J_{p,m}-INNER DILATIONS
OF MATRIX-VALUED FUNCTIONS
THAT BELONG TO THE CARATHÉODORY CLASS
AND ADMIT PSEUDOCONTINUATION

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Abstract. The class \( \ell^{p \times p} \) of matrix-valued functions \( c(z) \) holomorphic in the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \), having order \( p \), and satisfying \( \text{Re} \, c(z) \geq 0 \) in \( D \) is considered, as well as its subclass \( \ell^{p \times p} \Pi \) of matrix-valued functions \( c(z) \in \ell^{p \times p} \) that have a meromorphic pseudocontinuation \( c_-(z) \) to the complement \( D_e = \{ z \in \mathbb{C} : 1 < |z| \leq \infty \} \) of the unit disk with bounded Nevanlinna characteristic in \( D_e \).

For matrix-valued functions \( c(z) \) of class \( \ell^{p \times p} \Pi \) a representation as a block of a certain \( J_{p,m} \)-inner matrix-valued function \( \theta(z) \) is obtained. The latter function has a special structure and is called the \( J_{p,m} \)-inner dilation of \( c(z) \). The description of all such representations is given.

In addition, the following special \( J_{p,m} \)-inner dilations are considered and described: minimal, optimal, \( * \)-optimal, minimal and optimal, minimal and \( * \)-optimal. Also, \( J_{p,m} \)-inner dilations with additional properties are treated: real, symmetric, rational, or any combination of them under the corresponding restrictions on the matrix-valued function \( c(z) \). The results extend to the case where the open upper half-plane \( \mathbb{C}_+ \) is considered instead of the unit disk \( D \). For entire matrix-valued functions \( c(z) \) with \( \text{Re} \, c(z) \geq 0 \) in \( \mathbb{C}_+ \) and with Nevanlinna characteristic in \( \mathbb{C}_- \), the \( J_{p,m} \)-inner dilations in \( \mathbb{C}_+ \) that are entire matrix-valued functions are also described.

§1. Introduction

The class \( \ell^{p \times p} \) of matrix-valued functions \( c(z) \) of order \( p \) holomorphic in the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) and such that \( \text{Re} \, c(z) \geq 0 \) in \( D \) is of interest for function theory, as well as for the theory of Hilbert space operators, the theory of passive linear dynamic systems, control theory, and stochastic processes theory (see [4, 9, 12, 16]).

Darlington’s method is well known in the theory of passive linear circuits with lumped parameters. Development of this method required investigation of the subclass \( \ell^{p \times p} \Pi \) of matrix-valued functions \( c(z) \) that admit a meromorphic pseudocontinuation \( c_-(z) \) to the complement of the unit disk \( D_e = \{ z \in \mathbb{C} : 1 < |z| \leq \infty \} \) with bounded Nevanlinna characteristic in \( D_e \). The fact that \( c_-(z) \) is a pseudocontinuation of \( c(z) \) means that

\[
c(\zeta) = \lim_{r \to 1} c(r \zeta) = \lim_{r \to 1} c_-(r \zeta) \quad \text{almost everywhere on } |\zeta| = 1.
\]

In [3], the following Darlington representation of such matrix-valued functions was obtained:

\[
(1.1) \quad c(z) = [a_{11}(z) \tau + a_{12}(z)](a_{21}(z) \tau + a_{22}(z))^{-1},
\]

where \( \tau \) is a constant matrix of order \( p \) with \( \text{Re} \, \tau \geq 0 \), and \( A(z) = [a_{ij}(z)]_{i,j=1,2} \) is a \( J_p \)-inner matrix-valued function in \( D \) with the signature matrix \( J_p = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix} \); i.e., \( A(z) \) is

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a matrix-valued function of order 2p meromorphic in $D$ and taking $J_p$-contractive values on $\Omega^+_A$ (the domain of holomorphy of $A(z)$ in $D$) and having $J_p$-unitary nontangential boundary values $A(\zeta)$ almost everywhere on the circle $T = \{ \zeta \in \mathbb{C} : |\zeta| = 1 \}$:

$$A(z)^* J_p A(z) \leq J_p, \quad z \in \Omega^+_A,$$

$$A(\zeta)^* J_p A(\zeta) = J_p \quad \text{for a.e.} \quad \zeta \in T.$$ 

If a matrix-valued function $c(z)$ is of class $\ell^{p \times p} \Pi$, then its rank $m_c = \text{rank} \text{Re} c(\zeta)$ is constant for a.e. $\zeta \in T$, because $2 \text{Re} c(\zeta)$ is a nontangential boundary value of the function $c(z) + c_-(\frac{1}{z})$, which has bounded Nevanlinna characteristic in $D$. In (1.1), the matrix $\tau$ is such that $\text{rank} \tau = m_c$, and there exists a representation of $c(z)$ in the form (1.1) with $\tau = \begin{bmatrix} I_{m_c} & 0 \\ 0 & 0 \end{bmatrix}$ for $1 \leq m_c \leq p$ and with $\tau = 0_{p \times p}$ for $m_c = 0$. In the Darlington method, the number $m_c$ is interpreted as the minimal number of the reduced scattering channels (see [1, 2]).

In this paper, we consider another representation of a matrix-valued function $c(z)$ of class $\ell^{p \times p} \Pi$; namely, we represent it as a block of a $J_{p,m}$-inner dilation in $D$ matrix-valued function

$$\theta(z) = \begin{bmatrix} \alpha(\zeta) & \beta(\zeta) & 0 \\ \gamma(\zeta) & \delta(\zeta) & I_p \\ 0 & I_p & 0 \end{bmatrix}, \quad \delta(z) = c(z), \quad \text{for } m \geq m_c > 0$$

and

$$\theta_0(z) = \begin{bmatrix} c(z) & I_p \\ I_p & 0 \end{bmatrix} \quad \text{for } m = m_c = 0,$$

where

$$J_{p,m} = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_p \\ 0 & -I_p & 0 \end{bmatrix} \quad \text{for } m > 0,$$

$$J_{p,0} = J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix} \quad \text{for } m = 0.$$

The functions $\theta(z)$ of this type will be called $J_{p,m}$-inner dilations of $c(z)$. The following theorem is proved.

**Theorem 1.** A matrix-valued function $c(z)$ belongs to the class $\ell^{p \times p} \Pi$ if and only if there exists a $J_{p,m}$-inner dilation $\theta$ of $c(z)$ of the form (1.2). Moreover, if $c \in \ell^{p \times p} \Pi$, then for the corresponding $J_{p,m}$-inner dilation $\theta$ of the form (1.2) we have $m \geq m_c \geq 0$, and there exists a $J_{p,m}$-inner dilation with $m = m_c$.

The “if” part can easily be checked. To verify the “only if” part, we apply the method used in [1, 6] to obtain a representation of matrix-valued functions $s(z)$ of class $\ell^{p \times q}$ (see also [15]), i.e., a representation of $(p \times q)$-matrix-valued functions holomorphic and contractive in $D$ that admit a pseudocontinuation $s_-(z)$ in $D_c$ in the form of a block of an inner matrix-valued function $S(z)$ of order $n$, where

$$n = p + r_1 = q + r_2, \quad r_1 = \text{rank}(I_q - s(\zeta)s(\zeta)^*), \quad r_2 = \text{rank}(I_p - s(\zeta)^*s(\zeta))$$

for a.e. $\zeta \in T$, and

$$S(z) = \begin{bmatrix} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z) \end{bmatrix}, \quad s_{12} = s(z).$$

In the same way (see [7]), a representation was obtained for an arbitrary $(p \times q)$-matrix-valued function $f(z)$ of bounded Nevanlinna characteristic in $D$ and having a
pseudocontinuation \( f_\cdot(z) \) in \( D_c \) of bounded Nevanlinna characteristic in \( D_c \). Such a function can be represented as a block \( w_{12} \) of some \( j_{pq} \)-inner in \( D \) matrix-valued function \( W(z) \):

\[
W(z) = \begin{bmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{bmatrix}, \quad w_{12}(z) = f(z),
\]

where

\[
j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}.
\]

A representation of \( c(z) \in \ell^{p\times p}\Pi \) with \( m_c > 0 \) in the form \((1.2)\) is not unique. In the present paper, all such representations with \( m = m_c \) are described, much in the same way that the representations \((1.4)\) of \( s(z) \in S^{p\times q}\Pi \) were obtained earlier in \([6]\); see also the representations \((1.5)\) in \([7]\).

In §3, some special representations \((1.2)\) are considered and described: minimal, optimal, \(*\)-optimal, minimal and \(*\)-optimal, minimal and \(*\)-optimal, which will be useful in constructing passive realizations of the impedance matrices \( c(z) \). A separate paper will be devoted to this subject. In §4, the \( J_{p,m} \)-inner dilations with additional properties are described: real, symmetric, rational, and with various combinations of these properties, under the corresponding restrictions on \( c(z) \). All these results are transferred to the case where the open half-plane \( \mathbb{C}_+ \) is considered instead of the unit disk \( D \). In §5, for entire matrix-valued functions \( c(z) \) with \( \Re c(z) > 0 \) in \( \mathbb{C}_+ \) and with bounded Nevanlinna characteristic in the lower half-plane \( \mathbb{C}_- \), the \( J_{p,m} \)-inner dilations in \( \mathbb{C}_+ \) are described. They are still entire matrix-valued functions.

In subsequent papers we shall consider conservative and various passive (minimal, optimal, etc.) realizations of matrix-valued functions \( c(z) \) of class \( \ell^{p\times p}\Pi \) with \( m_c > 0 \) in the form of a resistance matrix of a dissipative system. Such realizations are constructed by considering the corresponding \( J_{p,m} \)-inner dilation \( \theta(z) \) of \( c(z) \) and a conservative transmission system with the transmission matrix \( \theta(z) \).

We are also planning to consider the relationship between \( J_{p,m} \)-inner dilations and the theory of stochastic realization of discrete time stationary processes, developed by A. Lindquist, D. Picci, and their followers (see \([18–20]\)). The analysis of precisely these papers brought the authors to the results presented here.

