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SHARP ESTIMATES FOR SOLUTIONS OF SYSTEMS WITH AFTEREFFECT

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Abstract. Sharp estimates are established for strong solutions of systems of differential-difference equations of both neutral and retarded type.

The approach is based on the study of the resolvent corresponding to the generator of the semigroup of shifts along the trajectories of a dynamical system. In the case of neutral type equations, the Riesz basis property of the subsystem of exponential solutions is used.

§1. Introduction. Statement of the problem and main results

In this paper, we consider the initial value problem for a differential-difference equation of the form

$$\sum_{k=0}^{n}\sum_{j=0}^{m} A_{kj}u^{(j)}(t-h_k) = f(t), \quad t>0, \tag{1.1}$$

$$u(t) = g(t), \quad t\in[-h,0]. \tag{1.2}$$

Here, A_{kj} is a matrix of size $r\times r$ with constant complex entries, and the numbers h_k are shifts, $0=h_0<h_1<\dots<h_n=h$.

We denote by $L(\lambda)$ the characteristic matrix of equation (1.1),

$$L(\lambda) = \sum_{k=0}^{n}\sum_{j=0}^{m} A_{kj}\lambda^j e^{-\lambda h_k},$$

by $l(\lambda)=\det L(\lambda)$ the characteristic quasipolynomial of equation (1.1), and by λ_q the zeros of the function $l(\lambda)$ in ascending order of their modules. Let ν_q be the multiplicity of λ_q as a zero of the function $l(\lambda)$, and let Λ be the set of all zeros of $l(\lambda)$.

We assume that the function $g(t)$ belongs to the Sobolev space $\mathbf{H}:=H^m((-h,0);\mathbb{C}^r)$ and that the right-hand side of $f(t)$ belongs to $L^2((0,T);\mathbb{C}^r)$ for every $T>0$.

Definition 1.1. A vector-valued function $u(t)$ that belongs to the space $H^m((-h,T),\mathbb{C}^r)$ for every $T>0$ is called a *strong solution of problem* (1.1), (1.2) if $u(t)$ solves equation (1.1) almost everywhere on the semiaxis $\mathbb{R}_+$ and satisfies condition (1.2) pointwise.

The following statement about quasipolynomials is well known.

Proposition 1.2. *If* $\det A_{0m}\neq 0$, *then the quantity* $\varkappa_+ := \sup_{\lambda_q\in\Lambda}\operatorname{Re}\lambda_q$ *is finite.*

Now we state the main result of the paper.

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Theorem 1.3. *Suppose* $\det A_{0m} \neq 0$. *Then problem* (1.1), (1.2) *is uniquely solvable, and, for some* M, *its solution* $u(t)$ *satisfies the estimate*

$$\|u\|_{H^m(t-h,\,t)} \le d_1\sqrt{t}\bigg(\int_0^t (t-s+1)^{2(M-1)} e^{2\varkappa_+(t-s)}\|f(s)\|^2_{\mathbb{C}^r}\,ds\bigg)^{1/2} + d_2(t+1)^{M-1}e^{\varkappa_+ t}\|g\|_{\mathbf{H}}, \quad t>h, \tag{1.3}$$

with constants d_1 *and* d_2 *independent of* f *and* g.

The positive integer M occurring in (1.3) is determined by the structure of the set of zeros of $l(\lambda)$. If the separability condition is fulfilled for Λ,

$$\inf_{\lambda_p\neq\lambda_q} |\lambda_p-\lambda_q| > 0, \tag{1.4}$$

then M can be replaced with $N = \max_{\lambda_q\in\Lambda}\nu_q$. For the reader's convenience, we recall the terminology; for more detailed discussion, see the books [1]–[4]. The classification below is given for the case where the term with the highest derivative and with zero lag is nondegenerate. An equation is of *retarded type* if it contains no terms with the highest derivative and a nonzero lag. For equation (1.1), this means that $A_{km}=0$ if $k>0$. An equation is of *neutral type* if it involves terms with the highest derivative and a nonzero lag. In our case, this means that the matrix A_{km} is nonsingular for some $k>0$. Finally, an equation is of *neutral-neutral type* if it contains a term with the highest derivative and maximal lag. In our case, this means that the matrix A_{nm} is nonsingular.

At present, there are many papers where various estimates for solutions of functional-differential equations are obtained (see [1]–[6], and the references therein).

Nevertheless, obtaining sharp (unimprovable) estimates for solutions of the above-mentioned equations still remains an important problem, which plays a significant role in control theory and dynamical systems. In the present paper, we establish such estimates for strong solutions of systems of differential-difference equations of neutral type.

Since equation (1.1) has constant coefficients and constant lags, it would be natural to use the Laplace transform and its inverse to estimate the solutions of (1.1). However, this would not lead to estimates of type (1.3). This is quite clear, because, inverting the Laplace transform, we must move the line of integration to a positive distance ε from the set Λ (of the poles of the function $l^{-1}(\lambda)$). This explains why in the estimates known before the quantity $\varkappa_+$ was replaced by $\varkappa_+ + \varepsilon$ ($\varepsilon>0$).

Our approach is different and, in principle, is of spectral nature. With a given problem (1.1), (1.2) we associate the semigroup of shifts along the trajectories of solutions of the homogeneous equation. The resolvent of the generator $\mathbf{D}$ of that semigroup is a finite-dimensional perturbation of the Volterra operator. Such operators attracted attention of many authors (see, e.g., [7] and the references therein).

The spectrum of the generator $\mathbf{D}$ coincides with the zero set Λ of the characteristic quasipolynomial $l(\lambda)$, and the exponential solutions of (1.1) are generalized eigenvectors of $\mathbf{D}$. Solutions are given in the from of an integral of the resolvent applied to the initial function; the integration contour is a line parallel to the imaginary axis and lying to the right of the spectrum of $\mathbf{D}$.

Next, we use the estimates obtained for the resolvent to represent the solution as the integral along a line lying to the left of the line mentioned above plus a series of exponential solutions corresponding to points of the spectrum Λ that belong to the strip between the two lines. The collection of subspaces formed by such exponential solutions is a Riesz basis in the closure of their linear span, while the integral gives an exponentially small contribution to the estimate of the solution.

For the first time, sharp estimates of solutions of homogeneous equations of neutral type were obtained in [8]–[12], where the case of neutral-neutral type equations

($\det A_{n1} \neq 0$) of the first differential order ($m = 1$) was considered. For solutions of homogeneous neutral-neutral type equations ($\det A_{mn} \neq 0$) of an arbitrary differential order m, sharp estimates were obtained in [13]–[15]. Sharp estimates for solutions of scalar homogeneous equations of an arbitrary differential order m in the scale of Sobolev spaces with an arbitrary index were obtained in [16]. We mention that, in [16], not only the basis with brackets property of the family of exponential solutions, but also the basis property of the family of divided differences constructed from the exponential solutions, was studied. Geometric properties of such families of divided differences were studied in [17]. We note that, in the case of Sobolev spaces with integral index, the estimates obtained in [16] for solutions of scalar homogeneous equations have recently been generalized in [18] to the case of equations with distributed lags representable as Stieltjes integrals without the atomicity condition at the point $-h$.

For nonhomogeneous equations, sharp estimates were obtained in [19] in the scalar case and in [10] in the vector case.

We note that estimate (1.3) is unimprovable in the following sense. For a given equation, the constant $\varkappa_+$ in (1.3) cannot be reduced if we want that the inequality be valid for all right-hand sides and initial conditions (the corresponding examples were given in [10, 13]; we briefly present such examples below). For the scalar case ($r = 1$) and a homogeneous equation, an exact value for the constant M was obtained in [16]; in the vector case, the value of M obtained below may be too large. If the support of the function $f(t)$ is unbounded, then the factor $\sqrt{t}$ cannot be dropped. The corresponding examples were presented in [19, 10].

The present paper is organized as follows. In §2, we give the required estimates for the characteristic quasipolynomial and also estimates for the resolvent of the generator $\mathbf{D}$ of the semigroup of shifts along the trajectories of solutions of the homogeneous equation. The proof of the main result (Theorem 1.3) is given in §3. Estimates for solutions of nonhomogeneous equations are obtained from those for homogeneous equations. Passage to nonhomogeneous equations is done exactly as in [19, 20].

