

LINEAR AND NONLINEAR EXTENSIONS OF LIPSCHITZ FUNCTIONS FROM SUBSETS OF METRIC SPACES

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ABSTRACT. A relationship is established between the linear and nonlinear extension constants for Lipschitz functions defined on subsets of metric spaces. Proofs of several results announced in our earlier paper are presented.

§1. INTRODUCTION

Let (M, d) be a metric space with the underlying set M and the metric d . In what follows we suppress the symbol d in this notation and simply refer to M as a metric space. Let B be a Banach space with norm $\|\cdot\|$. The space $\text{Lip}(M, B)$ of Banach-valued Lipschitz functions $f : M \rightarrow B$ is determined by the seminorm

$$|f|_{\text{Lip}(M, B)} := \sup_{m \neq m'} \frac{\|f(m) - f(m')\|}{d(m, m')}.$$

In case $B = \mathbb{R}$, we denote the corresponding space by $\text{Lip}(M)$. Regarding a subset $S \subset M$ as the metric space equipped with the induced metric, we introduce the family of spaces $\text{Lip}(S, B)$.

Definition 1.1. A subset $S \subset M$ admits a *simultaneous Lipschitz extension* if there exists a linear bounded operator $T : \text{Lip}(S) \rightarrow \text{Lip}(M)$ such that

$$Tf|_S = f, \quad f \in \text{Lip}(S).$$

The set of all such operators is denoted by $\text{Ext}(S, M)$, and the optimal extension constant is given by

$$\lambda(S, M) := \inf\{\|T\| : T \in \text{Ext}(S, M)\}.$$

(We assume that $\lambda(S, M) = \infty$ if $\text{Ext}(S, M) = \emptyset$.) Finally, we define the *global linear Lipschitz extension constant*:

$$\lambda(M) := \sup\{\lambda(S, M) : S \subset M\}.$$

We say that M satisfies the *linear Lipschitz extension property* if $\lambda(M) < \infty$.

In the same way we define the *nonlinear Lipschitz extension constant* $\nu(S, M, B)$ to be the infimum of the constants C such that every $f \in \text{Lip}(S, B)$ admits an extension to a function $\tilde{f} \in \text{Lip}(M, B)$ satisfying $|\tilde{f}|_{\text{Lip}(M, B)} \leq C|f|_{\text{Lip}(S, B)}$. Then we define

$$\nu(M, B) := \sup\{\nu(S, M, B) : S \subset M\}.$$

We are ready to present the main result of the paper. For its formulation we let \mathcal{B}_{fin} denote the category of all finite-dimensional Banach spaces. Also, we set

$$\nu(M) := \sup\{\nu(M, B) : B \in \mathcal{B}_{\text{fin}}\}.$$

2000 *Mathematics Subject Classification.* Primary 54E40.

Key words and phrases. Lipschitz function, linear extension operator, Banach space.

Supported in part by NSERC.

Theorem 1.2. *The following identity is true:*

$$\lambda(M) = \nu(M).$$

Let us formulate several consequences of this result.

A convex subset K of a Banach space B containing $0 \in B$ is said to be *generating* if the union of all its dilations λK , $\lambda \geq 1$, is dense in B .

Corollary 1.3. $\lambda(K) < \infty$ for a generating set $K \subset B$ if and only if $\dim B < \infty$.

In particular, $\lambda(B) = \infty$ if $\dim B = \infty$. This naturally leads to the following problem.

Problem 1.4. Find the asymptotics for $\lambda(l_p^n)$ as $n \rightarrow \infty$.

We recall that, for any finite-dimensional Banach space B , we have

$$\lambda(B) \leq c \dim B,$$

where c is a numerical constant bounded by about 4000 in [JLSch] and by 24 in [BB2]. The method of the latter paper also gives the following estimates for $p = 1, \infty$ and some numerical constant $c > 1/4$:

$$c\sqrt{n} \leq \lambda(l_p^n) \leq 24n, \quad n \in \mathbb{N}.$$

As a consequence of our Theorem 1.2 and [LN, Lemma 1.13] we shall obtain a lower estimate for the case where $p = 2$. Together with Theorem 1.11 of the above cited paper, this leads to the following statement.

