

ON FINITE SIMPLY REDUCIBLE GROUPS

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Dedicated to the centenary of D. K. Faddeev's birth

ABSTRACT. A finite group G is said to be *simply reducible* (SR -group) if it has the following two properties: 1) each element of G is conjugate to its inverse; 2) the tensor product of every two irreducible representations is decomposed as a sum of irreducible representations of G with multiplicities not exceeding 1. It is proved that a finite SR -group is solvable if it has no composition factors isomorphic to the alternating groups A_5 or A_6 .

§1. INTRODUCTION

Definition 1. A finite group G is called an SR -group if it has the following two properties: 1) each element of G is conjugate to its inverse; 2) the tensor product of every two irreducible representations is decomposed as a sum of irreducible representations of G with multiplicities not exceeding 1.

The class of SR -groups was introduced by E. Wigner [6] and turned out to be fairly nontrivial. We note that SR -groups are of some interest in the theory of relation schemes (see [1, Chapter 2]). As pointed out in the book [4], there is no complete description of all SR -groups. We mention the important paper [5], in which some properties of SR -groups were investigated. Results on finite nonsupersolvable SR -groups were obtained in [3]. S. P. Strunkov (private communication) suggested that the SR -groups must be solvable. The main result of the present paper is a theorem giving a positive answer to the Strunkov conjecture under the additional assumption that the groups in question have no composition factors isomorphic to A_5 or to A_6 .

In fact, we prove this statement for groups satisfying a weaker condition.

Definition 2. A finite group G is called an ASR -group if it satisfies the following condition:

1') the tensor square of any irreducible representation of this group is decomposed as a sum of irreducible representations of G with multiplicities not exceeding 1.

So, the definition of an SR -group is relaxed at two points: condition 1) (the groups satisfying this condition are said to be *real*) is lifted, and condition 2) is imposed only on the tensor squares of representations. Obviously, every SR -group is an ASR -group. The converse is false, because every Abelian group is an ASR -group whereas the Abelian SR -groups are elementary Abelian 2-groups. Examples of non-Abelian real ASR -groups are presented in §8.

The main result of the paper is as follows.

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Theorem 1. *Any finite ASR-group without composition factors isomorphic to A_5 or to A_6 is solvable.*

It seems plausible that all finite ASR-groups are solvable. Theorem 1 is a consequence of the following more general statement.

Theorem 2. *Let G be a nonsolvable ASR-group of the smallest order. Then G has a unique minimal normal subgroup N that is the direct product of $m > 1$ groups isomorphic to A_5 or to A_6 . The subgroup M of G consisting of the elements that normalize every minimal normal subgroup of N is normal in G , and the quotient group G/M is a solvable ASR-subgroup of the symmetric group S_m and acts transitively on the set of minimal normal subgroups of N .*

In particular, any nonsolvable finite ASR-group, if it exists, must be of order divisible by 3600.

One more result of this paper should be mentioned. Namely, Corollary 1, proved in §2, says that $k(G)^3 < |G|$ for almost every simple non-Abelian group G . The only exceptions are provided by groups isomorphic to $L_2(q)$ for even q . Lemma 8 and Corollary 2 imply that every ASR-group is nonsimple.

The outline of the paper is as follows. In §2, we present some auxiliary results needed in the sequel. Also, that section contains information about the conjugacy classes of elements in simple non-Abelian groups. The proof of Corollary 1 (in which the greater part of the calculation is related to the most difficult case of the groups $L_n^\varepsilon(q)$, $B_n(q)$, and $C_n(q)$) is used to present this information. In §3, we consider some general properties concerning the structure of ASR-groups and a minimal counterexample to Theorem 2. In particular, we prove that a minimal counterexample to Theorem 2 has a unique minimal normal subgroup N with solvable quotient group that is a direct product of isomorphic simple non-Abelian groups with certain restrictions on the degrees of irreducible representations. In §4, we use the results of [18] to exclude almost all simple groups that can arise as composition factors of N . The remaining set of groups consists of the families $L_3(q)$ and $U_3(q)$ for small values of q and of the infinite family $L_2(q)$. In §5, we present necessary information on the correspondence between the irreducible characters of a group G having a normal subgroup, the characters of this subgroup, and the characters induced from this subgroup (Clifford theory). Using this technique, we reduce the problem to the case where the composition factors of the group N are isomorphic to one of the groups of the family $L_2(q)$. In §6, we discuss the characters of the group $PGL_2(q)$ and the action of its automorphism group on the set of characters of degree $q + 1$. Finally, in §7, we prove Theorem 2, which implies Theorem 1. In §8, we give examples of groups illustrating the notion of an ASR-group and its relationship with the notion of an SR-group.

We adopt the standard notation, which is explained when required. All groups under consideration are finite. The terminology concerning group representations and characters follows [12]. We consider only ordinary representations, i.e., representations over the field of complex numbers. In particular, an irreducible character is a character of an irreducible representation over the field of complex numbers. All necessary information concerning finite simple groups, their automorphism groups, and the corresponding notation can be found in [17].

For the reader's convenience, we recall some notation. We denote by C_n a cyclic group of order n and by Q_8 the quaternion group. The letter p will denote a prime number. For a group X , we denote by $|X|_p$ the largest power of p dividing the order of X (the order of its Sylow p -subgroup).

For brevity, we use the unified notation for the groups $L_n(q)$ and $U_n(q)$ (respectively, for $SL_n(q)$ and $SU_n(q)$). Namely, we write $L_n^\varepsilon(q) = L_n(q)$ if the variable ε (sign) takes

the value + (that is, $SL_n^+(q) = SL_n(q)$) and $L_n^\varepsilon(q) = U_n(q)$ if ε takes the value - (that is, $SL_n^-(q) = SU_n(q)$). In the case of groups of Lie type, the number q is a power of a prime p .

Let n be a positive integer. We denote by $p(n)$ the number of (unordered) partitions of n , i.e., the number of integer solutions of the equation $n = x_1 + x_2 + \dots + x_n$, where $x_1 \geq x_2 \geq \dots \geq x_n \geq 0$. The greatest common divisor of two integers m and n is denoted by (m, n) . By $\text{Irr}(G)$ we denote the set of irreducible characters of G . If G is a simple group of Lie type, then its Steinberg character (see [9]) is denoted by St_G . The number of conjugacy classes of elements in G is denoted by $k(G)$. The character of G induced from a character ζ of a subgroup H of G is denoted by ζ^G . The restriction of a character χ of G to a subgroup H of G is denoted by $\chi|_H$. The inner product of characters χ and Θ of G is denoted by $[\chi, \Theta]$. We use an index at the inner product if we want to specify the group in which the inner product is calculated. For example, the Frobenius famous reciprocity law is stated as follows. For all characters χ of G and Θ of a subgroup H of G we have:

$$[\chi, \Theta^G]_G = [\chi|_H, \Theta]_H.$$

If the group in which the calculation is done is clear from the context, we drop the index at the inner product.

In Subsection 4.1 we shall need the formula for the dimension of irreducible representations of the groups A_n and S_n (the *hook formula*) corresponding to a given partition of n and the related Young diagram. Relevant information can be found in the book [2]; see Theorem 20.1 therein.

§2. AUXILIARY RESULTS

The following results were proved in [18] (see Theorem 1, Lemma 2.1, and Theorem 2).

Lemma 1. *Let G be a finite non-Abelian simple group of Lie type over the field $F = GF(q)$. Let l be the untwisted rank of G . Then*

$$k(G) \leq (6q)^l.$$

If $k(G) \geq |G|_p$, where p is the characteristic of F , then G is isomorphic to $L_2(q)$ for an even q with $k(G) = q + 1$ or to $L_2(5)$ (in the latter case, $k(G) = 5$).

Lemma 2. (i) *Let G be a solvable subgroup of S_n ($n \geq 3$). Then $k(G) \leq \sqrt{3}^{(n-1)}$, and the order of G does not exceed $24^{(n-1)/3}$.*

(ii) *Let G be a subgroup of S_n . Then $k(G) \leq 2^{n-1}$.*

Proof. (i) The statement concerning the number of conjugacy classes of a solvable subgroup of S_n is part of Theorem 2.2 in [16]. By Dixon's theorem [11], the largest order of a solvable subgroup of S_n does not exceed $24^{(n-1)/3}$.

(ii) This statement coincides with Theorem 2 in [18]. □

In some cases, we need sharper estimates for the groups $L_n^\varepsilon(q)$.

Lemma 3. *Let G be a finite non-Abelian simple group isomorphic to $L_n^\varepsilon(q)$, where $n \leq 6$. Then $k(G) \leq 2q^{n-1}$, with the exception of $G = U_4(2)$ ($k(G) = 20$) and $G = U_5(2)$ ($k(G) = 47$).*

Proof. The number of conjugacy classes of $L_n^\varepsilon(q)$ was calculated for $n \leq 8$ in [19]. For the groups $L_n(q)$, the corresponding numbers are given explicitly as polynomials in n and q (see Table 4 in [19]). Studying the values of these expressions proves the lemma for the groups $L_n(q)$.

For $U_n(q)$, we have a formula representing $k(U_n(q))$ in terms of the values of the polynomials $\gamma'_n = \gamma'_n(q)$. Therefore, in this case, a preliminary calculation is necessary. The expressions for $k(U_n(q))$ have the following form (formula (6.13) in [19]):

$$k(U_n(q)) = 1/(q + 1, n) \sum_{d_1, d_2} \phi(d_1)\phi_2(d_2)\gamma'_{n/(d_1d_2)},$$

where the summation is taken over all pairs of divisors d_1 and d_2 of $q + 1$ for which the product d_1d_2 divides n . Here, ϕ is the Euler function, and $\phi_2(n) = n^2 \prod (1 - 1/p^2)$ (where p runs through all prime divisors of n) is a generalization of the Euler function. We have $\phi_2(1) = \phi(1) = 1$. The polynomials γ'_n are given explicitly (see Table 5 in [19]). Using these tables and the formula mentioned above, by way of an example we present here the calculation of $k(U_3(q))$ and $k(U_4(q))$.