At the end of §3, we present examples showing that our results are sharp and also give some remarks concerning equations of the neutral-neutral type.

In §4 (see the Appendix), we prove the estimate for quasipolynomials stated in §2.

The basic results of the present paper were announced in [21].

§2. Construction of the resolvent and estimates

Before proving the main theorem, we present several auxiliary statements, which are of considerable interest in themselves.

We write

$$f(x) \asymp g(x), \quad x \in X,$$

if the inequalities

$$c\, g(x) \leq f(x) \leq C g(x), \quad x \in X,$$

are valid with positive constants c and C independent of x.

In one-sided estimates of this type we use the symbols $\prec$ and $\succ$. By $\|\cdot\|_{\mathbb{C}^r}$ we denote the vector and the matrix norm in the space $\mathbb{C}^r$.

The fact that the problem in question has a unique solution follows from [8].

Proposition 2.1 (see [8])**.** *Suppose $A_{0m} \neq 0$ and f is a compactly supported function that belongs to $L^2(0,\infty;\mathbb{C}^r)$. Then there exists γ such that, for $g \in \mathbf{H}$, the solution of problem* (1.1), (1.2) *exists, is unique, and satisfies the estimate*

$$\int_0^\infty (\|u^{(m)}(t)\|^2_{\mathbb{C}^r} + \|u(t)\|^2_{\mathbb{C}^r}) e^{-2\gamma t}\, dt \prec \|g\|^2_{\mathbf{H}} + \|f\|^2_{L^2(0,\infty;\mathbb{C}^r)}. \tag{2.1}$$

Consider the semigroup $\mathbf{V}_t$ of bounded operators acting in $\mathbf{H}$ by the rule

$$(\mathbf{V}_t)\, g(\theta) = u(t+\theta), \quad t \geq 0, \quad \theta \in [-h, 0],$$

where $u(t)$ is a solution of the homogeneous ($f(t) \equiv 0$) equation (1.1) corresponding to the initial function $g(\theta)$. Thus, $\mathbf{V}_t$ is the shift operator along the trajectories of solutions of the homogeneous equation (1.1).

We denote by $\mathcal{L}(f)$ the operator acting from the space $H^{m+1}((-h,0),\mathbb{C}^r)$ to $\mathbb{C}^r$ as follows:

$$\mathcal{L}(f) := \sum_{k=0}^{n} \sum_{j=0}^{m} A_{kj} f^{(j)}(-h_k). \tag{2.2}$$

Arguing as in [3, 31, 32], we can prove the following statement.

Proposition 2.2. *Suppose* $\det A_{0m} \neq 0$. *Then the family of operators* $\mathbf{V}_t$, $t > 0$, *forms a* C^0*-semigroup in the space* $\mathbf{H}$; *the generator* $\mathbf{D}$ *of this semigroup has the domain*

$$\operatorname{Dom} \mathbf{D} = \left\{ y \in H^{m+1}((-h,0);\mathbb{C}^r) \mid \mathcal{L}y = 0 \right\} \tag{2.3}$$

and acts by the rule

$$(\mathbf{D}y)\,(\theta) = \frac{d}{d\theta} y(\theta), \quad \theta \in [-h, 0]. \tag{2.4}$$

The proof of Proposition 2.2 is given in the Appendix.

Let G denote the set ρ-distant from the spectrum Λ:

$$G = G(\rho, \Lambda) := \mathbb{C} \backslash \{\lambda \mid \exists \lambda_q \in \Lambda : |\lambda - \lambda_q| < \rho\}.$$

In the sequel, we use the following notation. We introduce the Volterra operator

$$\big(R_V(\lambda) f\big)(\theta) = \int_0^\theta e^{\lambda(\theta-\tau)} f(\tau)\, d\tau, \quad \theta \in [-h, 0],$$

and the finite-dimensional operator

$$\big(R_1(\lambda) f\big)(\theta) = e^{\lambda\theta} L^{-1}(\lambda) \mathcal{L}(R_V f).$$

Lemma 2.3 (see [13, 15])**.** *For the resolvent* $R_{\mathbf{D}}(\lambda)$ *of* $\mathbf{D}$ *we have*

$$R_{\mathbf{D}}(\lambda) = R_V(\lambda) - R_1(\lambda).$$

Proof. The equation $(\mathbf{D} - \lambda) y = f$ has a solution

$$y(\theta) = \int_0^\theta e^{\lambda(\theta-\tau)} f(\tau)\, d\tau + e^{\lambda\theta} c.$$

Since $y \in \operatorname{Dom} \mathbf{D}$, we have

$$\mathcal{L}(y) = \mathcal{L}(R_V(f)) + \mathcal{L}(e^{\lambda\theta}) c = 0.$$

But $\mathcal{L}(e^{\lambda\theta}) = e^{\lambda\theta} L(\lambda)$, and we obtain the claim. □

An explicit expression for the resolvent $R_{\mathbf{D}}(\lambda)$ was obtained in [13], but here we use a different form of the resolvent.

We put

$$\alpha(f) := \sum_{p=0}^{m-1} \frac{f^{(p)}(0)}{\lambda^{p+1}}, \quad \beta(f) := -\sum_{k=0}^{n} \sum_{j=0}^{m} A_{kj} \lambda^j \sum_{p=j}^{m-1} \frac{f^{(p)}(-h_k)}{\lambda^{p+1}}, \tag{2.5}$$

$$\gamma(f) := \frac{1}{\lambda^m} \sum_{k=0}^{n} \sum_{j=0}^{m} A_{kj} \lambda^j e^{-\lambda h_k} \int_0^{-h_k} e^{-\lambda\tau} f^{(m)}(\tau)\, d\tau. \tag{2.6}$$

Theorem 2.4. *We have*

$$R_{\mathbf{D}}(\lambda)f = R_V(\lambda)f - e^{\lambda\theta}\alpha(f) + e^{\lambda\theta}L^{-1}(\lambda)\big(\beta(f)+\gamma(f)\big). \tag{2.7}$$

The proof of the theorem is given in the Appendix.

To integrate the resolvent $R_{\mathbf{D}}(\lambda)$ along a contour, we need the following estimates.

Lemma 2.5. *If* $|\operatorname{Re}\lambda| \prec 1$, *then*:

$$\|R_V f\|_{L^2} \prec \|f\|_{L^2}, \tag{2.8}$$

$$\|\beta(f)\|_{\mathbb{C}^r} \prec \frac{1}{|\lambda|+1}\|f\|_{\mathbf{H}}, \tag{2.9}$$

$$\|\gamma(f)\|_{\mathbb{C}^r} \prec \|f\|_{\mathbf{H}}. \tag{2.10}$$

Proof. Estimate (2.8) follows from the Cauchy–Bunyakovskii inequality. Relation (2.9) is a consequence of embedding theorems (see [22]).

Relation (2.10) follows from the identity

$$e^{-\lambda h_k}\int_0^{-h_k} e^{-\lambda\tau}\psi(\tau)\,d\tau = \int_0^{h_k} e^{-\lambda s}\psi(s-h_k)\,ds,$$

which can be checked by the change $s=\tau+h_k$. The lemma is proved. □

Lemma 2.6. *Suppose* $\det A_{0m} \neq 0$. *Then*:

a) *there exists* $\varkappa$ *such that*

$$\|L^{-1}(\lambda)\|_{\mathbb{C}^r} \prec |\lambda|^{-m} \tag{2.11}$$

for all λ *with* $\operatorname{Re}\lambda > \varkappa$;

b) *if* $|\operatorname{Re}\lambda| \asymp 1$, *then*

$$\|L^{-1}(\lambda)\|_{\mathbb{C}^r} \prec |\lambda|^{-m} \tag{2.12}$$

on G.

The proof of Lemma 2.6 is given in the Appendix.

In the sequel, we denote by $\|\cdot\|$ the norms of operators acting in $\mathbf{H}$, and by $\|\cdot\|_{L^2}$ the norm in $L^2(-h,0)$.

