in\Lambda} \lambda = 0$; c) $K + \Lambda \subset \Omega$.

It is clear that the notion of a 1-sufficient set coincides with that of a sufficient set. An open subset Ω of Γ is said to be *strictly n-sufficient* if for every compact subset K of Γ there exists an $(n+1)$-element subset Λ of Γ such that $K+\Lambda\subset\Omega$ and $\sum_{\lambda\in\Lambda}\lambda=0$.

Remark 10.1. Every n-sufficient open set Λ is strictly k-sufficient for some $k\le n$.

Proof. Let us prove that one can take as k the smallest integer m, $1\le m\le n$, such that Λ is m-sufficient. If $k=1$, then there is nothing to prove. Let $k>1$. It is clear that there exists a compact subset K_0 of Γ such that conditions b) and c) can be fulfilled for $K=K_0$ only if $\operatorname{card}(\Lambda)\ge k+1$. Now let K be an arbitrary compact subset of Γ. Then there exists a nonempty finite subset Λ of Γ such that $\operatorname{card}(\Lambda)\le k+1$, $\sum_{\lambda\in\Lambda}\lambda=0$, and $(K\cup K_0)+\Lambda\subset\Omega$. Now it is clear that $\operatorname{card}(\Lambda)=k+1$. $\square$

The following theorem generalizes Theorem 5.1.

Theorem 10.2. *Let Ω be an open subset of Γ. If Ω is n-sufficient, then $L^p_\Omega(G)=L^p(G)$ for all $p\in(0,\tau_n)$.*[7]

Proof. Since the sequence $\{\tau_j\}_{j=1}^\infty$ is strictly monotone decreasing by Theorem 9.8, Remark 10.1 allows us to restrict ourselves to the case where Λ is strictly n-sufficient. Since $p<\tau_n$, there exists a finite sequence $\{a_j\}_{j=0}^n$ such that

$$\int_{\mathbb{T}^n}\Big|1+a_0\prod_{j=1}^n z_j^{-1}+\sum_{j=1}^n a_jz_j\Big|^p\,d\boldsymbol{m}_n(z_1,\dots,z_n)=1-\sigma$$

for some $\sigma=\sigma(n,p)>0$. Then

$$\int_{\mathbb{T}^n}\Big|1+a_0\zeta_0\prod_{j=1}^n z_j^{-1}+\sum_{j=1}^n a_j\zeta_jz_j\Big|^p\,d\boldsymbol{m}_n(z_1,\dots,z_n)=1-\sigma \tag{10.1}$$

for every finite sequence $\{\zeta_j\}_{j=0}^n$ in $\mathbb{T}$ satisfying $\prod_{j=0}^n\zeta_j=1$. Let $f\in\mathcal{L}^p(G)\overset{\text{def}}{=}\mathcal{L}^p_G(G)$. Then there exists an $(n+1)$-element set $\Lambda\subset\Gamma$ such that $\sum_{\lambda\in\Lambda}\lambda=0$ and $\Lambda+\operatorname{supp}\mathcal{F}f\subset\Omega$. We arrange the points of Λ in a sequence $\{\lambda_j\}_{j=0}^n$. Identity (10.1) implies

$$\begin{aligned}\int_G\int_{\mathbb{T}^n}|f(x)|^p\Big|1+a_0(x,\lambda_0)\prod_{j=1}^n z_j^{-1}+\sum_{j=1}^n a_j(x,\lambda_j)z_j\Big|^p\,d\boldsymbol{m}_n(z_1,\dots,z_n)\,d\boldsymbol{m}_G(x)\\ =(1-\sigma)\|f\|^p_{L^p}.\end{aligned}$$

Hence, we can choose $z\in\mathbb{T}^n$ so that

$$\Big\|f(x)\Big(1+a_0(x,\lambda_0)\prod_{j=1}^n z_j^{-1}+\sum_{j=1}^n a_j(x,\lambda_j)z_j\Big)\Big\|^p_{L^p}=(1-\sigma)\|f\|^p_{L^p}.$$

Note that $f(x)\Big(a_0(x,\lambda_0)\prod_{j=1}^n z_j^{-1}+\sum_{j=1}^n a_j(x,\lambda_j)z_j\Big)\in L^p_\Omega(G)$. Thus, we have proved that

$$\operatorname{dist}_{L^p}(f,L^p_\Omega(G))\le(1-\sigma)^{\frac1p}\|f\|_{L^p}$$

for every $f\in\mathcal{L}^p(G)$ and so for every $f\in L^p(G)$, because $\mathcal{L}^p(G)$ is dense in $L^p(G)$. Now it is clear that $L^p_\Omega(G)=L^p(G)$. $\square$

As has already been mentioned, Theorem 5.1 is, in essence, a special case of Theorem 9.1 in [1]. In a similar sense, Theorem 10.2 is a special case of the following theorem.