**Notation**

- \( \mathbb{C} \) is the set of complex numbers;
- \( \mathbb{R} \) is the set of real numbers;
- \( \Re z = \frac{z + \overline{z}}{2} \) is the real part of \( z \in \mathbb{C} \);
- \( \Im z = \frac{z - \overline{z}}{2i} \) is the imaginary part of \( z \in \mathbb{C} \);
- \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) is the open unit disk;
- \( D_c = \{ z \in \mathbb{C} : 1 < |z| \leq \infty \} \) is the exterior of the open unit disk in the extended complex plane \( \overline{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \);
- \( \mathbb{C}_+ = \{ z \in \mathbb{C} : \Im z > 0 \} \) is the open upper half-plane;
- \( \mathbb{C}_- = \{ z \in \mathbb{C} : \Im z < 0 \} \) is the open lower half-plane;
- \( T = \{ \zeta \in \mathbb{C} : |\zeta| = 1 \} \) is the unit circle;
- \( \Re A \) is the real part of the matrix \( A \), i.e., \( \Re A = \frac{A + A^T}{2} \);
- \( A^T \) is the transpose of the matrix \( A \);
- \( \text{rank} A \) is the rank of \( A \);
- \( \text{trace} A \) is the trace of \( A \);
- \( ||A|| \) is the norm of \( A \), which is the maximal singular number of \( A \);
- \( I_m \) is the identity matrix of order \( m \);
§2. Preliminary information about matrix-valued functions belonging to the Nevanlinna class or to its subclasses

2.1. Basic classes of matrix-valued functions. A measurable \((p \times q)\)-matrix-valued function \(f(\zeta)\) on the unit circle \(T\) belongs to the space \(L^{p,q}_{\infty}\) with \(1 \leq r < \infty\) if

\[
\|f\|_{r} = \frac{1}{2\pi} \int_{|\zeta|=1} \text{trace}\{f(\zeta)^{*}f(\zeta)\}^{\frac{r}{2}} |d\zeta| < \infty,
\]

and to the space \(L^{p,q}_{\infty}\) if

\[
\text{ess sup}\{\|f(\zeta)\| : \zeta \in T\} < \infty.
\]

A \((p \times q)\)-matrix-valued function \(f(z)\) holomorphic in \(D\) belongs to the Hardy class \(H^{p,q}_{\infty}\) with \(1 \leq r < \infty\) if

\[
\|f\|_{r}^{\infty} = \sup_{\rho<1} \int_{|\zeta|=1} \text{trace}\{f(\rho\zeta)^{*}f(\rho\zeta)\}^{\frac{r}{2}} |d\zeta| < \infty,
\]

and to the class \(H^{p,q}_{\infty}\) if

\[
\|f\|_{\infty} = \sup\{\|f(z)\| : z \in D\} < \infty.
\]

A \((p \times q)\)-matrix-valued function \(s(z)\) holomorphic in \(D\) belongs to the Schur class \(S^{p,q}_{\infty}\) if \(s(z)^{*}s(z) \leq I_{q}\) for all \(z \in D\).

A \((p \times p)\)-matrix-valued function \(c(z)\) holomorphic in \(D\) belongs to the Carathéodory class \(\mathcal{L}^{p,p}\) if

\[
\text{Re} c(z) = \frac{c(z) + c(z)^{*}}{2} \geq 0 \quad \text{for all} \quad z \in D.
\]

A \((p \times q)\)-matrix-valued function \(f(z)\) meromorphic in \(D\) belongs to the Nevanlinna class \(N^{p,q}_{\infty}\) of matrix-valued functions of bounded characteristic if it can be represented in the form

\[
f = h^{-1} g,
\]

where \(g \in H^{p,q}_{\infty}\) and \(h \in H_{\infty} (= H^{1 \times 1}_{\infty})\).

It should be noted that the class \(N^{p,q}_{\infty}\) contains the classes \(H^{p,q}_{\infty}\), \(1 \leq r \leq \infty\), \(S^{p,q}_{\infty}\), and \(\mathcal{L}^{p,p}\) (for \(p = q\)). An arbitrary function \(f(z) \in N^{p,q}_{\infty}\) has non-tangential boundary values \(f(\zeta)\) almost everywhere on the circle \(T\). Therefore, in particular, the limit \(f(\zeta) = \lim_{\rho \uparrow 1} f(\rho \zeta)\) exists for a.e. \(\zeta \in T\), and \(f(z)\) is uniquely determined by the boundary values \(f(\zeta)\) on a set of positive Lebesgue measure of \(T\).

Observe that \(\text{Re} c(z) \in L^{1}_{1} \) for an arbitrary matrix-valued function \(c(z) \in \mathcal{L}^{p,p}\). For any class \(\mathcal{X}^{p,q}_{\infty}\) of matrix-valued functions, we shall write \(\mathcal{X}\) instead of \(\mathcal{X}^{1 \times 1}_{\infty}\) and \(\mathcal{X}^{p,q}\) instead of \(\mathcal{X}^{p,q}_{\infty}\).

A matrix-valued function \(f(z) \in S^{p,q}_{\infty}\) is inner (\(*\)-inner) if \(f(\zeta)^{*}f(\zeta) = I_{q}\) (respectively, \(f(\zeta)f(\zeta)^{*} = I_{p}\)) for a.e. \(\zeta \in T\). The class of inner matrix-valued functions is denoted by \(S^{p,q}_{\infty}\) and the class of \(*\)-inner functions by \(S^{p,q}_{\infty}\). Note that the classes \(S^{p,q}_{\infty}\) and \(S^{p,q}_{\infty}\) are not empty for \(p \geq q\) and for \(p \leq q\), respectively, and for \(p = q\) we have \(S^{p,q}_{\infty} = S^{p,q}_{\infty}\).

If \(f\) and \(h\) are matrix-valued functions, \(f \in H^{p,q}_{\infty}\) and \(h \in H^{p}_{\infty}\), then \(fh \in H^{p}_{\infty}\) and

\[
\|fh\|_{2} \leq \|f\|_{\infty}\|h\|_{2};
\]

therefore, the operator \(M_{f}\) of multiplication by a matrix-valued function \(f\),

\[
(M_{f}h)(z) = f(z)h(z),
\]
is a well-defined bounded operator from $H^2_M$ to $H^p_M$.

A matrix-valued function $f(z) \in H^{p \times q}_{\infty}$ is outer (+-outer) if $M_f H^2_M = H^p_M$ (respectively, $M_f - H^2_M = H^p_M$, where $f^+(z) = f(z^*)$).

The next lemma is important for what follows.

**Lemma 1.** A matrix-valued function $f(z) \in H^{p \times q}_{\infty}$ is outer if and only if

1. $\text{rank} f(z) = p$ for at least one point $z \in D$, and
2. every matrix-valued function $g \in H^{p \times q}_{\infty}$ such that $g(\zeta)^* g(\zeta) \leq f(\zeta)^* f(\zeta)$ for almost all $\zeta \in T$ satisfies the inequality $g(z)^* g(z) \leq f(z)^* f(z)$ for all $z \in D$.

Moreover, in that case rank $f(z) = p$ for all $z \in D$, rank $f(\zeta) = p$ for almost all $\zeta \in T$, and $g(z) = b(z) f(z)$ for some $b \in S^{r \times p}$.

The proof can be found in [11, p. 214, Proposition 2.4; p. 223, Proposition 4.1].

Every nonzero matrix-valued function $f(z) \in H^{p \times q}_\infty$ admits an inner-outer factorization of the form $f(z) = b(z) \varphi(z)$, where $b(z) \in S^{p \times r}$ and $\varphi(z)$ is an outer matrix-valued function of class $H^{r \times q}_\infty$ for some $r \leq \min \{p, q\}$. This representation is essentially unique; i.e., it is unique up to the replacement of $b(z)$ by $b(z) u$ and $\varphi(z)$ by $u^* \varphi(z)$ for some constant unitary matrix $u$ of order $r$. Moreover, $r = \text{rank} f(z)$ for all $z \in D$ except possibly a finite or countable set of points where rank $f(z) < r$. This set has no accumulation points in $D$.

Every nonzero matrix-valued function $f(z) \in H^{p \times q}_\infty$ admits also an essentially unique $*$-outer-$*$-inner factorization $f(z) = \varphi(z) b(z)$.

An arbitrary square matrix-valued function $f(z) \in H^{p \times p}_\infty$ with det $f(z) = 0$ in $D$ admits inner-outer and outer-inner factorizations $f = b_1 \varphi_1 = \varphi_2 b_2$, where $b_i \in S^{p \times p}$ and the $\varphi_i$ are outer matrix-valued functions of class $H^{p \times p}_\infty$ for $i = 1, 2$.

**2.2. The Smirnov class.** A $(p \times q)$-matrix-valued function $f(z)$ holomorphic in $D$ belongs to the Smirnov class $N^{p \times q}_+$ if it has a representation of the form $f = h^{-1} g$, where $g \in H^{p \times q}_\infty$ and $h$ is a scalar outer function in $H_\infty$. A matrix-valued function $f(z)$ is an outer function of class $N^{p \times q}_+$ if in the above representation $g$ is an outer function belonging to $H^{p \times q}_\infty$. In this case we shall write $f(z) \in N^{p \times q}_+$. It is clear that $N^{p \times q}_+ \subset N^{p \times q}_\infty \subset N^{p \times q}_\ast$.

A matrix-valued function $f$ of the Smirnov class $N^{p \times p}_+$ belongs to $N^{p \times p}_\ast$ if and only if det $f(z) = 0$ for all $z \in D$ and $f^{-1} \in N^{p \times p}_+$; therefore, $f \in N^{p \times p}_\ast \iff f^{-1} \in N^{p \times p}_\ast$.

The maximum principle is true in the Smirnov class.

**Lemma 2.** If $f \in N^{p \times q}_+$, then for $1 \leq r < \infty$ we have

$$\sup_{\rho < 1} \int_{|\zeta| = 1} (\text{trace} \{ f(\rho \zeta)^* f(\rho \zeta) \})^\frac{r}{2} |d\zeta| = \int_{|\zeta| = 1} (\text{trace} \{ f(\zeta)^* f(\zeta) \})^\frac{r}{2} |d\zeta| \leq \infty,$$

$$\sup_{z \in D} \|f(z)\| = \text{ess sup}_{\zeta \in T} \{\|f(\zeta)\|\} \leq \infty.$$
For scalar functions \( f \in N^+ \) this assertion was proved by Smirnov (see [13]), and for matrix-valued functions it was proved by Ginzburg [14] [17].

Let \( f(z) \in N^{p \times q} \). Then an ordered pair of matrix-valued functions \( \{b_1, b_2\} \), where \( b_1(z) \in S_{in}^{p \times p} \) and \( b_2(z) \in S_{in}^{q \times q} \) are such that \( b_1(z)f(z)b_2(z) \in N_+^{p \times q} \), is called a denominator of \( f(z) \). A denominator of \( f(z) \) of the form \( \{u, b\} \), where \( u \) is a unitary matrix of order \( p \), is said to be right, and a denominator of the form \( \{b, v\} \), where \( v \) is a unitary matrix of order \( q \), is said to be left. We denote by

\[
\text{Den}(f) = \{(b_1, b_2) : b_1 \in S_{in}^{p \times p}, b_2 \in S_{in}^{q \times q}, b_1f_b \in N_+^{p \times q}\}
\]

the set of all denominators of \( f(z) \), and by

\[
\text{Den}^r(f) = \{(u, b) \in \text{Den}(f), u = \text{const}\}, \\
\text{Den}^l(f) = \{(b, v) \in \text{Den}(f), v = \text{const}\}
\]

the sets of all right and left denominators of \( f(z) \), respectively. Note that for an arbitrary matrix-valued function \( f \in N^{p \times q} \) the sets \( \text{Den}^r(f) \) and \( \text{Den}^l(f) \) are not empty (see [2] [6]).