We have the following values of γ'_i for $i \leq 4$: $\gamma'_1 = 1$, $\gamma'_2 = q + 1$, $\gamma'_3 = q^2 + q + 2$, and $\gamma'_4 = q^3 + q^2 + 3q + 2$. In the case where $(n, q + 1) = 1$, the summation is taken over a unique pair of divisors, $d_1 = d_2 = 1$, and we see that γ'_n is the number of the conjugacy classes of the elements of $U_n(q)$, i.e., the above relation takes the form $k(U_n(q)) = \gamma'_n$. Thus, it remains to consider the cases where $(n, q + 1) \neq 1$.

Let $n = 3$ and $(3, q + 1) = 3$. Then

$$k(U_3(q)) = \frac{1}{3}(\gamma'_3 + \phi(1)\phi_2(3)\gamma'_1 + \phi(3)\gamma'(1)) = \frac{1}{3}(q^2 + q + 12),$$

which gives the value of $k(U_3(q))$ for $(q + 1, 3) = 3$.

Now, let $n = 4$. Here we have more possibilities: $(4, q + 1) = 1$, $(4, q + 1) = 2$, and $(4, q + 1) = 4$. In the first case, we obtain the polynomial γ'_4 . If $(4, q + 1) = 4$, then we have the following pairs $\{d_1, d_2\}$ of divisors of $(4, q + 1) = 4$ over which the summation is taken: $\{1, 1\}$, $\{1, 2\}$, $\{2, 1\}$, $\{2, 2\}$, $\{1, 4\}$, and $\{4, 1\}$. Therefore, for $(4, q + 1) = 4$ we obtain

$$\begin{aligned} k(U_4(q)) &= \frac{1}{4}(\gamma'_4 + (\phi(2) + \phi_2(2))\gamma'_2 + (\phi(4) + \phi_2(4) + \phi(2)\phi_2(2))\gamma'_1) \\ &= \frac{1}{4}((q^3 + q^2 + 3q + 2) + 4(q + 1) + 17) = \frac{1}{4}(q^3 + q^2 + 7q + 23). \end{aligned}$$

If $(4, q + 1) = 2$, then the same calculation gives

$$k(U_4(q)) = \frac{1}{2}(q^3 + q^2 + 7q + 9).$$

Now, we present the remaining formulas for $n = 5$ and $n = 6$. Let $n = 5 = (q + 1, 5)$. Then

$$k(U_5(q)) = \frac{1}{5}(q^4 + q^3 + 3q^2 + 4q + 31).$$

Finally, for $n = 6$, we have

$$\begin{aligned} k(U_6(q)) &= q^5 + q^4 + 3q^3 + 5q^2 + 6q + 4 && \text{if } (6, q + 1) = 1; \\ k(U_6(q)) &= \frac{1}{2}(q^5 + q^4 + 3q^3 + 9q^2 + 10q + 12) && \text{if } (6, q + 1) = 2; \\ k(U_6(q)) &= \frac{1}{3}(q^5 + q^4 + 3q^3 + 5q^2 + 16q + 14) && \text{if } (6, q + 1) = 3; \\ k(U_6(q)) &= \frac{1}{6}(q^5 + q^4 + 3q^3 + 9q^2 + 20q + 62) && \text{if } (6, q + 1) = 6. \end{aligned}$$

It is easy to prove that the conclusion of Lemma 3 is true for the groups $U_n(q)$, $3 \leq n \leq 6$. We show how to do this for $n = 6$. It is easy to show that for $q \geq 2$ the expression $\gamma'_6 = q^5 + q^4 + 3q^3 + 5q^2 + 6q + 4$ cannot be less than any of the remaining expressions for $k(U_6(q))$. Therefore, it suffices to prove that $k(U_5) \leq 2q^5$ for every integer

$q \geq 2$. This statement is obvious for $q \geq 7$ because γ'_6 is the q -adic expression of an integer less than $2q^5$. Substituting $q = 3, 4, 5, 6$, we see that the inequality is valid. In the case where $q = 2$, the inequality $\gamma'_6 \leq 2 \cdot 2^5 = 2^6$ is no longer true, but $2 + 1 = 3$, and therefore, we must prove that $\frac{1}{3}(2^5 + 2^4 + 3 \cdot 2^3 + 5 \cdot 2^2 + 16 \cdot 2 + 14) \leq 2^6$, which gives the result in the case in question. If $n = 4$ or $n = 5$, then the above possibility does not arise, which leads to the exceptions listed in the lemma. \square

Lemmas 1, 2, and 3 imply the following statement.

Corollary 1. *Let G be a non-Abelian simple group for which $|G| < k(G)^3$. Then G is isomorphic to the simple group $L_2(q)$ for some even q .*

Proof. Let G be a simple group of Lie type with untwisted Lie rank l , and let $k(G)^3 \geq |G|$. We use the estimate $k(G) \leq (6q)^l$ proved in Lemma 1.

1. If $G \simeq L_n^\varepsilon(q)$, then $l = n - 1$, and it is easy to check that $|G| \geq q^{n(n-1)+(n-2)/2}$ for $q \geq 3$, and $|G| \geq 2^{(n-1)^2} 3^{n-1}$ for $q = 2$ and $n \geq 2$. Therefore,

$$(6q)^{3n-3} \geq q^{n(n-1)+(n-2)/2} \quad \text{for } q \geq 3 \quad \text{and}$$

$$(12)^{3n-3} \geq 2^{n(n-1)} \quad \text{for } q = 2.$$

If $q \geq 3$, then $6 < q^{5/3}$, and so $n^2 - 8.5n + 7 < 0$, which gives $n \leq 7$. For $q \geq 4$, we have a sharper estimate, $6 < q^{4/3}$, whence $n \leq 6$.

For $n = 7$ and $q = 3$, we can use Tables 4 and 5 in [19] to obtain the estimate $k(G) \leq 2 \cdot 3^6$ for the group $G = L_{10}^\varepsilon(q)$, because $(7, 3 - \varepsilon \cdot 1) = 1$. Thus, we have $n \leq 7$ for $q \geq 3$. From the above and Lemma 3, we obtain $k(G) \leq 2 \cdot q^{n-1}$. Hence,

$$(2 \cdot q^{n-1})^3 \geq q^{n(n-1)+(n-2)/2}.$$

It is easily seen that $n \leq 3$. Applying sharp estimates contained in [19] and in Lemma 3, we prove the claim.

If $q = 2$, then $n \leq 11$. A more thorough calculation of the order of the group shows that $n \leq 10$. For the groups $L_n^\varepsilon(q)$ with $n = 10$, the number of conjugacy classes is given in Tables 2 and 5 in [19], because $(10, q - \varepsilon \cdot 1) = 1$. It can be proved directly that, in both cases (for $\varepsilon = +$ and $\varepsilon = -$), $k(G)^3$ is considerably less than the order of the Sylow 2-subgroup of G .

The group $L_n(2)$ coincides with $GL_n(2)$, and the values of the conjugacy classes are given in Table 2 in [19] for n as above. In all cases, we have $k(L_n(2)) < 2^n$. Thus, it remains to consider the groups $U_n(2)$ for $n \leq 9$. If $n = 9$, then we can use Table 5 in [19] (the computational algorithm is given in the proof of Lemma 3). If $n = 8$ or $n = 7$, then Table 5 gives exact values for $k(U_n(q))$. If $n \leq 6$, then we can use Lemma 3. In each of the remaining cases we have $k(G)^3 < |G|$.

2. If $G \simeq B_n(q)$ or $C_n(q)$, then $l = n$, and it is easy to check that $|G| \geq \frac{1}{2}q^{2n^2+(n-2)/2}$ for $q \geq 3$ and $|G| \geq 2^{2n^2} 3^n$ for $q = 2$ and $n \geq 2$.

Therefore,

$$(6q)^{3n} \geq \frac{1}{2}q^{2n^2+(n-2)/2} \quad \text{for } q \geq 3 \quad \text{and}$$

$$(12)^{3n} \geq 2^{2n^2} 3^n \quad \text{for } q = 2.$$

First, we consider the case where $q \geq 3$. Solving the above inequality for $q \geq 6$, we get $n \leq 3$. If $q \geq 4$, then $6q \leq q^{7/3}$. Therefore, we have $n \leq 4$ for $q \in \{4, 5\}$. Finally, for $q = 3$, we obtain $18 = 6q = 2 \cdot 3^2$, which implies $n \leq 4$.

The cases where $q \geq 7$ and $n \geq 3$ are ruled out by direct calculation. Thus, the remaining values of q belong to the set $\{3, 4, 5\}$, and $n \in \{2, 3, 4\}$. The case of $n = 4$ can be ruled out by an easy supplementary calculation. Finally, the case where $n = 3$ is

possible only if $q = 3$ and can be ruled out by [10], because $k(B_3(3))$ and $k(C_3(3))$ do not exceed 3^4 .

Applying the computer algebra system GAP with function `NrConjugacyClasses(G)`, in which we use the function `ProjectiveSymplecticGroup(4,q)` instead of `G`, we can easily rule out the remaining groups for $n = 2$ (the groups $B_2(q)$ and $C_2(q)$ are isomorphic).

If $q = 2$, then the inequality $k(G)^3 \geq |G|$ implies $n < 5$, and it remains to use [10].

3. Similarly, we can rule out the remaining groups of Lie type. For completeness, we note that $k(^2B_2(q)) = q + 3$, $k(^2G_2(q)) = q + 8$, $k(^2F_4(q)) = q^2 + 4q + 17$ for $q > 2$, and $k(^2F_4(2)') = 22$ (see [18]).

4. Let G be a sporadic simple group. By [10], the numbers of classes for such groups are as follows: $k(M_{11}) = 10$, $k(M_{22}) = 12$, $k(M_{23}) = 17$, $k(M_{24}) = 26$, $k(HS) = 24$, $k(J_1) = 15$, $k(J_2) = 21$, $k(J_3) = 21$, $k(J_4) = 124$, $k(Co_1) = 103$, $k(Co_2) = 60$, $k(Co_3) = 42$, $k(McL) = 24$, $k(Suz) = 47$, $k(He) = 38$, $k(HN) = 54$, $k(Th) = 86$, $k(Fi_{22}) = 69$, $k(Fi_{23}) = 103$, $k(Fi'_{24}) = 120$, $k(B) = 188$, $k(M) = 194$, $k(Ru) = 38$, $k(Ly) = 53$. In this case, Corollary 1 can be verified directly.