[7]Recall that the number τ_n was defined in the preceding section before Lemma 9.9.

Theorem 10.3. *Let $\{\Omega_j\}_{j=0}^n$ be a finite sequence of open subsets of Γ. Suppose that for every compact subset K of Γ there exists a finite sequence $\{\gamma_j\}_{j=0}^n$ in Γ such that $\sum_{j=0}^n \gamma_j = 0$ and $\gamma_j + K \subset \Omega_j$ for all $j \in \mathbb{Z}$, $0 \le j \le n$. Then for $p < \tau_n$ each function $f \in L^p(G)$ can be represented in the form $f = \sum_{j=0}^n f_j$, where $f_j \in L^p_{\Omega_j}(G)$ for $j = 0, 1, \dots, n$.*

We omit the proof of this theorem because it is much similar to the proof of Theorem 10.2; see also the proof of Theorem 9.1 in [1].

Let Ω be an open subset of $\mathbb{R}^n$, and let $a \in \mathbb{R}^n$. The set Ω is said to be *exhaustive in the direction of a* if for every compact set $K \subset \mathbb{R}^n$ there exists a positive number t_K such that $t\,a + K \subset \Omega$ for all $t > t_K$.

Corollary 10.4. *Let $\{\Omega_j\}_{j=0}^N$ be a finite sequence of open subsets of $\mathbb{R}^n$. Suppose that Ω_j is exhaustive in the direction of $a_j \in \mathbb{R}^n$ for all j, and $\sum_{j=0}^N a_j = 0$. Let $p < \tau_m$, where $m = \min(n, N)$. Then each function $f \in L^p(\mathbb{R}^n)$ can be represented in the form $f = \sum_{j=0}^N f_j$, where $f_j \in L^p_{\Omega_j}(\mathbb{R}^n)$ for $j = 0, 1, \dots, N$.*

Proof. It is clear that for $p < \tau_N$ the required result follows from Theorem 10.3. This proves the corollary in the case where $N \le n$. Suppose $N > n$. The origin belongs to the convex hull of the sequence $\{a_j\}_{j=0}^N$. Consequently, by the Carathéodory theorem, there exists a nonnegative integer $r \le n$, a subsequence $\{a_{j_k}\}_{k=0}^r$, and a sequence of positive numbers $\{\mu_k\}_{j=0}^r$ such that $\sum_{j=0}^r \mu_k a_{j_k} = 0$. Then the sequences $\{\Omega_{j_k}\}_{k=0}^r$ and $\{\mu_k a_{j_k}\}_{k=0}^r$ satisfy the assumptions of the corollary to be proved, and $r \le n$. □

Corollary 10.5. *Let $\{\Omega_j\}_{j=0}^N$ be a finite sequence of open convex cones in $\mathbb{R}^n$. Suppose that the convex hull of $\bigcup_{j=0}^N \Omega_j$ coincides with $\mathbb{R}^n$. Let $p < \tau_m$, where $m = \min(n, N)$. Then each function $f \in L^p(\mathbb{R}^n)$ can be represented in the form $f = \sum_{j=0}^N f_j$, where $f_j \in L^p_{\Omega_j}(\mathbb{R}^n)$ for $j = 0, 1, \dots, N$.*

Theorem 10.6. *Let Ω be an open subset of $\mathbb{R}^n$. Suppose that $t\Omega \subset \Omega$ for all $t > 1$,*[8] *and the convex hull of Ω contains the origin. Then $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for all $p < \tau_n$.*

Proof. Clearly, there exists a finite sequence $\{a_j\}_{j=0}^n$ of nonzero vectors in Ω whose convex hull contains the origin. It is easily seen that there is a sequence of positive numbers $\{\mu_j\}_{j=0}^n$ such that $\sum_{j=0}^n \mu_j a_j = 0$ and $\mu_j a_j \in \Omega$ for $j = 0, 1, \dots, n$. Then $\mu_j a_j + B \subset \Omega$ for $j = 0, 1, \dots, n$, where B is a sufficiently small ball centered at the origin. Hence, $t\mu_j a_j + tB \subset \Omega$ for $j = 0, 1, \dots, n$ and $t > 1$. It remains to refer to Theorem 10.2. □

The author does not know whether the statements proved above in this section are true for all $p < 1$. In particular, the following question is open.

Let Ω be an open subset of $\mathbb{R}^n$. Suppose that $t\Omega = \Omega$ for all $t > 0$ and that the convex hull of Ω coincides with $\mathbb{R}^n$. Is it true that $L^p_\Omega(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for all $p \in (0, 1)$?

10.2. Dependence on p. This subsection is devoted to construction of an open subset Ω of $\mathbb{R}$ such that $L^p_\Omega(\mathbb{R}) = L^p(\mathbb{R})$ for all sufficiently small p but not for all $p \in (0, 1)$.