A denominator \( \{b_1, b_2\} \in \text{Den}(f) \) is called a divisor of a denominator \( \{b_1, b_2\} \in \text{Den}(f) \) if \( b_1(z) = u(z)b_1(z) \) and \( b_2(z) = b_2(z)v(z) \), where \( u(z) \in S_{in}^{p \times p} \) and \( v(z) \in S_{in}^{q \times q} \). Such a divisor is said to be trivial if \( u(z) = \text{const} \) and \( v(z) = \text{const} \).

A denominator of a matrix-valued function \( f \in N^{p \times q} \) is minimal if it has no nontrivial divisors in \( \text{Den}(f) \).

**Lemma 3.** For an arbitrary matrix-valued function \( f(z) \in N^{p \times q} \) there exists a minimal right (left) denominator \( \{u, b\} \in \text{Den}^r(f) \) (\( \{b_1, v\} \in \text{Den}^l(f) \)). It is unique up to a right (left) unitary factor of \( b_2 \) (respectively, \( b_1 \)) and up to a unitary matrix \( u \) (respectively, \( v \)).

**Lemma 4.** (a) Suppose \( f(z) \in N^{p \times q} \) and \( \{b_1, b_2\} \in \text{Den}(f) \). Then there exists a minimal denominator \( \{b_1, b_2\} \) of \( f \) that is a divisor of \( \{b_1, b_2\} \).

(b) Let \( f(z) \) be a rational \((p \times q)\)-matrix-valued function. Then \( f(z) \in N^{p \times q} \), and the inner matrix-valued functions \( b_1 \) and \( b_2 \) in its arbitrary minimal denominator \( \{b_1, b_2\} \) are rational.

The proofs of Lemmas 3 and 4 can be found in [2] [6].

An arbitrary nonzero matrix-valued function \( f(z) \in N_+^{p \times q} \) has an essentially unique inner-outer factorization of the form

\[
f(z) = b_1(z)\varphi_1(z), \quad \text{where} \quad b_1 \in S_{in}^{p \times r} \quad \text{and} \quad \varphi_1 \in N_+^{r \times q},
\]

and an essentially unique -outer- -inner factorization of the form

\[
f(z) = \varphi_2(z)b_2(z), \quad \text{where} \quad \varphi_2 \in N_+^{r \times q} \quad \text{and} \quad b_2 \in S_{in}^{r \times q}.
\]

In these representations we have \( r = \text{rank} f(z) \) for all \( z \in D \) except possibly an at most countable set of points where \( \text{rank} f(z) < r \). This set has no accumulation points in \( D \).

The following inclusions are true:

\[
S_{in}^{p \times q} \subset H_2^{p \times q} \subset N_+^{p \times q} \quad \text{and} \quad \ell^{p \times p} \subset N_+^{p \times p}.
\]

### 2.3. The class \( \Pi^{p \times q} \)

A \((p \times q)\)-matrix-valued function \( f_- \) meromorphic in \( D_e = \{z \in \mathbb{C} : 1 < |z| \leq \infty\} \) is called a pseudocountinuation of \( f, f \in N^{p \times q} \), if \( f^\# \in N^{q \times p} \) and

\[
f(\zeta) := \lim_{\rho \uparrow 0} f(\rho \zeta) = \lim_{\rho \downarrow 0} f_-(\rho \zeta) \quad \text{for a.e.} \quad \zeta \in T.
\]

The subclass of all \( f \in N^{p \times q} \) that have a pseudocountinuation \( f_- \) to \( D_e \) will be denoted by \( \Pi^{p \times q} \). The intersection of \( \mathfrak{B}^{p \times q} \) and \( \Pi^{p \times q} \) will be denoted by \( \mathfrak{B}^{p \times q} \Pi \).

For a given \( f \in \Pi^{p \times q} \), a pseudocountinuation \( f_- \) is unique, because the function \( f_-^\# \) of class \( N^{q \times p} \) in \( D \) is uniquely determined by its boundary values \( f_-(\zeta)^\# = f(\zeta)^\# \).
If for a matrix-valued function $f \in \Pi^{p \times q}$ we consider its boundary values $f(\zeta)$ and its pseudocontinuation $f_-$, then as a result we get a matrix-valued function defined everywhere on the complex plane except possibly some set of Lebesgue measure zero on the circle $T$ and isolated singularities, namely, the poles of $f$ and $f_-$. This matrix-valued function will be denoted in the same way as the initial one, i.e., $f(z)$. The set where this function is holomorphic will be denoted by $\Omega_f$, and $\Omega^-_f := \Omega_f \cap D$, $\Omega^+_f := \Omega_f \cap D_e$.

We have

$$S_{in}^{p \times p} \subset \Pi^{p \times p},$$

and moreover, the pseudocontinuation $s_-$ of $s \in S_{in}^{p \times p}$ can be obtained by the symmetry principle,

$$s_-(z) = \left[\frac{1}{s_+(z)}\right]^{-1}, \quad z \in D, \quad \det s\left(\frac{1}{z}\right) \neq 0,$$

from the identity

$$s(\zeta)s(\zeta)^* = s(\zeta)^*s(\zeta) = I_p \quad \text{for a.e. } \zeta \in T.$$

The following fact, implied by the results by Douglas, Shapiro, and Shields [10], is very important.

**Lemma 5.** Let $f \in H_2^{p \times q}$. Then $f \in H_2^{p \times q}\Pi$ if and only if there exists a matrix-valued function $b \in S_{in}^{p \times p}$ such that $b(\zeta)^*f(\zeta) = g(\zeta)^*$ for a.e. $\zeta \in T$, where $g(\zeta)$ stands for the nontangential boundary values of some matrix-valued function $g(z) \in H_2^{p \times p}$. Moreover, $b(z)$ can be taken in the form $b(z) = \eta(z)I_p$, where $\eta \in S_{in}$.

**Proof.** For $p = q = 1$ the result is contained in [10].

Let $p \neq 1$ or $q \neq 1$. The matrix-valued function $f \in H_2^{p \times q}$ has a pseudocontinuation to the exterior of the unit disk $D_e$ if and only if each of its entries is of class $H_2\Pi$ and satisfies the conclusion of the lemma for $p = q = 1$. Therefore, the required $b \in S_{in}^{p \times p}$ of the form $b(z) = \eta(z)I_p$, $\eta \in S_{in}$, exists. The function $\eta(z)$ can be taken as the product of all functions of class $S_{in}$ given by the scalar version of the lemma for each entry of $f$.

Lemma 5 is proved. □

Let $f \in \Pi^{p \times q}$, and let

$$r_f = \max\{\text{rank } f(z) : z \in \Omega_f\};$$

then $\text{rank } f(z) = r_f$ for all $z \in \Omega_f$ except possibly a set of isolated points, and moreover, $\text{rank } f(\zeta) = r_f$ for a.e. $\zeta \in T$.

Note that all rational $(p \times q)$-matrix-valued functions belong to $\Pi^{p \times q}$.

**Rosenblum–Rovnyak Theorem.** Suppose $f \in \Pi^{p \times p}$, $r = r_f$, and $f(\zeta) \geq 0$ for a.e. $\zeta \in T$. Then:

1) the factorization problem

$$g(\zeta)^*g(\zeta) = f(\zeta) \quad \text{for a.e. } \zeta \in T$$

has a solution $g = \varphi \in N_{out}^{r \times p}$ unique up to a constant left unitary factor of order $r$; every solution $\varphi \in N_{out}^{r \times p}$ belongs to $\Pi^{r \times p}$;

2) the dual factorization problem

$$\omega(\zeta)\omega(\zeta)^* = f(\zeta) \quad \text{for a.e. } \zeta \in T$$

has a solution $\omega = \psi$ such that $\psi^* \in N_{out}^{r \times p}$, unique up to a constant right unitary factor of order $r$, and $\psi \in \Pi^{p \times r}$;

3) a matrix-valued function $f$ belongs to $L_1^{p \times p}$ if and only if $\varphi \in H_2^{r \times p}$ ($\psi \in H_2^{p \times r}$).
4) if \( f \in L_1^{r \times p} \), then the set of solutions \( g \in H_2^{r \times p} \) of the direct factorization problem can be described by the formula \( g = b_1 \varphi \), where \( b_1 \in S_m^{r \times r} \), and the set of solutions \( \omega \in H_2^{p \times r} \) of the dual factorization problem can be described by the formula \( \omega = \psi b_2 \), where \( b_2 \in S_m^{r \times r} \);  
5) a \((p \times p)\)-matrix-valued function \( f \) is rational if and only if the solutions \( \varphi \) and \( \psi \) of the direct and dual factorization problems are rational matrix-valued functions of size \( r \times p \) and \( p \times r \), respectively;  
6) for a rational \((p \times p)\)-matrix-valued function \( f \), the set of rational solutions \( g \in H_2^{r \times p} \) of the direct factorization problem can be described by the formula \( g = b_1 \varphi \), where \( b_1 \) is a rational inner matrix-valued function of order \( r \), and the set of rational solutions \( \omega \in H_2^{p \times r} \) of the dual factorization problem can be described by the formula \( \omega = \psi b_2 \), where \( b_2 \) is a rational inner matrix-valued function of order \( r \);  
7) if \( g \in N^{r \times p} \) is a solution of the direct factorization problem, then \( g \in \Pi^{r \times p} \) and  
\[
\Omega g = f(z) \quad \text{for all } z \in \Omega_g \cap \Omega_{g*},
\]
and if \( \omega \in N^{p \times r} \) is a solution of the dual factorization problem, then \( \omega \in \Pi^{p \times r} \) and  
\[
\Omega \omega = f(z) \quad \text{for all } z \in \Omega_\omega \cap \Omega_{\omega*}.
\]

**Proof.** The results stated in the theorem are contained, e.g., in the book [12]. \( \square \)

§3. \( J_{p,m} \)-INNER DILATIONS

### 3.1. Necessary information about matrix-valued functions of classes \( P(J) \) and \( U(J) \)

We let \( J \) denote a signature matrix, i.e., a matrix of order \( m \) such that  
\[
J^* = J, \quad J^2 = I_m.
\]

A matrix \( \theta \) of order \( m \) said to be \( J \)-contractive if  
\[
\theta^* J \theta \leq J,
\]
and it is \( J \)-unitary if  
\[
\theta^* J \theta = J.
\]

Equivalent conditions are \( \theta J \theta^* \leq J \) and \( \theta J \theta^* = J \), respectively.

Put  
\[
P = (I_m + J)/2, \quad Q = (I_m - J)/2.
\]

For a \( J \)-contractive matrix \( \theta \), the matrix  
\[
S = (Q + P \theta)(P + Q \theta)^{-1}
\]
is well defined. It is called the Potapov–Ginzburg transform of \( \theta \). Since  
\[
I_m - S^* S = (P + Q \theta)^{-1} (J - \theta^* J \theta)(P + Q \theta)^{-1},
\]
we have the following statement (see [2, 14]).

**Lemma 6.** A matrix \( \theta \) is \( J \)-contractive if and only if the matrix \( S \) defined by (3.2) is contractive \( (\| S \| \leq 1) \).

