Алгебра и анализ
Том 19 (2007), № 6

St. Petersburg Math. J.
Vol. 19 (2008), No. 6, Pages 953–959
S 1061-0022(08)01029-7
Article electronically published on August 21, 2008

A VARIANT OF A THEOREM BY SPRINGER

I. PANIN AND U. REHMANN

To the memory of D. K. Faddeev

Abstract. The theorem in question gives a sufficient condition for a quadratic space over a local ring R to contain a hyperbolic plane over R.

Our main aim in this paper is to prove the following result, which is a variant of the Springer theorem [La] for quadratic spaces over local rings.

Theorem. *Let R be a local Noetherian domain that has an infinite residue field of characteristic different from 2. Let $S = R[T]/(F(T))$ be an integral étale extension over R. Let (V, q) be a quadratic space over R such that the S-quadratic space $(V \otimes_R S, q \otimes_R S)$ contains a hyperbolic plane $\mathbb{H}_S$. If the degree of the polynomial $F(T)$ is odd, then the space (V, q) already contains a hyperbolic plane over R.*

This theorem is a main ingredient in the proof of the following result established in [P].

0.1. Theorem. *Let R be a regular local ring, K its field of fractions, and (V, φ) a quadratic space over R. Suppose R contains a field of characteristic zero. If $(V, \varphi) \otimes_R K$ is isotropic over K, then (V, φ) is isotropic over R; i.e., there exists a unimodular vector $v \in V$ with $\varphi(v) = 0$.*

It is well known that any finite *étale* extension S of R has the form $S = R[T]/(F(T))$, where $F(T)$ is a monic separable polynomial. If A is a semilocal ring and (W, ϕ) is a quadratic space over A, then W contains a hyperbolic plane if and only if W contains a unimodular isotropic vector w. A vector w is said to be unimodular if w can be taken as the first vector w_1 of a free A-base $w_1, \dots, w_n$ of the A-module W.

§1. Preliminaries

In this section we formulate two results to be used in the proof of the main theorem. These two results will be proved in §3. We need to fix some notation:

k is an infinite field ($\mathrm{char}(k) \neq 2$);
$f(t)$ is a monic separable polynomial of degree n over k;
$l = k[t]/(f(t))$ is a separable k-algebra;
$\theta = t \mod f(t)$ is an element of l;
(W, ϕ) is a quadratic space over k of rank ≥ 3;
(W_l, ϕ_l) is the quadratic space $(W \otimes_k l, \phi \otimes_k l)$ over l;
$W[t]^{(m)} = W \cdot 1 \oplus W \cdot t \oplus \cdots \oplus W \cdot t^{m-1} \subset W[t] = W \otimes_k k[t]$;
$k[t]^{(m)} = k \cdot 1 \oplus k \cdot t \oplus \cdots \oplus k \cdot t^{m-1} \subset k[t]$;

2000 *Mathematics Subject Classification.* Primary 53A04; Secondary 52A40, 52A10.
Key words and phrases. Quadratic forms, Springer's theorem, local domain.
This work is a part of the project SFB-701 at Fakultät für Mathematik, Universität Bielefeld. The first author is also supported by the Presidium of RAS Program "Fundamental Research", an RFBR-grant, and the INTAS-05-1000008-8118 grant.

$ev : W[t]^{(n)} \to W_l$ is the map given by

$$ev(v_0 + v_1 t + \cdots + v_{n-1}t^{n-1}) = v_0 + v_1\theta + \cdots + v_{n-1}\theta^{n-1};$$

$v(\theta) \stackrel{\text{def}}{=} ev(v(t))$ for any $v(t) \in W[t]^{(n)}$;
$\phi^{(n)} : W[t]^{(n)} \to k[t]^{(2n-1)}$ is the map given by

$$\phi^{(n)}(v_0 + v_1 t + \cdots + v_{n-1}t^{n-1}) = \phi(v_0) + 2(v_0, v_1)\cdot t + \phi(v_1)\cdot t^2 + \cdots + \phi(v_{n-1})\cdot t^{2n-2}.$$

1.1. Proposition. *Suppose the quadratic space (W_l, ϕ_l) contains a hyperbolic plane $\mathbb{H}_l$ as a direct summand. Then there exists a unimodular isotropic vector $w \in W_l$, and for any such vector w there exists an element $v(t) \in W[t]^{(n)}$ satisfying the following conditions*:

(1) *$\phi^{(n)}(v(t)) \in k[t]^{(2n-1)}$ is a polynomial of degree $2n-2$;*
(2) *$\phi^{(n)}(v(t))$ is a separable polynomial;*
(3) *$\langle w, v(\theta)\rangle \in l^\times$, where $l^\times \subset l$ is the group of units of l;*
(4) *$\phi_l(v(\theta)) = 0$ in l.*

1.2. Let A be a semilocal ring, let $\bar{A} = A/\mathrm{Rad}(A)$, and let (U, ψ) be a quadratic space over A. Suppose (U, ψ) contains a hyperbolic plane $\mathbb{H}_A$ as a direct summand. We shall use "bar" for reduction modulo the radical $\mathrm{Rad}(A)$. For example, $(\bar{U}, \bar{\psi})$ is the quadratic space $(U/\mathrm{Rad}(A)\cdot U, \bar{\psi})$ over $\bar{A}$.

1.3. Lemma. *Let $u \in U$ be a unimodular isotropic vector. Then for any isotropic vector $\mathrm{v} \in \bar{U}$ with $\langle \mathrm{v}, \bar{u}\rangle \in \bar{A}^\times$ there exists a vector $v \in U$ such that*

(1) *$\bar{v} = \mathrm{v}$ in $\bar{U}$, and*
(2) *$\psi(v) = 0$.*

Proof. Let v_0 be any vector in U with $\bar{v}_0 = \mathrm{v}$. Then

$$v = \frac{-\langle v_0, v_0\rangle}{2\langle v_0, u\rangle}\cdot u + v_0$$

is the required isotropic vector. Indeed, $\psi(v) = -\langle v_0, v_0\rangle + \langle v_0, v_0\rangle = 0$ and $\bar{v} = \bar{v}_0$, because $\langle \bar{v}_0, \bar{v}_0\rangle = \langle \mathrm{v}, \mathrm{v}\rangle = 0$. Since $\bar{v}_0 = \mathrm{v}$, we have $\bar{v} = \mathrm{v}$. The lemma is proved. □

1.4. Lemma. *Let $k[t]$ be the polynomial ring over the field k, and let (W, ϕ) be the quadratic space over k. Let $w(t) = w_0 + w_1\cdot t + \cdots + w_{n-1}\cdot t^{n-1}$ be an element of $W[t]$ with $w_i \in W$. Suppose that the polynomial $\phi^{(n)}(w(t)) \in k[t]$ is separable, and let $g(t) \in k[t]$ be an irreducible polynomial dividing $\phi^{(n)}(w(t))$. Then $w(t)$ does not vanish modulo $g(t)$.*

Proof. If $w(t)$ vanishes modulo $g(t)$, then $w(t) = g(t)\cdot u(t)$ for an element $u(t) \in W[t]$. In this case we have

$$\phi^{(n)}(w(t)) = \phi^{(n)}(g(t)\cdot u(t)) = g(t)^2\cdot\phi^{(n)}(u(t)) \in k[t].$$

This relation contradicts the separability of $\phi^{(n)}(w(t))$. Thus, $w(t)$ does not vanish modulo $g(t)$. The lemma is proved. □

1.5. Lemma. *Let A be a semilocal ring, and let (U, ψ) be a quadratic space over A. Let $v \in U$ be a unimodular isotropic vector. Then a hyperbolic plane can be split out of (U, ψ), i.e.,*

$$(U, \psi) \cong (U', \psi') \perp \mathbb{H}.$$

Proof. This is easy. □

§2. The proof of the main theorem

2.1. Theorem. *Let R be a local Noetherian domain that has an infinite residue field of characteristic different from 2. Let $S = R[T]/(F(T))$ be an integral étale extension over R. Let (V, q) be a quadratic space over R such that the S-quadratic space $(V \otimes_R S, q \otimes_R S)$ contains a hyperbolic plane $\mathbb{H}_S$. If $\deg F(T)$ is odd, then the space (V, q) contains a hyperbolic plane $\mathbb{H}_R$.*

Proof. The case of rank $(q) = 2$ is obvious. So we may assume rank $(q) \geq 3$. Let $\mathfrak{m}$ be the maximal ideal of R, and let k be the residue field $R/\mathfrak{m}$ of R. Let $l = S/\mathfrak{m}S = k[t]/(f(t))$, where $f(t) = F[T] \mod \mathfrak{m}$. Since S is *étale* over R, so is the k-algebra l. In particular, the k-algebra l is separable.