5. If G is isomorphic to the alternating group A_n , then, by Lemma 2, we have $k(G) \leq 2^{n-1}$. Since $|A_n| = n!/2 \leq 2^{3(n-1)}$, we obtain $n! \leq 2^{3n-2}$. A direct calculation shows that $15! > 2^{40}$. On the other hand, taking logarithms, we see that $\log_2 n! \leq 3n - 2$. For $n \geq 16$, we obviously have $\log_2 n! \geq 4(n - 15) + \log_2 15! > 4n - 60 + 40 = 4n - 20$. Therefore, $n \leq 17$. It is clear that $k(A_n) \leq 2k(S_n) = 2p(n)$, where $p(n)$ is the number of partitions of n . A table of values of $p(n)$ for $n \leq 100$ is contained in [7, p. 53]. For $5 \leq n \leq 17$, the function $2p(n)$ is majorized by the function $2^{n/2+2}$. Consequently, it suffices to prove the inequality $n! = 2|A_n| \geq 2^{3n/2+7}$ for $n \geq 6$. If $n \geq 11$, then it is easy to check that $n! > 2^{3n/2+7}$. Therefore, we can restrict ourselves to the case where $n \leq 10$. The cases where $n = 10$ and $n = 9$ can be ruled out by direct calculation using the estimate $k(A_n) \leq 2p(n)$. For $n = 6, 7$, or 8 , we need sharper values of $k(A_n)$.

Since A_n contains only even permutations and every two permutations of the same cycle type are conjugate in S_n , we have one of the following two possibilities. Either the permutations that belong to A_n and have the same cycle type are conjugate in A_n , or there exist two conjugacy classes of permutations in A_n of a given cycle type. In either case, a (rough) estimate from above for the number of conjugacy classes in A_n is given by twice the number of types of even permutations of an n -element set.

Let $n = 8$. Then we have 12 cycle types of even permutations. We obtain $40320 = 8! > (2 \cdot 12)^3 = 13824$, whence $n = 8$ is impossible.

Let $n = 7$. Then there are 8 cycle types of even permutations. We have $5040 = 7! > (2 \cdot 8)^3 = 4096$, a contradiction.

Let $n = 6$. We have $k(A_6) = 7$ and $720 = 6! > (7)^3 = 343$, and so this case is also ruled out.

If $n = 5$, then $k(A_5) = 5$ and $|A_5| < 5^3$. Since $A_5 \simeq L_2(4)$, we see that, in this case, our statement is valid. \square

The following useful result was obtained by Gallagher [13].

Lemma 4. *Let G be a group and H a subgroup of G . Then*

$$k(H)/|G : H| \leq k(G) \leq |G : H|k(H).$$

If H is a normal subgroup of G , then $k(G) \leq k(H)k(G/H)$.

Lemma 5. *Let G be a non-Abelian simple group. Then there exists an irreducible character χ of G such that $\chi(1)^3 \geq |G|$.*

Proof. See [15]. \square

Lemma 6. *Let L be a non-Abelian simple group of Lie type defined over a field $GF(q)$ of characteristic p . Then L has an irreducible character St_L , the degree of which is equal to the order of the Sylow p -subgroup of L .*

Proof. See [9]. □

§3. A MINIMAL COUNTEREXAMPLE TO THEOREM 1

The following two lemmas will be used repeatedly, and Lemma 8 plays a crucial role in the proofs of the main results of the present paper.

Lemma 7. (i) *A quotient group of an ASR-group is an ASR-group.*

(ii) *The direct product of two ASR-groups is an ASR-group.*

Proof. The first part follows from the fact that every representation of a quotient group can be regarded as a representation of the given group. Part (ii) follows from the fact that every irreducible representation of a direct product $G = A \times B$ is the tensor product of an irreducible representation of A and an irreducible representation of B (see [12, Theorem (4.21)]). In particular, every irreducible character of G is a product $\varphi\psi$, where $\varphi \in \text{Irr}(A)$ and $\psi \in \text{Irr}(B)$. The latter remark will be used repeatedly in the sequel. □

Lemma 8. *Let G be a non-Abelian ASR-group, and let χ be an irreducible character of G . Then $\chi(1) < k(G)$ and $|G| < k(G)^3$.*

Proof. Since G is an ASR-group, the tensor square of every irreducible representation has a decomposition as a sum of irreducible representations of G with multiplicities at most 1. In particular, for an irreducible representation of G with character χ_0 of the largest degree $\chi_0(1)$, we obtain $\chi_0^2 = \sum_k c_k \chi_k$, where the summation is taken over all irreducible characters of G , and $c_k \in \{0, 1\}$. Hence,

$$\chi_0(1)^2 = \sum_{\chi_k \in \text{Irr}(G)} c_k \chi_k(1) \leq k(G)\chi_0(1).$$

Since G is non-Abelian, the inequality is strict. Thus, $\chi_0(1) < k(G)$ for an irreducible character χ_0 of G of the greatest degree. However, in this case, we have $\chi(1) < k(G)$ for every irreducible character χ of G .

By the Frobenius formula and the above, we have

$$|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 < k(G)\chi_0(1)^2 \leq k(G)^3.$$

The lemma is proved. □

Corollary 2. *Let G be a nonsolvable ASR-group. Then for the degree of any irreducible representation of G we have $\chi(1) \leq k(G) - 2$.*

Proof. Indeed, choosing a character χ_0 of the largest degree and arguing as above, we obtain the inequality

$$\chi_0(1)^2 \leq \sum_{j=1}^{k(G)} \chi_j(1) \leq a\chi_0(1) + (k(G) - a - 1)(\chi_0(1) - 1) + 1,$$

where a is the number of irreducible characters χ of G of degree $\chi(1) = \chi_0(1)$. Since the number of different degrees of irreducible characters of a nonsolvable group is at least 3 (see [12, Corollary (12.6)]), we have $k(G) - a - 1 > 0$. Hence, $\chi_0(1)(\chi_0(1) - k(G) + 1) \leq 1 - (k(G) - a - 1)$. If $\chi_0(1) = k(G) - 1$, then $G = G'$, and therefore, there exists a quotient group H of G that is (by Lemma 7(i)) a simple non-Abelian ASR-group. By Corollary 1, the group H is isomorphic to $L_2(q)$ for some even q . However, the group

$L_2(q)$ has characters of degrees $q+1$, q , $q-1$, and 1 (see [20]). This contradiction shows that $\chi_0(1) - (k(G) - 1) < 0$, which proves our statement. \square

From this point on, we assume that the *ASR*-group G under consideration is a minimal counterexample to Theorem 1. We prove some properties of G , which are used in the subsequent sections to prove that such a counterexample does not exist.

Lemma 9. *Let G be a nonsolvable *ASR*-group of the smallest order, and let N be a minimal normal subgroup of G . Then G/N is a solvable group, and N is a socle of G , that is, the direct product $R_1 \times R_2 \times \cdots \times R_m$ of isomorphic non-Abelian simple groups. The group G is a subgroup of the wreath product $\mathfrak{G} = A \wr S$, where $A = \text{Aut}(L)$ for a simple non-Abelian group L isomorphic to R_1 , and S is a solvable subgroup of the symmetric group of degree m , acting transitively on the set of minimal normal subgroups of N .*

Proof. Lemma 7(i) implies that, for every normal subgroup N of G , the quotient group G/N is also an *ASR*-group. Since the group G is minimal, the group G/N is solvable if $N \neq 1$. We assume that G has two distinct minimal normal subgroups, say M and N . Since G/M and G/N are solvable by the above, G is also solvable, because

$$G \simeq G/(M \cap N) \leq G/M \times G/N.$$

Thus, G has a unique solvable minimal subgroup N , i.e., N is the socle of G . Corollary 1 shows that N is a proper nonsolvable subgroup of G .

Since $C_G(N)$ is normal in G and $C_G(N) \cap N = Z(N)$ is a solvable normal subgroup of G , we see that $C_G(N) \cap N = 1$. From the above, it follows that $C_G(N) = 1$, so that $G \subseteq \text{Aut}(N)$.

Since, obviously, N is a characteristically simple group, we have

$$N = R_1 \times R_2 \times \cdots \times R_m,$$

where the R_i are isomorphic non-Abelian simple groups ($i \in \{1, 2, \dots, m\}$, $m \geq 1$) and the quotient group G/N is a solvable *ASR*-group.

The structure of the automorphism group of a characteristically simple non-Abelian group is well known. This automorphism group is a subgroup of the wreath product $\mathfrak{G} = \text{Aut}(L) \wr S$, where $L \simeq R_1$ is a non-Abelian simple group and S is a subgroup of the symmetric group S_m , acting transitively on the set of minimal normal subgroups of N . Since G/N is solvable, we conclude that S is a solvable subgroup of S_m . The lemma is proved. \square

We preserve the notation of Lemma 9 until the end of the proof of Theorems 1 and 2.

Lemma 10. *Let $a = |\text{Out}(L)|$, and let χ_0 be an irreducible character of L having the greatest degree. Then*

$$\chi_0(1) \leq \sqrt{3}ak(L) \leq 1.733 \cdot ak(L).$$

Proof. It is easily seen that G has a normal subgroup M (the intersection of the base of the wreath product $\mathfrak{G} = A \wr S$ with G) that normalizes each subgroup R_i , $i \in \{1, 2, \dots, m\}$, and satisfies the conditions $|M : N| \leq a^m$ and $G/M \simeq S$.

By Lemma 4, we have

$$(1) \quad k(M) \leq |M : N|k(N) = a^m k(L)^m.$$

Using Lemma 2(i) and Lemma 4, from (1) we deduce that

$$(2) \quad k(G) \leq (2ak(L))^m.$$

Since χ_0 is an irreducible character of the group $L \simeq R_1$ of the largest degree, we see that $\Theta_0 = \chi_0^m$ is an irreducible character of $N = R_1 \times R_2 \times \cdots \times R_m$. By the Frobenius reciprocity theorem, we obtain

$$[\Theta_0^G|_N, \Theta_0]_N = [\Theta_0^G, \Theta_0^G]_G > 0.$$

Therefore, there exists an irreducible character Φ_0 of G such that the restriction $\Phi_0|_N$ contains Θ_0 . In particular,

$$(3) \quad \Phi_0(1) \geq \chi_0(1)^m.$$

Relations (2) and (3) and Lemma 8 yield

$$\chi_0(1)^m \leq \Phi_0(1) \leq k(G) \leq (2ak(L))^m.$$

Thus,

$$(4) \quad \chi_0(1) \leq 2ak(L),$$

where χ_0 is an irreducible greatest degree character of L and $a = |\text{Out}(L)|$.