The matrix \( \theta \) can be expressed in terms of \( S \) by the formula  
\[
\theta = (S Q - P)^{-1} (Q - SP).
\]
Consider the Potapov class $P(J)$ of matrix-valued functions $\theta(z)$ meromorphic in the unit disk $D$ and having $J$-contractive values at every point of $D$ where it is holomorphic, i.e.,
\[(3.3)\]
\[\theta(z)^*J\theta(z) \leq J, \quad z \in \Omega^+_0.\]

Such matrix-valued functions will also be called $J$-contractive.

By Lemma 6, $\theta(z)$ is $J$-contractive if and only if the matrix-valued function
\[S(z) = (Q + P\theta(z))(P + Q\theta(z))^{-1}, \quad z \in \Omega^+_0,\]
extends continuously to the entire $D$ so that $S(z) \in S^{m \times m}$. Here $P$ and $Q$ are defined by (3.1). Since
\[\theta(z) = (S(z)Q - P)^{-1}(Q - S(z)P),\]
any $J$-contractive matrix-valued function $\theta(z)$ can be represented as a ratio of bounded matrix-valued functions holomorphic in $D$. Therefore, the following is true.

**Lemma 7.** Any $J$-contractive matrix-valued function has bounded Nevanlinna characteristic.

In other words, $P(J) \subset N^{m \times m}$. This implies that any $J$-contractive matrix-valued function $\theta(z)$ has radial limit values almost everywhere on $T$,
\[\theta(\zeta) = \lim_{\rho \uparrow 1} \theta(\rho \zeta),\]
and these limit values on a subset of $T$ of positive Lebesgue measure determine the function $\theta(z)$ uniquely. Passing to the limit in (3.3), we get
\[(3.4)\]
\[\theta(\zeta)^*J\theta(\zeta) \leq J \quad \text{for a.e. } \zeta \in T.\]

We shall be interested in $J$-contractive matrix-valued functions $\theta(z)$ with $J$-unitary boundary values, i.e., the functions such that
\[(3.5)\]
\[\theta(\zeta)^*J\theta(\zeta) = J \quad \text{for a.e. } \zeta \in T.\]

Such matrix-valued functions are said to be $J$-inner. The class of $J$-inner matrix-valued functions will be denoted by $U(J)$. Clearly, $U(I_p) = S_{in}^{p \times p}$.

**Remark 1.** Condition (3.5) implies that $\det \theta(\zeta) \neq 0$ a.e. on $T$ for $\theta \in U(J)$; therefore, $\det \theta(z) \neq 0$ for $z \in \Omega^+_0$ except probably some subset of $\Omega^+_0$ without accumulation points in $D$. It follows that $\theta(z)^{-1} \in N^{m \times m}$, and the pseudocontinuation $\theta_-$ of $\theta$, defined by the “symmetry principle”
\[\theta_-(z) = J[\theta^*(z)]^{-1}J,\]
has bounded Nevanlinna characteristic in $D_e$. The boundary values
\[\theta_-(\zeta) = \lim_{\rho \downarrow 1} \theta_-(\rho \zeta) \quad \text{(for a.e. } \zeta \in T)\]
coincide almost everywhere with the boundary values $\theta(\zeta)$ of $\theta(z)$. Thus, $U(J) \subset \Pi^{m \times m}$.

**Remark 2.** Let $J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix}$. For a matrix-valued function $c(z)$ of order $p$ we define $\theta_0(z)$ by (1.2*). It is easy to check that
1) $c \in \ell^{p \times p} \iff \theta_0 \in P(J_p)$;
2) $c \in \ell^{p \times p}$ and $\text{Re} \ c(z) = 0$ for a.e. $\zeta \in T \iff \theta_0 \in U(J_p)$.

**3.2. Proof of Theorem 1.** Let $m \geq 0$. Consider the signature matrix $J_{p,m}$ defined by (1.3).

A matrix-valued function $\theta(z) \in U(J_{p,m})$ is called a $J_{p,m}$-unitary dilation of $c(z) \in \ell^{p \times p}$ if it has the block structure of type (1.2) for $m > 0$ and of type (1.2*) for $m = 0$.

Now, we start proving Theorem 1 (see the Introduction).
Proof of the “only if” part. Let \( c \in \ell^p \times \Pi \). If \( m_c = 0 \), then the matrix-valued function \( \theta_0 \) defined in (1.2*) is a unique \( J_p \)-inner dilation of \( c(z) \).

Now, let \( m_c > 0 \). The matrix-valued function \( \text{Re} \, c(\zeta) \), which is nonnegative for a.e. \( \zeta \in T \), is the boundary value of the function \( c(\zeta) + c^\#(\zeta) \), belonging to the Nevanlinna class \( N_p \times \Pi \). Since \( \text{Re} \, c(\zeta) \in \mathcal{I}^{p} \times \Pi \) for \( c \in \ell^p \times \Pi \), the Rosenblum–Rovnyak theorem shows that the factorization problem

\[
2 \text{Re} \, c(\zeta) = g(\zeta)^* g(\zeta) \quad \text{for a.e. } \zeta \in T
\]

is solvable in \( H_2^{m \times p} \), and its solution satisfies the condition \( g \in H_2^{m \times p} \Pi \) for \( m = m_c \). Therefore, by Lemma 5, there exists a matrix-valued function \( b \in S_m \) such that \( b(\zeta)^* g(\zeta) = \omega(\zeta)^* \), where \( \omega(\zeta) \) is the boundary value of \( \omega \in H_2^{p \times m} \). Put

\[
\alpha = b, \quad \beta = g, \quad \gamma = \omega, \quad \delta = c, \quad \theta = \begin{bmatrix} \alpha & \beta & 0 \\ \gamma & \delta & I_p \\ 0 & I_p & 0 \end{bmatrix}.
\]

Then the following identity is true:

\[
\theta(\zeta)^* J_{p,m} \theta(\zeta) = J_{p,m} \quad \text{for a.e. } \zeta \in T.
\]

Indeed, for a.e. \( \zeta \in T \) we have

\[
\theta(\zeta)^* J_{p,m} \theta(\zeta) = \begin{bmatrix} \alpha(\zeta)^* \gamma(\zeta)^* 0 \\ \beta(\zeta)^* \delta(\zeta)^* I_p \\ 0 I_p 0 \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & -I_p & 0 \\ 0 & I_p & 0 \end{bmatrix} \begin{bmatrix} \alpha(\zeta) & \beta(\zeta) & 0 \\ \gamma(\zeta) & \delta(\zeta) & I_p \\ 0 & I_p & 0 \end{bmatrix} = \begin{bmatrix} \alpha(\zeta)^* \alpha(\zeta) & \alpha(\zeta)^* \beta(\zeta) - \gamma(\zeta)^* & 0 \\ \beta(\zeta)^* \alpha(\zeta) - \gamma(\zeta)^* & \beta(\zeta)^* \beta(\zeta) - 2 \text{Re} \, \delta(\zeta) & -I_p \\ 0 & -I_p & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} \alpha(\zeta)^* \alpha(\zeta) & \alpha(\zeta)^* \beta(\zeta) - \gamma(\zeta)^* & 0 \\ \beta(\zeta)^* \alpha(\zeta) - \gamma(\zeta)^* & \beta(\zeta)^* \beta(\zeta) - 2 \text{Re} \, \delta(\zeta) & -I_p \\ b(\zeta)^* b(\zeta) & b(\zeta)^* g(\zeta) - \omega(\zeta)^* & 0 \\ g(\zeta)^* b(\zeta) - \omega(\zeta) & g(\zeta)^* g(\zeta) - 2 \text{Re} \, c(\zeta) & -I_p \\ 0 & -I_p & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_p \\ 0 & -I_p & 0 \end{bmatrix} = J_{p,m}.
\]

Therefore, \( \theta \) is \( J_{p,m} \)-unitary almost everywhere on \( T \). It remains to show that \( \theta \) is \( J_{p,m} \)-contractive in \( D \). For this, we use the Potapov–Ginzburg transform of \( \theta(z) \):

\[
P = (I_{2p+m} + J_{p,m})/2, \quad Q = (I_{2p+m} - J_{p,m})/2, \quad \tilde{S}(z) = (Q + P\theta(z))(P + Q\theta(z))^{-1}.
\]

We have

\[
\tilde{S}(z) = \begin{bmatrix} \alpha(z) - \frac{1}{2} \beta(z)(I_p + \frac{1}{2} \delta(z))^{-1} \gamma(z) & \beta(z)(I_p + \frac{1}{2} \delta(z))^{-1} 0 \\ 0 & 0 \end{bmatrix}.
\]

Clearly, the matrix-valued function \( \tilde{s} = (I_p - \frac{1}{2} \delta(z))^{-1} \mu(z) \) belongs to the Schur class \( S_p^{m \times p} \), because \( \frac{1}{2} \delta \in \ell^p \times \Pi \). The function \( (I_p + \frac{1}{2} \delta(z))^{-1} (= \frac{1}{2} (I_p + \mu(z))) \) is of class \( H_2^{p \times m} \). This implies that all blocks of \( \tilde{S} \) belong to the Smirnov classes of appropriate sizes. Therefore, \( \tilde{S} \in N_+^{(2p+m) \times (2p+m)} \), and (3.7) is equivalent to the identity

\[
\tilde{S}(z)^* \tilde{S}(z) = I_{2p+m} \quad \text{for a.e. } \zeta \in T.
\]

By the maximum principle for the Smirnov class, \( \tilde{S} \) is contractive in \( D \):

\[
\tilde{S}(z)^* \tilde{S}(z) \leq I_{2p+m}, \quad z \in D.
\]
Now, we can use Lemma 6 to conclude that the resulting matrix-valued function \( \theta \) is \( J_{p,m} \)-contractive in \( D \):

\[
(3.8) \quad \theta(z)^* J_{p,m} \theta(z) \leq J_{p,m}, \quad z \in \Omega^+_0.
\]

Thus, the function \( \theta \), meromorphic in \( D \), belongs to \( U(J_{p,m}) \). Moreover, it has a block structure as in (1.2), so that \( \theta \) is a \( J_{p,m} \)-inner dilation of \( c(z) \) with \( m = m_c \).

**Proof of the “if” part.** Suppose a matrix-valued function \( c(z) \) has a \( J_{p,m} \)-inner dilation \( \theta \) of type (1.2) or (1.2*) with \( m \geq 0 \). In accordance with Remark 1, the condition \( \theta \in U(J_{p,m}) \) implies \( \theta \in \Pi^{(2p+m) \times (2p+m)} \). Therefore, the matrix-valued function \( c(z) \) is in \( \Pi^{p \times p} \), being a block of its dilation \( \theta(z) \). The property \( c \in \ell^{p \times p} \) follows from Remark 2 if \( m = 0 \). If \( m > 0 \), then the same property follows from the inequality \( \Re c(z) \geq \beta(z)^* \beta(z) \) for \( z \in \Omega^+_p \). The latter inequality is a consequence of (3.8). If a matrix-valued function \( c(z) \) of order \( p \) is meromorphic in \( D \) and \( \Re c(z) \geq 0 \) for \( z \in \Omega^+_p \), then \( c(z) \) may have only removable singularities, so that we can view it as holomorphic in \( D \). Therefore, \( c \in \ell^{p \times p} \).