Corollary 1.5. *For some numerical constants $0 < c_1 < c_2$ we have*

$$c_1 n^{1/8} \leq \lambda(l_2^n) \leq c_2 n^{1/2}.$$

In Theorem 2.18 of [BB2] we constructed a metric graph G of bounded geometry such that $\text{Ext}(S, G) = \emptyset$ for some subspace $S \subset G$; in particular, $\lambda(G) = \infty$. Our next result makes it possible to construct one-dimensional metric spaces with the latter property that “almost coincide” with the real line (i.e., \mathbb{R} equipped with the standard metric). To formulate the result we recall several definitions.

A map $f : (M_0, d_0) \rightarrow (M_1, d_1)$ is called a *C-isometric embedding*, $C > 0$, if

$$C^{-1}d_0(m', m'') \leq d_1(f(m'), f(m'')) \leq Cd_0(m', m'')$$

for all $m', m'' \in M_0$. We also call such an f a *quasi-isometric embedding* if the constant C is inessential.

Next, if for every $m \in M_0$ there is an open ball $B \subset M_0$ centered at m such that the above inequality is fulfilled for some $C = C(B) > 0$ and all $m', m'' \in B$, then the map f is called a *locally quasi-isometric embedding*.

Finally, a map $f : (M_0, d_0) \rightarrow (M_1, d_1)$ is called an *L-rough similarity*, $L > 0$, if

$$d_0(m', m'') - L \leq d_1(f(m'), f(m'')) \leq d_0(m', m'') + L$$

for all $m', m'' \in M_0$.

Corollary 1.6. *Let (M, d) be a complete separable length space.¹ Then for every $\epsilon > 0$ there is a metric space (M_ϵ, d_ϵ) with the following properties:*

- (a) $\lambda(M) \leq \lambda(M_\epsilon)$;
- (b) M_ϵ is homeomorphic and locally quasi-isometric to the real line. Moreover, there is a closed subset $F \subset \mathbb{R}$ of measure ϵ such that $\mathbb{R} \setminus F$ admits an isometric embedding in M_ϵ . If, in addition, $\text{diam } M < \infty$, then M_ϵ is ϵ -roughly similar to the real line;

¹I.e., $d(m, m') = \inf_\gamma l(\gamma)$, where the infimum is taken over all curves in M joining m and m' , and $l(\gamma)$ stands for the length of γ .

(c) M_ϵ is a proper metric space (i.e., every closed ball in M_ϵ is compact).

Though $\dim M_\epsilon = 1$, this space is metrically as large as the space M (which may be of dimension infinity). In a sense, M_ϵ may be viewed as a kind of Peano type curve. The following assertion confirms this claim.

Corollary 1.7. *Assume that M_ϵ admits a quasi-isometric embedding into a proper metric space \widetilde{M} with a cocompact action of the group of isometries.² Then M also admits a quasi-isometric embedding in \widetilde{M} .*

In particular, M_ϵ cannot be embedded in any such space if M is a closed ball of an infinite-dimensional Banach space. Also, in this case there is a subspace $S \subset M_\epsilon$ such that $\text{Ext}(S, M_\epsilon) = \emptyset$.

§2. FINITENESS PROPERTY

In the proof of Theorem 1.2 we use some results on the linear Lipschitz extension constants, established in [BB2]; we formulate these results in a form suitable for our consideration.

Definition 2.1. Let $\{(M_i, d_i)\}_{i \in \mathbb{N}}$ be a sequence of metric spaces. This sequence is said to be γ -convergent to a metric space (M, d) if for every compact subspace K of M there is a sequence $\{K_i \subset M_i : i \in I \subset \mathbb{N}\}$ of compact subspaces that converges to K in the Gromov–Hausdorff metric.

For the reader’s convenience, we recall the definition of the Gromov–Hausdorff distance $d_{\mathcal{GH}}(M_0, M_1)$ between metric spaces (M_i, d_i) , $i = 0, 1$ (see, e.g., [BH, pp. 70–75] for more details).