Lemma 2.7. *Suppose* $\det A_{0m} \neq 0$. *Then, for* $|\operatorname{Re}\lambda| \prec 1$ *and* $\lambda \in G$, *the operator-valued function* $R_{\mathbf{D}}(\lambda)$ *is bounded,*

$$\|R_{\mathbf{D}}\| \le \text{const}\,.$$

Proof of Lemma 2.7. Obviously,

$$\|R_{\mathbf{D}}f\|_{\mathbf{H}}^2 \asymp \|(R_{\mathbf{D}}f)^{(m)}\|_{L^2}^2 + \|R_{\mathbf{D}}f\|_{L^2}^2.$$

The first term provides the main contribution. By Lemma 4.1 (iii) with $j=m$, we obtain

$$(R_{\mathbf{D}}f)^{(m)} = \int_0^\theta e^{\lambda(\theta-\tau)}f^{(m)}(\tau)\,d\tau - \lambda^m e^{\lambda\theta}L^{-1}(\lambda)\big(\beta(f)+\gamma(f)\big). \tag{2.13}$$

By (2.13), we have

$$\begin{aligned}\|(R_{\mathbf{D}}f)^{(m)}\|_{L^2}^2 \le{}& \left\|\int_0^\theta e^{\lambda(\theta-\tau)}f^{(m)}(\tau)\,d\tau\right\|_{L^2}\\ &+ |\lambda^m|\big\|L^{-1}(\lambda)\big\|_{L^2}\big\|e^{\lambda\theta}\big\|_{L^2}(\|\beta(f)\|_{\mathbb{C}^r}+\|\gamma(f)\|_{\mathbb{C}^r}).\end{aligned} \tag{2.14}$$

Relation (2.11) implies

$$\|(R_{\mathbf{D}}f)^{(m)}\|_{L^2} \le K(\lambda)\|f\|_{\mathbf{H}} + K(\lambda)(\|\beta(f)\|_{\mathbb{C}^r}+\|\gamma(f)\|_{\mathbb{C}^r}),$$

where

$$K(\lambda) = \Big[\frac{1 - e^{-2\Re\lambda h}}{2\operatorname{Re}\lambda}\Big]^{1/2}.$$

Applying estimates (2.9) and (2.10), we see that

$$\|(R_{\mathbf{D}} f)^{(m)}\|_{L^2} \le K(\lambda)\|f\|_{\mathbf{H}} + K(\lambda)(1/|\lambda| + 1)\|f\|_{\mathbf{H}}.$$

The lemma is proved. □

§3. Proof of the main theorem

First, we estimate a solution u of problem (1.1), (1.2) with homogeneous equation (1.1) ($f \equiv 0$). Let $U(t)$ be the following family of elements of $\mathbf{H}$:

$$\big[U(t)\big](\theta) = [\mathbf{V}_t g](\theta) = u(t+\theta), \quad \theta \in [-h, 0], \ \ t \ge 0,$$

where u is a solution of the homogeneous equation (1.1) corresponding to a given function $g \in \mathbf{H}$. Following [3], we write the initial homogeneous problem (1.1), (1.2) as the abstract Cauchy problem

$$\frac{dU}{dt} = \mathbf{D}U(t), \quad t > 0, \ \ U|_{t=0} = g \tag{3.1}$$

in the Hilbert space $\mathbf{H}$.

It is well known [23]–[25] that a solution $U(t)$ of problem (3.1) can be represented in the form

$$U(t) = \text{P.V.}\,\frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} R_{\mathbf{D}}(\lambda) e^{\lambda t} g\, d\lambda, \quad t > 0, \ \ \varkappa > \varkappa_+. \tag{3.2}$$

First, we consider the "smooth case": $g = R^2_{\mathbf{D}}(\lambda_0)\varphi$, $\varphi \in \mathbf{H}$, where λ_0 belongs to the resolvent set of the operator $\mathbf{D}$ in the half-plane $\{\operatorname{Re}\lambda > \max(\varkappa, 0)\}$. We note (see, e.g., [24]) that the set of $g = R^2_{\mathbf{D}}(\lambda_0)\varphi$, where φ runs through the space $\mathbf{H}$, is dense in $\mathbf{H}$.

The Hilbert identity (see, e.g., [24]) yields

$$R_{\mathbf{D}}(\lambda) g = \frac{R_{\mathbf{D}}(\lambda)}{(\lambda - \lambda_0)^2}\varphi - \frac{R_{\mathbf{D}}(\lambda_0)}{(\lambda - \lambda_0)^2}\varphi - \frac{R^2_{\mathbf{D}}(\lambda_0)}{\lambda - \lambda_0}\varphi. \tag{3.3}$$

Substituting the right-hand side of (3.3) in (3.2) and using the Jordan lemma, we obtain

$$U(t) = \frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} \frac{R_{\mathbf{D}}(\lambda)}{(\lambda - \lambda_0)^2} e^{\lambda t}\varphi\, d\lambda, \quad t > 0. \tag{3.4}$$

It is well known [1, Chapter 12] that if $\det A_{0m} \ne 0$, then the zeros λ_q of the quasipolynomial $l(\lambda)$ lie in the union of the strip $-a < \operatorname{Re}\lambda < \varkappa$ and logarithmic strips, i.e., sets of the form

$$-\ln|\operatorname{Im}\lambda| - \alpha < \operatorname{Re}\lambda < -\ln|\operatorname{Im}\lambda| + \alpha.$$

We choose $a > 0$ so large that the line $\operatorname{Re}\lambda = -a$ contains no points of the spectrum Λ and every strip of the form $-b < \operatorname{Re}\lambda < -a$ contains only a finite number of points of the spectrum. In other words, the line $\operatorname{Re}\lambda = a$ separates the neutral branches from logarithmic ones.

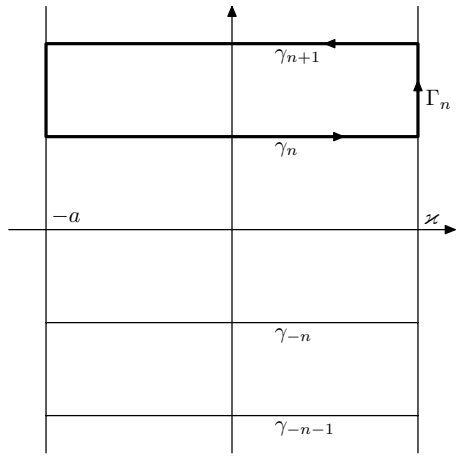

FIGURE 1

Proposition 3.1. *Suppose* $\det A_{0m} \neq 0$. *Then there exists a family of rectangular contours* $\Gamma_n \in G(\rho, \Lambda)$ *of the form*

$$\begin{aligned}\Gamma_n = \{\lambda \,\big|\, \operatorname{Im}\lambda = \gamma_n, -a \le \operatorname{Re}\lambda \le \varkappa\}\\ \cup \{\lambda \,\big|\, \operatorname{Re}\lambda = \varkappa, \gamma_n \le \operatorname{Im}\lambda \le \gamma_{n+1}\}\\ \cup \{\lambda \,\big|\, \operatorname{Im}\lambda = \gamma_{n+1}, -a \le \operatorname{Re}\lambda \le \varkappa\}\\ \cup \{\lambda \,\big|\, \operatorname{Re}\lambda = -a, \gamma_n \le \operatorname{Im}\lambda \le \gamma_{n+1}\}\end{aligned}$$

such that the sequence γ_n *satisfies*

$$0 < \delta \le \gamma_{n+1} - \gamma_n \le \Delta \le \infty,$$

the constants δ *and* Δ *do not depend on* $n \in \mathbb{Z}$, *and the multiplicities* m_n *of the roots* λ_q *lying in the rectangles* G_n *with boundaries* Γ_n *are bounded uniformly in* n:

$$M := \max_n m_n < \infty. \tag{3.5}$$

Moreover, estimate (2.12) *is fulfilled on* Γ_n *uniformly in* n.

The proof of Proposition 3.1 is given in the Appendix.

Note that, in the case of equations of retarded type (e.g., for the first order scalar equation $y'(t) + \sum_0^n A_{k0} y(t - h_k) = f$), it may happen that only finitely many zeros λ_q of the function $l(\lambda)$ lie in the strip $\{\lambda : -a < \operatorname{Re}\lambda < \varkappa\}$; in other words, there are no neutral branches.

Most interesting and meaningful is the case of "general position", where both neutral and retarded branches are present. This is certainly the case if there is an index k_0, $1 \le k_0 \le n$, such that $\det A_{k_0 m} \neq 0$.