Now, we prove that \( m \geq m_c \). If \( m > 0 \) for \( \theta \), then

\[
2 \Re c(\zeta) = \beta(\zeta)^* \beta(\zeta) \quad \text{for a.e. } \zeta \in T,
\]

and \( m_c = \text{rank } \beta(\zeta) \) on \( T \), whence \( m \geq m_c \). If \( m = 0 \), then \( \theta = \theta_0 \), and therefore, by Remark 2, \( c(z) \) is of class \( \ell^{p \times p} \) and \( \Re c(\zeta) = 0 \) a.e. on \( T \). In this case, \( c(z) \) has a pseudocontinuation \( c_z(z) = -c^*(z) \), whence \( c \in \ell^{p \times p} \) and \( m = m_c \left(= 0 \right) \). The theorem is proved.

In what follows, we shall consider only \( J_{p,m} \)-inner dilations of \( c \in \ell^{p \times p} \) with the minimal possible \( p \), i.e., with \( m = m_c \).

**3.3. Description of the set of \( J_{p,m} \)-inner dilations with \( m = m_c \).** A \( J_{p,m} \)-inner dilation \( \theta \) of a matrix-valued function \( c \in \ell^{p \times p} \) with \( m = m_c > 0 \) is not unique. Indeed, if \( b_1 \in S^{m \times m}_m \) and \( b_2 \in S^{m \times m}_m \), then the function

\[
\theta_1 = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \theta \begin{bmatrix} b_2 & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix}
\]

is also a \( J_{p,m} \)-inner dilation of \( c(z) \).

Let

\[
\theta(z) = \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \gamma(z) & \delta(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}
\]

be an arbitrary \( J_{p,m} \)-inner dilation of \( c \in \ell^{p \times p} \), and let \( m = m_c > 0 \). Since \( \theta(\zeta) \) is \( J_{p,m} \)-unitary, (3.7) is fulfilled almost everywhere on \( T \). This is equivalent to the following family of scalar identities for a.e. \( \zeta \in T \):

\[
(3.9) \quad \alpha(\zeta)^* \alpha(\zeta) = I_m, \quad \alpha(\zeta)^* \beta(\zeta) = \gamma(\zeta)^*,
\]

\[
(3.10) \quad 2 \Re c(\zeta) = \beta(\zeta)^* \beta(\zeta).
\]

By (3.8), we have

\[
(3.11) \quad \begin{bmatrix} I_m - \alpha(z)^* \alpha(z) & \gamma(z)^* - \alpha(z)^* \beta(z) & 0 \\ \gamma(z) - \beta(z)^* \alpha(z) & 2 \Re c(z) - \beta(z)^* \beta(z) & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0
\]

for \( z \in \Omega^+_p \), whence

\[
(3.12) \quad \alpha(z)^* \alpha(z) \leq I_m, \quad \beta(z)^* \beta(z) \leq 2 \Re c(z), \quad z \in \Omega^+_p.
\]
Furthermore, (3.12) implies that the matrix-valued functions $\alpha$ and $\beta$, meromorphic in $D$, may have only removable singularities. Therefore, we may assume that $\alpha$ and $\beta$ are holomorphic in $D$. In particular, for $\beta$ we have

$$
\frac{1}{2\pi} \int_{|\xi| = 1} \| \beta(\rho \xi) \xi \|^2 \, |d\xi| \leq \frac{1}{2\pi} \int_{|\xi| = 1} 2(\Re c(\rho \xi) \xi, \xi) \, |d\xi| = 2(\Re c(0) \xi, \xi) < \infty,
$$

where $\xi \in \mathbb{C}^p$. Hence, $\beta$ is an $H_2^{m \times p}$-solution of the factorization problem (3.10), and $\alpha$ belongs to $S_m^{m \times m}$ by (3.9) and (3.12).

Relation (3.7) is equivalent to

$$
(3.13) \quad \theta(\zeta) J_{p,m} \theta(\zeta)^* = J_{p,m} \quad \text{for a.e. } \zeta \in T.
$$

Therefore,

$$
(3.14) \quad \alpha(\zeta) \alpha(\zeta)^* = I_m, \quad \alpha(\zeta) \gamma(\zeta)^* = \beta(\zeta),
$$

$$
(3.15) \quad 2 \Re c(\zeta) = \gamma(\zeta) \gamma(\zeta)^* \quad \text{for a.e. } \zeta \in T.
$$

Since (3.8) is equivalent to

$$
(3.16) \quad \theta(z) J_{p,m} \theta(z)^* \leq J_{p,m}, \quad z \in \Omega^+_\theta,
$$
in a similar way we show that $\gamma$ is a solution of the factorization problem (3.15), and it belongs to $H_2^{p \times m}$.

The matrix-valued functions $\beta$ and $\gamma$ can be represented in the form

$$
(3.17) \quad \beta(z) = b_1(z) \varphi(z), \quad \gamma(z) = \psi(z) b_2(z),
$$

where $\varphi$ and $\psi$ are an outer and a $*$-outer solution of the factorization problems (3.10) and (3.15). They have rank $m$ and belong to the classes $H_2^{m \times p}$ and $H_2^{p \times m}$, respectively; $b_1$ and $b_2$ are of class $S_m^{m \times m}$. The Rosenblum–Rovnyak theorem ensures that such “maximal” solutions $\varphi$ and $\psi$ of problems (3.10) and (3.15) exist. The functions $\varphi(z)$ and $\psi(z)$ are uniquely determined by $c(z)$, up to a unitary left or right constant matrix, respectively.

Since $c \in \mathcal{P}^{p \times p}$ and $m = m_c > 0$, the matrix-valued function $\Re c(\zeta)$ (nonnegative for a.e. $\zeta \in T$) is the boundary value of a matrix-valued function of class $N^{p \times p}$ in the disk $D$. Therefore, since rank $\Re c(\zeta) = m$ a.e. on $T$, the function $\Re c(\zeta)$ has a principal minor of order $m$ different from zero a.e. on $T$, whereas any principal minor of order exceeding $m$ is identically zero. Without loss of generality we may assume that such a principal minor of order $m$ is at the upper left corner of the matrix $\Re c(\zeta)$. To arrive at this case, we can always make a permutation of the rows of $\Re c(\zeta)$, together with the same permutation of the columns. As a result, we get a matrix-valued function $h(\zeta) = K \Re c(\zeta) K^*$ that is determined now by $\tilde{c} = KcK^*$; hence $h(\zeta) = \Re \tilde{c}(\zeta)$, where $K$ is a constant orthogonal matrix. A dilation $\theta$ of $c$ can be obtained from the dilation $\tilde{\theta}$ of $\tilde{c}$ by multiplying $\tilde{\theta}$ from the left and from the right by $J_{p,m}$-unitary matrices $\begin{bmatrix} I_m & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix}$ and $\begin{bmatrix} I_m & 0 & 0 \\ 0 & K^* & 0 \\ 0 & 0 & K^* \end{bmatrix}$, respectively. Under our assumption, an outer and a $*$-outer solution of the factorization problems (3.10) and (3.15) have the following form:

$$
\begin{bmatrix} \varphi_1 & \varphi_2 \\ \psi_1 & \psi_2 \end{bmatrix},
$$

where $\varphi_1$ and $\psi_1$ are matrix functions of order $m$ with $\det \varphi_1 \neq 0$ and $\det \psi_1 \neq 0$. They could be determined uniquely by imposing the normalization conditions $\varphi_1(0) > 0$ and $\psi_1(0) > 0$. These solutions will be called the normalized outer and $*$-outer solutions of (3.10) and (3.15), respectively. They will be denoted by $\varphi_N$ and $\psi_N$. In the general
case, we assume that \( \varphi_N = \left[ \begin{array}{c} \varphi_1 \\ \varphi_2 \end{array} \right] K \) and \( \psi_N = K^* \left[ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right] \), where \( K \) is the orthogonal matrix considered above. The normalization conditions are \( \varphi_1(0) > 0 \) and \( \psi_1(0) > 0 \).

In what follows, a description of all \( J_{p,m} \)-inner dilations of the form \( \begin{bmatrix} \varphi & \star \psi \\ \star \psi & \varphi \end{bmatrix} \) with \( m = m_c \) will be given for \( c \in \ell^p \times \Pi \) with \( m_c > 0 \). This description will involve the normalized outer \( (\varphi_N) \) and \(*\)-outer \( (\psi_N) \) solutions of the factorization problems (3.10) and (3.15), as well as the matrix-valued function \( s_c(\zeta) \) defined by the relation

\[
(3.18) \quad s_c(\zeta) \psi_N(\zeta)^* = \varphi_N(\zeta) \quad \text{for a.e. } \zeta \in T.
\]

This function was introduced in [8] in a more general setting. There it was called a scattering suboperator, because precisely this suboperator arises when we consider interior scattering channels in conservative resistance systems with resistance matrix equal to \( c(z) \). In the same paper, it was shown that the matrix-valued function \( s_c(\zeta) \) plays a role in investigating minimal passive resistance systems with the impedance matrix \( c(z) \).

Note that if \( s_c(\zeta) \) is defined by (3.18), then it has unitary values, i.e.,

\[
(3.19) \quad s_c(\zeta)^* s_c(\zeta) = I_m \quad \text{for a.e. } \zeta \in T.
\]

Moreover, if \( c \in \ell^p \times \Pi \), then \( s_c(\zeta) \) is the nontangential boundary value of a matrix-valued function \( s_c(z) \) of class \( N^{p,x} \). The latter function is defined by

\[
(3.20) \quad s_c(z) \psi_N^z(\zeta) = \varphi_N(\zeta), \quad z \in \Omega_{\varphi_N} \cap \Omega_{\psi_N} \cap \Omega_s.
\]

Relation (3.20) is equivalent to the formula

\[
(3.21) \quad s_c(z) = \varphi_1(z) \psi_1^z(\zeta)^{-1}, \quad z \in \Omega_{\varphi_1} \cap \Omega_{\psi_1}.
\]

which in turn is equivalent to the identity

\[
(3.22) \quad s_c(z) = \varphi_1^z(z)^{-1} \varphi_1(z), \quad z \in \Omega_{\varphi_1} \cap \Omega_{\psi_1}.
\]

We show that (3.20) and (3.21) are indeed equivalent. Let (3.20) be fulfilled. Then

\[
 s_c(z) \left[ \begin{array}{cc} \psi_1^z(z) & \psi_2^z(z) \end{array} \right] = \left[ \begin{array}{cc} \varphi_1(z) & \varphi_2(z) \end{array} \right],
\]

whence

\[
 s_c(z) \varphi_1^z = \varphi_1(z), \quad s_c(z) \varphi_2^z = \varphi_2(z).
\]

Therefore, (3.21) is true.