By Lemma 2(i), we have $k(G/M) \leq 3^{(m-1)/2}$. Therefore, we can replace the constant 2^m by $3^{m/2}$ in inequality (2). The resulting inequality takes the form

$$\chi_0(1) \leq \sqrt{3}ak(L) \leq 1.733 \cdot ak(L),$$

which proves the lemma. □

§4. REDUCTION

In this section, we reduce the general situation to the case where the simple normal subgroup $R_1 \simeq L$ of the minimal normal subgroup N of G is a classical simple group of small dimension.

By the classification theorem, a non-Abelian simple group is either a simple group of Lie type, or an alternating group, or one of the 26 sporadic simple groups. We use Lemma 10 and relation (4) obtained in the preceding section and consider each of these possibilities for L separately.

4.1. Alternating groups. For $n \geq 7$, inequality (4) takes the following form:

$$(5) \quad \chi_0(1) \leq 4p(n),$$

where $p(n)$ is the number of partitions of n and χ_0 is an irreducible character of A_n of largest degree. Let $L = A_n$ be an alternating group satisfying (4). Then $n \leq 6$.

Lemma 11. *Let $L = A_n$ be an alternating group satisfying inequality (4). Then $n \leq 6$.*

Proof. First, we find all alternating groups satisfying (4). By Lemma 6, the group L has a character with degree greater than $|L|^{1/3} = (n!/2)^{1/3}$. Therefore, we obtain the inequality $n! \leq 128p(n)^3$. Since $4p(n) < 2^n$ for $n \geq 5$, we obtain $n! < 2^{3n+1}$, which implies $n \leq 21$. Using the fact that, for $n > 7$, the group $L = A_n$ has an irreducible character φ of degree $n(n-1)(n-3)(n-6)/8$ corresponding to the partition $n = (n-4) + 3 + 1$ (see [2, Theorem 20.1]), we see that $\varphi(1) > 4p(n)$ for $9 \leq n \leq 21$. Therefore, $n \leq 8$.

The group $L = A_8$ is isomorphic to $L_4(2)$ and has 14 irreducible characters. We have $\chi_0(1) = 70$ and $a = 2$. Therefore, $\chi_0(1) > 4 \cdot 14 = 2ak(L)$, and we may assume that $n = 7$.

The group $L = A_7$ has 9 irreducible characters and $\chi_0(1) = 35$. Applying Lemma 10, we obtain $35 = \chi_0(1) \leq 1.733 \cdot 2 \cdot 9 < 34.2$, a contradiction. The lemma is proved. □

Remark. The group A_5 is isomorphic to $L_2(4)$, and A_6 is isomorphic to $L_2(9)$. Therefore, all possible exceptions are classical groups.

4.2. Sporadic simple groups.

Lemma 12. *The group L is not a sporadic simple group.*

Proof. Using the Frobenius theorem, we represent inequality (4) in the form

$$(6) \quad |L| = \sum_{\chi \in \text{Irr}(L)} \chi(1)^2 \leq k(L)\chi_0(1)^2 \leq 4a^2k(L)^3.$$

We recall that for sporadic groups the parameter a does not exceed 2. The results of [10] show that the groups $L = M_{11}$ ($k(L) = 10$), $L = M_{22}$ ($k(L) = 12$), $L = M_{23}$ ($k(L) = 17$), and $L = M_{24}$ ($k(L) = 26$) do not satisfy inequality (6). For the remaining sporadic simple groups, except the Mathieu groups, the calculation is obvious. \square

4.3. Exceptional simple groups of Lie type.

Lemma 13. *Let L be an exceptional simple group of Lie type satisfying the inequality of Lemma 10. Then $L \simeq G_2(2)'$.*

Proof. We consider the Steinberg character to exclude various possibilities.

1. Let L be isomorphic to the group ${}^2B_2(q)$, $q = 2^t$. By [18], we have $k(L) = q + 3$. By Lemma 10, we obtain

$$q^2 \leq \chi_0(1) \leq 1.733t(q + 3).$$

It is easy to show that this inequality is not fulfilled for any $t \geq 3$.

2. Let L be isomorphic to ${}^2G_2(q)$, $q = 3^t$, $t \geq 3$. In this case, the inequality has the form

$$q^3 \leq \chi_0(1) \leq 1.733t(q + 8)$$

(see [18, p. 500]). It is easily seen that this inequality is not fulfilled for any $t \geq 3$.

3. Let L be isomorphic to ${}^2F_4(q)'$, $q = 2^t$, $t \geq 1$. In this case, the inequality takes the form

$$q^{12} \leq \chi_0(1) \leq 1.733t(q^2 + 4q + 17)$$

(see [18, p. 500]). For $q = 2$, we have $k({}^2F_4(2)') = 22$ and $a = 2$. It is easy to check that this inequality is not fulfilled for any $t \geq 1$.

4. Let L be isomorphic to ${}^3D_4(q)$, $q = p^t$, $t \geq 1$. We use Lemma 3. In this case, the inequality has the form

$$q^{12} \leq \chi_0(1) \leq 2 \cdot 3t(6q)^4 \leq 6tq^8$$

with $q \geq 7$. Therefore, $p^{4t} \leq 6t$, which, obviously, is not true. Thus, $q \leq 5$. For $q = 5$, we have $(6q)^4 \leq q^9$ and $t = 1$. Hence, $5^3 \leq 6$, a contradiction. Let $q = 4$ and $t = 2$. In this case we obtain $2^{24} \leq 12(24)^4$, i.e., $2^{10} \leq 3^5$, a contradiction. Thus, $q = 2$ or $q = 3$ and $t = 1$. Consequently, the possible candidates are the groups ${}^3D_4(2)$ and ${}^3D_4(3)$. However, $k({}^3D_4(2)) = 35$, which rules out this group by inequality (4). By [21], we have $k({}^3D_4(3)) = 37$. Using Lemma 10, we obtain a contradiction also in this case.

5. Let L be isomorphic to $G_2(q)$, $q = p^t$, $t \geq 1$. By Lemma 1, the inequality takes the form

$$q^6 \leq \chi_0(1) \leq 1.733t(3, p)(6q)^2,$$

which is equivalent to

$$p^{4t} \leq 1.733 \cdot 36 \cdot (3, p)t < 64 \cdot (3, p) \cdot t.$$

If $p \neq 3$, then $p^{4t} < 64t$ is valid only for $p = 2$ and $t = 1$, i.e., $L = G_2(2)' \simeq U_3(3)$, as stated above.

Thus, $p = 3$. It is easily seen that $t = 1$ in this case, whence $L \simeq G_2(3)$. By [10], we have $k(L) = 23$. Obviously, in this case the inequality is not fulfilled.

6. Let L be isomorphic to $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, or $E_8(q)$ with $q = p^t$ and $t \geq 1$. By using Lemmas 1 and 10, these cases can easily be ruled out. The lemma is proved. \square

4.4. Classical simple groups of Lie type.

Lemma 14. *Let L be a classical simple group of Lie type for which the inequality of Lemma 10 is valid. Then $L \simeq L_3^\varepsilon(q)$, where $3 \leq q \leq 8$ and $q \neq 7$, with the exception of $L_3(5)$, or $L \simeq L_2(q)$.*

Proof. We prove the lemma by considering each series of classical groups separately and, as a rule, applying the Steinberg character as a test. Below, $q = p^t$, where p is a prime.

1. The case of the groups $L_n^\varepsilon(q)$, $n \geq 3$. First, we prove that $n \leq 6$. For this, we use the estimate of Lemma 3. Inequality (4) has the following form:

$$q^{n(n-1)/2} \leq 2 \cdot 2(n, q - \varepsilon \cdot 1)t(6q)^{n-1},$$

where $q = p^t$. It is easily seen that $2t \leq q$. For $q \geq 3$, we have $6q < q^3$ and $q^{n(n-1)/2} < q^{3n}$, whence $n \leq 6$. Now, assume that $q = 2$. Then, using similar considerations, we can easily show that $n \leq 7$. Table 5 in [19] gives the value of $k(U_7(2))$, and Table 4 gives the value of $k(L_7(2))$, which show that inequality (4) fails in this case.

Let $n \leq 6$. By Lemma 3, we have $k(L) \leq 2 \cdot q^{n-1}$ for $q \geq 3$. Therefore, inequality (4) takes the form

$$q^{n(n-1)/2} \leq 4(n, q - \varepsilon \cdot 1) \cdot 2q^{n-1}t.$$

For $n \geq 7$, we have $8t(n - \varepsilon \cdot 1) < q^2$. Therefore, inequality (4) implies that

$$n^2 - n < 2n + 2.$$

Hence, $n \leq 3$. Using Lemma 3 (and exact values for the number of conjugacy classes), we prove Lemma 14 in the case under consideration. If $q = 2$, we again use Lemma 3 and [19].

Considering the exact values for the remaining groups, we obtain the following list of groups isomorphic to $L_3^\varepsilon(q)$ and satisfying the inequality of Lemma 10: $L_3(2)$, $L_3(3)$, $L_3(4)$, $L_3(8)$, $U_3(3)$, $U_3(4)$, $U_3(5)$, and $U_3(8)$.

Among the remaining groups in question, only one group $U_4(2)$ satisfies the inequality of Lemma 10 with the Steinberg character in the role of χ_0 . However, there is a character of degree 81 for which the inequality fails. Therefore, the final list of groups satisfying the inequality is exhausted by the above-mentioned groups $L_3(q)$ and $U_3(q)$ for $3 \leq q \leq 8$, with the exception of $q = 7$ and $L_3(5)$. The group $L_3(2)$ is ruled out because it is isomorphic to $L_2(7)$, and the group $U_3(2)$ is solvable.