We denote by $x_{q,j,0}$ the elements of the canonical system of generalized eigenvectors of $L(\lambda)$ corresponding to the eigenvalue λ_q and by $x_{q,j,s}$ the generalized eigenvectors of order s (j shows the index of the vector $x_{q,j,0}$ in the special basis of the subspace of solutions for the equation $L(\lambda_q)x = 0$).

We consider the following sequence of exponential solutions of the homogeneous equation (1.1):

$$e_{q,j,s}(t) = e^{\lambda_q t}\Big(\frac{t^s}{s!} x_{q,j,0} + \frac{t^{s-1}}{(s-1)!} x_{q,j,1} + \dots + x_{q,j,s}\Big).$$

It can be proved that, for $t \in [-h, 0]$, the solutions $e_{q,j,s}(t)$ form a family of eigenfunctions and generalized eigenfunctions of the operator $\mathbf{D}$.

We note that the exponential solutions $e_{q,j,s}(t)$ of homogeneous ($f \equiv 0$) equation (1.1) satisfy this equation for all $t \in \mathbb{R}$.

Consider the following set of functions:

$$U_n(t) = \frac{1}{2\pi i} \int_{\Gamma_n} R_{\mathbf{D}}(\lambda) e^{\lambda t} g \, d\lambda$$

(for a fixed t, this is an element of $\mathbf{H}$). It is well known [25] that

$$U_n(t) = e^{t\mathcal{P}_n} \mathcal{P}_n g,$$

where

$$\mathcal{P}_n = \frac{1}{2\pi i} \int_{\Gamma_n} R_{\mathbf{D}}(\lambda) \, d\lambda$$

is the Riesz projection onto the generalized eigenspace $\mathcal{P}_n\mathbf{H}$ of $\mathbf{D}$,

$$\mathcal{P}_n\mathbf{H} = \operatorname{Span}_{\lambda_q \in G_n} \{e_{q,j,s}(\tau),\ -h \le \tau \le 0\}.$$

Here $m_n = \dim \mathcal{P}_n\mathbf{H}$. We use the following result [26]:

$$\|e^{t\mathcal{P}_n}\| \le C(m_n) e^{\varkappa_n t} (1+t)^{m_n - 1} \|\mathcal{P}_n\|^{m_n - 1}, \quad t > 0,$$

where $\varkappa_n := \sup_{\lambda_q \in G_n} \operatorname{Re} \lambda_q$ and $C(m_n)$ is a constant depending only on m_n. In our case, we have $m_n \le M$, $\varkappa_n \le \varkappa_+$, so that $C(m_n) < \text{const}$,

$$\|e^{t\mathcal{P}_n}\| \le \text{const}\ e^{\varkappa_+ t} (1+t)^{M-1} \|\mathcal{P}_n\|^{m_n - 1}, \quad t > 0. \tag{3.6}$$

The following result was proved in [13, 15].

Theorem 3.2. *Suppose* $\det A_{0m} \ne 0$. *Then the family of subspaces* $\{\mathcal{P}_n\mathbf{H}\}$ *forms a Riesz basis in the closure of its linear span.*

The proof of Theorem 3.2 is quite cumbersome and technical; it consists in checking the assumptions of Lemma 5 in [27] for the resolvent $R_{\mathbf{D}}(\lambda)$.

This theorem implies an important estimate for the part of the solution that corresponds to the part of the spectrum Λ lying in the strip $\{\lambda : -a < \operatorname{Re} \lambda < \varkappa\}$.

Lemma 3.3. *Suppose* $\det A_{0m} \ne 0$. *Then*

$$\Big\| \sum_n U_n(t) \Big\|_{\mathbf{H}} \prec (t+1)^{(M-1)} e^{\varkappa_+ t} \|g\|_{\mathbf{H}}. \tag{3.7}$$

Proof. Since the function $e^{t\mathcal{P}_n}\mathcal{P}_n g$ lies in $\mathcal{P}_n\mathbf{H}$ for each t, we obtain

$$\Big\| \sum_n U_n(t) \Big\|_{\mathbf{H}}^2 = \Big\| \sum_n e^{t\mathcal{P}_n}\mathcal{P}_n g \Big\|_{\mathbf{H}}^2 \asymp \sum_n \|e^{t\mathcal{P}_n}\mathcal{P}_n g\|_{\mathbf{H}}^2$$

by the Riesz basis property of the family $\{\mathcal{P}_n\mathbf{H}\}$. Next, using (3.6), we have

$$\Big\| \sum_n U_n(t) \Big\|_{\mathbf{H}}^2 \prec (t+1)^{2(M-1)} e^{2\varkappa_+ t} \sum_n \|\mathcal{P}_n g\|_{\mathbf{H}}^2.$$

Applying the basis property of $\{\mathcal{P}_n\mathbf{H}\}$ once again, we get

$$\Big\| \sum_n U_n(t) \Big\|_{\mathbf{H}}^2 \prec (t+1)^{2(M-1)} e^{2\varkappa_+ t} \|g\|_{\mathbf{H}}^2.$$

The lemma is proved. □

In the representation (3.4), we want to pass from integration along the line $\varkappa + i\mathbb{R}$ to integration along the line $-a + i\mathbb{R}$. For this, we introduce the following system of contours:

$$L_n = L_n^{(1)} \cup L_n^{(2)} \cup L_n^{(3)} \cup L_n^{(4)} \cup L_n^{(5)},$$

where

$$\begin{aligned}
L_n^{(1)} &= \{\lambda \,|\, \lambda = \varkappa + i\mu,\ -\infty \le \mu \le \gamma_{-n}\},\\
L_n^{(2)} &= \{\lambda \,|\, \lambda = \xi + i\gamma_{-n},\ -a \le \xi \le \varkappa\},\\
L_n^{(3)} &= \{\lambda \,|\, \lambda = -a + i\mu,\ \gamma_{-n} \le \mu \le \gamma_n\},\\
L_n^{(4)} &= \{\lambda \,|\, \lambda = \xi + i\mu,\ -a \le \xi \le \varkappa\},\\
L_n^{(5)} &= \{\lambda \,|\, \lambda = \varkappa + i\mu,\ \gamma_n \le \mu \le +\infty\}.
\end{aligned}$$

Let K_n be the contour

$$K_n = L_n^{(2)} \cup L_n^{(3)} \cup L_n^{(4)} \cup L_n^{(6)},$$

where

$$L_n^{(6)} = \{\lambda \,|\, \lambda = \varkappa + i\mu,\ \gamma_{-n} \le \mu \le \gamma_n\}.$$

Here, the contour K_n is oriented counterclockwise and the line L_n is directed upwards (see Figure 2).

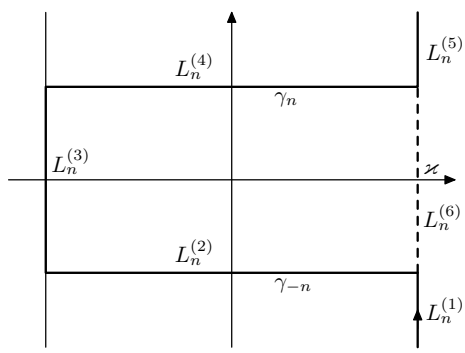

FIGURE 2

It is easy to check the formula

$$\frac{1}{2\pi i}\int_{\varkappa - i\infty}^{\varkappa + i\infty} e^{\lambda t}\frac{R_{\mathbf{D}}(\lambda)}{(\lambda-\lambda_0)^2}\varphi\, d\lambda = \frac{1}{2\pi i}\int_{L_n} e^{\lambda t}\frac{R_{\mathbf{D}}(\lambda)}{(\lambda-\lambda_0)^2}\varphi\, d\lambda + \frac{1}{2\pi i}\int_{K_n} e^{\lambda t}\frac{R_{\mathbf{D}}(\lambda)}{(\lambda-\lambda_0)^2}\varphi\, d\lambda\,.$$

We recall that φ is an element of $\mathbf{H}$. The Cauchy theorem and (3.4) yield

$$\frac{1}{2\pi i}\int_{K_n} e^{\lambda t}\frac{R_{\mathbf{D}}(\lambda)}{(\lambda-\lambda_0)^2}\varphi\, d\lambda = \frac{1}{2\pi i}\int_{K_n} e^{\lambda t}R_{\mathbf{D}}(\lambda) g\, d\lambda = \sum_{k=-n}^{n} U_k(t). \tag{3.8}$$

Fixing t and letting $n \to +\infty$, we see that

$$\int_{L_n} e^{\lambda t}\frac{R_{\mathbf{D}}(\lambda)}{(\lambda-\lambda_0)^2}\varphi\, d\lambda \ \to\ \int_{-a-i\infty}^{-a+i\infty} e^{\lambda t}\frac{R_{\mathbf{D}}(\lambda)}{(\lambda-\lambda_0)^2}\varphi\, d\lambda,$$

because the integrals along $L_n^{(1)}$, $L_n^{(2)}$, $L_n^{(4)}$, and $L_n^{(5)}$ tend to zero. Indeed, $R_{\mathbf{D}}(\lambda)$ is uniformly bounded on the contours Γ_n by Lemma 2.7.