Conversely, assume (3.21). The factorization problems (3.10) and (3.15) ensure that \( \varphi_N \varphi_N = \psi_N \psi_N^z \), i.e.,

\[
 \left[ \begin{array}{cc} \varphi_1^z & \varphi_2 \end{array} \right] = \left[ \begin{array}{cc} \psi_1^z & \psi_2 \end{array} \right],
\]

whence \( \varphi_1^z \varphi_2 = \psi_1^z \psi_2 \) and \( \varphi_2 = \varphi_2 \). Since \( \varphi_1^z \varphi_1 = \psi_1^z \psi_1 \), we have \( \varphi_1 = \varphi_1 \). Therefore, (3.20) is true.

Now, we state and prove a theorem that yields a complete description of the set of all \( J_{p,m} \)-inner dilations of \( c(z) \).

**Theorem 2.** Let \( c \in \ell^p \times \Pi \), and let \( m = m_c > 0 \). Consider the matrix-valued functions \( \varphi_N \in H^p_{2 \times 2} \) and \( \psi_N \in H^p_{2 \times m} \) that are the normalized outer and \(*\)-outer solutions of the factorization problems

\[
 2 \text{Re } c(\zeta) = \varphi(\zeta)^* \varphi(\zeta) \quad \text{and} \quad 2 \text{Re } c(\zeta) = \psi(\zeta) \psi(\zeta)^* \quad \text{for a.e. } \zeta \in T,
\]

respectively. Let \( s_c \) be defined by (3.20) and \( \vartheta \) by the formula

\[
(3.23) \quad \vartheta(z) = \begin{bmatrix} s_c(z) & \varphi_N(z) & 0 \\ \psi_N(z) & c(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}.
\]
Let \( \{b_1, b_2\} \) be a denominator of the matrix-valued function \( s_c \in \mathbb{R}^{p \times p} \). Put

\[
\alpha(z) = b_1(z)s_c(z)b_2(z), \quad \beta(z) = b_1(z)\varphi_N(z),
\]

and

\[
\gamma(z) = \psi_N(z)b_2(z), \quad \delta(z) = c(z),
\]

then we can construct functions \( \vartheta \) where

\[
\vartheta(z) = \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \gamma(z) & \delta(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}.
\]

Then \( \vartheta(z) \) is a \( J_{p,m} \)-unitary dilation of \( c(z) \), and it has a unique representation of the form

\[
\vartheta(z) = \begin{bmatrix} b_1(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \vartheta(z) \begin{bmatrix} b_2(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix},
\]

where \( \{b_1, b_2\} \in \text{Den}(s_c) \). All \( J_{p,m} \)-inner dilations of \( c \in \mathbb{R}^{p \times p} \) can be obtained in this way.

Proof. Let \( \vartheta(z) \) be a \( J_{p,m} \)-inner dilation of \( c \in \mathbb{R}^{p \times p} \) of the form \( \text{(1.2)} \). Then, as has been shown above, its blocks \( \beta \) and \( \gamma \) are solutions of the factorization problems \( \text{(3.10)} \) and \( \text{(3.15)} \). They belong to \( H_2^{m \times p} \) and \( H_2^{p \times m} \), respectively, and \( \alpha \) belongs to \( S_m^{m \times m} \). The functions \( \beta \) and \( \gamma \) can be represented as in \( \text{(3.17)} \), where \( \varphi = \varphi_N \) and \( \psi = \psi_N \) are the normalized outer and \( * \)-outer solutions of the factorization problems \( \text{(3.10)} \) and \( \text{(3.15)} \), belonging to \( H_2^{m \times p} \) and \( H_2^{p \times m} \), respectively, and \( b_1 \in \mathcal{S}_m^{m \times m}, b_2 \in \mathcal{S}_m^{m \times m} \). The functions \( \varphi_N \) and \( \psi_N \) determine the function \( s_c(z) \) uniquely by one of the formulas \( \text{(3.20)} \)–\( \text{(3.22)} \). The function \( s_c(z) \) satisfies \( \text{(3.19)} \).

Next, since \( \gamma(z)^* = \alpha(z)^* \beta(z) \) a.e. on \( T \), we have \( b_2(z)^* \psi_N(z) = \alpha(z)^* b_1(z) \varphi_N(z) \), and with the help of \( \text{(3.18)} \) we get \( \alpha = b_1 s_c b_2 \).

This leads to the following parametrization for the blocks of the dilation \( \vartheta \):

\[
\alpha = b_1 s_c b_2, \quad \beta = b_1 \varphi_N, \quad \gamma = \psi_N b_2.
\]

Therefore,

\[
\vartheta(z) = \begin{bmatrix} b_1(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \vartheta(z) \begin{bmatrix} b_2(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix},
\]

where \( \vartheta \) is uniquely determined by \( c \), by formula \( \text{(3.23)} \).

Thus, the freedom in the choice of the dilation reduces to the inner matrix-valued functions \( b_1 \) and \( b_2 \). Nevertheless, these functions are not arbitrary, because the following condition must be fulfilled:

\[
b_1 s_c b_2 \in S_m^{m \times m},
\]

where \( s_c \in N_m^{m \times m} \) is uniquely determined by \( c \) in accordance with one of the formulas \( \text{(3.20)} \)–\( \text{(3.22)} \). The boundary condition \( \text{(3.19)} \) yields the equivalence \( \text{(3.27)} \) and the relation

\[
b_1 s_c b_2 \in N_m^{m \times m}.
\]

Therefore, the dilation \( \vartheta \in U(J_{p,m}) \) of \( c \in \mathbb{R}^{p \times p} \) has the form \( \text{(3.20)} \), where \( \vartheta \) is uniquely determined by \( c(z) \), and \( \{b_1, b_2\} \) is a denominator of \( s_c \), i.e., \( \{b_1, b_2\} \in \text{Den}(s_c) \).

Conversely, if an arbitrary matrix-valued function \( c \in \mathbb{R}^{p \times p} \) with \( m_c > 0 \) is given, then we can construct functions \( s_c \) and \( \vartheta \) by \( \text{(3.20)} \)–\( \text{(3.22)} \). Furthermore, we
take an arbitrary denominator \( \{b_1, b_2\} \) of \( s_c \) and consider the function

\[
\theta(z) = \begin{bmatrix}
    b_1(z) & 0 & 0 \\
    0 & I_p & 0 \\
    0 & 0 & I_p
\end{bmatrix} \vartheta(z) \begin{bmatrix}
    b_2(z) & 0 & 0 \\
    0 & I_p & 0 \\
    0 & 0 & I_p
\end{bmatrix}.
\]

A direct calculation shows that (3.24) and (3.25) are fulfilled and that \( \theta \) has \( J_{p,m} \)-unitary boundary values a.e. on \( T \).

The matrix-valued function \( \vartheta(z) \) defined by (3.23) has \( J_{p,m} \)-unitary boundary values almost everywhere on the unit circle \( T \), but in general it may fail to be \( J_{p,m} \)-unitary. It is such if and only if \( s_c \in N_{+}^{m \times m} \). However, by the choice of \( b_1 \) and \( b_2 \), the Potapov–Ginzburg transform of \( \theta \) is contractive in \( D \), which can be checked in the same way as was done in the proof of (3.8) in Theorem 1. Using Lemma 6, we conclude that \( \theta \) is \( J_{p,m} \)-contractive in the disk \( D \). Therefore, \( \theta \in U(J_{p,m}) \). The theorem is proved. \( \Box \)

3.4. Minimal and optimal \( J_{p,m} \)-inner dilations. The result obtained allows us to describe the set of all minimal (in a sense) \( J_{p,m} \)-inner dilations of the form (1.2) with \( m = m_c \).

If two \( J_{p,m} \)-inner matrix-valued functions \( \theta \) and \( \theta_1 \) satisfy \( \theta = \theta_1 \theta_2 \), where \( \theta_2 \) is also a \( J_{p,m} \)-inner matrix-valued function, then \( \theta_1 \) is called a left divisor of \( \theta \). We say that such a divisor is trivial if \( \theta = \theta_1 U \), where \( U \) is a \( J_{p,m} \)-unitary matrix. A right divisor and a right trivial divisor are defined in a similar way.

A dilation \( \theta \in U(J_{p,m}) \) is said to be minimal if it admits no nontrivial \( J_{p,m} \)-inner left and right divisors of the form

\[
\begin{bmatrix}
    w(z) & 0 & 0 \\
    0 & I_p & 0 \\
    0 & 0 & I_p
\end{bmatrix},
\]

i.e., if \( \theta \) cannot be represented in the form

\[
(3.29) \quad \theta(z) = \begin{bmatrix}
    u(z) & 0 & 0 \\
    0 & I_p & 0 \\
    0 & 0 & I_p
\end{bmatrix} \vartheta(z) \begin{bmatrix}
    v(z) & 0 & 0 \\
    0 & I_p & 0 \\
    0 & 0 & I_p
\end{bmatrix},
\]

where \( \vartheta \in U(J_{p,m}) \), \( u \in S_{in}^{m \times m} \), \( v \in S_{in}^{m \times m} \), and at least one of the functions \( u(z) \) and \( v(z) \) is not constant.

We are going to find out for what denominators \( \{b_1, b_2\} \) of \( s_c \) formula (3.26) produces minimal \( J_{p,m} \)-inner dilations.

Theorem 3. A \( J_{p,m} \)-inner dilation \( \theta \) of a matrix-valued function \( c \in \mathfrak{P}^{p \times p} \Pi \) with \( m = m_c > 0 \) is minimal if and only if the corresponding denominator \( \{b_1, b_2\} \) of \( s_c \), occurring in (3.26), is minimal.

Proof. Let \( \theta \) be a \( J_{p,m} \)-inner dilation of \( c \in \mathfrak{P}^{p \times p} \Pi \), and let \( \{b_1, b_2\} \) be the denominator of \( s_c \) that corresponds to \( \theta \) by formula (3.26). Suppose that this dilation is minimal, i.e., (3.26) is fulfilled, where \( \vartheta \in U(J_{p,m}) \), \( u \in S_{in}^{m \times m} \), \( v \in S_{in}^{m \times m} \), and either \( u(z) \) or \( v(z) \) is not constant. Then \( \theta(z) \) is a \( J_{p,m} \)-inner dilation of \( c(z) \) with some denominator \( \{b_1, b_2\} \) of \( s_c \). Since the representation of \( \theta(z) \) in the form (3.26) is unique, we have \( b_1(z) = u(z) b_1(z) \) and \( b_2(z) = b_2(z) v(z) \); i.e., \( \{b_1, b_2\} \) is a nontrivial divisor of the denominator \( \{b_1, b_2\} \). Conversely, if we have such a divisor, we get relation (3.29) in which at least one of the inner functions, either \( u(z) \) or \( v(z) \), is not constant. The theorem is proved. \( \Box \)

A dilation \( \theta \in U(J_{p,m}) \) of \( c \in \mathfrak{P}^{p \times p} \Pi \) of the form (1.2) with \( m = m_c \) is said to be optimal if \( \beta \) in (1.2) is an outer matrix-valued function; it is \( * \)-optimal if \( \gamma \) in (1.2) is a \( * \)-outer matrix-valued function.