2. The case of the groups $B_n(q)$ and $C_n(q)$. We consider the groups $L = P\Omega_{2n+1}(q)$ and $L = PSp_{2n}(q)$. In the Lie notation, these are $B_n(q)$ and $C_n(q)$. By Lemma 1, $k(L) \leq (6q)^n$ in both cases, and the inequality of Lemma 10 has the form

$$q^{n^2} \leq 1.733 \cdot 2 \cdot t(6q)^n.$$

If $n \geq 4$, then there are no solutions. Let $n = 3$. Then $q^9 \leq 3.466 \cdot 6^3 \cdot q^3$, i.e., $q^6 \leq 3.466 \cdot t \cdot 6^3$. It is easily seen that $q \leq 3$, i.e., the possible exceptions are $PSp_6(q)$ or $P\Omega_7(q)$ for $q \in \{2, 3\}$.

We prove that the groups $B_3(q)$ and $C_3(q)$ for $q \in \{2, 3\}$ do not satisfy the inequality in question. The results of [10] show that $k(G) \leq 98$ for each of the above-mentioned groups over $GF(3)$, and the inequality $3^9 \leq 3.466 \cdot 98 \leq 346$ obviously fails. If $L \simeq B_3(2)$ or $C_3(2)$, then $k(L) = 30$. Here, $|\text{Out}(L)| = 1$ for both groups, and the inequality takes the form $2^9 \leq 1.733 \cdot 30$, which is obviously false.

Let $n = 2$. Then $q^2 \leq 3.466 \cdot t \cdot 6^2$. We recall that $q = p^t$. Let $t = 1$. Then $q = p \leq 11$. For $t = 2$, the solutions are $q \in \{4, 9\}$. Finally, if $t \geq 3$, then only $q = 2^t$ with $t \in \{3, 4\}$ can be a solution of the above inequality.

Now, we rule out the remaining simple groups for $n = 2$. We know that the groups $C_2(q)$ and $B_2(q)$ coincide. We apply the computer algebra system GAP with the function `NrConjugacyClasses(G)`, where we use the function `ProjectiveSymplecticGroup(4, q)` instead of `G`. Now the inequality can be rewritten as

$$q^4 \leq 1.733 \cdot |\text{Out}(C_2(q))|k(C_2(q)).$$

The resulting values for the classes are summarized in the table below.

q	2	3	4	5	7	8	9	11	16
$k(C_2(q))$	11	20	27	34	52	83	74	100	291
$ \text{Out}(C_2(q)) $	2	2	4	2	2	6	4	2	8

In all cases except for $C_2(2)$, the inequality fails. We have $C_2(2) \simeq S_6$ and $C_2(2)' \simeq L_2(9)$. Thus, the group $C_2(2)$ satisfies the conclusion of the lemma.

3. The case of the groups $D_n(q)$ and ${}^2D_n(q)$. In both cases, we have $n \geq 4$, because the groups of smaller Lie rank are isomorphic to the groups considered above. Since the automorphism group of $D_4(q) \simeq P\Omega_8^+(q)$ is calculated in an absolutely different way, the case where $n = 4$ will be studied separately. For both groups, the inequality in question can be somewhat sharpened with the help of Lemma 1; this yields

$$q^{n(n-1)} \leq 1.733 \cdot 4 \cdot 2t \cdot (6q)^n \leq 14 \cdot t \cdot (6q)^n,$$

where $n \geq 5$ and $q = p^t$. It is easily seen that $(6q)^n \leq q^{4n}$. Therefore, if $n \geq 6$, then the inequality admits no solutions that are natural powers of p . Let $n = 5$. Since $3 < 2^{8/5}$, we have $q^{20} \leq 14 \cdot t(6q)^5 \leq 14 \cdot tq^{18}$ if $n = 5$. Hence, $q^2 = p^{2t} \leq 14t$. The following values of q are possible: 2, 3, and 4. Returning to the initial inequality, we can easily check that $q = 2$ and $q = 3$ do not fit. In the case of $q = 2$, the order of the outer automorphism group of L is $2(4, q^5 \pm 1) = 2$, and so the inequality is not fulfilled.

Let $n = 4$ and $L \simeq D_4(q)$. Since $|\text{Out}(L)| = 6(q^4 - 1, 4)t$, the inequality implies the relation $p^{8t} \leq q^8 \leq \chi_0(1) \leq 5392.232t$, which is impossible for $q \geq 3$. Therefore, $q = 2$. By [10], we have $k(D_4(2)) = 53$. The group L has an irreducible character χ_0 of degree 6075. Thus, this possibility is also ruled out.

If $L \simeq {}^2D_4(q)$, then $|\text{Out}(L)| = 2(q^4 + 1, 4)t$. Thus, we have the following inequality: $q^{12} \leq 1.733 \cdot 2(q^4 + 1, 4)t \cdot (6q)^4$. Hence, $q^8 \leq 14 \cdot 6^4t = 18144t$. If $p \geq 5$, then this inequality has no solutions. It is easily seen that the only possible solutions are $q = 2$ and $q = 3$. By [10], we have $k({}^2D_4(2)) = 39$. Therefore, $2^{12} \leq 1.733 \cdot 39 < 80$, a contradiction. If $q = 3$, then $3^{12} \leq 1.733 \cdot 4 \cdot k({}^2D_4(3)) = 1.733 \cdot 4 \cdot 190 < 1600$, a contradiction.

Thus, there are no simple orthogonal groups satisfying the conditions of Lemma 14. The lemma is proved. □

§5. CHARACTERS AND NORMAL SUBGROUPS

We consider the following situation. The group $G \leq \mathfrak{G} = A \wr S$ is a smallest order counterexample to the conclusion of Theorem 2, where $A = \text{Aut}(L)$ and L is a non-Abelian simple group. The subgroup $N = R_1 \times R_2 \times \cdots \times R_m$ is a minimal normal subgroup of G that is a direct product of conjugate non-Abelian simple groups isomorphic to L , and S is a solvable subgroup of S_m .

By the results of the preceding section, we may assume that L is one of the following groups: $L_3^\varepsilon(3)$, $L_3^\varepsilon(4)$, $L_3^\varepsilon(8)$, $U_3(5)$, or $L_2(q)$. Moreover, there exists a normal subgroup \mathfrak{M} of \mathfrak{G} containing N such that $\mathfrak{M} = A_1 \times A_2 \times \cdots \times A_m$ with $A_i = \text{Aut}(R_i)$ and

$S = \mathfrak{G}/\mathfrak{M} \simeq G/M$. Here, G acts transitively on the set $\{R_1, R_2, \dots, R_m\}$, and the kernel of this action is the subgroup $M = G \cap \mathfrak{M}$.

In the case where $L \simeq L_2(q)$ for q odd, we need yet another subgroup M_0 of G , which is defined as follows. We denote by \mathfrak{M}_0 the subgroup of \mathfrak{G} lying in \mathfrak{M} and isomorphic to the group $D_1 \times D_2 \times \dots \times D_m$, where $D_i \simeq PGL_2(q)$, and put $M_0 = \mathfrak{M}_0 \cap G$.

We are interested in irreducible characters of G , the restriction of which to N contains a certain special character of N . By the Clifford theory (see Chapter 6 in [12]), if $\Theta \in \text{Irr}(N)$ and χ is an irreducible character of G , then the relation $[\chi|_N, \Theta]_N \neq 0$ implies that

$$(7) \quad \chi_N = e \sum_{i=1}^l \Theta_i,$$

where the Θ_i are characters conjugate to Θ . We recall that the character conjugate to Θ by an element $g \in G$ is defined as follows: $\Theta^g(x) = \Theta(gxg^{-1})$ for every $x \in G$. The number l of distinct characters conjugate to Θ is equal to the index of the *inertia subgroup* $I_G(\Theta)$ of Θ (consisting of all $g \in G$ such that $\Theta^g = \Theta$), and the integer $e = e(\chi)$ divides $|I_G(\Theta) : N|$ if G/N is a solvable group. The character $\Theta \in \text{Irr}(N)$ is said to be *invariant* if $I_G(\Theta) = G$.

In the following lemma, we state briefly some facts about characters induced from normal subgroups. The proofs can be found in [12, Chapter 6].

Lemma 15. *Let K be a normal subgroup of a finite group X with solvable quotient group X/K , and let ψ be an irreducible character of K . Then $\psi^X = \sum_{i=1}^s e_i \chi_i$, where $\chi_i \in \text{Irr}(X)$, each e_i divides $|I_X(\psi)/K|$, $\chi_i|_K = e_i \sum_{j=1}^l \psi_j$, where the ψ_j are characters conjugate to $\psi = \psi_1$, $l = |X : I_X(\psi)|$, and $\sum_{i=1}^s e_i^2 = |I_X(\psi) : K|$. If $\chi \in \text{Irr}(X)$ and $[\chi|_K, \psi] \neq 0$, then $\chi = \chi_i$ for some $i \leq s$.*

The following result is due to Gallagher [13] and is given below in the same form as in [16, (A'_2)]. To state this result, we need the following definition.

Definition 3. For every group T , we define the parameter $k_0(T)$ as follows. For each subgroup of T (including T), we consider the number of conjugacy classes of its elements; the largest among these numbers is $k_0(T)$.

If T is Abelian, then $k_0(T) = k(T) = |T|$. In most applications below, this number is not much different from $k(T)$ and gives an upper bound for $k(T)$. The following property makes the parameter $k_0(T)$ convenient: if Y is a subgroup of X , then $k_0(Y) \leq k_0(X)$. Furthermore, Lemma 4 shows that if Y is a normal subgroup of X , then $k_0(X) \leq k_0(Y)k_0(X/Y)$.

Lemma 16. *Let K be a normal subgroup of a finite group X , and let $k_0(X/K) \leq r$. Let ψ be an irreducible character of K . Then the number of irreducible characters of X lying over (conjugate to) ψ (we mean $\chi \in \text{Irr}(X)$ such that $[\chi|_K, \psi] \neq 0$) does not exceed $k_0(I_G(\psi)/K)$. In particular, this number does not exceed r .*

Now, we describe the characters used in the proof of Theorem 2.