Passing to the limit as $n \to \infty$ in (3.8), we obtain

$$\frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty} e^{\lambda t}\frac{R_{\mathbf{D}}(\lambda)}{(\lambda-\lambda_0)^2}\varphi\,d\lambda = \frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} e^{\lambda t}\frac{R_{\mathbf{D}}(\lambda)}{(\lambda-\lambda_0)^2}\varphi\,d\lambda + \sum_{k=-\infty}^{k=\infty} U_k(t). \tag{3.9}$$

Thus, we arrive at the following formula for the solution:

$$U(t) = U_B(t) + W(t) = U_B(t) + \frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} e^{\lambda t}\frac{R_{\mathbf{D}}(\lambda)}{(\lambda-\lambda_0)^2}\varphi\,d\lambda, \tag{3.10}$$

where the first term $U_B(t)$ corresponding to the neutral branches can be represented as the series

$$U_B(t) = \sum_n U_n(t).$$

To study $[W(t)](\theta)$, we need the following statement.

Lemma 3.4. *We have*

$$\text{(i)} \qquad \text{P.V.}\int_{-a-i\infty}^{-a+i\infty} e^{\lambda t}\frac{R_{\mathbf{D}}(\lambda_0)\varphi}{(\lambda-\lambda_0)^2}\,d\lambda = 0, \quad t>0;$$

$$\text{(ii)} \qquad \text{P.V.}\int_{-a-i\infty}^{-a+i\infty} e^{\lambda t}\frac{R^2_{\mathbf{D}}(\lambda_0)\varphi}{(\lambda-\lambda_0)}\,d\lambda = 0, \quad t>0.$$

Proof. We prove (ii); relation (i) is proved in the same way. By the Cauchy theorem, we have

$$\int_{-a-iR}^{-a+iR} e^{\lambda t}\frac{R^2_{\mathbf{D}}(\lambda_0)\varphi}{(\lambda-\lambda_0)}\,d\lambda + \int_{C_R} e^{\lambda t}\frac{R^2_{\mathbf{D}}(\lambda_0)\varphi}{(\lambda-\lambda_0)}\,d\lambda = 0, \quad t>0,$$

where C_R is the semicircle $\{\lambda \,|\, \lambda = Re^{i\varphi} - a,\ \frac{\pi}{2} \le \varphi \le \frac{3\pi}{2}\}$.

By the Jordan lemma, we have

$$\lim_{R\to+\infty}\int_{C_R} e^{\lambda t}\frac{R^2_{\mathbf{D}}(\lambda_0)\varphi}{(\lambda-\lambda_0)}\,d\lambda = 0,$$

and the required statement follows. The lemma is proved. □

Thus, relation (3.10), Lemma 3.4, and relation (3.3) imply

$$W(t) = \text{P.V.}\,\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} e^{\lambda t}R_{\mathbf{D}}(\lambda)g\,d\lambda. \tag{3.11}$$

Lemma 3.5. *We have*

$$\text{(i)}\ \ \text{P.V.}\,\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} e^{\lambda t}R_V(\lambda)g\,d\lambda = 0, \quad t>h;$$

$$\text{(ii)}\ \ \text{P.V.}\,\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} e^{\lambda(t+\theta)}\alpha(g)\,d\lambda = 0, \quad t>h.$$

The proof of this lemma is given in the Appendix.

Thus, the representation (3.11) leads to the formula

$$[W(t)](\theta) = \text{P.V.}\,\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} e^{\lambda(t+\theta)}L^{-1}(\lambda)(\beta(g)+\gamma(g))\,d\lambda. \tag{3.12}$$

Lemma 3.6. *For the solution $U(t)$ of the problem in question, we have*

$$u(t) = \lim_{\theta\to+0} U(t)(\theta) = u_B(t) + w(t),$$

where

$$u_B(t) = \lim_{\theta\to+0} U_B(t)(\theta), \tag{3.13}$$

and

$$w(t) := \lim_{\theta\to+0} W(t)(\theta) = \frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} e^{\lambda t}\Psi(\lambda, g)\,d\lambda, \tag{3.14}$$

where $\Psi(\lambda, g) = L^{-1}(\lambda)(\beta(g) + \gamma(g))$.

Proof of Lemma 3.6. We can pass to the limit in (3.13) because, for a fixed t, the function $U_B(t)(\theta)$ of θ is an element of the space $\mathbf{H} = H^m((-h, 0), \mathbb{C}^r)$. The proof of the fact that the integral in (3.12) converges uniformly is given is the Appendix. □

Now, we estimate the function $w(t)$.

Lemma 3.7. *The function* $e^{at}w(t)$ *belongs to* $W_2^m((-h, +\infty), \mathbb{C}^r)$ *and satisfies the inequality*

$$\|e^{at}w(t)\|_{W_2^m((-h,+\infty),\mathbb{C}^r)} \prec \|g\|_{\mathbf{H}}. \tag{3.15}$$

Proof of Lemma 3.7. By estimates (2.12), (2.9), and (2.10), the function $\Psi(-a + i\mu, g)$ of the variable μ belongs to the space $L^2(\mathbb{R}, \mathbb{C}^r)$. Since the Fourier transformation is unitary, the function

$$\Xi(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\mu t}\Psi(-a + i\mu, g)\,d\mu$$

belongs to $L^2(\mathbb{R}, \mathbb{C}^r)$.

Note that $e^{at}w(t) = \Xi(t)$ for $t > h$, and

$$\Xi^{(m)}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\mu t}(i\mu)^m\Psi(-a + i\mu, g)\,d\mu.$$

Estimates (2.12), (2.9), and (2.10) imply that $(i\mu)^m\Psi(-a+i\mu, g) \in L^2(\mathbb{R}, \mathbb{C}^r)$, whence $\Xi^{(m)}(t) \in L^2(\mathbb{R}, \mathbb{C}^r)$.

Now, we prove (3.15). Since

$$(i\mu)^m\Psi(-a + i\mu, g) = (i\mu)^m L^{-1}(-a + i\mu)\beta(g) + (i\mu)^m L^{-1}(-a + i\mu)\gamma(g),$$

we estimate each term on the right-hand side of the latter relation.

By (2.12) and (2.9),

$$\|(i\mu)^m L^{-1}(-a + i\mu)\beta(g)\|_{L^2(\mathbb{R},\mathbb{C}^r)} \prec \|g\|_{\mathbf{H}}. \tag{3.16}$$

Observe that the functions

$$\tilde{f}(\lambda) = \int_{-h}^{0} e^{\lambda t}f(t)\,dt, \quad f \in L^2[-h, 0],$$

satisfy the estimate

$$\|\tilde{f}(-a + i\mu)\|_{L^2(\mathbb{R},\mathbb{C}^r)} \prec \|f\|_{L^2[-h,0]}.$$

Therefore, formula (2.6) for $\gamma(g)$ yields the inequality

$$\|\gamma(g)(-a + i\mu)\|_{L^2(\mathbb{R},\mathbb{C}^r)} \prec \|g\|_{\mathbf{H}}.$$

Thus, by (2.12) we have

$$\|(i\mu)^m L^{-1}(-a + i\mu)\gamma(g)(-a + i\mu)\|_{L^2(\mathbb{R},\mathbb{C}^r)} \prec \|g\|_{\mathbf{H}}. \tag{3.17}$$