The next assertion follows from Theorem 3 and Lemma 3.
Theorem 4. All optimal \( J_{p,m} \)-inner dilations of a matrix-valued function \( c \in \ell^p \times \Pi \) are described by the formula

\[
\theta(z) = \begin{bmatrix}
    u & 0 & 0 \\
    0 & I_p & 0 \\
    0 & 0 & I_p
\end{bmatrix}
\begin{bmatrix}
    b(z) & 0 & 0 \\
    0 & I_p & 0 \\
    0 & 0 & I_p
\end{bmatrix}
\theta(z),
\]

where \( \{u,b\} \in \text{Den}'(s_c) \) and \( u \) is a unitary matrix of order \( m \). Moreover, an optimal dilation \( \theta \) is minimal if and only if the corresponding right denominator \( \{u,b\} \) of \( s_c \) is minimal. Such a denominator exists and is essentially unique.

All \( * \)-optimal \( J_{p,m} \)-inner dilations of a matrix-valued function \( c \in \ell^p \times \Pi \) are described by the formula

\[
\theta(z) = \begin{bmatrix}
    b(z) & 0 & 0 \\
    0 & I_p & 0 \\
    0 & 0 & I_p
\end{bmatrix}
\begin{bmatrix}
    v & 0 & 0 \\
    0 & I_p & 0 \\
    0 & 0 & I_p
\end{bmatrix}
\theta(z),
\]

where \( \{b,v\} \in \text{Den}'(s_c) \) and \( v \) is a unitary matrix of order \( m \). Moreover, a \( * \)-optimal dilation \( \theta \) is minimal if and only if the corresponding left denominator \( \{b,v\} \) of \( s_c \) is minimal. Such a denominator exists and is essentially unique.

\[\text{§4. Real, Symmetric, and Rational } J_{p,m} \text{-inner Dilations}\]

4.1. Real \( J_{p,m} \)-inner dilations. Matrix-valued functions \( f(z) \) real in the disk \( D \), i.e., satisfying \( f(\overline{z}) = f(z) \), appear very often in applications. For this reason, we consider real matrix-valued functions \( c(z) \in \ell^p \times \Pi \) and their \( J_{p,m} \)-inner dilations.

A denominator \( \{b_1,b_2\} \) of \( f \in \ell^p \times \Pi \) is said to be real if \( b_1(z) \) and \( b_2(z) \) are real matrix-valued functions.

Lemma 8. For a real matrix-valued function \( f(z) \in \ell^p \times \Pi \), its minimal right and minimal left denominators can be chosen to be real.

Proof. See Theorem 6.1 in [2].

Theorem 5. A real matrix-valued function \( c(z) \in \ell^p \times \Pi \) with \( m_c > 0 \) has a real \( J_{p,m} \)-inner dilation \( \theta(z) \) of the form (1.2) with \( m = m_c \). All such \( J_{p,m} \)-inner dilations are described by formula (3.28) with real denominators of \( s_c(z) \). In particular, minimal optimal as well as minimal \( * \)-optimal \( J_{p,m} \)-inner dilations can be chosen to be real.

Proof. Since \( c(z) \in \ell^p \times \Pi \) is real, \( 2 \Re c(\zeta) \) is also real. Therefore, the functions \( \varphi_N(\overline{\zeta}) \) and \( \psi_N(\overline{\zeta}) \) are solutions of the factorization problems (3.10) and (3.11), together with \( \varphi_N(z) \) and \( \psi_N(z) \). Since they are outer and \( * \)-outer matrix-valued functions, respectively, and they satisfy the normalization conditions \( \varphi_N(0) > 0 \), \( \psi_N(0) > 0 \), we have \( \varphi_N(z) = \overline{\varphi_N(\overline{z})} \) and \( \psi_N(z) = \overline{\psi_N(\overline{z})} \) because such solutions are unique. Therefore, \( \varphi_N(z) \) and \( \psi_N(z) \) are real, and hence, \( s_c(z) \) and \( \theta(z) \) are also real. Moreover, the \( J_{p,m} \)-inner dilation \( \theta(z) \) is real if and only if the denominator \( \{b_1(z),b_2(z)\} \) of \( s_c(z) \) has real inner matrix-valued functions \( b_1 \) and \( b_2 \). Combined with Theorem 3 and Lemma 8, this fact shows that a minimal optimal and a minimal \( * \)-optimal \( J_{p,m} \)-inner dilation of the real matrix-valued function \( c(z) \) can be chosen to be real.

4.2. Symmetric \( J_{p,m} \)-inner dilations. In applications, the case of symmetric matrix-valued functions \( f(z) \), i.e., those for which \( f(z)^T = f(z) \) for all \( z \in \Omega_j \), is also important. In this subsection we consider symmetric \( J_{p,m} \)-inner dilations of symmetric matrix-valued functions \( c(z) \in \ell^p \times \Pi \).

The denominators of the form \( \{b(z),b(z)^T\} \) for a function \( f(z) \) belonging to the Nevanlinna class are said to be symmetric. Such denominators always exist, because for a scalar
function \( \eta \in S_m \) such that \( \eta f \in N^{p \times q}_m \) the pair \( \{ \eta I_p, \eta I_p \} \) is a symmetric denominator of \( f \).

**Theorem 6.** Any symmetric matrix-valued function \( c(z) \in \ell^{p \times p} \Pi \) with \( m_c > 0 \) has a symmetric \( J_{p,m} \)-inner dilation \( \theta \). Moreover, all symmetric \( J_{p,m} \)-inner dilations are described by formula (3.26), where \( \{ b_1, b_2 \} \) is a symmetric denominator of \( s_c \), i.e., \( b_2(z) = b_1(z)^T \).

**Proof.** Since \( c(z) \in \ell^{p \times p} \Pi \) is symmetric, the outer function \( \psi_N(z)^T \) and the *-outer function \( \varphi_N(z)^T \) are, respectively, solutions of the factorization problems (3.10) and (3.15), together with the normalized solutions \( \varphi_N(z) \) and \( \psi_N(z) \). Moreover, they satisfy the same normalization conditions. Therefore, \( \psi_N(z) = \varphi_N(z)^T \), which implies that the matrix-valued functions \( s_c(z) \) and \( \vartheta(z) \) are symmetric. Moreover, the dilation \( \theta(z) \) given by (3.26) is symmetric if and only if the denominator \( \{ b_1, b_2 \} \) of \( s_c(z) \) satisfies the condition \( b_2(z) = b_1(z)^T \).

We say that a symmetric denominator \( \{ b(z), b(z)^T \} \) of \( f \in N^{p \times p} \) is minimal symmetric if it has no nontrivial symmetric divisors. Such denominators of \( s_c \) correspond to minimal symmetric \( J_{p,m} \)-inner dilations of \( c(z) \), i.e., to those admitting no representation of the form (3.26) with \( \vartheta(z)^T = \vartheta(z) \) and with nonconstant \( u(z) \) and \( v(z) (= u(z)^T) \). The existence of such dilations follows from the next statement.

**Lemma 9.** For every symmetric denominator \( \{ b(z), b(z)^T \} \) of \( f \in N^{p \times p} \), there exists a minimal symmetric denominator \( \{ \hat{b}(z), \hat{b}(z)^T \} \) that is a divisor of the initial one. In the scalar case \( (m = 1) \) the function \( f(z) \) has a unique minimal symmetric denominator (up to a constant factor \( \kappa \) with \( |\kappa| = 1 \)).

Note that the minimal symmetric denominator \( \{ \hat{b}(z), \hat{b}(z)^T \} \) of \( f \in N^{p \times p} \) may fail to be its minimal denominator. For the proof of Lemma 9, see [2] (Lemma 6.1 and the Remark after it).

Now, we consider the case where \( c(z) \in \ell^{p \times p} \Pi \) is both real and symmetric. The following results are consequences of Theorems 6.3 and 6.4 in [2].

**Theorem 7.** For a real symmetric matrix-valued function \( c(z) \in \ell^{p \times p} \Pi \) with \( m_c > 0 \), there exists a real symmetric \( J_{p,m} \)-inner dilation \( \theta(z) \) of the form (1.2). All such dilations are described by formula (3.26) with real symmetric denominators \( \{ b_1, b_2 \} \) of \( s_c(z) \).

Any real symmetric \( J_{p,m} \)-inner dilation \( \theta(z) \) can be represented as in (3.29), where \( \hat{\theta}(z) \) is a real minimal symmetric \( J_{p,m} \)-inner dilation of \( c(z) \). Moreover, the functions \( u(z) \) \( \in S_m^{m \times m} \) and \( v(z) \in S_m^{m \times m} \) satisfy \( u(z)^T = u(z) \) and \( v(z)^T = v(z) \).

Note that, for a given real symmetric matrix-valued function \( c(z) \in \ell^{p \times p} \Pi \), its real minimal symmetric \( J_{p,m} \)-inner dilation is not unique. However, for a scalar \( c(z) \in \Pi \), its real minimal symmetric \( J_{p,m} \)-inner dilation is unique, because we have an essentially unique minimal symmetric denominator of \( s_c(z) \).

### 4.3. Rational \( J_{p,m} \)-inner dilations.

In this subsection we consider the case where \( c(z) \in \ell^{p \times p} \Pi \) is a rational matrix-valued function.

We say that a denominator \( \{ b_1, b_2 \} \) of \( f \in N^{p \times q} \) is rational if \( b_1(z) \) and \( b_2(z) \) are rational.

If \( c \in \ell^{p \times p} \Pi \) is rational, then so is \( c(z) + c\#(z) \). Therefore, the normalized solutions \( \varphi_N(z) \) and \( \psi_N(z) \) of the factorization problems (3.10) and (3.15) are rational matrix-valued functions of size \( m \times p \) and \( p \times m \), respectively (see assertions 5 and 6 of the Rosenblum–Rovnyak theorem). This implies that the functions \( s_c(z) \) and \( \vartheta(z) \) defined by (3.21) and (3.24) are also rational. The \( J_{p,m} \)-inner dilation \( \theta \) is rational if and only if
so is the denominator \( \{b_1(z), b_2(z)\} \) of \( s_c(z) \). Assertion (b) of Lemma 4 and Theorem 3 imply that the minimal \( J_{p,m} \)-inner dilations of a rational \( c \in \ell^{p \times p} \Pi \) are rational.

Thus, the following theorem is true.