Let η_1 be an irreducible character of the subgroup R_1 invariant under A_1 . The choice of η_1 will be specified later, depending on the type of the group $L \simeq R_1$. Since the group G is transitive on the set $\{R_1, R_2, \dots, R_m\}$, there exist elements x_1, x_2, \dots, x_m of G such that $R_1^{x_i} = R_i$. We can define a character η_i on the subgroup R_i by $\eta_i(x_i^{-1}gx_i) = \eta_1(g)$ for all $g \in R_1$. If η_1 is an irreducible character of R_1 and x_1, x_2, \dots, x_m are elements of G such that $R_1^{x_i} = R_i$, then $\eta_i = \eta_1^{x_i^{-1}}$ is an irreducible character of R_i . The character $\Theta = \eta_1\eta_2 \cdots \eta_m$ is defined on N as follows. If $g = g_1g_2 \cdots g_m \in N$, where $g_i \in R_i$, then

$\Theta(g) = \eta_1(g_1)\eta_2(g_2)\cdots\eta_m(g_m)$. If η_1 is invariant under A_1 , then Θ is invariant under the group M .

Let $\Theta_1, \Theta_2, \dots, \Theta_l$ be distinct characters conjugate to $\Theta = \Theta_1$ under the action of G . Each of these characters can be written as $\Theta_i = \Theta_1^{a_i}$ for an appropriate element $a_i \in \mathfrak{M} = A_1 \times A_2 \times \cdots \times A_m$. Since $G \leq \mathfrak{G} = A \wr S = \mathfrak{M} \rtimes S$, every element $y \in G$ can be represented in the form $y = \sigma a$ for appropriate $a \in \mathfrak{M}$ and $\sigma \in S$. Since σ permutes the R_i and also the η_i , we obtain $\Theta^\sigma = \Theta$, and therefore, $\Theta^y = \Theta^a$ for some $a \in \mathfrak{M}$.

Thus, the number of characters conjugate to $\Theta = \Theta_1$ in G satisfies the inequality $l = |G : I_G(\Theta)| \geq |M : I_M(\Theta)|$.

Now, we assume that the character Θ is invariant in G . Applying Lemma 15, we conclude that $\Theta^G = \sum_{i=1}^r e_i \chi_i$, where the χ_i are irreducible characters of G such that $\chi_i|_N = e_i \Theta$ and r is the number of irreducible characters such that the decompositions of their restrictions to N contain Θ . By Lemmas 2, 4, and 16, we obtain $r \leq k_0(M/N) \cdot 3^{(m-1)/2}$, where $r_1 = k_0(M/N)$ is a number such that every subgroup of M/N has at most r_1 classes of conjugate elements.

Since G is an *ASR*-group, we have $\chi_i^2 = \sum_k c_{k,i} \chi_k$, where $c_{k,i} \in \{0, 1\}$ for all i and k .

We choose an irreducible character $\psi = \chi_{j_0}$ of G for which the coefficient $[\psi|_N, \Theta] = e_{j_0} = e$ is maximal. Then $\psi^2 = \sum_k u_k \chi_k$, where the coefficients $u_k = c_{k,j_0}$ belong to the set $\{0, 1\}$ and the summation is taken over all irreducible characters of G . It is obvious that ψ^2 is a class function. Thus, by the Frobenius reciprocity law, we obtain

$$(8) \quad [\psi^2|_N, \Theta]_N = [\psi^2, \Theta^G] = \left[\sum_k u_k \chi_k, \sum_{j=1}^r e_j \chi_j \right] = \sum_{j=1}^r u_j e_j.$$

It is easily seen that, for every $i \leq m$, the character η_i^2 on the subgroup R_i has the expansion $\eta_i^2 = d\eta_i + \varpi_i$, where d is an integer depending only on the type of the group and on the choice of η_1 , and ϖ_i is the sum of the remaining characters of R_i entering this expansion with nonnegative integral coefficients (or is identically zero). Since $\psi^2|_N = \psi|_N \psi|_N = e^2 \Theta^2 = e^2 \eta_1^2 \eta_2^2 \cdots \eta_m^2$ and $\eta_i^2 = d\eta_i + \varpi_i$, we can write

$$\psi^2|_N = e^2 d^m \Theta + \mathcal{Y},$$

where \mathcal{Y} is a character of N (or the function that vanishes identically on N). In particular, we arrive at the inequality

$$e^2 d^m \leq [\psi^2|_N, \Theta]_N = \sum_{j=1}^r u_j e_j \leq r e \leq k_0(M/N) \cdot 3^{m/2} e.$$

Finally, we obtain the estimate

$$(9) \quad e \cdot d^m \leq k_0(M/N) \cdot 3^{m/2}.$$

Lemma 17. *Let G be a minimal counterexample to Theorem 2. Then a minimal normal subgroup N of G is the direct product of simple groups isomorphic to $L_2(q)$ for some q .*

Proof. We consider each of the possible situations separately.

Let $L \simeq L_3(3)$. Then $|\text{Out}(L)| = 2$. Therefore, $k_0(M/N) \leq 2^m$. By [10], this group has an invariant character η of degree 39, and the multiplicity d of $\eta = \chi_{12}$ (in the notation of [10]) in η^2 is equal to 11. Substituting this in (9), we get a contradiction.

Now, assume that $L \simeq L_3(4)$. The Steinberg character $\eta = St_L$ of L has degree 64. Using the character table of this group given in [10], we see that $St_L^2 = 13 \cdot St_L + \varsigma$, where ς is the sum of the remaining characters involved in the decomposition of St_L^2 . The group of outer automorphisms of L is isomorphic to the dihedral group of order 12. It is easy to check that the number of conjugacy classes of any subgroup of this group does not exceed

6. Using induction based on Lemma 4, we can easily prove that $k_0(M/N) \leq 6^m$. In this case, inequality (9) has the form $e \cdot 13^m \leq 6^m 3^{m/2}$. Hence, $13 \leq 6\sqrt{3}$, a contradiction.

Let $L \simeq U_3(3)$. By [10], we have $|\text{Out}(L)| = 2$, and the character $\eta = St_L$ has multiplicity $d = 4$ in the expansion into irreducible characters of L . Applying formula (9), we conclude that $e4^m \leq 2^m 3^{m/2}$, and this rules out the case under consideration.

Let $L \simeq U_3(5)$. By [10], we have $|\text{Out}(L)| = 6$, and the character $\eta = St_L$ has multiplicity $d = 16$ in the expansion into irreducible characters of L . Applying formula (9), we conclude that $e \cdot 16^m \leq 6^m 3^{m/2}$, a contradiction.

Let $L \simeq U_3(8)$. By [10], we have $|\text{Out}(L)| = 18$, and the group $\text{Out}(L)$ is isomorphic to the direct product of S_3 and a cyclic group of order 3. Using induction and Lemma 4, we see that the number $k_0(M/N)$ does not exceed 9^m . The character $\eta = St_L$ has multiplicity $d = 25$ in the expansion into irreducible characters of L . Applying formula (9), we obtain $e \cdot 25^m \leq 9^m 3^{m/2}$, i.e., $25 \leq 9\sqrt{3}$, a contradiction.

In the remaining cases $L \simeq L_3(8)$ and $U_3(4)$, we use a noninvariant character η of the corresponding group.

In more detail, let η be an irreducible character of L of the greatest degree. For $L \simeq L_3(8)$ this is the character χ_{72} of degree 657, and for $U_3(4)$ this is the character χ_{22} of degree 75. By [10], these characters of L have the property $\eta^2 = d\eta + \varpi$, where ϖ is the sum of the remaining characters involved in the expansion of η^2 . Simple calculations, based on the character tables of the above-mentioned groups and the GAP system, show that $d = 18$ for $L \simeq L_3(8)$ and $d = 7$ for $L \simeq U_3(4)$.

Let η_1 be the character of R_1 corresponding to the character η of the group L isomorphic to R_1 . As before, we consider the character $\Theta = \eta_1 \eta_2 \cdots \eta_m$ of N . Let $\Theta_1, \Theta_2, \dots, \Theta_l$ be distinct characters conjugate to $\Theta = \Theta_1$ under the action of the subgroup G .

As noted above, the number of characters conjugate to $\Theta = \Theta_1$ in G satisfies the inequality $|G : I_G(\Theta)| \geq |M : I_M(\Theta)|$. We also note that, for all $i \leq m$ and $a \in A_i$, the character η_i^a is an irreducible character of R_i and $(\eta_i^a)^2 = d\eta_i^a + \varpi^a$, i.e., the multiplicity of η_i^a in $(\eta_i^a)^2$ is the same as the multiplicity of η_i in η_i^2 . In particular, for all $b \in A$ we have $(\Theta^b)^2 = d^m \Theta^b + \Gamma$, where Γ is the sum of the remaining irreducible characters in the expansion of $(\Theta^b)^2$. By Lemma 15, we have $\Theta^G = \sum_{i=1}^r e_i \chi_i$.

Now, we find the multiplicity of the character $\chi = \chi_{i_0}$ for the index i_0 such that $e = e_{i_0} = \max_{i=1}^r e_i$. Since G is an *ASR*-group, we see that $\chi^2 = \sum_k c_k \psi_j$ is the sum of distinct irreducible characters of G . Therefore,

$$[\chi^2, \Theta^G]_G = \sum_{j=1}^r c_j e_j,$$

where $c_j \in \{0, 1\}$. Using the Frobenius reciprocity law, we obtain

$$[\chi^2, \Theta^G]_G = [\chi^2|_N, \Theta]_N = e^2 \left[\left(\sum_{j=1}^l \Theta_j \right)^2, \Theta \right].$$

Since $(\sum_{j=1}^l \Theta_j)^2 = \Theta_1^2 + \Xi$, where Ξ is a sum of characters with nonnegative integral coefficients, we obtain $[(\sum_{j=1}^l \Theta_j)^2, \Theta] \geq [\Theta^2, \Theta] = d^m$. Thus,

$$(10) \quad e^2 d^m \leq \sum_{j=1}^r c_j e_j \leq er \leq e|M : N|k_0(G/M).$$

Applying inequality (10) to the case where $R_1 \simeq L \simeq U_3(4)$, we get $e \cdot 7^m \leq 4^m 3^{m/2}$, whence $7 \leq 1.733 \cdot 4$, a contradiction.

If $R_1 \simeq L \simeq L_3(8)$, then $18^m \leq 6^m 3^{m/2}$, which is also false. The lemma is proved. \square

§6. THE CHARACTERS OF THE GROUP $PGL_2(q)$

If $q \geq 4$ is even, then the group $PGL_2(q)$ coincides with $L = L_2(q)$.

The character table of $D = PGL_2(q)$, where q is odd, is taken from [8].