Using inequalities (3.16) and (3.17), we see that

$$\|(i\mu)^m\Psi(-a + i\mu, g)\|_{L^2(\mathbb{R},\mathbb{C}^r)} \prec \|g\|_{\mathbf{H}}.$$

Recalling that the Fourier transformation is unitary, we obtain the inequality

$$\|\Xi\|_{L^2(\mathbb{R},\mathbb{C}^r)} + \|\Xi^{(m)}\|_{L^2(\mathbb{R},\mathbb{C}^r)} \prec \|g\|_{\mathbf{H}},$$

which implies (3.15). □

Now, we complete the proof of the main theorem. We note that inequality (3.15) implies the following estimate:

$$\|w(t)\|_{W_2^m(t-h,t)} \prec e^{-at}\|g\|_{\mathbf{H}}.$$

Indeed,

$$\int_{t-h}^{t}(\|w(\tau)\|^2+\|w^{(m)}(\tau)\|^2)\,d\tau=\int_{t-h}^{t}(\|e^{-a\tau}\Xi(\tau)\|^2+\|\left(e^{-a\tau}\Xi(\tau)\right)^{(m)}\|^2)\,d\tau$$
$$\prec\int_{t-h}^{t}e^{-2a\tau}\left[\sum_{j=0}^{m}\|\Xi^{(j)}(\tau)\|^2_{\mathbb{C}^r}\right]d\tau\prec e^{-2a(t-h)}\|\Xi\|^2_{W^m(t-h,t;\mathbb{C}^r)}$$
$$\prec e^{-2a(t-h)}\|\Xi\|^2_{W^m(\mathbb{R};\mathbb{C}^r)}\prec e^{-at}\|g\|_{\mathbf{H}}.$$

Finally, combining the estimates for the basic part with the estimate for the norm of $w(t)$, we obtain

$$\|u\|_{W_2^m(t-h,t)}\prec(t+1)^{M-1}e^{\varkappa_+t}\|g\|_{\mathbf{H}}+e^{-at}\|g\|_{\mathbf{H}}\prec(t+1)^{M-1}e^{\varkappa_+t}\|g\|_{\mathbf{H}}.$$

This proves the theorem for the case of a homogeneous equation. □

Now, we outline the proof of the main estimate (1.3) for the case of a nonhomogeneous equation ($f\neq 0$). Suppose $g=0$ and $t=Rh$, $R\in\mathbb{N}$. We represent the right-hand side f of equation (1.1) in the form

$$f=\sum_0^\infty f_j=\sum_0^\infty\chi_{[jh,jh+h]}f,$$

where $\chi_{[jh,jh+h]}$ is the characteristic function of the interval $[jh,jh+h]$. Then the solution is the sum $u=\sum_{j=1}^R u_j$ of the solutions u_j corresponding to the right-hand sides f_j. An elementary inequality implies the estimate

$$\|u\|^2_{H^m(Rh-h,Rh)}\le\left(\sum_{j=1}^R\|u_j\|_{H^m(Rh-h,Rh)}\right)^2\le R\sum_{j=1}^R\|u_j\|^2_{H^m(Rh-h,Rh)}.$$

For each j, the system is "nonhomogeneous" only on the interval $[jh,jh+h]$ of length h. On this interval, we estimate the solution with the help of (2.1); on $[0,jh]$ and on $[jh+h,\infty)$, we use the estimates proved for the homogeneous equation. The factor $\sqrt{t}$ is obtained from the factor R because $R=t/h$. A complete proof is presented in [19].

We give examples showing that estimate (1.3) is sharp.

Example 1 ([10])**.** Consider the following homogeneous equation:

$$\frac{du(t)}{dt}-au(t)-\frac{du(t-1)}{dt}+au(t-1)=0,\ t>0.$$

It is well known [3] that the roots of the corresponding characteristic quasipolynomial are simple, separated (see (1.4)), and lie on the imaginary axis. Then $\varkappa_+=0$, $M=1$, and (1.3) gives

$$\|u\|_{H^1(t-1,t)}\le\text{const}\,\|g\|_{H^1(-1,0)}.$$

The sharpness of the above estimate follows from the fact that the norm of any exponential solution $e^{\lambda_q t}$ is constant.

Example 2 ([8])**.** Consider the following nonhomogeneous equation:

$$u'(t)+u'(t-1)=1,\quad t>0, \tag{3.18}$$

$$u(t)=0,\quad t\in[-1,0]. \tag{3.19}$$

Constructing the solution of problem (3.18) and (3.19) step by step, we obtain

$$u(t) = \begin{cases} k & \text{if } \ t \in [2k-1, 2k], \\ t-k & \text{if } \ t \in [2k, 2k+1]. \end{cases}$$

It is easily seen that

$$\|u\|_{H^1[n-1,n]} \asymp n, \quad \|f\|_{L_2[0,n]} \asymp \sqrt{n}, \ n \in N.$$

Since, obviously, the characteristic quasipolynomial $L(\lambda) = \lambda(1 + e^{-\lambda})$ has simple roots lying on the imaginary axis, we have $\varkappa_+ = 0$ and $M = 1$. For $T = n$, the right-hand side of the main inequality (1.3) behaves like n, which proves that the estimate is sharp.

Remark 1. Several remarks concerning equations of neutral-neutral type studied in [8]–[10], [13]–[16], and [19, 20] are in order. For equations of the form (1.1) this situation occurs if $\det A_{0m} \neq 0$, $\det A_{mn} \neq 0$. Under this condition, the spectrum Λ of $\mathbf{D}$ lies in the strip $\{\lambda : \varkappa_- \leq \operatorname{Re}\lambda \leq \varkappa_+\}$, $\varkappa_- = \inf_{\lambda_q \in \Lambda} \operatorname{Re}\lambda_q$, and there are no logarithmic branches. In the half-plane $\{\lambda : \operatorname{Re}\lambda < \omega\}$, $\omega < \varkappa_-$, the matrix-valued function $L^{-1}(\lambda)$ admits the estimate

$$\|L^{-1}(\lambda)\|_{\mathbb{C}^r} \leq \text{const}\, |\lambda|^{-m} e^{\operatorname{Re}\lambda h}. \tag{3.20}$$

Taking into account the form of the functions $\beta(f)$ and $\gamma(f)$ (see (2.5) and (2.6)) and estimate (3.20), it is not hard to check, with the help of Jordan's lemma, that $[W(t)](\theta) = 0$, $t > h$.

Thus, in the case in question, we have $\mathbf{V}_t(g)(\theta) = U(t)(\theta) = U_\beta(t)(\theta)$, and it suffices to estimate the basic part of the solution.

We note also that the system of subspaces $\{\mathcal{P}_n \mathbf{H}\}$ forms a Riesz basis in the entire space $\mathbf{H}$ (for the proofs, see [13, 15].

Remark 2. The results of the present paper can be extended to more general equations of the form

$$\sum_{k=0}^{n} \sum_{j=0}^{m} A_{kj} u^{(j)}(t - h_k) + \sum_{j=0}^{m} \int_0^h B_j(s) u^{(j)}(t-s)\, ds = f(t), \quad t > 0,$$

where the entries of the matrix-valued functions $B_j(s)$ belong to $L_2(0, h)$.

Here it is assumed that the characteristic matrix

$$L(\lambda) = \sum_{k=0}^{n} \sum_{j=0}^{m} A_{kj} \lambda^j e^{-\lambda h_k} + \sum_{j=0}^{m} \int_0^h B_j(s) \lambda^j e^{-\lambda s}\, ds$$

has the following property: there is $a > 0$ such that no points of the spectrum Λ lie in the strip $\Pi_\delta = \{\lambda : -a - \delta < \operatorname{Re}\lambda < -a + \delta\}$ and

$$\|L^{-1}(\lambda)\|_{\mathbb{C}^r} \leq \frac{\text{const}}{(|\lambda| + 1)^m}, \quad \lambda \in \Pi_\delta.$$

The existence of such a strip Π_δ allows us to get dichotomy for the solution, i.e., to represent it in the form

$$u(t) = u_\beta(t) + w(t),$$

where $u_\beta(t)$ corresponds to the part of the spectrum in the strip $\{\lambda : -a < \operatorname{Re}\lambda < \varkappa_+\}$, and $w(t)$ gives an exponential contribution to the solution. A similar situation for close first order ($m = 1$) equations was studied in [6].