We shall write (V_R, q_R) for (V, q) and $(\bar{V}, \bar{q})$ for the reduction modulo $\mathfrak{m}$ of the R-quadratic space (V_R, q_R). Let (V_S, q_S) be the scalar extension of (V_R, q_R) up to S, and let $(\bar{V}_l, \bar{q}_l)$ be the reduction modulo $\mathfrak{m}S$ of the S-quadratic space (V_S, q_S). Clearly, $(\bar{V}_l, \bar{q}_l) = (\bar{V}, \bar{q}) \otimes_k l$. Now, set $(W, \phi) = (\bar{V}, \bar{q})$. Then $(W \otimes_k l, \phi \otimes_k l) = (\bar{V}_l, \bar{q}_l)$, and we shall write (W_l, ϕ_l) for $(\bar{V}_l, \bar{q}_l)$.

By assumption, the space (W_l, ϕ_l) contains a hyperbolic plane $\mathbb{H}_l$ as a direct summand. Thus, we can apply Proposition 1.1. So, using the notation of §1, we can find a vector $w \in W_l$ (unimodular and isotropic) and an element $v(t) = v_0 \cdot 1 + v_1 \cdot t + \cdots + v_{n-1} \cdot t^{n-1} \in W[t]^{(n)}$ satisfying the following conditions:

(1) $\phi^{(n)}(v(t)) \in k[t]$ has degree $2n - 2$;
(2) $\phi^{(n)}(v(t))$ is a separable polynomial over k;
(3) $\langle w, v(\theta) \rangle \in l^\times$ is an invertible element of l;
(4) $\phi_l(v(\theta)) = 0 \in l$, where $\theta = t \mod f(t)$ is the element of l.

Recall that the quadratic space (W_l, ϕ_l) is the reduction modulo $\mathfrak{m}S$ of the quadratic space (V_S, q_S). By Lemma 1.3, the element $v = v(\theta) \in W_l$ can be lifted up to a unimodular isotropic vector $\mathrm{v} \in V_S$.

Since $S = R[T]/(F(T))$, we have $V_S = V_R \otimes_R R[T]/(F(T))$. Set $\Theta = T \mod F(T)$. There are elements $\mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1} \in V_R$ such that $\mathrm{v} = \mathrm{v}_0 \cdot 1 + \mathrm{v}_1 \cdot \Theta + \cdots + \mathrm{v}_{n-1} \cdot \Theta^{n-1}$. Consider the element $\mathrm{v}(T) = \mathrm{v}_0 \cdot 1 + \mathrm{v}_1 \cdot T + \cdots + \mathrm{v}_{n-1} \cdot T^{n-1}$ in $V_R \otimes_R R[T]$, and consider the diagram

$$\begin{tikzcd}
W_l & W \times \cdots \times W \arrow[l, "ev"'] \arrow[r, "\phi^{(n)}"] & k \times \cdots \times k \arrow[r, "\mathrm{Disc}"] & k \\
\bar{V}_l \arrow[u, equal] & \bar{V} \times \cdots \times \bar{V} \arrow[l, "ev"'] \arrow[r, "\bar{q}^{(n)}"] \arrow[u, equal] & k \times \cdots \times k \arrow[r, "\mathrm{Disc}"] \arrow[u, equal] & k \arrow[u, equal] \\
V_S \arrow[u] & V_R \times \cdots \times V_R \arrow[l, "Ev"'] \arrow[r, "q_R^{(n)}"] \arrow[u] & R \times \cdots \times R \arrow[r, "\mathrm{Disc}"] \arrow[u] & R \arrow[u]
\end{tikzcd}$$

where $Ev(u_0, \ldots, u_{n-1}) = u_0 \cdot 1 + u_1 \cdot \Theta + \cdots + u_{n-1} \cdot \Theta^{n-1} \in V_S$,

$$q_R^{(n)}(u_0, \ldots, u_{n-1}) = q_R(u_0) + 2 \cdot (u_0, u_1) \cdot T + q_R(u_1) \cdot T^2 + \cdots + q_R(u_n) \cdot T^{2n-2},$$

and $\mathrm{Disc}(a_0, a_1, \ldots, a_{2n-2})$ denotes the discriminant of the polynomial $a_0 + a_1 T + \cdots + a_{2n-2} T^{2n-2}$. The vertical arrows are canonical maps.