Let d be a metric on the disjoint union $M_0 \sqcup M_1$ whose restriction to M_i coincides with d_i , $i = 0, 1$. Then each M_i is a subspace of the space $(M_0 \sqcup M_1, d)$, and the Hausdorff distance

$$d_{\mathcal{H}}(M_0, M_1) := \sup\{\max_{i=0,1} d(m_i, M_{1-i}) : m_i \in M_i\}$$

is well defined. Then the Gromov–Hausdorff metric is given by

$$d_{\mathcal{GH}}(M_0, M_1) := \inf_d d_{\mathcal{H}}(M_0, M_1),$$

where d runs over all such metrics on $M_0 \sqcup M_1$.

Our first formulation is based on [BB2, Theorem 2.2 and Remark 6.1].

Theorem A. (a) *Let $\{(M_i, d_i)\}_{i \in \mathbb{N}}$ be a sequence of metric spaces γ -convergent to a metric space (M, d) . Then*

$$\lambda(M) \leq \limsup_{i \rightarrow \infty} \lambda(M_i).$$

(b) *Let S be a subspace of (M, d) . Then*

$$\lambda(S, M) = \sup_F \lambda(F, M),$$

where F runs over all finite subsets of S .

Now, let S be a subset of M satisfying the following property.

For some bi-Lipschitz map $\delta : M \rightarrow M$,

- (A) $S \subset \delta(S)$;
- (B) $\bigcup_{j=0}^\infty \delta^j(S)$ is dense in M ;

²This means that for some compact set $K \subset \widetilde{M}$ and every point $m \in \widetilde{M}$ there is an isometry I of \widetilde{M} such that $I(m) \in K$.

(C) for the linear operator $\Delta : \text{Lip}(M) \rightarrow \text{Lip}(M)$ given by $(\Delta f)(m) := f(\delta(m))$, $m \in M$, we have

$$\|\Delta\| \cdot \|\Delta^{-1}\| = 1.$$

Then the following is true (see Corollary 2.5 in [BB2]).

Theorem B. *If $S \subset M$ satisfies conditions (A)–(C), then*

$$\lambda(S) = \lambda(M).$$

§3. AUXILIARY RESULTS

The proof of Theorem 1.2 is based on several auxiliary results from [BB2, §5]; see also [K, §3]. We present them below, referring to the cited papers for the proofs. In their formulations, we use the notion of a *pointed metric space*, that is, a metric space M with a distinguished point m_0 . A (*pointed*) *metric subspace* of such a space is a subset of M containing m_0 and endowed with the induced metric. We denote by $\text{Lip}_0(M, B)$ the subspace of $\text{Lip}(M, B)$ distinguished by the condition $f(m_0) = 0$, $f \in \text{Lip}(M, B)$. The notation $\text{Lip}_0(S, B)$ for a metric subspace S of M has a similar meaning. Note that all these spaces are Banach.

Now, let $K(M)$ be the *canonical predual* space to $\text{Lip}_0(M)$ (the Kantorovich–Rubinshtein space). Recall that $K(M)$ is the closed linear span of the set of the point evaluations

$$\delta_m(f) := f(m), \quad f \in \text{Lip}_0(M),$$

in the dual space $\text{Lip}_0(M)^*$ (see, e.g., [BB2, Appendix]).

Lemma 3.1. *Let (M, d, m_0) be a pointed metric space. Then:*

- (1) *there is an isometric embedding i_M of M in $\text{Lip}_0(M)^*$ such that $i_M(m_0) = 0$;*
- (2) *the map $m \mapsto \delta_m$ is an isometric embedding of M in $K(M)$.*

This lemma allows us to regard M as a metric subspace of either $\text{Lip}_0(M)^*$ or $K(M)$.

Lemma 3.2. *Suppose $E \in \text{Ext}(S, M)$ and $\text{card } S < \infty$. There exists a linear bounded operator $E_0 : K(M) \rightarrow K(S)$ such that*

- (1) *E_0 is predual to E ; in particular, $\|E_0\| = \|E\|$;*
- (2) *E_0 is a projection onto $K(S)$.*

Lemma 3.3. *If $f \in \text{Lip}_0(M, B)$, then there is a linear bounded operator $T_f : K(M) \rightarrow B$ such that*

- (1) *$T_f|_M = f$ (here M is identified with its isometric copy in $K(M)$);*
- (2) *$\|T_f\| = \|f\|_{\text{Lip}(M, B)}$.*

§4. PROOFS

Proof of Theorem 1.2. First, we prove

Claim I.

$$\nu(M) \leq \lambda(M).$$

The proof is based on the following two inequalities, the first of which asserts that

$$(4.1) \quad \nu(F, M, B) \leq \lambda(F, M)$$

provided $\text{card } F < \infty$.