**Theorem 8.** For a rational matrix-valued function \( c(z) \in \ell^{p \times p} \Pi \) with \( m_c > 0 \), there exists a rational \( J_{p,m} \)-inner dilation \( \theta(z) \) of the form \( \text{(1.2)} \). All rational \( J_{p,m} \)-inner dilations are described by formula \( \text{(3.26)} \) with rational denominators of \( s_c(z) \). All minimal \( J_{p,m} \)-inner dilations of \( c(z) \) are rational. Moreover,

1) for a rational real matrix-valued function \( c(z) \in \ell^{p \times p} \Pi \) with \( m_c > 0 \), there exists a rational real \( J_{p,m} \)-inner dilation \( \theta(z) \) of the form \( \text{(1.2)} \). All rational \( J_{p,m} \)-inner dilations are described by formula \( \text{(3.26)} \) with rational real denominators of \( s_c(z) \);

2) an arbitrary rational and symmetric matrix-valued function \( c(z) \in \ell^{p \times p} \Pi \) with \( m_c > 0 \) admits a rational symmetric \( J_{p,m} \)-inner dilation \( \theta(z) \), and all rational and symmetric \( J_{p,m} \)-inner dilations are described by formula \( \text{(3.26)} \) with rational and symmetric denominators \( \{b, h^T\} \in \text{Den}(s_c) \). In the class of all rational and symmetric \( J_{p,m} \)-inner dilations there exist rational, minimal, and symmetric \( J_{p,m} \)-inner dilations of \( c(z) \);

3) for a rational, real, and symmetric matrix-valued function \( c(z) \in \ell^{p \times p} \Pi \) with \( m_c > 0 \), there exists a rational, real, and symmetric \( J_{p,m} \)-inner dilation \( \theta(z) \) of the form \( \text{(1.2)} \). All such dilations are described by formula \( \text{(3.26)} \) with rational, real, and symmetric denominators \( \{b_1, b_2\} \) of \( s_c(z) \). An arbitrary rational, real, and symmetric \( J_{p,m} \)-inner dilation \( \theta(z) \) of \( c(z) \) can be represented in the form \( \text{(3.26)} \), where \( \tilde{\theta}(z) \) is a rational, real, minimal, and symmetric \( J_{p,m} \)-inner dilation of \( c(z) \); moreover, \( u(z) \in S_{\text{in}}^{m \times m} \) and \( v(z) \in S_{\text{in}}^{m \times m} \) are rational, \( u(\overline{z}) = u(z) \), and \( v(z) = v(z)^T \).

§5. Entire matrix-valued functions of class \( \ell^{p \times p} \Pi(\mathbb{C}_+) \)

Instead of the unit disk \( D \), we can consider the open upper half-plane

\[ \mathbb{C}_+ = \{z : \text{Im } z > 0\} \]

Then a class \( \mathcal{X} \) of matrix-valued functions on \( D \) turns into the corresponding class \( \mathcal{X}(\mathbb{C}_+) \) of functions on \( \mathbb{C}_+ \), with the exception of the Hardy space \( H_2^{p \times q} \), which will be discussed below.

The class \( \ell^{p \times p}(\mathbb{C}_+) \) consists of all matrix-valued functions \( c(z) \) of order \( p \) holomorphic in \( \mathbb{C}_+ \) and such that \( \text{Re } c(z) \geq 0 \) in \( \mathbb{C}_+ \). Its subclass \( \ell^{p \times p} \Pi(\mathbb{C}_+) \) consists of all \( c(z) \) belonging to \( \ell^{p \times p}(\mathbb{C}_+) \) and having a pseudocontinuation \( c_-(z) \) of bounded Nevanlinna characteristic in the lower half-plane \( \mathbb{C}_- = \{z : \text{Im } z < 0\} \):

\[ c(\mu) = \lim_{\nu \downarrow 0} c(\mu + i\nu) = \lim_{\nu \uparrow 0} c_-(\mu - i\nu) \quad \text{for a.e. } \mu \in \mathbb{R}, \]

where \( c_-(z) = h_2(z)^{-1}h_1(z) \), and \( h_1 \) is a matrix-valued function of order \( p \) holomorphic and bounded in \( \mathbb{C}_- \), and \( h_2 \) is a scalar bounded function holomorphic in \( \mathbb{C}_- \). For \( c(z) \in \ell^{p \times p}(\mathbb{C}_+) \), the condition \( (1 + \mu^2)^{-1} \text{Re } c(\mu) \in L_1^{p \times p} \Pi(\mathbb{C}_+) \) is fulfilled, and the rank \( m_c \) of \( c(z) \in \ell^{p \times p} \Pi(\mathbb{C}_+) \), i.e., \( \text{rank } \text{Re } c(\mu) \), is constant a.e. on \( \mathbb{R} \). If \( m_c > 0 \), then the factorization problems on \( \mathbb{R} \) of the types \( \text{(3.10)} \) and \( \text{(3.12)} \) are solvable in the class of all \( g \) and \( \omega \) such that \( (z + i)^{-1}g(z) \in H_2^{m \times m}(\mathbb{C}_+) \) and \( (z + i)^{-1}\omega(z) \in H_2^{p \times m}(\mathbb{C}_+) \), where \( H_2^{k \times l}(\mathbb{C}_+) \) is the Hardy class of \( (k \times l) \)-matrix-valued functions \( h(z) \) that are holomorphic in \( \mathbb{C}_+ \) and satisfy

\[ \sup_{\nu > 0} \int_{-\infty}^{\infty} \text{trace} \{h(\mu + i\nu)^*h(\mu + i\nu)\} d\mu < \infty. \]
A solution \( g = \varphi \) of the factorization problem of the form (3.10) is outer if and only if the closed linear span of the vector-valued functions \( e^{zt}(z + i)^{-1}\varphi(z)\xi \) with \( t > 0 \) and \( \xi \in \mathbb{C}^p \) is the entire space \( H^p_{\infty}(\mathbb{C}^+). \)

A matrix-valued function \( \psi(z) \) is \(*\)-outer in \( \mathbb{C}^+ \) if \( \psi(z)^T \) is outer in \( \mathbb{C}^+ \).

All results of the preceding sections have their analogs for matrix-valued functions on the upper half-plane. They can easily be deduced from the corresponding results in the unit disk by the change of variables \( \lambda = i\frac{1-z}{1+z} \), which transfers \( D \) onto \( \mathbb{C}^+ \).

Instead of real matrix-valued functions, we consider functions \( f(z) \) defined on \( \mathbb{C}^+ \) and possessing the property

\[
\overline{f(-\bar{z})} = f(z).
\]

These functions will be called \( I\)-real, because this is the class of function invariant under the operator \( I \) mapping \( f(z) \) to \( f(-\bar{z}) \).

We restrict ourselves to the case where \( c \in \ell^{p\times p}\Pi(\mathbb{C}^+) \) is an entire function.

**Lemma 10.** If \( f \in N^{p\times q}(\mathbb{C}^+) \) is an entire matrix-valued function, then its minimal denominators \( \{b_1, b_2\} \) are pairs of entire functions.

See [3].

By theorems of M. G. Krein and of Rosenblum–Rovnyak (see [12]), if \( f(\mu) \geq 0 \) and \( f(z) \) is an entire \((p \times p)\)-matrix-valued function of bounded Nevanlinna characteristic in \( \mathbb{C}^+ \) and \( \mathbb{C}^- \), i.e., \( f \in \Pi^{p\times p}(\mathbb{C}^+) \), then an outer solution \( g = \varphi \) of the factorization problem of the type (3.10) is an entire matrix-valued function of class \( \Pi^{m\times p}(\mathbb{C}^+) \). Therefore, \( \varphi^{-}(z) \) is an entire matrix-valued function of class \( \Pi^{p\times m}(\mathbb{C}^+) \). Hence, by Lemma 10, if \( \{I_m, b\} \) is its right minimal denominator, then the function \( b \in S^{m\times m}(\mathbb{C}^+) \) is entire, whence

\[
(5.1)
\]

where \( \omega(\mu) \) is a boundary value of an entire matrix-valued function \( \omega(z) \in N^{p\times m}(\mathbb{C}^+) \).

Put

\[
\alpha(z) = b(z), \quad \beta(z) = \varphi(z), \quad \gamma(z) = \omega(z), \quad \delta(z) = c(z),
\]

\[
\theta(z) = \begin{bmatrix}
\alpha(z) & \beta(z) & 0 \\
\gamma(z) & \delta(z) & I_p \\
0 & I_p & 0
\end{bmatrix}.
\]

Clearly, the matrix-valued function \( \theta(z) \) constructed in this way is an entire, optimal, and \( J_{p,m}\)-inner dilation of \( c(z) \). Now, we show that it is minimal.

Relation (5.1) implies that

\[
(5.2)
\]

for some \( d \in S^{m\times m}_\infty(\mathbb{C}^+) \). Consequently, \( \varphi^{-}(z)b(z) = \psi_1(z)d(z) \), whence \( \alpha(z) = b(z) = s_c(z)d(z) \), i.e., \( \{I_m, d\} \in \text{Den}^r(s_c) \). We prove that this right denominator of \( s_c \) is minimal. Let \( d(z) = w_1(z)w_2(z) \), where \( w_1, w_2 \in S^{m\times m}_\infty(\mathbb{C}^+) \) and \( \{I_m, w_1(z)\} \in \text{Den}^r(s_c) \). Then \( b(z) = [s_c(z)w_1(z)]w_2(z) \), so that \( w_2(z) \) is a right divisor of \( b(z) \). Being divisors of the entire matrix-valued function \( b \), the matrix-valued functions \( s_c w_1 \) and \( w_2 \) are entire.

We have \( \varphi^{-}(z)b(z) = \psi(z)[s_c(z)w_1(z)]w_2(z) \), whence \( \varphi^{-}(z)[s_c(z)w_1(z)] = \psi(z)w_1(z) \). This implies that \( \{I_m, s_c w_1^z\} \) is a right denominator of \( \varphi^{-}(z) \). Since \( \{I_m, b\} \) is the minimal right denominator of \( \varphi^{-}(z) \), we have \( w_2(z) = \text{const} \); i.e., \( \{I_m, d\} \) is the minimal right denominator of \( s_c(z) \). Therefore, by Theorem 3, the optimal \( J_{p,m}\)-inner dilation \( \theta(z) \) of \( c(z) \) is minimal.

Similar arguments apply to a \(*\)-outer solution of the factorization problem of type (3.10) on \( \mathbb{R} \). Thus, the following theorem is true.
Theorem 9. Let \( c(z) \in \ell^{p \times p}(\mathbb{C}_+) \) be an entire matrix-valued function with \( m = m_\ast \). Then any optimal minimal \( J_{p,m} \)-inner dilation of \( c(z) \) in \( \mathbb{C}_+ \) is entire, and any \( \ast \)-optimal minimal \( J_{p,m} \)-inner dilation of \( c(z) \) is entire. Moreover,

1) any entire \( I \)-real matrix-valued function \( c(z) \) admits an entire, \( I \)-real, minimal, and optimal \( J_{p,m} \)-inner dilation, as well as an entire, \( I \)-real, minimal, and \( \ast \)-optimal \( J_{p,m} \)-inner dilation;

2) if \( c(z) \in \ell^{p \times p}(\mathbb{C}_+) \) is an entire symmetric matrix-valued function, then it admits an entire, minimal, and symmetric \( J_{p,m} \)-inner dilation;

3) any entire, \( I \)-real, and symmetric matrix-valued function \( c(z) \) admits an entire, \( I \)-real, minimal, and symmetric \( J_{p,m} \)-inner dilation.

References


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