We need several auxiliary statements.

Lemma 10.7. *Let $\alpha < \beta < \gamma$. Suppose that $\alpha + \gamma \ge 2\beta$. Let $f \in L^p_{(\alpha,\beta)}(\mathbb{R})$, $g \in L^p_{(\gamma,+\infty)}(\mathbb{R})$. Then $\|f\|^p_{L^p} + \|g\|^p_{L^p} \le C(p)\|f + g\|^p_{L^p}$.*

[8] In the Russian version of this paper there is a mistake here: it is written "for some $t > 1$".

Proof. It suffices to prove the required estimate for integrable functions f and g. A homogeneity argument allows us to restrict ourselves to the case where $\alpha = 0$, $\beta = 1$, $\gamma \geq 2$. It is well known that $\|\Phi * h\|_{L^p} \leq C(p,\Phi)\|h\|_{L^p}$ for every $h \in L^1(\mathbb{R}) \cap L^p_{(0,+\infty)}(\mathbb{R})$ and $\Phi \in \mathcal{S}(\mathbb{R})$ (this follows, e.g., from the results of [12]). Take a function $\Phi \in \mathcal{S}(\mathbb{R})$ such that $(\mathcal{F}\Phi)(x) = 1$ for $x \in [0,1]$ and $(\mathcal{F}\Phi)(x) = 0$ for $x \in [2,\infty)$. Clearly, $f = \Phi * (f+g)$. Hence, $\|f\|_{L^p} \leq C(p,\Phi)\|f+g\|_{L^p}$. It remains to note that $\|g\|^p_{L^p} \leq \|f\|^p_{L^p} + \|f+g\|^p_{L^p}$. □

Corollary 10.8. *Let $\lambda \geq 4a > 0$. Let $f, g, h \in L^p_{(-a,a)}(\mathbb{R})$. Then*

$$\|f\|^p_{L^p} + \|g\|^p_{L^p} + \|h\|^p_{L^p} \leq C(p)\big\|f(x) + g(x)e^{-2\pi i\lambda x} + h(x)e^{4\pi i\lambda x}\big\|^p_{L^p}.$$

As was mentioned in §4, the space $L^p_{(\alpha,\beta)}(\mathbb{R})$ embeds in the space of entire functions in a natural way. Keeping this in mind, we view each function $f \in L^p_{(\alpha,\beta)}(\mathbb{R})$ as an entire function.

Lemma 10.9. *Let $p \in (0,+\infty)$. Then for every $f \in L^p_{(-1,1)}(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} \max\{|f(w)|^p : w \in \mathbb{C}, |w-x| \leq 1\}\, dx \leq C^{1+p}\|f\|^p_{L^p},$$

where C is an absolute constant.

Proof. It is easily seen that $\int_{\mathbb{R}} |f(x+iy)|^p\, dx \leq e^{2\pi p|y|}\int_{\mathbb{R}} |f(x)|^p\, dx$ for all $y \in \mathbb{R}$. Put $g(z) \stackrel{\text{def}}{=} e^{2\pi iz} f(z - i\sqrt{2})$ for $z \in \mathbb{C}_+$. Clearly, $g \in H^p$ and $\|g\|_{H^p} \leq e^{2\sqrt{2}\pi}\|f\|_{L^p}$. Consider the angular maximal function $(Mg)(x) \stackrel{\text{def}}{=} \sup\{|g(t+iy)| : |x-t| < y\}$. It is well known that $\|Mg\|_{L^p} \leq C^{1/p}\|g\|_{H^p}$, where C is an absolute constant. It remains to note that

$$\begin{aligned}(Mg)(x) &\geq \max\{|g(w)| : |w - x - i\sqrt{2}| \leq 1\}\\ &\geq e^{-2(1+\sqrt{2})\pi}\max\{|f(w)| : |w-x| \leq 1\}.\end{aligned}$$ □

Corollary 10.10. *Let $p \in (0,+\infty)$. Then for every $f \in L^p_{(-1,1)}(\mathbb{R})$ we have*

$$\sum_{n\in\mathbb{Z}} \max_{[n,n+1]} |f|^p \leq C^{1+p}\|f\|^p_{L^p},$$

where C is an absolute constant.