To formulate the second inequality, we define a “finite-dimensional” analog of the constant $\nu(S, M, B)$ by the formula

$$(4.2) \quad \nu_{\text{fin}}(S, M, B) := \sup\{\nu(F, M, B) : F \subset S, \text{card } F < \infty\}.$$

Then, for $\dim B < \infty$, we have

$$(4.3) \quad \nu(S, M, B) \leq \nu_{\text{fin}}(S, M, B).$$

We shall prove these inequalities later; now we derive Claim I from them.

In (4.1), we choose a Banach space B of class \mathcal{B}_{fin} and take the supremum over all $F \subset S$ with $\text{card } F < \infty$. Using (4.2), we get

$$\nu_{\text{fin}}(S, M, B) \leq \sup\{\lambda(F, M) : F \subset S, \text{ card } F < \infty\}.$$

By Theorem A, the right-hand side equals $\lambda(S, M)$. Together with (4.3), this leads to the inequality

$$\nu(S, M, B) \leq \lambda(S, M).$$

Taking the supremum over all $B \in \mathcal{B}_{\text{fin}}$ and $S \subset M$, we prove Claim I.

Now we establish inequalities (4.1) and (4.3). In their proofs, M is a pointed metric space with a distinguished point m_0 , and all subsets of M are assumed to be pointed subspaces of M (with the distinguished point m_0).

To prove (4.1), pick $F \subset M$ with $\text{card } F < \infty$. For $f \in \text{Lip}_0(F, B)$, let T_f be the linear operator of Lemma 3.3. Hence, $T_f : K(F) \rightarrow B$ and

$$\|T_f\| = L(f) := |f|_{\text{Lip}(F, B)}.$$

Given an extension operator $E \in \text{Ext}(F, M)$, we find the corresponding predual operator $E_0 : K(M) \rightarrow K(F)$ of Lemma 3.2 and put

$$E_f := T_f E_0.$$

Then $E_f : K(M) \rightarrow B$, and the norm of E_f is at most $\|E_0\| \cdot \|T_f\| = L(f)\|E\|$. By Lemma 3.3, $E_f|_M : M \rightarrow B$ is a Lipschitz extension of f from F to the whole of M . Moreover,

$$L(E_f|_M) \leq \|E_f\| \leq L(f)\|E\|.$$

By the definition of the nonlinear Lipschitz extension constant, this implies

$$\nu(F, M, B) \leq \|E\|.$$

Passing to the infimum over all $E \in \text{Ext}(F, M)$, we get (4.1).

To prove (4.3), for a given $\epsilon > 0$ we find a function $f \in \text{Lip}_0(S, B)$ with $L(f) = 1$ that admits an ϵ -optimal extension to M ; this means that there exists $\tilde{f} \in \text{Lip}_0(M, B)$ such that

$$(4.4) \quad -2\epsilon + \nu(S, M, B) \leq -\epsilon + L(\tilde{f}) \leq \inf\{L(g) : g \in \text{Lip}_0(M, B), g|_S = f\}.$$

Let \mathcal{F} be the family of all finite subsets of S , partially ordered by inclusion. For every $F \in \mathcal{F}$, we define g_F to be an ϵ -optimal extension of the trace $f|_F$ to M , that is,

$$(4.5) \quad -\epsilon + L(g_F) \leq \inf\{L(g) : g \in \text{Lip}_0(M, B), g|_F = f|_F\}.$$

