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REPRESENTATION THEORY OF (MODIFIED) REFLECTION EQUATION ALGEBRA OF $GL(m|n)$ TYPE

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Abstract. Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be a Hecke type solution of the quantum Yang–Baxter equation (a Hecke symmetry). Then, the Hilbert–Poincaré series of the associated R-exterior algebra of the space V is the ratio of two polynomials of degrees m (numerator) and n (denominator).

Under the assumption that R is skew-invertible, a rigid quasitensor category $\mathrm{SW}(V_{(m|n)})$ of vector spaces is defined, generated by the space V and its dual V^*, and certain numerical characteristics of its objects are computed. Moreover, a braided bialgebra structure is introduced in the modified reflection equation algebra associated with R, and the objects of the category $\mathrm{SW}(V_{(m|n)})$ are equipped with an action of this algebra. In the case related to the quantum group $U_q(sl(m))$, the Poisson counterpart of the modified reflection equation algebra is considered and the semiclassical term of the pairing defined via the categorical (or quantum) trace is computed.

§1. Introduction

The reflection equation algebra is a very useful tool of the theory of integrable systems with boundaries. It derives its name from an equation describing factorized scattering on a half-line (see [C], where the reflection equation depending on a spectral parameter was introduced for the first time).

By definition (see [KS]), the reflection equation algebra (REA for short) is an associative unital algebra over a ground field[1] $\mathbb{K}$ generated by elements l_i^j, $1 \le i, j \le N$, subject to the following quadratic commutation relations:

$$R_{12}L_1R_{12}L_1 = L_1R_{12}L_1R_{12}.$$

Here $L_1 = L \otimes I$, $L = \|l_i^j\|$ is the matrix composed of REA generators, while the linear operator $R : V^{\otimes 2} \to V^{\otimes 2}$ is an invertible solution of the quantum Yang–Baxter equation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}\,. \tag{1.1}$$

Here V is a finite-dimensional vector space over the field $\mathbb{K}$, $\dim_{\mathbb{K}} V = N$, and the indices of R correspond to the space (or spaces) in which the operator is applied. Thus, R_{12} and R_{23} denote the following operators in the space $V^{\otimes 3}$: $R_{12} = R \otimes I$, $R_{23} = I \otimes R$. Such an operator R will be called a *braiding* in what follows.

Nowadays, different types of the REA are known to have applications in mathematical physics and noncommutative geometry (cf. [KS]).

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[1] Mainly we are dealing with $\mathbb{K} = \mathbb{C}$, but sometimes $\mathbb{K} = \mathbb{R}$ is allowed.

The REA related to the Drinfeld–Jimbo Quantum Group (QG) $U_q(sl(m))$ arises in the construction of a q-analog of differential calculus on the groups $GL(m)$ and $SL(m)$, where it was treated as a q-analog of the exponential of vector fields (cf. [FP]).

In the case related to the QG $U_q(\mathfrak{g})$, an appropriate quotient of the REA can be treated as a deformation of the coordinate ring $\mathbb{K}[G]$, where G is the Lie group corresponding to a classical Lie algebra $\mathfrak{g}$. The Poisson bracket corresponding to this deformation was introduced by M. Semenov-Tian-Shansky.[2]

In the present paper we deal with *Hecke type* solutions of the Yang–Baxter equation (1.1) that satisfy the condition

$$(R - q\,I)(R + q^{-1}\,I) = 0\,, \tag{1.2}$$

where the nonzero parameter $q \in \mathbb{K}$ is assumed to be *generic*. By definition, this means that the values of q do not belong to the countable set formed by the nontrivial roots of unity: $q^k \neq 1$, $k = 2, 3, \dots$ (whereas the value $q = 1$ is not excluded). Consequently,

$$k_q := \frac{q^k - q^{-k}}{q - q^{-1}} \neq 0, \qquad k \in \mathbb{N}\,,$$

k_q being a q-analog of an integer k. In what follows, a braiding that satisfies relation (1.2) will be called a *Hecke symmetry*.

Especially, we are interested in families of Hecke symmetries R_q analytically depending on the parameter q in a neighborhood of $1 \in \mathbb{K}$ and such that for $q = 1$ the symmetry $R = R_1$ is involutive: $R^2 = I$.

The well-known example of such a family is the $U_q(sl(m))$ Drinfeld–Jimbo braidings

$$R_q = \sum_{i,j=1}^{m} q^{\delta_{ij}}\, h_i^j \otimes h_j^i + \sum_{i<j}^{m} (q - q^{-1})\, h_i^i \otimes h_j^j, \tag{1.3}$$

where the elements h_i^j form the natural basis in the space of left endomorphisms of V, that is, $h_i^j(x_k) = \delta_k^j\, x_i$ in a fixed basis $\{x_k\}$ of the space V. Note that for $q = 1$ the above braiding R equals the usual flip P.

The Hecke symmetry (1.3) and all related objects will be called *standard*. However, a large number of Hecke symmetries different from the standard one are known, even those that are not deformations of the usual flip (see [G3]).