Clearly, this diagram commutes, and the maps Ev, ev are isomorphisms. In particular, $v_i = \mathrm{v}_i \mod \mathfrak{m}$ for the components $v_0, \ldots, v_{n-1}$ (respectively, $\mathrm{v}_0, \ldots, \mathrm{v}_{n-1}$) of the element $v(t) \in W[t]$ (respectively, $\mathrm{v}(T) \in V_R[T]$). Thus, the reduction modulo $\mathfrak{m}$ of the polynomial $q_R^{(n)}(\mathrm{v}(T))$ coincides with the polynomial $\phi^{(n)}(v(t))$. The latter polynomial is separable and has degree $2n - 2$ by the choice of $v(t)$.

Since $\mathrm{v}(\Theta) = \mathrm{v}$ by the very choice of the element $\mathrm{v}(T)$, and since v is q_S-isotropic, we have $q_R(\mathrm{v}(\Theta)) = 0$. Thus, $q_R^{(n)}(\mathrm{v}(T))$ vanishes modulo $F(T)$ in the ring $R[T]$; i.e., there exists a polynomial $H(T)$ in $R[T]$ such that

$$(*) \qquad q_R^{(n)}(\mathrm{v}(T)) = F(T) \cdot H(T).$$

After reduction modulo $\mathfrak{m}$, we get the relation

$$(**) \qquad \phi^{(n)}(v(t)) = f(t) \cdot h(t),$$

where $h(t)$ is the reduction of $H(t)$ modulo $\mathfrak{m}$. Since $\deg \phi^{(n)}(v(t)) = 2n-2$ and $\deg f(t) = n$, we see that $\deg h(t) = n-2$. Since $\deg H(T) \leq n-2$, we conclude that $\deg H(T) = n-2$ and $H(T)$ is monic (the highest coefficient is invertible). Next, $\phi^{(n)}(v(t))$ is separable. Thus, $h(t)$ is also separable. This shows that the R-algebra $S' = R[T]/(H(T))$ is an *étale* R-algebra.

Let A denote the class of T modulo $H(T)$ in the ring S'. Then the vector $\mathrm{v}(A) = \mathrm{v}_0 \cdot 1 + \mathrm{v}_1 \cdot A + \cdots + \mathrm{v}_{n-1} \cdot A^{n-1}$ of the quadratic S'-space $(V_{S'}, q_{S'})$ is isotropic. In fact, $q_{S'}(\mathrm{v}(A)) = F(A) \cdot H(A) = 0$ in S'. If the vector $\mathrm{v}(A)$ is unimodular, then a hyperbolic plane $\mathbb{H}_{S'}$ can be split out of the quadratic space $(V_{S'}, q_{S'})$ over S' (see Lemma 1.5). So, in this case we have constructed a finite *étale* extension $S' = R[T]/(H(T))$ such that

$$\deg H(T) = \deg F(T) - 2$$

and the space $(V_{S'}, q_{S'})$ contains a hyperbolic plane $\mathbb{H}_{S'}$ as a direct summand. Repeating this procedure several times, finally we get a direct hyperbolic summand of the quadratic space (V_R, q_R) itself. Thus, to complete the proof of the theorem it remains to check that the vector $\mathrm{v}(A) \in V_{S'}$ is unimodular.

For this, we denote the ring $S'/\mathfrak{m}S'$ by k' and observe that $k' = k[t]/(h(t))$, where, as above, $h(t)$ is the reduction modulo $\mathfrak{m}$ of the polynomial $H(t)$. We denote by $\bar{V}_{k'}$ the k'-module $\bar{V} \otimes_k k'$ and consider the commutative diagram (with the same elements $\mathrm{v}(T)$ and $v(t)$ as above in this proof)

$$\begin{array}{ccccccc}
v(\alpha) & \in & \bar{V}_{k'} & \xleftarrow{\overline{Ev}} & \bar{V}[t]^{(n)} & \ni & v(t) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{v}(A) & \in & V_{S'} & \xleftarrow{Ev} & V_R[T]^{(n)} & \ni & \mathrm{v}(T)
\end{array}$$