When describing the character table of $D = PGL_2(q)$ for q odd, we assume that $q \geq 7$. The following notation is used also in the case where q is even, but in the latter case the group is denoted by L : the groups D and L have cyclic subgroups generated by an element a of order $q - 1$, by an element b of order $q + 1$, and by an element u of order p , where p is the characteristic of $GF(q)$. Here, every element of D is conjugate to one of the elements $1, u, a^r$ ($1 \leq r \leq (q - 3)/2$) or b^s ($1 \leq s \leq (q - 1)/2$), or to one of the involutions $z_- = a^{(q-1)/2}$ or $z_+ = b^{(q+1)/2}$.

Every element of $L = L_2(q) = PGL_2(q)$ with q odd is conjugate to one of the elements $1, u, a^r$ ($1 \leq r \leq (q - 2)/2$), or b^s ($1 \leq s \leq q/2$).

	1	u	a^r	z_-	b^s	z_+
1_D	1	1	1	1	1	1
Sgn	1	1	$(-1)^r$	$(-1)^{(q-1)/2}$	$(-1)^s$	$(-1)^{(q+1)/2}$
St	q	0	1	1	-1	-1
$Sgn \cdot St$	q	0	$(-1)^r$	$(-1)^{(q-1)/2}$	$(-1)^{s+1}$	$(-1)^{(q-1)/2}$
φ_i	$q + 1$	1	$\alpha^{ir} + \alpha^{-ir}$	$2(-1)^i$	0	0
ζ_j	$q - 1$	-1	0	0	$\beta^{js} + \beta^{-js}$	$2(-1)^{j+1}$

Thus, $k(D) = q + 2$ and $k(L) = q + 1$. We denote by α a primitive $(q - 1)$ st root of unity and by β a primitive $(q + 1)$ st root of unity in the field of complex numbers.

Now, we proceed directly to the description of the character table of D . There are two linear characters (i.e., characters of degree 1), namely, 1_D (the principal character) and a character denoted by Sgn . Besides, there is a Steinberg character St and a character $Sgn \cdot St$, both of degree q . The family of the characters φ_i consists of $(q - 3)/2$ characters of degree $q + 1$, and the family of the characters ζ_j consists of $(q - 1)/2$ characters each of degree $q - 1$.

Now, we present the character table of the group $L = L_2(q)$, where $q \geq 8$ is even. The group L has a unique linear character 1_L (the principal character). Besides, there is a Steinberg character St of degree q . The family of the characters φ_i consists of $(q - 2)/2$ characters of degree $q + 1$, and the family of the characters ζ_j , each of degree $q - 1$, consists of $q/2$ characters. The character table is taken from [20]:

	1	u	a^r	b^s
1_L	1	1	1	1
St	q	0	1	-1
φ_i	$q + 1$	1	$\alpha^{ir} + \alpha^{-ir}$	0
ζ_j	$q - 1$	-1	0	$\beta^{js} + \beta^{-js}$

The following lemma can be deduced from the orthogonality relations. We give an elementary proof.

Lemma 18. *Let α be a primitive $(q - 1)$ st root of unity in the field of complex numbers, and let $1 \leq j < q - 1$. Let q be odd. If j is odd, then $S = \sum_{r=1}^{(q-3)/2} (\alpha^{jr} + \alpha^{-jr}) = 0$. If j is even, then $S = -2$. If q is even, then $S_1 = \sum_{r=1}^{(q-2)/2} (\alpha^{jr} + \alpha^{-jr}) = -1$.*

Proof. Let q be odd. It is easily seen that S is the sum of the j th powers of all $(q - 1)$ st roots of unity except for 1^j and $(-1)^j$. Since the sum of the j th powers of all $(q - 1)$ st roots of unity is zero for $1 \leq j < q - 1$, we obtain the claim.

The statement concerning the sum S_1 for even q is proved similarly. □

Lemma 19. *Let φ_i be an irreducible character of $D = PGL_2(q)$ that has degree $q + 1$, where $1 \leq i \leq (q-3)/2$ if q is odd and $1 \leq i \leq (q-2)/2$ if q is even. Then the multiplicity of the Steinberg character St in φ_i^2 is equal to 2.*

Proof. Let $D = PGL_2(q)$, where q is odd. Using the character table, we calculate $[\varphi_i^2, St]$. We take into account the fact that the sizes of the conjugacy classes in D are as follows: $|u^G| = q^2 - 1$, $|(a^r)^G| = q(q+1)$ (r takes the values from 1 to $q-3$), and $|z_-^G| = q(q+1)/2$. We denote by $S^{(2)}$ the sum $\sum_{r=1}^{(q-3)/2} (\alpha^{jr} + \alpha^{-jr})^2$. Then

$$[\varphi_i^2, St] = \frac{(q+1)^2q + q(q+1)S^{(2)} + 2q(q+1)}{q(q^2 - 1)}.$$

By Lemma 17, we have

$$S^{(2)} = \sum_{r=1}^{(q-3)/2} (\alpha^{jr} + \alpha^{-jr})^2 = \sum_{r=1}^{(q-3)/2} (\alpha^{jr} + \alpha^{-jr}) + 2 \sum_{r=1}^{(q-3)/2} 1 = q - 5.$$

Substituting this in the above relation, we obtain $[\varphi_i^2, St] = 2$.

Let $D = L = L_2(q)$, where q is even. Then the calculation is similar, but now the sum of squares of the character values $S_1^{(2)} = \sum_{r=1}^{(q-2)/2} (\alpha^{jr} + \alpha^{-jr})^2$ required in the calculation of the dot product is equal to $q-3$, and there is no term corresponding to the contribution of the conjugacy class consisting of involutions. Thus, the result remains the same. □

Lemma 20. (i) *Let φ_l be an irreducible character of $D = PGL_2(q)$ (q is odd) of degree $q+1$ ($1 \leq l \leq (q-5)/4$), and let D_0 be a subgroup of index 2 in D (isomorphic to $L_2(q)$). Then $\varphi_l|_{D_0} \in \text{Irr}(D_0)$.*

(ii) *If γ is an automorphism of $PGL_2(q)$ induced by an automorphism of the field $GF(q)$ and $q \notin \{4, 5, 9\}$, then γ has an exact orbit on the set of characters of $L = L_2(q)$ that have degree $q+1$.*

Proof. (i) The description of the irreducible characters of $D_0 = L_2(q)$ for odd q (see [20]) shows that the restriction of φ_l to D_0 can be the sum of two distinct characters of D_0 only in the case where $q \equiv 1 \pmod{4}$, and, for every $r \leq (q-3)/2$, the value of φ_l at the element $(a^2)^r$ of L must be equal to $2 \cos(4\pi lr/(q-1)) = 2(-1)^r$. It is easy to check that this is possible only if l is a multiple of $(q-1)/4$. Hence, we obtain (i).

(ii) An automorphism σ of order e of the group $P = PGL_2(q)$ (here we consider the cases of even q and odd q simultaneously) induced by an automorphism of $GF(q)$, where $q = p^e$, takes the character φ_j to the character φ_{pj} . Every automorphism of P induced by an automorphism of the field is a power of σ . Now, it suffices to take $j = 1$ and to verify that all e characters $\varphi_1, \varphi_p, \dots, \varphi_{p^{e-1}}$ are distinct.

In the case where q is odd, it is important for us that the automorphism σ have an exact orbit on the set of irreducible characters of the subgroup D_0 , which has index 2 in D . Since D_0 has index 2 in D , we have $a^2 \in D_0$. Considering the values of φ_{p^i} and φ_{p^l} at a^2 , we see that if $\varphi_{p^i} = \varphi_{p^l}$ for some $i < l$, then $\cos(4\pi p^i/(q-1)) = \cos(4\pi p^l/(q-1))$, whence $2(p^l \pm p^i)/(q-1) = k$ for some integer k . Since $q-1 = p^e - 1$ and p is prime to $q-1$, we conclude that $2p^i(p^s \pm 1)/(p^e - 1)$ is an integer for some $s \leq e-1$. Since in this case we have $2(p^{e-1} + 1) \geq p^e$, it follows that $p = 3$ and $e = 2$. Consequently, if $q \neq 9$, then the automorphism σ of order e has an exact orbit on the set of characters of D_0 that have degree $q+1$. This implies the statement for an arbitrary automorphism γ . We note that if $q = 5$, then the set of characters of D_0 that have degree $q+1 = 6$ is empty.

If q is even, then we compare the values of φ_{2^i} and φ_{2^l} at the element $a \in P = L$. As in the preceding case, we conclude that φ_{2^i} and φ_{2^l} can be equal only if $\cos(4\pi 2^i / (2^e - 1)) = \cos(4\pi 2^l / (2^e - 1))$. In other words, $2^{i+1}(2^s - 1) / (2^e - 1)$ is an integer for some $s \leq e - 1$. Since the denominator of this fraction is odd, we see that $(2^s \pm 1) / (2^e - 1)$ is an integer, which is possible only if $e = 2$, i.e., if $q = 4$. The lemma is proved. \square

§7. PROOF OF THEOREMS 1 AND 2

By Lemma 9, a minimal counterexample to Theorem 1 is given by a subgroup G of the wreath product $A \wr S = \mathfrak{G}$ such that G contains the socle N of \mathfrak{G} , where $A = \text{Aut}(L)$ is the automorphism group of the non-Abelian simple group L and S is a solvable subgroup of the symmetry group of degree m .

By Lemma 17, a minimal normal subgroup N of G is the direct product $R_1 \times R_2 \times \dots \times R_m$ of simple groups, each one being isomorphic to the group $L = L_2(q)$ for some q , and the group G acts transitively on the set $\{R_1, R_2, \dots, R_m\}$. Now, we assume that $q \neq 4, 5$, or 9 .

We recall that \mathfrak{M} is a subgroup of \mathfrak{G} normalizing each subgroup R_i . Let \mathfrak{M}_0 be a subgroup of $D_1 \times D_2 \times \dots \times D_m$ lying in \mathfrak{M} , where $D_i \simeq PGL_2(q)$. Put $M = \mathfrak{M} \cap G$ and $M_0 = \mathfrak{M}_0 \cap G$. Next, let x_1, x_2, \dots, x_m be elements of G such that $R_1^{x_i} = R_i$.