We consider the REA corresponding to the standard $U_q(sl(m))$ Hecke symmetry (1.3) in more detail. This algebra possesses some very important properties, in contrast with the REA related to other quantum groups $U_q(\mathfrak{g})$, $\mathfrak{g} \neq sl(m)$.

First of all, it is a q-deformation of the commutative algebra $\mathrm{Sym}(gl(m)) = \mathbb{K}[gl(m)^*]$ (so, we get a deformation algebra without taking any additional quotient). Second, by a linear shift of REA generators (proportional to a parameter $\hbar$), we arrive at quadratic-linear commutation relations for the shifted generators. In this basis, the REA can be treated as a "double deformation" of the initial commutative algebra $\mathbb{K}[gl(m)^*]$. We refer to this form of the REA as the *modified Reflection Equation Algebra* (mREA) (see §5) and we denote it by $\mathcal{L}(R_q, \hbar)$. Putting $\hbar = 0$, we return to the (nonmodified) REA $\mathcal{L}(R_q)$.

The specialization of the algebra $\mathcal{L}(R_q, \hbar)$ at $q = 1$ gives the enveloping algebra $U(gl(m)_\hbar)$, where the notation $\mathfrak{g}_\hbar$ means that the bracket $[\ ,\ ]$ of a Lie algebra $\mathfrak{g}$ is replaced by $\hbar[\ ,\ ]$. (Note that this fact was observed in [IP].) The commutative algebra

[2]Note that any classical Lie group G admits another Poisson bracket, discovered by E. Sklyanin. Its quantization is given by an appropriate quotient of the so-called RTT algebra (see [FRT]). These two quantum analogs of the space $\mathbb{K}[G]$ are related by a transmutation procedure introduced by S. Majid (see [M] and the references therein). Nowadays, there is a universal treatment, based on pairs of so-called compatible braidings (see [IOP, GPS1, GPS2]).

$\mathbb{K}[gl(m)^*]$ is obtained by the double specialization of the algebra $\mathcal{L}(R_q, \hbar)$ at $\hbar = 0$ and $q = 1$.

Being equipped with the $U_q(sl(m))$-module structure, the algebra $\mathcal{L}(R_q, \hbar)$ (as well as $\mathcal{L}(R_q)$) is $U_q(sl(m))$-equivariant (or covariant). This means that

$$M(x \cdot y) = M_{(1)}(x) \cdot M_{(2)}(y)\,, \quad M \in U_q(sl(m))\,, \quad x, y \in \mathcal{L}(R_q, \hbar),$$

where we use Sweedler's notation for the quantum group coproduct $\Delta(M) = M_{(1)} \otimes M_{(2)}$.

The Poisson counterpart of the above double deformation of the algebra $\mathbb{K}[gl(m)^*]$ is the Poisson pencil

$$\{\,,\}_{PL,r} = a\,\{\,,\}_{PL} + b\,\{\,,\}_r\,, \quad a, b \in \mathbb{K}, \tag{1.4}$$

where $\{\,,\}_{PL}$ is the linear Poisson–Lie bracket related to the Lie algebra $gl(m)$ and $\{\,,\}_r$ is a natural extension of the Semenov-Tian-Shansky bracket to the linear space $gl(m)^*$.

We consider these Poisson structures and briefly discuss their role in defining a "quantum orbit" $\mathcal{O} \subset gl(m)^*$ in §7. Taking a two-dimensional sphere as an example, we suggest a method of constructing such quantum orbits. In contrast with other definitions of quantum homogeneous spaces, our quantum orbits are some quotients of the algebra $\mathcal{L}(R_q, \hbar)$. They look like the "fuzzy sphere"

$$\mathcal{S}L^c(\hbar) = U(su(2)_\hbar)/\langle C - c\rangle,$$

where C is the quadratic Casimir element. As is known, there exists a discrete series of numbers $c_k \in \mathbb{K}$ such that any algebra $\mathcal{S}L^{c_k}(\hbar)$ has a finite-dimensional representation in a linear space V_k and the corresponding map $\mathcal{S}L^{c_k}(\hbar) \to \mathrm{End}(V_k)$ is an $su(2)$-morphism.

A similar statement is valid for the quotients of the algebra $\mathcal{L}(R_q, \hbar)$ mentioned above. However, the corresponding spaces V_k become objects of a quasitensor category. In such a category, an object is characterized by its quantum dimension, which is defined via the categorical (quantum) trace. A deformation of the usual trace is a main feature of our approach to the quantum homogeneous spaces. In §7 we describe the semiclassical term of the paring defined via the quantum trace in the case of the standard Hecke symmetry.

In a similar way, we treat other quasitensor categories generated by skew-invertible Hecke symmetries. Roughly speaking, we are dealing with three problems in the present paper. The first problem is the classification of all (skew-invertible) Hecke symmetries R. A principal tool for studying this problem is the Hilbert–Poincaré (HP) series $P_-(t)$ corresponding to the "R-exterior algebra" of the space V (its definition is presented in §3). Though a classification of all possible forms of the HP series $P_-(t)$ has not been found yet, it is known that the HP series $P_-(t)$ of any Hecke symmetry is a rational function; see [H, D].[3] The ordered pair of integers $(m|n)$, where m (respectively, n) is the degree of the numerator $N(t)$ (respectively, denominator $D(t)$) of $P_-(t)$, plays an important role in the sequel and will be called the *birank* of the Hecke symmetry R (or of the corresponding space V). This pair enters our notation of the quasitensor Schur–Weyl category $\mathrm{SW}(V_{(m|n)})$ generated by V.