Here $Ev(u_0 + u_1 T + \cdots + u_{n-1}T^{n-1}) = u_0 + u_1 \cdot A + \cdots + u_{n-1} \cdot A^{n-1}$ and $\overline{Ev}(w_0 + w_1 \cdot t + \cdots + w_{n-1} \cdot t^{n-1}) = w_0 + w_1 \cdot \alpha + \cdots + w_{n-1} \cdot \alpha^{n-1}$ and $\alpha = t \mod h(t) \in k'$. To check that $\mathrm{v}(A)$ is unimodular, it suffices to verify that $v(\alpha)$ is unimodular. Observe that $\bar{V}_{k'} = \bar{V} \otimes_k k[t]/(h(t))$. Let $h(t) = h_1(t) \cdot \cdots \cdot h_r(t)$ be the factorization of $h(t)$ into irreducible polynomials. Since $h(t)$ is separable, we have $h_i(t) \neq h_j(t)$ for $i \neq j$. Then $\bar{V}_{k'} = \prod_{i=1}^r \bar{V} \otimes_k (k[t]/(h_i(t)))$, so that $v(\alpha)$ is unimodular in $\bar{V}_{k'}$ if and only if the elements $v(t) \in \bar{V}[t]$ do not vanish modulo any of the $h_i(t)$ $(i = 1, \ldots, r)$.

The polynomial $h_i(t)$ divides $h(t)$, and the polynomial $h(t)$ divides $\phi^{(n)}(v(t))$ (see $(*)$). Therefore, $h_i(t)$ divides $\phi^{(n)}(v(t))$. Since $h_i(t)$ is irreducible and $\phi^{(n)}(v(t))$ is separable, Lemma 1.4 shows that $v(t)$ does not vanish modulo $h_i(t)$. Thus, the element $v(\alpha)$ is indeed a unimodular vector in $\bar{V}_{k'}$, and the element $\mathrm{v}(A)$ is a unimodular vector in $V_{S'}$. This completes the proof of the theorem. □

§3. Proof of Proposition 1.1

3.1. Since the quadratic space (W_l, ϕ_l) contains a hyperbolic plane $\mathbb{H}_l$ as a direct summand, we can find a unimodular isotropic vector $w \in W_l$. We choose and fix such a vector.

Now, set $X(l) = \{v \in W_l \mid \phi(v) = 0\}$ and consider the map $\rho_w : X(l) \to l$ taking v to the scalar product $\langle w, v\rangle \in l$. Then, consider the following diagram of sets and their polynomial maps:

$$\begin{tikzcd}
l & X(l) \arrow[l, "\rho_w"'] \arrow[d, hook, "i"] & & Y \arrow[ll] \arrow[d, hook, "j"] & & & & \\
& W_l & W \times \cdots \times W = W[t]^{(n)} \arrow[l, "ev"'] \arrow[r, "\phi^{(n)}"] & k[t]^{(2n-1)} \arrow[r, "\mathrm{Disc}"] & k,
\end{tikzcd}$$

where ev is the map defined at the beginning of §1 (clearly, this map is an isomorphism), and where $Y = ev^{-1}(X(l))$, the maps i, j are inclusions, and $\phi^{(n)}$ is defined at the beginning of §1. The map Disc takes a polynomial $g(t) \in k[t]^{(2n-1)}$ to its discriminant. It is well known that $\mathrm{Disc}(g(t))$ has a polynomial expression in terms of the coefficients of $g(t)$.

This diagram is the diagram of k-rational points and their maps induced by the following diagram:

$$\begin{tikzcd}
R_{l/k}(\mathbb{A}^1_l) & R_{l/k}(X_l) \arrow[l, "\mathcal{E}_w"'] \arrow[d, hook, "i"] & & \mathbb{Y} \arrow[ll, "\cong"'] \arrow[d, hook, "j"] & & & & \\
& R_{l/k}(\mathbb{W}_l) & \mathbb{W} \times \cdots \times \mathbb{W} \arrow[l, "ev"', "\cong"] \arrow[r, "\phi^{(n)}"] & \mathbb{A}^1_k \times \cdots \times \mathbb{A}^1_k \arrow[r, "\mathrm{Disc}"] & \mathbb{A}^1_k
\end{tikzcd}$$