Proof. It is clear that

$$\sum_{n\in\mathbb{Z}} \max_{[n,n+1]} |f|^p \leq \sum_{n\in\mathbb{Z}} \int_n^{n+1} \max\{|f(w)|^p : |w-x| \leq 1\}\, dx \leq C^{1+p}\|f\|^p_{L^p}.$$ □

Corollary 10.11. *Let $a, p \in (0,+\infty)$. Then for every $f \in L^p_{(-a,a)}(\mathbb{R})$ we have*

$$\|f'\|_{L^p} \leq C^{1+p^{-1}}\, a\|f\|_{L^p}.$$

Proof. Dilations allow us to restrict ourselves to the case where $a = 1$. It remains to observe that $|f'(x)| \leq \max\{|f(w)| : |w-x| \leq 1\}$ by the Cauchy inequality. □

Of course, the above auxiliary statements are not new. For example, concerning Corollary 10.11, it should be noted that the inequality

$$\|f'\|_{L^p} \leq 2\pi\, a\|f\|_{L^p}$$

is true. For $p = +\infty$ this was proved by Bernstein, the case of $1 \leq p < +\infty$ was considered by Zygmund (see [34]), and the case where $0 < p < 1$ was studied by Arestov [5]; see also [16].[9]

[9] Rigorously speaking, in [5] and [16] a periodic version of this inequality was proved, from which the required inequality can be obtained.

Lemma 10.12. *Let $a, p \in (0,1)$. Then for every $f \in L^p_{(-a,a)}(\mathbb{R})$ we have*

$$\sum_{n\in\mathbb{Z}} \int_n^{n+1} |f(x) - f(n)|^p \, dx \le C(p)\, a^p \|f\|^p_{L^p}.$$

Proof. Applying Corollaries 10.10 and 10.11, we obtain

$$\sum_{n\in\mathbb{Z}} \int_n^{n+1} |f(x) - f(n)|^p \, dx \le \sum_{n\in\mathbb{Z}} \max_{[n,n+1]} |f'|^p \le C^{1+p} \|f'\|^p_{L^p} \le C(p) a^p \|f\|^p_{L^p}. \qquad \square$$

Lemma 10.13. *Let $\lambda \ge 4a > 0$, and let $f, g, h \in L^p_{(-a,a)}(\mathbb{R})$ with $r_{\mathbb{T}}(\{-1,2\}) \le p \le 1$. Then*

$$\|f(x) + g(x)e^{-2\pi i\lambda x} + h(x)e^{4\pi i\lambda x}\|^p_{L^p} \ge (1 - C(p)a^p\lambda^{-p})\|f\|^p_{L^p}.$$

Proof. A homogeneity argument allows us to restrict ourselves to the case where $\lambda = 1$. Applying Lemma 10.12 and Corollary 10.8, we obtain

$$\begin{aligned}
\|f(x) + g(x)e^{-2\pi ix} + h(x)e^{4\pi ix}\|^p_{L^p} &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} |f(x) + g(x)e^{-2\pi ix} + h(x)e^{4\pi ix}|^p \, dx \\
&\ge \sum_{n\in\mathbb{Z}} \int_n^{n+1} |f(n) + g(n)e^{-2\pi ix} + h(n)e^{4\pi ix}|^p \, dx \\
&- \sum_{n\in\mathbb{Z}} \int_n^{n+1} (|f(x) - f(n)|^p + |g(x) - g(n)|^p + |h(x) - h(n)|^p) \, dx \\
&\ge \sum_{n\in\mathbb{Z}} |f(n)|^p - \sum_{n\in\mathbb{Z}} \int_n^{n+1} (|f(x) - f(n)|^p + |g(x) - g(n)|^p + |h(x) - h(n)|^p) \, dx \\
&\ge \|f\|^p_{L^p} - \sum_{n\in\mathbb{Z}} \int_n^{n+1} (2|f(x) - f(n)|^p + |g(x) - g(n)|^p + |h(x) - h(n)|^p) \, dx \\
&\ge (1 - C(p)a^p)\|f\|^p_{L^p}.
\end{aligned}$$

$\square$

Theorem 10.14. *There exists an open set $\Omega \subset \mathbb{R}$ such that $L^p_\Omega(\mathbb{R}) = L^p(\mathbb{R})$ for all $p < r_{\mathbb{T}}(\{-1,2\})$ and $L^p_\Omega(\mathbb{R}) \ne L^p(\mathbb{R})$ for all $p \ge r_{\mathbb{T}}(\{-1,2\})$.*

Proof. Let $[K]$ denote the absolutely convex hull of K. Put $A_t(I) \stackrel{\text{def}}{=} (-t|I| + I) \cup (2t|I| + I)$, where I is a bounded interval of the real line. We can use induction to construct a sequence of open sets $\{U_n\}$ by putting $U_1 \stackrel{\text{def}}{=} (1,2)$, $U_{n+1} = A_{2^n}([U_n])$. Put $\Omega \stackrel{\text{def}}{=} \bigcup_{n=1}^\infty U_n$.