Constructing the category $\mathrm{SW}(V_{(m|n)})$ is the second problem we are dealing with in this paper. The objects of this category are direct sums of vector spaces $V_\lambda \otimes V_\mu^*$. Here V is the basic vector space equipped with a skew-invertible Hecke symmetry R, V^* is its dual, and λ and μ stand for arbitrary partitions (Young diagrams) of positive integers. The map $V \to V_\lambda$ is nothing but the Schur functor corresponding to the Hecke symmetry

[3]The HP series corresponding to a skew-invertible Hecke symmetry is described in §3. When $P_-(t)$ is a polynomial (in this case we say that R is even), it can differ drastically from the classical polynomial $(1+t)^n$, $n = \dim V$. We mention that all skew-invertible Hecke symmetries with $P_-(t) = 1 + nt + t^2$ were classified in [G3]. Also in [G3], a way of "gluing" such symmetries was suggested, which gives rise to skew-invertible Hecke symmetries with other nonstandard HP series.

R (for its classical version, see [FH]). The map $V^* \to V_\mu^*$ can be defined in a similar way. Note that $\mathrm{SW}(V_{(m|n)})$ is a monoidal quasitensor rigid category (as defined in [CP]), but it is not Abelian.

We compute some numerical characteristics of objects of this category. Namely, we are interested in their dimensions (classical and quantum). In contrast with the classical dimensions, which depend substantially on a specific form of the initial Hecke symmetry and are expressed via the roots of the above polynomials $N(t)$ and $D(t)$, the quantum dimensions depend only on the birank $(m|n)$. Moreover, in a sense, the category $\mathrm{SW}(V_{(m|n)})$ looks like the tensor category of $U(gl(m|n))$-modules.

The third problem elaborated below is in constructing the representations of the mREA $\mathcal{L}(R_q,\hbar)$ in the category $\mathrm{SW}(V_{(m|n)})$. Since for $q\neq 1$ the algebra $\mathcal{L}(R_q,\hbar)$ is isomorphic to the nonmodified REA $\mathcal{L}(R_q)$ (in fact, we have the same algebra written in two different bases), we automatically get a representation category of the latter algebra.[4] Note that certain representations of the REA have already been known, mainly for the even case (the birank $(m|0)$); see [K, Mu1, GS2, S]. In contrast with those papers, here we consider the mREA $\mathcal{L}(R_q,\hbar)$ related to a general type skew-invertible Hecke symmetry R of birank $(m|n)$ and equip objects of the category $\mathrm{SW}(V_{(m|n)})$ with the $\mathcal{L}(R_q,\hbar)$-module structure. Note that all the corresponding representations are equivariant (see §6).

A particular example we are interested in is the "adjoint" representation. By this we mean a representation ρ_{ad} of the mREA $\mathcal{L}(R_q,\hbar)$ in the linear span of its generators. In the case where a Hecke symmetry is a super-flip in a Z_2-graded near space V,

$$R:\ V^{\otimes 2}\to V^{\otimes 2}\,,\qquad R(x\otimes y)=(-1)^{\overline{x}\,\overline{y}}y\otimes x\,,$$

where x and y are homogeneous elements of V and $\bar z$ denotes the parity (grading) of a homogeneous element z, the mREA becomes the enveloping algebra $U(gl(m|n))$, and the representation ρ_{ad} coincides with the usual adjoint representation. This is one of the reasons why we treat the mREA $\mathcal{L}(R_q,\hbar)$ as a suitable analog of the enveloping algebra. Moreover, in the case of involutive skew-invertible Hecke symmetry, the corresponding mREA becomes the enveloping algebra of a generalized Lie algebra $\mathrm{End}(V)$; this is explained in §5. Such algebras were introduced in [G1].

Another property that makes the mREA similar to the enveloping algebra of a generalized Lie algebra (in particular, a Lie superalgebra) is its braided bialgebra structure. Such a structure is determined by a coproduct Δ and a counit ε. On the generators of mREA (organized into a matrix L; see §6) the coproduct reads

$$\Delta(L)=L\otimes 1+1\otimes L-(q-q^{-1})L\otimes L$$

and coincides with the coproduct of the enveloping algebra of the (generalized) Lie algebra at $q=1$. Note that, though we do not define an antipode in the algebra $\mathcal{L}(R_q,\hbar)$, the category $\mathrm{SW}(V_{(m|n)})$ of its representations is closed.