By the definition of \tilde{f} , for all $F \in \mathcal{F}$ we get

$$L(g_F) \leq L(\tilde{f}) + \epsilon.$$

Hence, the family $\{g_F\}_{F \in \mathcal{F}}$ is equicontinuous. It is also uniformly bounded on every ball in M , because $g_F(m_0) = f(m_0) = 0$ for all F . Finally, every g_F maps M into the finite-dimensional Banach space B . Therefore, a standard compactness argument involving the Tychonov theorem (see, e.g., [BB2, §5]), allows us to find a subnet of the net $\{g_F\}_{F \in \mathcal{F}}$ that converges pointwise to a Lipschitz function $g : M \rightarrow B$. By the definition of net convergence, we have

$$(4.6) \quad g|_S = f.$$

Next, passing to the limit in (4.5) and using (4.2), we also obtain

$$\begin{aligned} -\epsilon + L(g) &\leq -\epsilon + \sup_{F \in \mathcal{F}} L(g_F) \\ &\leq \sup_{F \in \mathcal{F}} \inf\{L(g) : g \in \text{Lip}_0(M, B), g|_F = f|_F\} \leq \nu_{\text{fin}}(S, M, B). \end{aligned}$$

Since \tilde{f} is an ϵ -optimal extension of f (see (4.4)), from (4.6) we see that

$$L(\tilde{f}) \leq L(g) + \epsilon.$$

Together with the preceding inequality and (4.4), this yields

$$\nu(S, M, B) \leq L(\tilde{f}) + \epsilon \leq \nu_{\text{fin}}(S, M, B) + 3\epsilon.$$

This completes the proofs of (4.3) and Claim I.

Claim II.

$$\lambda(M) \leq \nu(M).$$

We derive this fact from the inequality

$$(4.7) \quad \lambda(F, M) \leq \nu(F, M) := \sup\{\nu(F, M, B) : B \in \mathcal{B}_{\text{fin}}\},$$

valid for $\text{card } F < \infty$.

Taking here the supremum over all such F and applying Theorem A with $S = M$, we get

$$\lambda(M) = \sup_F \lambda(F, M) \leq \sup_F \nu(F, M) \leq \nu(M),$$

as required.

It remains to establish (4.7).

Let $\delta : F \rightarrow K(F)$ be the isometric embedding described in Lemma 3.1. Then $\delta \in \text{Lip}(F, K(F))$, and by the definition of $\nu(F, M)$ for a given $\epsilon > 0$ there is an extension $\tilde{\delta} \in \text{Lip}(M, K(F))$ such that

$$-\epsilon + L(\tilde{\delta}) \leq \nu(F, M)L(\delta) (= \nu(F, M)).$$

Let $f \in \text{Lip}_0(F)$. Since $\text{Lip}_0(F) = K(F)^*$, the function f can be regarded as a linear bounded functional on $K(F)$. We define a linear operator $E : \text{Lip}_0(F) \rightarrow \text{Lip}_0(M)$ by

$$(Ef)(m) := \langle f, \tilde{\delta}(m) \rangle_{K(F)}.$$

Then the norm $\|Ef\|_{\text{Lip}_0(M)}$ is bounded by

$$\|f\|_{\text{Lip}_0(F)} \sup_{m \neq m'} \frac{\|\tilde{\delta}(m) - \tilde{\delta}(m')\|_{K(F)}}{d(m, m')} = L(f)L(\tilde{\delta}) \leq (\nu(F, M) + \epsilon)L(f).$$

Hence,

$$\|E\| \leq \nu(F, M) + \epsilon,$$

and moreover,

$$(Ef)|_F = f,$$

i.e., $E \in \text{Ext}(F, M)$. By the definition of $\lambda(F, M)$, this implies inequality (4.7):

$$\lambda(F, M) \leq \nu(F, M).$$

Finally, from the inequalities of Claims I and II we get the required statement of Theorem 1.2. \square

Proof of Corollary 1.5. The following fact was established in the proof of Lemma 1.13 of the paper [LN]; see Remark 5.3 there.

Let $2 < p < \infty$ and $m \in \mathbb{N}$. There is a finite subset $F \subset l_2^{2m}$ with $\text{card } F \leq m^{2m}$ and a numerical constant $c > 0$ such that

$$\nu(F, l_2^{2m}, l_p^{2m}) \geq c \left(\frac{2m \log 2m}{\log(2m \log 2m)} \right)^{\frac{p-2}{p^2}}.$$