Let $p < r_{\mathbb{T}}(\{-1,2\})$. Then there exist numbers $\alpha = \alpha_p$ and $\beta = \beta_p$ satisfying $\|\alpha z^{-1} + 1 + \beta z^2\|_{L^p(\mathbb{T})} = \eta_p < 1$. Thus, in essence, we can repeat the proof of Theorem 5.1. Let $\varphi \in \mathcal{S}^0(\mathbb{R})$. There exists a positive integer n such that $\operatorname{supp} \mathcal{F}\varphi \subset [U_n]$. Put $f_t \stackrel{\text{def}}{=} -(\alpha e^{-2\pi i(l_n x + t)} + \beta e^{4\pi i(l_n x + t)})\varphi$, where $l_n = 2^n|[U_n]|$. Clearly, $f_t \in L^p_\Omega(\mathbb{R}^n)$ for all $t \in [0,1)$. By the Fubini theorem, we have

$$\int_0^1 \|\varphi - f_t\|^p_{L^p} \, dt = \eta_p^p \|\varphi\|^p_{L^p},$$

whence $\|\varphi - f_{t_0}\|^p_{L^p} = \eta_p^p \|\varphi\|^p_{L^p}$ for some t_0 in $[0,1)$. We have proved that $\operatorname{dist}_{L^p}(\varphi, L^p_\Omega(\mathbb{R})) \stackrel{\text{def}}{=} \inf\{\|\varphi - f\|_{L^p} : f \in L^p_\Omega(\mathbb{R}^n)\} \le \eta_p \|\varphi\|_{L^p}$ for all $\varphi \in \mathcal{S}^0(\mathbb{R})$. Hence, this inequality is true for all $\varphi \in L^p(\mathbb{R})$ because $\mathcal{S}^0(\mathbb{R})$ is dense in $L^p(\mathbb{R})$. Now it is clear that $L^p_\Omega(\mathbb{R}) = L^p(\mathbb{R})$.

Suppose $p \geq r_{\mathbb{T}}(\{-1,2\})$. Let $\mathcal{P}_n$ denote the orthogonal projection of $L^2_{[U_n]\cup U_{n+1}}(\mathbb{R})$ onto $L^2_{[U_n]}(\mathbb{R})$. Lemma 10.13 asserts that $\|\mathcal{P}_n f\|_{L^p} \leq (1+C(p)2^{-np})\|f\|_{L^p}$ for all $f \in \mathcal{S}_{[U_n]\cup U_{n+1}}(\mathbb{R})$. Let $f \in \mathcal{S}^0_\Omega(\mathbb{R})$. Then $f \in \mathcal{S}_{[U_n]\cup U_{n+1}}(\mathbb{R})$ for sufficiently large n. We put $\mathcal{P}f \stackrel{\text{def}}{=} (\prod_{j=1}^n \mathcal{P}_j)f$. Clearly, this definition of the operator $\mathcal{P}$ does not depend on the number n such that $f \in \mathcal{S}_{[U_n]\cup U_{n+1}}(\mathbb{R})$, and $\|\mathcal{P}f\|_{L^p} \leq \big(\prod_{j=1}^n(1+C(p)2^{-jp})\big)\|f\|_{L^p} \leq \big(\prod_{j=1}^\infty(1+C(p)2^{-jp})\big)\|f\|_{L^p}$. Hence, $\mathcal{P}$ can be extended up to a continuous projection of $L^p_\Omega(\mathbb{R})$ onto $L^p_{U_1}(\mathbb{R})$. This projection allows us to construct a nonzero linear continuous functional on $L^p_\Omega(\mathbb{R})$. Consequently, $L^p_\Omega(\mathbb{R}) \neq L^p(\mathbb{R})$. □

Corollary 10.15. *Let Ω denote the set constructed in the proof of Theorem* 10.14. *Then $L^p_{\Omega_+}(\mathbb{R}) \cap L^p_{\Omega_-}(\mathbb{R}) \neq \{0\}$ if and only if $p < r_{\mathbb{T}}(\{-1,2\})$, where $\Omega_+ \stackrel{\text{def}}{=} \Omega \cap (0,+\infty)$ and $\Omega_- \stackrel{\text{def}}{=} \Omega \cap (-\infty,0)$.*

Proof. The proof of Theorem 10.14 (see also the proof of Theorem 9.1 in [1]) shows that $L^p_{\Omega_+}(\mathbb{R}) + L^p_{\Omega_-}(\mathbb{R}) = L^p(\mathbb{R})$ for $p < r_{\mathbb{T}}(\{-1,2\})$. It follows that $L^p_{\Omega_+}(\mathbb{R}) \cap L^p_{\Omega_-}(\mathbb{R}) \neq \{0\}$. Indeed, suppose that $L^p_{\Omega_+}(\mathbb{R}) \cap L^p_{\Omega_-}(\mathbb{R}) = \{0\}$. To get a contradiction, we note that in this case the space $L^p(\mathbb{R})$ is isomorphic to $L^p_{\Omega_+}(\mathbb{R}) \times L^p_{\Omega_-}(\mathbb{R})$, and therefore, has a nontrivial dual.