In addition to the $\mathcal{L}(R_q,\hbar)$-module structure, the objects of the Schur–Weyl category corresponding to the standard Hecke symmetry (1.3) can be equipped with the action of the QG $U_q(sl(m))$. Moreover, the q-analogs of supergroups (see [KT]) can also be represented in the corresponding Schur–Weyl category. (Another way of constructing the representations of q-deformed algebras $U(gl(m|n)$, which is based on the triangular decomposition, was suggested in [Z].) Nevertheless, in general we know of no explicit construction of the QG type algebra for a skew-invertible Hecke symmetry,[5] whereas the mREA can be defined for any skew-invertible Hecke symmetry.

[4]Since for $q=1$ the isomorphism $\mathcal{L}(R_q,\hbar)\cong\mathcal{L}(R_q)$ breaks, we prefer to consider these algebras separately and use different names for them.

[5]An attempt of explicit description of such an object for some even nonquasiclassical Hecke symmetries was undertaken in [AG].

The mREA has one more advantage compared with the QG or its superanalogs. It is a more convenient tool for the explicit construction of projective modules over quantum orbits in the framework of the approach suggested in [GS1, GS3]. We plan to turn to these objects in a general (not necessarily even) case in our subsequent publications.

To complete the Introduction, we would like to emphasize a difference between the Hecke type braidings and the Birman–Murakami–Wenzl ones (in particular, those coming from the QG of B_n, C_n, and D_n series). In the latter case it is not difficult to define a "braided Lie bracket" in the space $\mathrm{End}(V)$ (see [DGG]) and introduce the corresponding "enveloping algebra". But this "enveloping algebra" is not a deformation of its classical counterpart, and therefore, is not an interesting object from our viewpoint.

The paper is organized as follows. In the next section we reproduce some elements of R-technique that form the base of the subsequent computations of some interesting numerical characteristics of the objects involved (the most cumbersome part of the computations is placed in the Appendix). §3 is devoted to the classification of (skew-invertible) Hecke symmetries. In §4 we construct the Schur–Weyl category $\mathrm{SW}(V_{(m|n)})$ generated by the space V. Our main object, the mREA $\mathcal{L}(R_q, \hbar)$, is introduced in §5, where we also study its deformation properties. In §6 we equip the mREA with a braided bialgebra structure, which allows us to define an equivariant action of the algebra $\mathcal{L}(R_q, \hbar)$ on each object of the category $\mathrm{SW}(V_{(m|n)})$. There we also present our viewpoint on the definition of braided (quantum) Lie algebras. §7 is devoted to the study of some semiclassical structures.

§2. Elements of R-technique

By R-technique we mean computational methods based on general properties of braidings (in particular, Hecke symmetries) regardless of their specific form. We are mostly interested in the so-called skew-invertible braidings, because they enable us to define numerical characteristics of Hecke symmetries and related objects.

A braiding R (see (1.1)) is said to be *skew-invertible* if there exists an endomorphism $\Psi : V^{\otimes 2} \to V^{\otimes 2}$ such that

$$\mathrm{Tr}_{(2)}\, R_{12}\, \Psi_{23} = P_{13} = \mathrm{Tr}_{(2)}\, \Psi_{12}\, R_{23}, \tag{2.1}$$

where the symbol $\mathrm{Tr}_{(2)}$ means calculating the trace in the second factor of the tensor product $V^{\otimes 3}$. Hereafter, P stands for the usual flip $P(x \otimes y) = y \otimes x$.

Fixing bases $\{x_i\}$ and $\{x_i \otimes x_j\}$ in V and $V^{\otimes 2}$ (respectively), we identify R (respectively, Ψ) with a matrix $\|R_{ij}^{kl}\|$ (respectively, $\|\Psi_{ij}^{kl}\|$):

$$R(x_i \otimes x_j) = x_k \otimes x_l\, R_{ij}^{kl}, \tag{2.2}$$

where the upper indices mark the rows of the matrix, and from now on summation over the repeated indices is assumed.

Being written in terms of matrices, relation (2.1) reads

$$R_{jb}^{ia}\, \Psi_{ak}^{bl} = \delta_k^i \delta_j^l = \Psi_{jb}^{ia}\, R_{ak}^{bl}\,.$$

Using Ψ, we define two endomorphisms B and C of the space V:

$$B(x_i) = x_j B_{\ i}^{j}, \qquad C(x_i) = x_j C_{\ i}^{j},$$

where

$$B_{\ i}^{j} := \Psi_{\ ki}^{kj}, \qquad C_{\ i}^{j} := \Psi_{\ ik}^{jk}, \tag{2.3}$$

that is,

$$B := \mathrm{Tr}_{(1)}\ \Psi, \qquad C := \mathrm{Tr}_{(2)}\ \Psi.$$

If the operator B (or C) is invertible, then we say that the corresponding braiding R is *strictly* skew-invertible. As was shown in [O], R is strictly skew-invertible if and only if R^{-1} is skew-invertible; moreover, the invertibility of B leads to the invertibility of C and *vice versa*.

A well-known important example of a strictly skew-invertible braiding is the superflip R on a superspace $V = V_0 \oplus V_1$, where V_0 and V_1 are (respectively) the even and odd components of V. In this case the operators B and C are called the *parity operators*, and their explicit form is

$$B(z) = C(z) = z_0 - z_1, \quad z \in V,$$

where $z_0(z_1)$ is the even (odd) component of $z = z_0 + z_1$.