of algebraic varieties. Here $R_{l/k}$ is the Weil restriction functor. Next, $\mathbb{W}$ (respectively, $\mathbb{W}_l$) is the k-vector space (respectively, the l-vector space) viewed as a k-variety (respectively, as an l-variety), X_l is the affine quadric in $\mathbb{W}_l$ given by the equation $\phi_l = 0$, $\mathcal{E}_w$ is the Weil restriction of the morphism $\rho_w : X_l \to \mathbb{A}^1_l$ that takes v to $\langle w, v\rangle$, the variety $\mathbb{Y}$ is the preimage of $R_{l/k}(X_l)$ under the evaluation isomorphism $ev : \mathbb{W}\times\cdots\times\mathbb{W} \to R_{l/k}(\mathbb{W}_l)$, and the morphisms i and j are closed embeddings. The product $\mathbb{W} \times \cdots \times \mathbb{W}$ consists of n factors and the product $\mathbb{A}^1_k \times \cdots \times \mathbb{A}^1_k$ consists of $2n - 1$ factors.

We define open subsets of the variety $\mathbb{Y}$ as follows:
$U_1 = j^{-1}((\phi^{(n)})^{-1}(\mathbb{A}^1_k \times \cdots \times \mathbb{A}^1_k \times (\mathbb{A}^1_k - \{0\})))$,
$U_2 = j^{-1}((\phi^{(n)})^{-1}(\mathrm{Disc}^{-1}(\mathbb{A}^1_k - \{0\})))$,
$U_3 = ev^{-1}(\mathcal{E}_w^{-1}(R_{l/k}(\mathbb{A}^1_l - \{0\}_l)))$.

We need the following lemmas.

3.2. Lemma. *$\mathbb{Y}$ is a k-rational variety.*

3.3. Lemma. *$U_1 \cap U_2 \neq \varnothing$.*

3.4. Lemma. *$U_3 \neq \varnothing$.*

With these three lemmas at hand, we can prove Proposition 1.1 as follows. Since $\mathbb{Y}$ is a k-rational variety, any one of its nonempty open subsets has a k-rational point (even infinitely many points). Thus, by Lemmas 3.3 and 3.4, we can find a point

$$v \in (U_1 \cap U_2 \cap U_3)(k) \subset \mathbb{Y}(k) \subset (\mathbb{W} \times \cdots \times \mathbb{W})(k).$$

Set $v(t) = j(v)$. Then the element $v(t)$ satisfies conditions (1), (2), and (3) of Proposition 1.1 by the very definition of the open sets U_1, U_2, and U_3. Since $v(\theta) \overset{\mathrm{def}}{=} ev(v(t))$, we see that $v(\theta)$ is a k-rational point of the variety $R_{l/k}(X_l)$, i.e., $v(\theta) \in X(l)$. Thus, $\phi_l(v(\theta)) = 0$, and condition (4) is also satisfied. To complete the proof of Proposition 1.1, it remains to prove Lemmas 3.2–3.4.

3.5. Proof of Lemma 3.2. Since $\mathbb{Y}$ is isomorphic to the variety $R_{l/k}(X_l)$, it suffices to check the k-rationality of the latter. The quadric X_l has an l-rational point, so that X_l is an l-rational variety. This implies the existence of nonempty open subvarieties V_1 in X_l and V_2 in $\mathbb{A}_l^{r-1}$ ($r = \dim_k W$) and an isomorphism $\alpha : V_1 \xrightarrow{\cong} V_2$ of l-varieties. Consider the diagram of l-varieties

$$X_l \supset V_1 \xrightarrow{\cong} V_2 \subset \mathbb{A}_l^{r-1}$$

and apply the Weil restriction functor to this diagram. This leads to a diagram

$$R_{l/k}(X_l) \supset R_{l/k}(V_1) \xrightarrow{\cong} R_{l/k}(V_2) \subset R_{l/k}(\mathbb{A}_l^{r-1})$$

of k-varieties. Since the left-hand and the right-hand side inclusions are open embeddings and since the variety $R_{l/k}(\mathbb{A}_l^{r-1})$ is an affine space over k, we conclude that $R_{l/k}(X_l)$ is k-rational. The lemma is proved.