Now, let $p \geq r_{\mathbb{T}}(\{-1,2\})$. In the proof of Theorem 10.14 we constructed a continuous projection $\mathcal{P}$ of $L^p_\Omega(\mathbb{R})$ onto $L^p_{U_1}(\mathbb{R})$. Similarly, we can construct a continuous projection $\mathcal{P}^{[n]}$ of $L^p_\Omega(\mathbb{R})$ onto $L^p_{\Omega\cap[U_n]}(\mathbb{R})$, which is formally equal to the infinite product $\prod_{j=n}^\infty \mathcal{P}_j$. It is easily seen that

$$\mathcal{P}^{[n]}(L^p_{\Omega_+}(\mathbb{R}) \cap L^p_{\Omega_-}(\mathbb{R})) \subset (\mathcal{P}^{[n]}(L^p_{\Omega_+}(\mathbb{R}))) \cap (\mathcal{P}^{[n]}(L^p_{\Omega_-}(\mathbb{R}))) = \{0\}$$

for all $n \in \mathbb{N}$. It remains to observe that $\lim_{n\to\infty} \mathcal{P}^{[n]} f = f$ for every $f \in L^p_\Omega(\mathbb{R})$. □

Remark 10.16. It can be seen from the proof of Theorem 10.14 that for $p \geq r_{\mathbb{T}}(\{-1,2\})$, the linear continuous functionals separate the points of the space $L^p_\Omega(\mathbb{R})$.

Remark 10.17. A similar construction works for every finite set $\Lambda \subset \mathbb{Z}\setminus\{0\}$. This allows us to get an example of a set $\Omega \subset \mathbb{R}$ such that $L^p_\Omega(\mathbb{R}) = L^p(\mathbb{R})$ if and only if $p < r_{\mathbb{T}}(\Lambda)$.

Remark 10.18. Similar examples can be constructed also for the group $\mathbb{T}$. In particular, there exists a set $\Omega \subset \mathbb{Z}$ such that $L^p_\Omega(\mathbb{T}) = L^p(\mathbb{T})$ if and only if $p < r_{\mathbb{T}}(\{-1,2\})$.

§11. Subsets Ω of $\mathbb{Z}$ and arithmetic progressions

The well-known Szemerédi theorem (see [30]) says that each subset of $\mathbb{Z}$ not containing arbitrarily long arithmetic progressions is of density zero. We are going to construct an example of a set $\Omega \subset \mathbb{Z}$ that contains no three-element arithmetic progressions[10] but is not rarefied enough for the relation $L^p_\Omega(\mathbb{T}) = L^p(\mathbb{T})$ to fail for any p.

Theorem 11.1. *There exists a set $\Lambda \subset \mathbb{Z}$ such that Λ contains no three-element arithmetic progressions and $L^p_\Lambda(\mathbb{T}) = L^p(\mathbb{T})$ for all $p < r_{\mathbb{T}}(\{-1,2\})$.*

Let A be a finite set. Put

$$\operatorname{dist}^*_{L^p}(1, L^p_A(\mathbb{T})) = \inf\{\|1-f\|_{L^p} : f \in \mathcal{L}^1_A(\mathbb{T}), (1-f)*\overline{f} = 0\}.$$

Clearly, $\operatorname{dist}_{L^p}(1, L^p_A(\mathbb{T})) \leq \operatorname{dist}^*_{L^p}(1, L^p_A(\mathbb{T})) \leq 1$ for every finite set $A \subset \mathbb{Z}$.

[10] As to three-element arithmetic progressions, it should be noted that the corresponding special case of the Szemerédi theorem was proved by Roth [25] as long ago as in 1952.

Lemma 11.2. *Let $0 < p < r_{\mathbb{T}}(\{-1,2\})$. Let A be a finite set of integers containing no three-element arithmetic progressions. Then there exists a finite set $B \subset \mathbb{Z}$ such that*

a) $A \subset B$;

b) *B contains no three-element arithmetic progressions*;

c) $\operatorname{dist}^*_{L^p}(1, L^p_B(\mathbb{T})) \le \eta_p \operatorname{dist}^*_{L^p}(1, L^p_A(\mathbb{T}))$ *for some $\eta_p \in (0,1)$ depending on p only.*

Proof. If $A \ni 0$, then the claim is true for $B = A$. Let $A \not\ni 0$. A compactness argument shows that there exists a trigonometric polynomial $h \in \mathcal{L}^1_A(\mathbb{T})$ satisfying $(1-h) * \overline{h} = -h * \overline{h} = 0$ and $\|1-h\|_{L^p} = \operatorname{dist}^*_{L^p}(1, L^p_A(\mathbb{T}))$. Put $A_0 = \operatorname{supp} \mathcal{F}(1-h) = \{0\} \cup \operatorname{supp} \mathcal{F}h$. We shall seek B in the form $B = (-n + A_0) \cup A \cup (2n + A_0)$, where n is a sufficiently large positive integer. Put $f_t \stackrel{\text{def}}{=} (\alpha z^{-n} e^{-2\pi i t} + \beta z^{2n} e^{4\pi i t})(h-1)$, where α and β denote the same numbers as in the proof of Theorem 10.14. Next, as in the proof of Theorem 10.14, the Fubini theorem implies the identity