Let R be a skew-invertible braiding. Some useful properties of the corresponding endomorphisms Ψ, B, and C are listed below.

1)
$$\mathrm{Tr}\, B = \mathrm{Tr}\, C,$$

$$\mathrm{Tr}_{(2)}\, B_2\, R_{21} = \mathrm{Tr}_{(2)}\, C_2\, R_{12} = I_1, \tag{2.4}$$

where I is the identity automorphism of V. These relations follow directly from the definitions (2.1) and (2.3).

2) The endomorphisms B and C commute and their product is a scalar operator,

$$B\,C = C\,B = \nu\, I, \tag{2.5}$$

where the numerical factor ν is nonzero if and only if the braiding R is strictly skew-invertible (in particular, if R is a skew-invertible Hecke symmetry).

3) The matrix elements of B and C realize a one-dimensional representation of the so-called RTT algebra associated with R (see [FRT]), that is

$$R_{12}B_1B_2 = B_1B_2R_{12}, \qquad R_{12}C_1C_2 = C_1C_2R_{12}\,. \tag{2.6}$$

As a direct consequence of the above relations, we have

$$\mathrm{Tr}_{(12)}(B_1B_2R_{12}X_{12}R_{12}^{-1}) = \mathrm{Tr}_{(12)}(B_1B_2R_{12}^{-1}X_{12}R_{12}) = \mathrm{Tr}_{(12)}(B_1B_2X_{12}),$$
$$\mathrm{Tr}_{(12)}(C_1C_2R_{12}X_{12}R_{12}^{-1}) = \mathrm{Tr}_{(12)}(C_1C_2R_{12}^{-1}X_{12}R_{12}) = \mathrm{Tr}_{(12)}(C_1C_2X_{12}),$$

where $X \in \mathrm{End}(V^{\otimes 2})$ is an arbitrary endomorphism and

$$\mathrm{Tr}_{(12)}(\dots) = \mathrm{Tr}_{(1)}(\mathrm{Tr}_{(2)}(\dots)).$$

4) The following important relations were proved in [I, O]:

$$\begin{aligned} B_1\Psi_{12} &= R_{21}^{-1}B_2, \qquad & \Psi_{12}B_1 &= B_2R_{21}^{-1},\\ C_2\Psi_{12} &= R_{21}^{-1}C_1, \qquad & \Psi_{12}C_2 &= C_1\,R_{21}^{-1}, \end{aligned} \tag{2.7}$$

where $R_{21} = PR_{12}P$. In the case $\nu \neq 0$, only one of the two lines above is independent, due to (2.5).

Therefore, for an arbitrary endomorphism $X \in \mathrm{End}(V)$, we obtain

$$\begin{aligned} \mathrm{Tr}_{(1)}(B_1R_{12}X_2R_{12}^{-1}) &= \mathrm{Tr}_{(1)}(B_1R_{12}^{-1}X_2R_{12}) = \mathrm{Tr}(BX)\,I_2,\\ \mathrm{Tr}_{(2)}(C_2R_{12}X_1R_{12}^{-1}) &= \mathrm{Tr}_{(2)}(C_2R_{12}^{-1}X_1R_{12}) = \mathrm{Tr}(CX)\,I_1. \end{aligned} \tag{2.8}$$

This completes the list of technical facts to be used in what follows.

§3. The general form of a Hecke symmetry

In this section we study the classification problem for (skew-invertible) Hecke symmetries. Our presentation is based on the theory of the A_{k-1} series Hecke algebras and their R-matrix representations. As a survey of the subject, we can recommend the work [OP1]. Some necessary facts of that theory are given in the Appendix for the reader's convenience.

Given a Hecke symmetry $R : V^{\otimes 2} \to V^{\otimes 2}$, we consider the R-symmetric algebra $\Lambda_+(V)$ and the R-skew-symmetric algebra $\Lambda_-(V)$ of the space V; by definition, these are the following quotients:

$$\Lambda_\pm(V) := T(V)/\langle(\mathrm{Im}(q^{\pm 1} I_{12} \mp R_{12})\rangle, \qquad I_{12} = I \otimes I. \tag{3.1}$$

Hereafter, $T(V)$ stands for the free tensor algebra of the space V, and $\langle J\rangle$ denotes the two-sided ideal generated in this algebra by a subset $J \subset T(V)$.

Then, we consider the Hilbert–Poincaré (HP) series of the algebras $\Lambda_\pm(V)$

$$P_\pm(t) := \sum_{k\ge 0} t^k \dim \Lambda^k_\pm(V), \tag{3.2}$$

where $\Lambda^k_\pm(V) \subset \Lambda_\pm(V)$ is the homogeneous component of degree k.

The following proposition plays the decisive role in the classification of all possible forms of the Hecke symmetries.