Choosing $p = 4$, we get

$$\nu(F, l_2^{2m}, l_4^{2m}) \geq cm^{1/8}.$$

The definition implies that

$$\sup\{\nu(l_2^{2m}, B) : B \in \mathcal{B}_{\text{fin}}\} \geq \nu(l_2^{2m}, l_4^{2m}) \geq \sup \nu(\Gamma, l_2^{2m}, l_4^{2m}),$$

where Γ runs over all finite point metric subspaces of l_2^{2m} . These two inequalities show that

$$\sup\{\nu(l_2^{2m}, B) : B \in \mathcal{B}_{\text{fin}}\} \geq cm^{1/8}.$$

Finally, by Theorem 1.2, the left-hand side equals $\lambda(l_2^{2m})$, and this yields the required lower bound

$$\lambda(l_2^n) \geq c_1 n^{1/8}, \quad n \in \mathbb{N},$$

with some constant $c_1 > 0$ independent of n . The upper bound

$$\lambda(l_2^n) \leq c_2 n^{1/2}, \quad n \in \mathbb{N},$$

was in fact proved in [LN], since these authors used a linear extension operator to prove the estimate

$$\nu(l_2^n, B) \leq cn^{1/2}, \quad n \in \mathbb{N}$$

(see Theorem 1.11 and part 5 of Theorem 5.1 in [LN]). □

Proof of Corollary 1.3. Take $\delta(x) := 2x$, $x \in B$, and let S be the generating set K of the corollary. We check that the conditions (A)–(C) of Theorem B are fulfilled. Since K is convex and contains 0, we have

$$K \subset \delta(K) := 2K.$$

For the same reason, $\bigcup_{i=1}^\infty \delta^i(K) = \bigcup_{\lambda \geq 1} \lambda K$, and the latter set is dense in B by the definition of a generating set. Finally, the operator $\Delta : \text{Lip}(B) \rightarrow \text{Lip}(B)$ is now defined by $(\Delta f)(x) := f(2x)$, $x \in B$, so that $\|\Delta\| = 2$ and $\|\Delta^{-1}\| = 1/2$. Hence, Theorem B yields

$$(4.8) \quad \lambda(K) = \lambda(B).$$

Let, in particular, U_B be the closed unit ball of the Banach space B . Clearly, this is a generating set; therefore, (4.8) implies

$$(4.9) \quad \lambda(U_B) = \lambda(B).$$

Applying this to the Banach space l_2^n and using the lower bound mentioned in Corollary 1.5, we get

$$(4.10) \quad \lambda(U_{l_2^n}) \geq c_1 n^{1/8}, \quad n \in \mathbb{N}.$$

Now, let B be an infinite-dimensional Banach space. By the Dvoretzky theorem [D], for every sufficiently large n , say $n \geq n_0$, there is an n -dimensional linear subspace $B_n \subset B$ such that the ball $U_B \cap B_n$ is 2-isometric to $U_{l_2^n}$. In other words, there is a bi-Lipschitz map ϕ of $U_{l_2^n}$ onto $U_B \cap B_n$ such that

$$\max\{L(\phi), L(\phi^{-1})\} \leq 2.$$

This immediately implies the inequality

$$\lambda(U_B) \geq \lambda(U_B \cap B_n) \geq \frac{1}{4}\lambda(U_{I_2^n}).$$

Together with (4.10) and (4.9), this shows that, for $n \geq n_0$,

$$\lambda(B) = \lambda(U_B) \geq \frac{1}{4}c_1 n^{1/8} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Consequently,

$$\lambda(B) = \infty \text{ if } \dim B = \infty.$$

In the case where $n := \dim B < \infty$, the classical Whitney theorem [W] implies that $\lambda(B) \leq c(n) < \infty$; see also the above-cited inequality [JLSch]. \square

Proof of Corollary 1.6. Given a complete separable length metric space (M, d) , we choose a monotone increasing sequence of finite subspaces $M_n \subset M$, $n \in \mathbb{N}$, such that $\text{card } M_n = n$, $\bigcup_{n \in \mathbb{N}} M_n$ is dense in M and, moreover, all M_n contain a distinguished point $m^* \in M$.