$$\int_0^1 \|1 - h - f_t\|^p_{L^p}\, dt = \eta_p^p \|1-h\|^p_{L^p}.$$

Hence, there exists a number $t_0 \in [0,1)$ satisfying $\|1 - h - f_{t_0}\|_{L^p} \le \eta_p \|1-h\|_{L^p}$. Note that $\operatorname{supp} \mathcal{F}(h + f_{t_0}) \subset (-n + A_0) \cup A \cup (2n + A_0)$. It is easily seen that if n is sufficiently large, then the set $(-n + A_0) \cup A \cup (2n + A_0)$ contains no three-element arithmetic progressions and $(h + f_{t_0}) * (\overline{h} + \overline{f}_{t_0}) = 0$. □

Corollary 11.3. *Suppose $0 < p < r_{\mathbb{T}}(\{-1,2\})$ and $n \in \mathbb{Z}$. Let A be a finite set containing no three-element arithmetic progressions. Then for every positive number $\varepsilon > 0$ there exists a finite set $B \subset \mathbb{Z}$ such that*

a) $A \subset B$;

b) *B contains no three-element arithmetic progressions*;

c) $\operatorname{dist}_{L^p}(z^n, L^p_B(\mathbb{T})) < \varepsilon$.

Proof. It suffices to consider the case where $n = 0$, which immediately reduces to Lemma 11.2. □

Proof of Theorem 11.1. We enumerate $\mathbb{Z}$ by a sequence $\{n_k\}_{k=1}^\infty$ in such a way that each integer occurs in this sequence infinitely many times. We can take a strictly monotone increasing sequence of positive numbers $\{p_k\}_{k=1}^\infty$ tending to $r_{\mathbb{T}}(\{-1,2\})$. Corollary 11.3 allows us to construct an increasing sequence $\{A_k\}_{k=1}^\infty$ of finite subsets of $\mathbb{Z}$ such that each A_k contains no three-element arithmetic progressions and $\operatorname{dist}_{L^{p_k}}(z^{n_k}, L^{p_k}_{A_k}(\mathbb{T})) < 2^{-k}$ for all positive integers k. Now it is clear that $L^p_A(\mathbb{T}) = L^p(\mathbb{T})$ for $p < r_{\mathbb{T}}(\{-1,2\})$, where $A = \bigcup_{k=1}^\infty A_k$. □

The author does not know whether the condition $p < r_{\mathbb{T}}(\{-1,2\})$ in Theorem 11.1 can be replaced with the condition $p < 1$.

Theorem 11.4. *There exists a set $\Omega \subset \mathbb{Z}$ containing no four-element arithmetic progressions and such that $L^p_\Omega(\mathbb{T}) = L^p(\mathbb{T})$ for all $p \in (0,1)$.*

Lemma 11.5. *Suppose that $\sum_{k=0}^\infty \varepsilon_k 5^k = 0$, where $\varepsilon_k \in \mathbb{Z}$ for all k, $|\varepsilon_k| \le 4$ for all k, and $\varepsilon_k = 0$ for all sufficiently large k. Then $\varepsilon_k = 0$ for all k.*

Proof. It suffices to observe that if $\varepsilon_k = 0$ for all $k > k_0$ and $\varepsilon_{k_0} \ne 0$, then $\left|\sum_{k=0}^\infty \varepsilon_k 5^k\right| \ge 5^{k_0} - 4\sum_{k=0}^{k_0-1} 5^k = 1$. □

Corollary 11.6. *Let Λ be the set of all integers n representable in the form $n = \sum_{k=0}^\infty \varepsilon_k 5^k$, where $\varepsilon_k \in \{-1,0,1\}$ for all k and $\varepsilon_k = 0$ for all sufficiently large k. Then Λ contains no four-element arithmetic progressions.*

Proof. Suppose that the numbers $\{n_j\}_{j=1}^4$ form an arithmetic progression, where $n_j = \sum_{k=0}^{\infty} \varepsilon_k^{(j)} 5^k$. Lemma 11.5 shows that for each k the sequence $\{\varepsilon_k^{(j)}\}_{j=1}^4$ is an arithmetic progression. Hence, $\varepsilon_k^{(1)} = \varepsilon_k^{(2)} = \varepsilon_k^{(3)} = \varepsilon_k^{(4)}$ for each k. □

Lemma 11.7. *Let Λ be the same as in Corollary* 11.6. *Then* $1 \in L^p_{\Lambda\setminus\{0\}}(\mathbb{T})$.