Proposition 1. *Consider an arbitrary Hecke symmetry R satisfying* (1.1) *and* (1.2) *at a generic value of the parameter q. Then the following statements hold true.*

1. *The HP series $P_\pm(t)$ obey the relation*

$$P_+(t)\, P_-(-t) = 1.$$

2. *The HP series $P_-(t)$ (and hence $P_+(t)$) is a rational function of the form*

$$P_-(t) = \frac{N(t)}{D(t)} = \frac{1 + a_1\, t + \cdots + a_m\, t^m}{1 - b_1\, t + \cdots + (-1)^n\, b_n\, t^n} = \frac{\prod_{i=1}^m (1 + x_i t)}{\prod_{j=1}^n (1 - y_j t)}\,, \tag{3.3}$$

where the coefficients a_i and b_i are positive integers, the polynomials $N(t)$ and $D(t)$ are relatively prime, and all real numbers x_i and y_i are positive.

3. *If, moreover, the Hecke symmetry is skew-invertible, then the polynomials $N(t)$ and $D(-t)$ are reciprocal.*[6]

The first item of the above list was proved in [G2]; the second and the third were proved in [H, Da] and [DH].

Definition 2. Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be a skew-invertible Hecke symmetry, and let m (respectively, n) be the degree of the numerator $N(t)$ (respectively, of the denominator $D(t)$) of the HP series $P_-(t)$. The ordered pair of integers $(m|n)$ will be called *the birank* of R. If $n = 0$ (respectively, $m = 0$), the Hecke symmetry will be called *even* (respectively, *odd*). Otherwise we say that R is of the general type.

Remark 3. In the sense of the above definition, any skew-invertible Hecke symmetry is a generalization of the superflip for which $P_-(t) = (1+t)^m(1-t)^{-n}$, where $m = \dim V_0$, $n = \dim V_1$. Such a treatment of Hecke symmetries is also motivated by similarity of the corresponding Schur–Weyl categories (see below).

[6]Recall that a polynomial $p(t) = c_0 + c_1 t + \cdots + c_n t^n$ with real coefficients c_i is called *reciprocal* if $p(t) = t^n p(t^{-1})$ or, equivalently, $c_i = c_{n-i}$, $0 \le i \le n$.

Now we obtain some important consequences of Proposition 1. Let R be a Hecke symmetry of birank $(m|n)$. As is known, the Hecke symmetry R makes it possible to define representations ρ_R of the A_{k-1} series Hecke algebras $H_k(q)$, $k\ge 2$, in homogeneous components $V^{\otimes p}\subset T(V)$, for all $p\ge k$:

$$\rho_R:\ H_k(q)\to \mathrm{End}(V^{\otimes p}),\quad p\ge k\,.$$

Explicitly, these representations are given in formula (A.3) in the Appendix.

Under the representation ρ_R, the primitive idempotents $e_a^\lambda\in H_k(q)$, $\lambda\vdash k$, turn into the projection operators

$$E_a^\lambda(R)=\rho_R(e_a^\lambda)\in \mathrm{End}(V^{\otimes p}),\quad p\ge k\,, \tag{3.4}$$

where the index a enumerates the standard Young *tableaux* (λ,a) that can be constructed for a given partition $\lambda\vdash k$. The total number of the standard Young *tableaux* corresponding to the partition λ is denoted by d_λ.

Under the action of these projectors, the spaces $V^{\otimes p}$, $p\ge 2$, expand into the direct sum

$$V^{\otimes p}=\bigoplus_{\mu\vdash p}\bigoplus_{a=1}^{d_\mu}V_{(\mu,a)},\qquad V_{(\mu,a)}=\mathrm{Im}(E_a^\mu). \tag{3.5}$$

Relation (A.2) shows that the projectors E_a^μ with different a are related by invertible transformations, and therefore, all spaces $V_{(\mu,a)}$ with fixed μ and different a are isomorphic.

At a generic value of q, the Hecke algebra $H_k(q)$ is known to be isomorphic to the group algebra $\mathbb{K}[\mathfrak{S}_k]$; see [We]. On the basis of this fact, we can prove the following result [GLS1, H]:

$$V_{(\lambda,a)}\otimes V_{(\mu,b)}=\bigoplus_{\nu}\bigoplus_{d_{ab}\in I_{ab}}V_{(\nu,d_{ab})}\cong\bigoplus_{\nu}c_{\lambda\mu}^{\nu}\,V_{(\nu,d_0)}\,,\quad \lambda\vdash p,\ \mu\vdash k,\ \nu\vdash(p+k)\,, \tag{3.6}$$

where the integers $c_{\lambda\mu}^\nu$ are the Littlewood–Richardson coefficients, and the *tableau* index d_{ab} takes values in a subset $I_{ab}\subset\{1,2,\dots,d_\nu\}$ that depends on the values of the indices a and b. The number d_0 (3.6) stands for the index of an arbitrary fixed *tableau* from the set (ν,d), $1\le d\le d_\nu$. Identity (3.6) has the following meaning. Though the summands $V_{(\nu,d_{ab})}$ do depend on the values of a and b, the *total* number of these summands (the cardinality of I_{ab}) depends only on the partitions λ, μ, and ν and is equal to the Littlewood–Richardson coefficient $c_{\lambda\mu}^\nu$. Therefore, due to isomorphism $V_{(\nu,d_{ab})}\cong V_{(\nu,d_0)}$, we can replace the sum over d_{ab} by the space $V_{(\nu,d_0)}$ with the corresponding multiplicity $c_{\lambda\mu}^\nu$ (see [GLS1]).