Remark 3. We note that, in some other spaces (largely, in L_p and C), the spectral properties of the differentiation operator $\mathbf{D}$ were studied by many authors. See the detailed survey [28] by Sedletskiĭ and the references therein.

On the other hand, the study of the differentiation operator in the scale of Sobolev spaces with integral index can be related to the study of the differentiation operator $\mathbf{D}$ with spectral parameter in boundary conditions. We mention the paper [29], where the case of nth order differential operators with spectral parameter in boundary conditions was studied.

The methods used in the present paper can also be applied to the study of the unconditional basis property of the spaces $\{\mathcal{P}_n\mathbf{H}\}$ in the space $\mathbf{H}$.

§4. Appendix

Proof of Theorem 2.4. Let η denote the function $R_V f$. We need the following statement.

Lemma 4.1. *In the above notation, the following representations are valid.*

(i) *For* $0 \le s \le m$ *we have*

$$\eta(\theta) = -\sum_{p=0}^{s-1} \frac{f^{(p)}(\theta)}{\lambda^{p+1}} + e^{\lambda\theta} \sum_{p=0}^{s-1} \frac{f^{(p)}(0)}{\lambda^{p+1}} + \lambda^{-s} \int_0^\theta e^{\lambda(\theta-\tau)} f^{(s)}(\tau)\, d\tau;$$

(ii) $$\eta^{(j)}(\theta) = \lambda^j \sum_{p=0}^{j-1} \frac{f^{(p)}(\theta)}{\lambda^{p+1}} + \lambda^j \int_0^\theta e^{\lambda(\theta-\tau)} f(\tau)\, d\tau, \quad j = 0, 1, \dots;$$

(iii) $$\eta^{(j)}(\theta) = -\lambda^j \sum_{p=j}^{m-1} \frac{f^{(p)}(\theta)}{\lambda^{p+1}} + e^{\lambda\theta}\lambda^j \sum_{p=0}^{m-1} \frac{f^{(p)}(0)}{\lambda^{p+1}} + \lambda^{-m+j} \int_0^\theta e^{\lambda(\theta-\tau)} f^{(m)}(\tau)\, d\tau.$$

Proof of Lemma 4.1. (i) Integrating by parts, we obtain

$$\lambda^{-s} \int_0^\theta e^{\lambda(\theta-\tau)} f^{(s)}(\tau)\, d\tau = -\lambda^{-s-1} f^{(s)}(\theta) + \lambda^{-s-1} e^{\lambda\theta} f^{(s)}(0) + \lambda^{-s-1} \int_0^\theta e^{\lambda(\theta-\tau)} f^{(s+1)}(\tau)\, d\tau.$$

Statement (ii) of the lemma can easily be obtained by induction,

$$\eta^{(j+1)}(\theta) = \lambda^j \sum_{p=0}^{j-1} \frac{f^{(p+1)}(\theta)}{\lambda^{p+1}} + \lambda^j \left[f(\theta) + \lambda \int_0^\theta e^{\lambda(\theta-\tau)} f(\tau)\, d\tau \right].$$

Statement (iii) follows from (ii) and (i) with $s = m$. The lemma is proved. □

We return to the proof of the theorem. By definition and Lemma 4.1(iii), we have

$$\mathcal{L}(R_v f) = -\sum_{k=0}^{n} \sum_{j=0}^{m} A_{kj} \lambda^j \sum_{p=j}^{m-1} \lambda^{-p-1} f^{(p)}(-h_k) + L(\lambda) \sum_{p=0}^{m-1} \lambda^{-p-1} f^{(p)}(0) + \sum_{k=0}^{n} \sum_{j=0}^{m} A_{kj} \lambda^j \lambda^{-m} e^{-\lambda h_k} \int_0^{-h_k} e^{-\lambda\tau} f^{(m)}(\tau)\, d\tau.$$

This gives (2.7). The theorem is proved. □

Proof of Lemma 2.6. (a) We represent $L(\lambda)$ in the form

$$L(\lambda) = A_{0m}\lambda^m \left(I + \sum_{k=1}^{n} A_{0m}^{-1} A_{km} e^{-\lambda h_k} + \sum_{k=1}^{n} \sum_{j=0}^{m-1} A_{0m}^{-1} A_{kj} \lambda^{j-m} e^{-\lambda h_k} \right). \tag{4.1}$$

The second term on the right in (4.1) is arbitrarily small for sufficiently large $\varkappa$ and $\operatorname{Re}\lambda > \varkappa$. The norm of the third term is at most $\frac{\text{const}}{|\lambda|}$.

Thus, for sufficiently large $\varkappa > 0$ we have

$$\|L^{-1}(\lambda)\|_{\mathbb{C}^r} = |\lambda|^{-m}\|A_{0m}^{-1}\left(I + B(\lambda)\right)^{-1}\|_{\mathbb{C}^r},$$

where

$$B(\lambda) = \sum_{k=1}^{n} A_{0m}^{-1}A_{km}e^{-\lambda h_k} + \sum_{k=1}^{n}\sum_{j=0}^{m-1} A_{0m}^{-1}A_{kj}\lambda^{j-m}e^{-\lambda h_k}$$

and

$$\|B(\lambda)\| \le \frac{1}{2} \quad \text{for} \quad \operatorname{Re}\lambda > \varkappa.$$

Consequently,

$$\|L^{-1}(\lambda)\|_{\mathbb{C}^r} \le \frac{\text{const}}{|\lambda|^m}, \quad \operatorname{Re}\lambda > \varkappa.$$

(b) By the results of [1, Chapter 12], in the region $\{\lambda : A < \operatorname{Re}\lambda < B\} \cap G$ the quasipolynomial $l(\lambda)$ satisfies

$$|l(\lambda)| \ge \text{const}\,|\lambda|^{mr}. \tag{4.2}$$

The elements of $L^{-1}(\lambda)$ are the cofactors of the matrix $L(\lambda)$ divided by $l(\lambda)$. The largest degree of λ in those cofactors does not exceed $\lambda^{m(r-1)}$. Combining this with (4.2), we obtain (b). □

Proof of Proposition 3.1. We represent the quasipolynomial $l(\lambda)$ in the form

$$l(\lambda) = D_1(\lambda) + D_2(\lambda),$$

where

$$D_1(\lambda) = \lambda^{mr}\sum_{\tau=0}^{\tau_0} c_\tau e^{-\lambda\tau}, \quad D_2(\lambda) = l(\lambda) - D_1(\lambda),$$

i.e., each summand in $D_1(\lambda)$ involves the maximal power λ^{mr}, and the summands in $D_2(\lambda)$ involve powers at most $mr - 1$.

Observe that $\frac{D_1(iz)}{(iz)^{mr}}$ is an entire function of sine type. Consequently (see [30]), there exists a system of contours of the form Γ_n such that the function $\frac{D_1(\lambda)}{\lambda^{mr}} = m(\lambda)$ satisfies the following condition:

$$\sum_{\mu_q \in G_n} \tau_q \le M, \quad n \in \mathbb{Z},$$

where the μ_q are the zeros of $m(\lambda)$, the τ_q are their multiplicities, and G_n is the rectangle with boundary Γ_n. Moreover,

$$|m(\lambda)|_{\lambda\in\Gamma_n} \ge \text{const}\,.$$

Hence, for $D_1(\lambda)$ we have:

$$|D_1(\lambda)| \ge \text{const}\,|\lambda|^{mr}, \quad \lambda \in \Gamma_n.$$

Therefore, if $\lambda \in \Gamma_n$ has a sufficiently large absolute value, then

$$\frac{|D_2(\lambda)|}{|D_1(\lambda)|} \le \frac{\text{const}}{|\lambda|}.$$

Applying Rouche's theorem, we see that for the characteristic quasipolynomial $l(\lambda) = D_1(\lambda) + D_2(\lambda)$ we have

$$m_n = \sum_{\lambda_q \in \Gamma_n} \nu_q \le M,$$

where ν_q is the multiplicity of the zero λ_q of $l(\lambda)$. □

Proof of Proposition 2.2. It seems that Proposition 2.2 is not new; here we present a short proof of it for completeness. Close results were given in [3, 6, 31, 32].

By the results of [6, 31, 32], the differentiation operator is a generator of the semigroup of shifts. We prove that the multipoint boundary conditions (2.3) are fulfilled.