Let m_1, \dots, m_n be points of M_n enumerated in some way so that $m_1 = m^*$. We also set $m_{n+1} := m^*$. We join m_i and m_{i+1} by a curve γ_i of length $l(\gamma_i) \leq 2d(m_i, m_{i+1})$, $1 \leq i \leq n$. Let f_n denote the concatenation (union) of all γ_i , $1 \leq i \leq n$. Then f_n is a closed curve in M with the endpoint m^* , having length $\sum_{i=1}^n l(\gamma_i)$, and passing through all points of M_n . The reparametrization of f_n by arclength divided by an appropriate positive number turns this map into a Lipschitz map $g_n : [n-1, n-1 + \frac{\epsilon}{2^n}] \rightarrow \text{image } f_n \subset M$ passing through all the points of M_n and such that

$$(4.11) \quad g_n(n-1) = g_n(n-1 + \epsilon/2^n) = m^*.$$

Now, let $(\widehat{M}, \widehat{d})$ be the metric space with the underlying set $M \times \mathbb{R}$ and the metric \widehat{d} given by the formula $\widehat{d}((m, t), (m', t')) := \max\{d(m, m'), |t - t'|\}$. We denote the graph of g_n by $h_n := \{(g_n(t), t) : n-1 \leq t \leq n-1 + \frac{\epsilon}{2^n}\} \subset \widehat{M}$. Using the family $\{h_n\}_{n \in \mathbb{N}}$, we define a locally Lipschitz embedding $h : \mathbb{R} \rightarrow \widehat{M}$ by

$$h(t) := \begin{cases} (m^*, t) & \text{if } t \leq 0, \\ h_n(t) & \text{if } t \in J_n^-, \ n \in \mathbb{N}, \\ (m^*, t) & \text{if } t \in J_n^+, \ n \in \mathbb{N}. \end{cases}$$

Here $J_n^- := [n-1, n-1 + \frac{\epsilon}{2^n}]$ and $J_n^+ := [n-1 + \frac{\epsilon}{2^n}, n]$. By (4.11), this determines a homeomorphism of \mathbb{R} onto the metric subspace $M_\epsilon := h(\mathbb{R})$ of the space \widehat{M} . Clearly, M_ϵ is also locally quasi-isometric to \mathbb{R} (equipped with the standard metric). Moreover, in accordance with the definition, the set $\mathbb{R} \setminus (\bigcup_{n \in \mathbb{N}} J_n^-)$ equipped with the standard metric is isometric under h to a subspace of M_ϵ . If, in addition, $\text{diam } M < \infty$, then we replace the metric d on M by the equivalent metric $\widetilde{d} := \epsilon d / (2 \text{diam } M)$ and then construct the metric space M_ϵ as above for the space (M, \widetilde{d}) . Clearly, this space is ϵ -roughly similar to \mathbb{R} .

This completes the proof of assertion (b) of the corollary.

Now we prove assertion (a). Let T_n be an n -pointed subset of the interval J_n^- such that $h(T_n) = M_n \times T_n$ and $n-1 \in T_n$. Then

$$(4.12) \quad \bigcup_{n \in \mathbb{N}} M_n \times T_n \subset M_\epsilon.$$

We translate $M_n \times T_n$ along \mathbb{R} by $1-n$, and denote the resulting set by S_n . Then $S_n \subset M_n \times [0, \frac{\epsilon}{2^n}] \subset \widehat{M}$. We shall show that the sequence $\{S_n\}_{n \in \mathbb{N}}$ γ -converges to

$M \times \{0\} \subset \widehat{M}$. Then, since S_n is isometric to $M_n \times T_n$, Theorem A and (4.12) yield the desired inequality:

$$\lambda(M) = \lambda(M \times \{0\}) \leq \limsup_{n \rightarrow \infty} \lambda(M_n \times T_n) \leq \lambda(M_\epsilon).$$

To establish the γ -convergence of $\{S_n\}_{n \in \mathbb{N}}$, consider an arbitrary compact set $\widehat{K} \subset M \times \{0\}$; then $\widehat{K} = K \times \{0\}$ for some compact set K of M . Let K_n denote the intersection of the $\epsilon/2^n$ -neighbourhood of K in M with M_n , and let $\widehat{K}_n := (K_n \times [0, \frac{\epsilon}{2^n}]) \cap S_n$. Since $\bigcup_{n \in \mathbb{N}} M_n$ is dense in M and the sequence $\{M_n\}_{n \in \mathbb{N}}$ is monotone increasing, $\widehat{K}_n \neq \emptyset$ for sufficiently large n . Then, for such an n , the Hausdorff distance from \widehat{K} to \widehat{K}_n is at most $\frac{\epsilon}{2^n}$. Hence, the compact sets $\widehat{K}_n \subset S_n$ with sufficiently large n form a sequence convergent in the Hausdorff metric to \widehat{K} . By Definition 2.1, this means that $\{S_n\}_{n \in \mathbb{N}}$ γ -converges to $M \times \{0\}$.