A particular example of the spaces $V_{(\lambda,a)}$ is the homogeneous components $\Lambda_+^k(V)$ and $\Lambda_-^k(V)$ of the algebras $\Lambda_\pm(V)$; see (3.1). They are the images of the projectors $E^{(k)}$ and $E^{(1^k)}$ that correspond to the one-row and one-column partitions (k) and (1^k), respectively. This important fact allows us to calculate the dimensions (over the ground field $\mathbb{K}$) of all spaces $V_{(\lambda,a)}$, provided that the Hilbert–Poincaré series $P_-(t)$ is known. Since all the spaces $V_{(\lambda,a)}$ corresponding to one and the same partition λ are isomorphic, we denote their $\mathbb{K}$-dimensions by the symbol $\dim V_\lambda$.

In the sequel, the following corollary of Proposition 1 will be useful.

Corollary 4. *Let R be a Hecke symmetry of birank $(m|n)$, and let the Hilbert–Poincaré series of $\Lambda_-(V)$ be given by* (3.3). *Then for the dimensions of the spaces $V_{(k)}$ and $V_{(1^k)}$*

determined by partitions (k) *and* (1^k), $k \in \mathbb{N}$, *we have*

$$\dim V_{(k)} = s_{(k)}(x|y) := \sum_{i=0}^{k} h_i(x) e_{k-i}(y), \tag{3.7}$$

$$\dim V_{(1^k)} = s_{(1^k)}(x|y) := \sum_{i=0}^{k} e_i(x) h_{k-i}(y), \tag{3.8}$$

where the h_i *and* e_i *are (respectively) the complete symmetric and elementary symmetric functions of their arguments.*

Proof. We only prove the first of the above formulas because the second can be proved in the same way. Since $V_{(k)} = \Lambda_+^k(V)$, the dimension of $V_{(k)}$ can be found as an appropriate derivative of the Hilbert–Poincaré series $P_+(t)$,

$$\dim V_{(k)} = \frac{1}{k!}\frac{d^k}{dt^k} P_+(t)_{|t=0}.$$

Using the relation $P_+(t)P_-(-t) = 1$ (see Proposition 1) and (3.3), we present $P_+(t)$ in the form

$$P_+(t) = \prod_{i=1}^{n}(1 + y_i t) \prod_{j=1}^{m} \frac{1}{(1 - x_j t)} = \mathcal{E}(y|t)\mathcal{H}(x|t),$$

where $\mathcal{E}(\cdot)$ and $\mathcal{H}(\cdot)$ denote the generating functions of the elementary and complete symmetric functions in the finite set of variables [Mac]:

$$e_k(y) = \sum_{1 \le i_1 < \cdots < i_k \le n} y_{i_1} \cdots y_{i_k} = \frac{1}{k!}\frac{d^k}{dt^k}\mathcal{E}(y|t)_{|t=0},$$

$$h_k(x) = \sum_{1 \le j_1 \le \cdots \le j_k \le m} x_{j_1} \cdots x_{j_k} = \frac{1}{k!}\frac{d^k}{dt^k}\mathcal{H}(x|t)_{|t=0}.$$

Calculating the kth derivative of $P_+(t)$ at $t = 0$, we get (3.7). □

Note that the polynomials $s_{(k)}(x|y)$ and $s_{(1^k)}(x|y)$ defined in (3.7) and (3.8) belong to the class of supersymmetric polynomials in $\{x_i\}$ and $\{y_j\}$. By definition (see [St]), a polynomial $p(u|v)$ in two sets of variables is said to be *supersymmetric* if it is symmetric with respect to any permutation of the arguments $\{u_i\}$ as well as of the arguments $\{v_j\}$ and if, additionally, setting $u_1 = v_1 = t$ in $p(u|v)$ yields a result independent of t. Evidently, the polynomials in question satisfy this definition if we set, for example, $u = x$ and $v = -y$.

Actually, the polynomials $s_{(k)}(x|y)$ (respectively, $s_{(1^k)}(x|y)$), $k \in \mathbb{N}$, are supersymmetric analogs of the complete symmetric (respectively, elementary symmetric) functions in a finite number of variables. In particular, they generate the entire ring of supersymmetric polynomials in the variables $\{x_i\}$ and $\{y_j\}$. The $\mathbb{Z}$-basis of this ring is formed by the Schur supersymmetric functions $s_\lambda(x|y)$, which can be expressed in terms of $s_{(k)}$ (or $s_{(1^k)}$) through the Jacobi–Trudi relations [Mac]. The Schur supersymmetric functions determine the dimensions $\dim V_\lambda$. In order to formulate the corresponding result, we need yet another definition.