The subgroup \mathfrak{M}_0 has the character $\Theta = \eta_1 \eta_2 \dots \eta_m$, where $\eta_1 = \varphi_1$ is an irreducible character of degree $q + 1$ (see the notation in the preceding section) of $D_1 \simeq PGL_2(q)$, and the characters η_i of D_i are defined by the formula

$$\eta_i(x_i^{-1} g x_i) = \eta_1(g)$$

for all $g \in R_1$.

The character Θ , defined at every element $y = y_1 y_2 \dots y_m \in \mathfrak{M}_0$ with $y_i \in A_i$ by the equation $\Theta(y) = \eta_1(y_1) \eta_2(y_2) \dots \eta_m(y_m)$, is an irreducible character of \mathfrak{M}_0 . Since $\Theta|_N$ is an irreducible character of $N = R_1 \times R_2 \times \dots \times R_m$ by Lemma 20(i) and from what has been said above, we see that $\Theta|_N$ is also an irreducible character of the intermediate subgroup M_0 , which lies in \mathfrak{M}_0 and is normal in M .

Let ζ_1 be the Steinberg character (of degree q) of D_1 . As above, we construct an irreducible character $\Xi = \zeta_1 \zeta_2 \dots \zeta_m$ of \mathfrak{M}_0 , where the ζ_i are irreducible characters of degree q of the group D_i . By construction, $\zeta_i = \zeta_1^{x_i^{-1}}$. Using Lemma 20(ii), we can easily check that Ξ is a \mathfrak{G} -invariant character.

The special choice $\eta_1 = \varphi_1$ of the character and the condition $q \notin \{4, 5, 9\}$ ensure that $\eta_1^\gamma \neq \eta_1$ for every field automorphism γ of $PGL_2(q)$. It follows that, for every $a \in M \setminus M_0$, the character Θ^a is distinct from Θ . As noted above, we have $|G : I_G(\Theta)| = l \geq |M : M_0| = t$.

We establish yet another important property:

$$I_G(\Theta)M/M \simeq I_G(\Theta)/(I_G(\Theta) \cap M) = I_G(\Theta)/M_0.$$

Indeed, $M_0 \leq I_G(\Theta) \cap M$. Since every automorphism of M_0 induced by an element of $M \setminus M_0$ does not fix Θ , we have $I_G(\Theta) \cap M = M_0$. The fact that $I_G(\Theta)M/M \leq G/M$ and $I_G(\Theta)/M_0$ are isomorphic has an important consequence.

By Theorem 15, if χ is an irreducible character of G such that $[\chi|_{M_0}, \Theta] \neq 0$, then $\chi|_N = e(\chi) \sum_{i=1}^l \Theta_i$, where $\Theta_1, \Theta_2, \dots, \Theta_l$ are all characters of M_0 conjugate to $\Theta = \Theta_1$ and $e(\chi)$ is a nonnegative integer. We have $\Theta^G = \sum_{i=1}^s e_i \chi_i$, where $s = k_0(I_G(\Theta)/M)$ (see Lemma 16). Also, we have $\sum_{i=1}^s e_i^2 = |I_G(\Theta) : M_0| \leq |G : M|$.

Since G/M is a solvable subgroup of $S \leq S_m$, we see that s does not exceed $3^{m/2}$ by Lemma 2.

Let χ_j be an irreducible character of G involved in the expansion of Θ^G . Since G is an *ASR*-group, we have $\chi_j^2 = \sum_k \delta_{k,j} \chi_k$, where $\chi_k \in \text{Irr}(G)$ and $\delta_{k,j} \in \{0, 1\}$ for all k . On the other hand, $\Xi^G = \sum_{k=1}^d b_k \chi_k$, where $\chi_k \in \text{Irr}(G)$ and the coefficients b_k of the expansion play the same role as the e_i in the expansion of Θ^G . In particular, by the invariance of Ξ , we obtain

$$\sum_{k=1}^d b_k^2 = |G : M_0| = t|G/M|.$$

Here d is the total number of irreducible characters in the expansion of Ξ^G .

Hence, $[\chi_j^2, \Xi^G] = \sum_{i=1}^d \delta_{i,j} b_i$, where $\delta_{i,j} \in \{0, 1\}$. On the other hand, by the Frobenius reciprocity theorem, we have

$$[\chi_j^2, \Xi^G]_G = [\chi_j^2|_{M_0}, \Xi]_{M_0} = e_j^2 \left[\sum_{i=1}^l \sum_{k=1}^l \Theta_i \Theta_k, \Xi \right]_{M_0}.$$

By Lemma 19, $\eta_i^2 = 2\zeta_i + \psi_i$ for some character ψ_i of D_i . Therefore, if $f \leq l$, then $\Theta_f^2 = 2^m \Xi + \Upsilon_f$ for some character Υ_f of M_0 . Hence,

$$(11) \quad \sum_{i=1}^d \delta_{i,j} b_i = [\chi_j^2, \Xi^G]_G \geq e_j^2 l 2^m.$$

By Lemma 16, the number d of irreducible characters of G whose restriction to M_0 contains Ξ does not exceed a number r ($= k_0(G/M_0)$) such that each subgroup of G/M_0 has at most r conjugacy classes.

The group G/M_0 is an extension of M/M_0 by the group S , which is a solvable subgroup of S_m . Therefore, Lemmas 2(i) and 4 imply that $k_0(G/M_0) \leq t3^{m/2}$. Since each subgroup of G/M_0 is an extension of an Abelian group of order dividing t by a solvable subgroup of S_m , we have $d \leq t3^{m/2}$.

Applying the Cauchy–Bunyakovski–Schwarz inequality to (11), we obtain

$$e_j^2 l 2^m \leq t^{1/2} 3^{m/4} \left(\sum_{i=1}^d b_i^2 \right)^{1/2} = t^{1/2} 3^{m/4} |G/M_0|^{1/2}.$$

However, $|G/M_0| = l|I_G(\Theta) : M_0| = l \sum_{i=1}^s e_i^2 \leq l s e^2$, where $e = \max_{i=1}^s e_i$. We recall that $s \leq 3^{m/2}$.

Consequently,

$$(12) \quad e_j^2 l 2^m \leq 3^{m/2} t^{1/2} l^{1/2} e.$$

This inequality is valid for all j . Choosing e_j equal to e , from (12) we deduce that

$$e(l/t)^{1/2} 2^m \leq 3^{m/2}.$$

Since $e \geq 1$ and $l \geq t$, we have $2^m \leq 3^{m/2}$. The contradiction obtained shows that G cannot be an *ASR*-group unless R_1 is isomorphic to $L_2(5) \simeq L_2(4)$ or to $L_2(9)$. This completes the proof of Theorem 2.

As pointed out in the Introduction, Theorem 1 follows directly from Theorem 2.

Almost all simple groups are not *ASR*-groups (the character tables for A_5 , S_5 , and A_6 are well known, and no other composition factors can occur, by Theorem 1). The subgroups of the automorphism group of A_6 that strictly contain A_6 are the following: S_6 , $PGL_2(9)$, M_{10} (the stabilizer of a point in the Mathieu group M_{11}), and the group $P\Gamma L_2(9)$. The number of classes of all such groups, excluding the last of them, does not exceed 11, and each group has an irreducible character of degree 10. By Corollary 2, this implies that none of these groups is an *ASR*-group. The group $P\Gamma L_2(9)$ has 13

conjugacy classes and an irreducible character of degree 20, which rules out this group as well. Thus, the order of a nonsolvable *ASR*-group (if it exists) must be divisible by 3600.

§8. EXAMPLES

In the following examples, we used the computer algebra system GAP [14]. GAP contains the library `SmallGroups`, which consists of small finite groups sorted by order. The call instruction looks like this: `G:=SmallGroup(m,n)`; where `m` is the order of a group and `n` is its number in the set of groups of order `m`. The command `IdSmallGroup(G)`; gives the order and the number of `G` in the form `[m,n]`. The command `Display(CharacterTable(G))`; outputs the character table of `G`.

In what follows, $G_{[m,n]}$ means the group `SmallGroup(m,n)`.

Example 1. The group $G_{[64,190]}$ has the following presentation:

$$G_{[64,190]} = \langle a, b, c \mid a^{16} = b^2 = c^2 = [b, c] = 1, a^b = a^{-1}, a^c = a^7 \rangle.$$

In other words, $G_{[64,190]} = \langle a \rangle \rtimes \langle b, c \rangle \simeq C_{16} \rtimes E_4$. Next, we have $k(G_{[64,190]}) = 16$ and $k(G_{[64,190]}/Z(G_{[64,190]})) = k(C_2 \times D_{16}) = 14$, where $Z(G_{[64,190]}) = \langle a^8 \rangle$. All irreducible characters of $G_{[64,190]}$ have degrees 1, 2, or 4. The two irreducible characters of this group that are not characters of $G_{[64,190]}/Z(G_{[64,190]})$ have degree 4.

All characters of $G_{[64,190]}$ are real, and the squares of irreducible characters can be expanded as sums of irreducible characters with coefficients 0 and 1. At the same time, some characters enter the expansion of the tensor product of two irreducible characters of degree 4 with the coefficient 2. Thus, this group is a real *ASR*-group but not an *SR*-group.

Example 2. The group $G_{[72,24]}$ has the following presentation:

$$G_{[72,24]} = \langle a, b, c, d \mid a^3 = b^3 = c^4 = d^4 = [a, b] = [a, d] = [b, c] = 1, \\ a^c = a^{-1}, b^d = b^{-1}, c^d = c^{-1}, c^2 = d^2 \rangle.$$

Thus, $G_{[72,24]} = \langle a, b \rangle \rtimes \langle c, d \rangle \simeq E_9 \rtimes Q_8$. The group $G_{[72,24]}$ is supersolvable. It can be proved that this group is also an *ASR*-group but not an *SR*-group. It has three pairs of distinct irreducible characters such that the tensor product of characters in each pair contains characters of multiplicity 2.

To obtain an example of a nonreal *ASR*-group, it suffices to consider the direct product of a group from the above examples by the semidihedral group SD_{2^n} , where $n \geq 4$.

Examples 1 and 2 show that the absence of multiplicities greater than 1 in the decomposition of the tensor square of an arbitrary irreducible representation does not imply that the same property is valid for the tensor product of two arbitrary irreducible representations.

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