Proof. Let $\varphi \in \mathcal{L}^1_{\Lambda\setminus\{0\}}(\mathbb{T})$. We fix a positive integer N and put $f_t = \frac{1}{2}(e^{2\pi i t} z^{5^N} + e^{-2\pi i t} z^{-5^N})(1-\varphi)$. In the same way as in the proof of Theorem 5.1, we see that $\|1-\varphi-f_{t_0}\|^p_{L^p} \le \eta_p^p \|1-\varphi\|^p_{L^p}$ for some $t_0 \in [0,1)$. Note that $f_t \in \mathcal{L}^1_{\Lambda\setminus\{0\}}(\mathbb{T})$ for all $t \in [0,1)$ if the number N is sufficiently large. Consequently, $\operatorname{dist}_{L^p}(1, L^p_{\Lambda\setminus\{0\}}(\mathbb{T})) \stackrel{\text{def}}{=} \inf\{\|1-g\|_{L^p} : g \in L^p_{\Lambda\setminus\{0\}}(\mathbb{T})\} \le \eta_p \operatorname{dist}_{L^p}(1, L^p_{\Lambda\setminus\{0\}}(\mathbb{T}))$, whence $1 \in L^p_{\Lambda\setminus\{0\}}(\mathbb{T})$. □

Remark 11.8. The set Λ defined in Corollary 11.6 is the support of the Fourier transform of a singular measure μ: for the role of μ we can take the Riesz product

$$\mu = \prod_{k=0}^{\infty} (1 + \operatorname{Re}(z^{5^k}));$$

see [17] or [34]. Thus, Lemma 11.7 follows from Shapiro's results [27]; see also [1].

Corollary 11.9. *Suppose $0 < \varepsilon, p < 1$. Then there exists a positive integer N such that $\operatorname{dist}_{L^p}(1, L^p_{\Lambda_N\setminus\{0\}}(\mathbb{T})) < \varepsilon$, where Λ_N is the set of all integers n representable in the form $n = \sum_{k=0}^{N} \varepsilon_k 5^k$, where $\varepsilon_k \in \{-1,0,1\}$ for all k.*

Proof. It suffices to note that

$$\lim_{N\to+\infty} \operatorname{dist}_{L^p}(1, L^p_{\Lambda_N\setminus\{0\}}(\mathbb{T})) = \operatorname{dist}_{L^p}(1, L^p_{\Lambda\setminus\{0\}}(\mathbb{T})) = 0.$$

□

Lemma 11.10. *Suppose $0 < \varepsilon, p < 1$ and $n \in \mathbb{Z}$. Let A be a finite set of integers that contains no four-element arithmetic progressions. Then there exists a finite set $B \subset \mathbb{Z}$ such that*

a) $A \subset B$;
b) *B contains no four-element arithmetic progressions*;
c) $\operatorname{dist}_{L^p}(z^n, L^p_B(\mathbb{T})) < \varepsilon$.

Proof. It suffices to consider the case where $n = 0$. By Corollary 11.9, there exists a positive integer N such that $\operatorname{dist}_{L^p}(1, L^p_{\Lambda_N\setminus\{0\}}(\mathbb{T})) < \varepsilon$. Observe that $\operatorname{dist}_{L^p}(1, L^p_{\Lambda_N\setminus\{0\}}(\mathbb{T})) = \operatorname{dist}_{L^p}(1, L^p_{5^k\Lambda_N\setminus\{0\}}(\mathbb{T}))$ for all nonnegative integers k, because the transformation $z \mapsto z^5$ preserves Lebesgue measure on $\mathbb{T}$. It is easily seen that the set $B = A \cup 5^k\Lambda_N$ satisfies all required conditions if k is sufficiently large. □

Proof of Theorem 11.4. We can repeat the proof of Theorem 11.1 almost word by word. We can enumerate the set $\mathbb{Z}$ by a sequence $\{n_k\}_{k=1}^{\infty}$ in such a way that each integer occurs in this sequence infinitely many times. Take a strictly monotone increasing sequence of positive numbers $\{p_k\}_{k=1}^{\infty}$ tending to 1. Lemma 11.10 allows us to construct an increasing sequence $\{A_k\}_{k=1}^{\infty}$ of finite subsets of $\mathbb{Z}$ such that each set A_k contains no four-element arithmetic progressions and $\operatorname{dist}_{L^{p_k}}(z^{n_k}, L^{p_k}_{A_k}(\mathbb{T})) < 2^{-k}$ for all positive integers k. Now it is clear that $L^p_A(\mathbb{T}) = L^p(\mathbb{T})$ for all $p < 1$, where $A = \bigcup_{k=1}^{\infty} A_k$. □

Remark 11.11. There exists a set Ω containing arbitrarily long arithmetic progressions and such that continuous linear functionals separate the points of $L^p_\Omega(\mathbb{T})$; see Theorem 8.9 in [1].

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St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023, Russia

E-mail address: alex@pdmi.ras.ru

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