Consider the function

$$\Psi(\theta) = \int_0^{+\infty} e^{-\lambda t}(\mathbf{V}_t g)(\theta)\,dt = \int_0^{+\infty} e^{-\lambda t} u(t+\theta)\,dt, \quad \theta \in [-h, 0],\ \operatorname{Re}\lambda > \varkappa_+, \tag{4.3}$$

where u is a strong solution of the homogeneous equation (1.1) corresponding to the initial function g. By the general theory (see [23, 24]), we obtain

$$\Psi(\theta) = (R_{\mathbf{D}}(\lambda)g)(\theta),$$

so that $\Psi \in \operatorname{Dom}(\mathbf{D})$.

Since u is a strong solution of the homogeneous equation (1.1), we have

$$\sum_{k=0}^{n}\sum_{j=0}^{m} A_{kj} u^{(j)}(t-h_k) = 0, \quad t > 0.$$

Applying the operator $\mathcal{L}$ to $\Psi(\theta)$, we obtain

$$(\mathcal{L}\Psi)(\theta) = \int_0^{+\infty} e^{-\lambda t}\left(\sum_{k=0}^{n}\sum_{j=0}^{m} A_{kj} u^{(j)}(t-h_k)\right) dt = 0,$$

i.e.,

$$(\mathcal{L}\Psi)(\theta) = \sum_{k=0}^{n}\sum_{j=0}^{m} A_{kj}\Psi^{(j)}(-h_k) = 0. \tag{4.4}$$

The representation (4.3) shows that Ψ belongs to $H^{m+1}((-h,0),\mathbb{C}^r)$, and, by (4.4), it satisfies condition (2.3).

We prove that the domain of $\mathbf{D}$ coincides with $\{\psi : \psi \in H^{m+1}((-h,0),\mathbb{C}^r), \mathcal{L}\psi = 0\}$. For this, we consider the operator J given by the formula

$$J\varphi = \frac{d}{dt}\varphi(\theta), \quad \theta \in [-h, 0],$$

with the domain

$$\operatorname{Dom}(J) = \left\{\varphi : \varphi \in H^{m+1}((-h,0),\mathbb{C}^r), \mathcal{L}\varphi = 0\right\}.$$

Since $\Psi \in \{\psi \in H^{m+1}((-h,0),\mathbb{C}^r), \mathcal{L}\psi = 0\}$ and conditions (4.4) are fulfilled, we obtain $\operatorname{Dom}(\mathbf{D}) \subseteq \operatorname{Dom}(J)$.

Now, we check the inclusion $\operatorname{Dom}(J) \subseteq \operatorname{Dom}(\mathbf{D})$. Let $\varphi \in \operatorname{Dom}(J)$. We prove that $\varphi \in \operatorname{Dom}(\mathbf{D})$. Consider the functions

$$\psi_\lambda = (\lambda I - J)\varphi, \quad \chi_\lambda = (\lambda I - \mathbf{D})^{-1}\psi_\lambda.$$

Then $\chi_\lambda \in \operatorname{Dom}(\mathbf{D})$ and $(\lambda I - \mathbf{D})\chi_\lambda = \psi_\lambda$. In other words,

$$\psi_\lambda = \lambda\varphi - \varphi', \quad \psi_\lambda = \lambda\chi_\lambda - \chi_\lambda'.$$

Hence,

$$\lambda(\varphi - \chi_\lambda) - (\varphi' - \chi_\lambda') = 0$$

or

$$\omega = \varphi - \chi_\lambda = e^{\lambda\theta}c,$$

where c is a constant vector in $\mathbb{C}^r$.

On the other hand, the function $\omega(\theta)$ satisfies conditions (4.3), i.e.,

$$\mathcal{L}(\omega)(\theta) = L(\lambda)e^{\lambda\theta}c = 0.$$

By Proposition 1.2, the matrix-valued function $L(\lambda)$ is invertible in the half-plane $\{\lambda : \operatorname{Re}\lambda > \varkappa\}$.

Consequently, $c = 0$, whence $\omega = \varphi - \chi_\lambda \equiv 0$, i.e., $\varphi = \chi_\lambda \in \operatorname{Dom}(\mathbf{D})$.

The proposition is proved. □

Proof of Lemma 3.5. (i) We note that the function g lies in $\mathbf{H} = H^m((-h,0),\mathbb{C}^r)$. Integrating by parts, we obtain

$$(R_V(\lambda)g)(\theta) = \int_0^\theta e^{\lambda(\theta-\tau)}g(\tau)\,d\tau = \frac{g(\theta)}{\lambda} - \frac{e^{\lambda\theta}g(0)}{\lambda} + \frac{1}{\lambda}e^{\lambda\theta}\int_0^\theta e^{-\lambda\tau}g'(\tau)\,d\tau.$$

The function

$$G(\lambda) = \int_0^\theta e^{-\lambda\tau}g'(\tau)\,d\tau, \quad \theta \in [-h,0],$$

is bounded in the half-plane $\{\lambda : \operatorname{Re}\lambda < -a\}$. Consequently, for $t > h$, by Jordan's lemma, we obtain

$$\lim_{R\to+\infty}\int_{C_R} e^{\lambda t}R_V(\lambda)g\,d\lambda = 0,$$

which implies the required statement.

To prove (ii), it suffices (see (2.5)) to show that

$$\text{P.V.}\,\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} e^{\lambda(t+\theta)}\frac{1}{\lambda^j}\,d\lambda = 0, \quad j \in \mathbb{N}.$$

However, this relation follows directly from Jordan's lemma,

$$\lim_{R\to+\infty}\frac{1}{2\pi i}\int_{C_R} e^{\lambda(t+\theta)}\frac{1}{\lambda^j}\,d\lambda = 0, \quad t > h,\ j \in \mathbb{N}.$$ □

To prove that the integral (3.12) converges uniformly, we denote by $\Psi(\lambda,g)$ the function

$$\Psi(\lambda,g) = L^{-1}(\lambda)(\beta(g) + \gamma(g)).$$

We prove that, for a fixed t,

(a) the integral (3.12) converges uniformly with respect to $\theta \in [-h,0]$;

(b) for $\theta \to +0$, the function $e^{\lambda(t+\theta)}\Psi(\lambda,g)$ in the integrand converges uniformly with respect to $\lambda \in [-a-iR_1, -a+iR_2]$ to the function $e^{\lambda t}\Psi(\lambda,g)$.

Proof. (a) Observe that

$$|e^{\lambda(t+\theta)}\Psi(\lambda,g)|_{\mathbb{C}^r} \le e^{-a(t-h)}|\Psi(\lambda,g)|_{\mathbb{C}^r}.$$

The function $\Psi(\lambda,g)$ is integrable on $\lambda \in [-a-i\infty, -a+i\infty]$, because $L^{-1}(\lambda)$ admits estimate (2.12),

$$\|L^{-1}(\lambda)\|_{\mathbb{C}^r} \prec |\lambda|^{-m},$$

and, by the trace theorem (see [22]), the function $\beta(g)(-a+i\mu)$ satisfies

$$|\beta(g)(\lambda)|_{\mathbb{C}^r} \prec \frac{1}{|\lambda|}\|g\|_{\mathbf{H}}.$$

In turn, the function $\gamma(\lambda) = \gamma(g)(-a+i\mu)$ belongs to the space $L^2(\mathbb{R},\mathbb{C}^r)$ with respect to μ. Thus, the function $e^{-a(t-h)}|\Psi(\lambda,g)|_{\mathbb{C}^r}$ is an integrable majorant for $e^{\lambda(t+\theta)}\Psi(\lambda,g)$. Statement (b) follows from the inequality

$$\|e^{\lambda(t+\theta)}\Psi(\lambda,g) - e^{\lambda t}\Psi(\lambda,g)\|_{\mathbb{C}^r} \le \sup_{\lambda\in[-a-iR_1,-a+iR_2]}\|e^{\lambda t}\Psi(\lambda,g)\|_{\mathbb{C}^r}\cdot|e^{\lambda\theta}-1|$$

and the fact that, on each interval $[-a-iR_1, -a+iR_2]$, the function $e^{\lambda\theta}$ converges to 1 uniformly in λ as $\theta \to +0$. □

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