3.6. Proof of Lemma 3.4. The morphism $ev : \mathbb{Y} \to R_{l/k}(X_l)$ is an isomorphism by the very definition of the variety $\mathbb{Y}$. Therefore, it suffices to check that the variety

$$\mathcal{E}_w^{-1}(R_{l/k}(\mathbb{A}_l^1 - \{0\}_l))$$

is nonempty. We show that this variety has k-rational points. For this, we recall that the morphism

$$\mathcal{E}_w : R_{l/k}(X_l) \to R_{l/k}(\mathbb{A}_l^1)$$

induces a map of k-rational points, which coincides with

$$\rho_w : X(l) \to l \quad (v \mapsto \langle v, w\rangle).$$

Take any $v \in X(l)$ with $\langle w, v\rangle \in l^\times$ and observe that $\rho_w(v) \in l^\times$ for this element v and moreover, the group $l^\times$ coincides with the set of k-rational points of the variety $R_{l/k}(\mathbb{A}_l^1 - \{0\}_l)$. Thus, the element v is a k-rational point of the variety $\mathcal{E}_w^{-1}(R_{l/k}(\mathbb{A}_l^1 - \{0\}_l))$. The lemma is proved.

3.7. Proof of Lemma 3.3. We show that already $(U_1 \cap U_2)(l) \neq \varnothing$. For this, we consider an l-basis $e_1, e_2, \dots, e_r$ of the free l-module W_l such that $e_1^2 = 0$, $e_2^2 = 0$, $(e_1, e_2) = 1$, and $(e_1, e_i) = (e_2, e_i) = 0$ for $i \geq 3$. For any polynomial $h(t) \in l[t]^{(n)}$ and any element $e \in W_l$, we set

$$h(t) \cdot e = (a_0 \cdot e) \cdot 1 + (a_1 \cdot e) \cdot t + \cdots + (a_{n-1} \cdot e) \cdot t^{n-1} \in l \otimes_k W[t]^{(n)} = W_l[t]^{(n)}.$$

Since $l = k[t]/(f(t))$ and $\theta = t \mod f(t)$ (see §1 for the definition of θ), we have a unique decomposition

$$f(t) = f^{(1)}(t)(t - \theta) \text{ in } l[t].$$

Let $g(t) \in k[t]$ be a separable polynomial of degree $n-2$ that is coprime to $f(t)$. Consider the element

$$v(t) = f^{(1)}(t) \cdot e_1 + (t - \theta)g(t) \cdot e_2 \in l \otimes_k W[t]^{(n)} = W_l[t]^{(n)}.$$

3.8. *Claim.* $v(t) \in (U_1 \cap U_2)(l)$.

To check this, first we observe that

$$(*) \qquad \phi^{(n)}(v(t)) = 2 \cdot f^{(1)}(t) \cdot (t - \theta) \cdot g(t) \cdot \langle e_1, e_2\rangle = 2f(t) \cdot g(t).$$

Thus,

$$\phi(v(\theta)) = 2 \cdot f(\theta) \cdot g(\theta) = 0$$

in l, and therefore $v(t) \in \mathbb{Y}(l)$.

Next, $(*)$ shows that $\phi^{(n)}(v(t)) \in k[t] \subset l[t]$ and it has degree $2n-2$. So, $v(t) \in U_1(l)$.

Finally, $\phi^{(n)}(v(t))$ is separable, because $f(t)$ and $g(t)$ are separable and coprime polynomials in $k[t]$. Thus, $\mathrm{Disc}(\phi^{(n)}(v(t))) \in k^\times \hookrightarrow l^\times$, whence $v(t) \in U_2(l)$. So, $(U_1 \cap U_2)(l) \ni v(t)$, and consequently, $(U_1 \cap U_2)(l) \neq \varnothing$. The lemma is proved.

References

[La] T. Y. Lam, *The algebraic theory of quadratic forms*, Math. Lecture Note Ser., W. A. Benjamin, Inc., Reading, MA, 1973. MR0396410 (53:277)

[P] I. Panin, *Rationally isotropic quadratic spaces are locally isotropic*, www.math.uiuc.edu/K-theory/0671/2003

St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023, Russia

E-mail address: panin@pdmi.ras.ru

Current address: SFB-701 at Fakultät für Mathematik, Universität Bielefeld, Germany

E-mail address: panin@math.uni-bielefeld.de

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

E-mail address: rehmann@math.uni-bielefeld.de

Received 30/JUL/2007

Originally published in English