Thus, assertion (a) is also proved.

It remains to check that the space M_ϵ is proper (assertion (c)). Let $\overline{B}_r(m) \subset M_\epsilon$ be the closed ball of M_ϵ centered at m and of radius r . Let $p_{\mathbb{R}} : \widehat{M} \rightarrow \mathbb{R}$ denote the canonical projection onto the second component of $\widehat{M} = M \times \mathbb{R}$. By our definition of M_ϵ , this ball is contained in the image $h([p_{\mathbb{R}}(m) - r, p_{\mathbb{R}}(m) + r])$ of the compact interval $[p_{\mathbb{R}}(m) - r, p_{\mathbb{R}}(m) + r]$ of \mathbb{R} . Since h is continuous, this image is compact. Therefore, the closed subset $\overline{B}_r(m)$ of this image is also compact.

This proves assertion (c). □

Proof of Corollary 1.7. Let $f : M_\epsilon \rightarrow \widetilde{M}$ be a C -isometric embedding in a proper metric space \widetilde{M} with the cocompact action of its group of isometries. Let $\{S_n\}_{n \in \mathbb{N}} \subset \widetilde{M}$ be the sequence defined above. By the construction of S_n , the map f determines a C -isometric embedding $f_n : S_n \rightarrow \widetilde{M}$. We choose an isometry I_n of \widetilde{M} such that, for some fixed compact set $K \subset \widetilde{M}$, the point $(m^*, 0) \in S_n$ goes to a point of K under the map $\widehat{f}_n := I_n \circ f_n$. For $k \geq n$, let S_{nk} denote a subset of S_k whose canonical projection onto the space M coincides with M_n . Since $M_n \subset M_k$ for $k \geq n$, such a set exists and is unique. Since the point $(m^*, 0)$ belongs to every S_{nk} with $k \geq n$, the point $\widehat{f}_n(m^*, 0)$ belongs to K . Therefore, $\widehat{f}_n(S_{nk})$ lies in the λ -neighbourhood of K with $\lambda := C \max\{\text{diam } M_n, \frac{\epsilon}{2^n}\}$. Since the space \widetilde{M} is proper, the closure of this neighbourhood, denoted by K_n , is compact. Now we identify S_{nk} for $k \geq n$ with M_n by the canonical projection $p_M : \widetilde{M} \rightarrow M$ and denote by $\phi_{nk} : M_n \rightarrow K_n$ the map $\widehat{f}_k \circ p_M^{-1}$, where p_M^{-1} is the map inverse to $p_M|_{S_{nk}}$. Since the sequence of finite sets $\{S_{nk}\}_{k \geq n}$ converges to $M_n \times \{0\}$ in the Hausdorff metric of the space \widetilde{M} , the map ϕ_{nk} is a C_k -isometric embedding and C_k tends to C as $k \rightarrow \infty$. Also, we clearly have

$$(4.13) \quad \phi_{mk}|_{M_n} = \phi_{nk} \quad \text{for all } k \geq m \geq n.$$

Since the target space of each ϕ_{nk} is the compact set K_n , we can use (4.13) and the Cantor diagonal process to find a subsequence $\{\phi_{nk(n)}\}_{n \in \mathbb{N}}$ that converges pointwise on the dense subset $\bigcup_{n \in \mathbb{N}} M_n$ of M , and its limit, say \widehat{f} , is a C -isometric embedding in \widetilde{M} . Extending \widehat{f} to the entire M by continuity, we obtain the desired quasi-isometric embedding of M in \widetilde{M} .

The remaining statements of the corollary follow straightforwardly from its first part and [BB2, Theorem 2.8]. □

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Received 20/APR/2006
Originally published in English