Definition 5 ([BR]). Given two arbitrary integers $m \ge 0$ and $n \ge 0$, consider a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ satisfying the restriction $\lambda_{m+1} \le n$. The (infinite) set of all such partitions will be denoted by $\mathsf{H}(m,n)$, and any partition $\lambda \in \mathsf{H}(m,n)$ will be called a hook partition of type $\mathsf{H}(m,n)$.

Proposition 6 ([H]). *Let R be a Hecke symmetry of birank $(m|n)$. Then the dimensions* $\dim V_\lambda$ *of the spaces in* (3.5) *are determined by the following rules.*

1. *For any $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathsf{H}(m,n)$, the dimension $\dim V_\lambda$ is nonzero and is given by the formula*

$$\dim V_\lambda = s_\lambda(x|y)\,. \tag{3.9}$$

   Here

$$s_\lambda(x|y) = \det \| s_{(\lambda_i - i + j)}(x|y) \|_{1 \le i,j \le k}\,,$$

   where $s_{(k)}(x|y)$ is defined in (3.7) *for $k \ge 0$ and $s_{(k)} := 0$ for $k < 0$.*
2. *For an arbitrary partition λ we have*

$$\dim V_\lambda = 0 \;\Leftrightarrow\; \lambda \notin \mathsf{H}(m,n).$$

Proof. Since

$$\dim(U \otimes W) = \dim U \dim W, \qquad \dim(U \oplus W) = \dim U + \dim W,$$

calculation of the dimensions of the spaces on both sides of (3.6) yields

$$\dim V_\lambda \dim V_\mu = \sum_\nu c^\nu_{\lambda\mu} \dim V_\nu.$$

Now, (3.9) is a direct consequence of an inductive procedure based on Corollary 4 (cf., e.g., [GPS2]).

The second claim can be deduced from the properties of the Schur functions $s_\lambda(x|y)$ established in [BR] (see also [H]). □

To finish this section, we present yet another important numerical characteristic of the Hecke symmetry that can be expressed in terms of its birank.

Proposition 7. *Let R be a skew-invertible Hecke symmetry with birank $(m|n)$. Then*

$$\mathrm{Tr}\; B = \mathrm{Tr}\; C = q^{n-m}(m-n)_q. \tag{3.10}$$

The proof of this is rather technical and is placed in the Appendix.

Corollary 8. *For a skew-invertible Hecke symmetry with birank $(m|n)$, the factor ν in* (2.5) *equals $q^{2(n-m)}$, i.e.,*

$$BC = CB = q^{2(n-m)}I\,.$$

Proof. First, observe that if R is a skew-invertible Hecke symmetry, then the same is true for the operator $R_{21} = PR_{12}P$, and therefore we have

$$R^{-1}_{21} = R_{21} - (q - q^{-1})\, I_{21}.$$

Applying $\mathrm{Tr}_{(2)}$ to the first formula in (2.7), we obtain

$$\begin{aligned} B_1 C_1 &= \mathrm{Tr}_{(2)}\,(B_1 \Psi_{12}) = \mathrm{Tr}_{(2)}(R^{-1}_{21} B_2) \\ &= \mathrm{Tr}_{(2)}((R_{21} - (q - q^{-1})\, I_{21}) B_2) = I_1 - (q - q^{-1})\, I_1\, \mathrm{Tr}(B) = q^{2(n-m)}\, I_1. \end{aligned}$$ □

§4. The quasitensor category $\mathrm{SW}(V_{(m|n)})$

Our next goal is to construct the quasitensor *Schur–Weyl category* $\mathrm{SW}(V_{(m|n)})$ of vector spaces, generated by the space V equipped with a skew-invertible Hecke symmetry R of birank $(m|n)$. The objects of this category possess a module structure over the *reflection equation algebra*, which will be considered in detail in the next sections.

In constructing this category, we proceed by analogy with the paper [GLS1], where such a category was constructed for an even Hecke symmetry of birank $(m|0)$. A peculiarity of the even case is that the space V^*, dual to V, can be identified with a specific object $V_{(1^{m-1})}$ of the category (see (3.5) for the definition of V_λ). This property ensures that the category constructed in [GLS1] is rigid.[7]

This is not so in the case of a general birank $(m|n)$, and we need to properly enlarge the category by adding the dual spaces to all objects. In turn, this requires a consistent extending of the categorical braidings to the dual objects and defining the invariant pairings. In the present section we elaborate on these problems in detail.

So, let R be a skew-invertible Hecke symmetry of birank $(m|n)$. Upon fixing a basis $\{x_i\}_{1\le i\le N}$ of the space V, $\dim V = N$, we represent R by the matrix (2.2). Also, we introduce the dual vector space V^* and fix the basis $\{x^i\}_{1\le i\le N}$ in V^* dual to $\{x_i\}$ with respect to the nondegenerate bilinear form

$$\langle\, ,\rangle_r : \; V\otimes V^* \to \mathbb{K}\,, \qquad \langle x_i, x^j\rangle_r = \delta_i^j\,. \tag{4.1}$$

The subscript r ("right") refers to the order of arguments in the form $\langle\, ,\rangle_r$: the vectors of the dual space V^* stand to the right of the vectors of V.

By definition, the dual space to the tensor product $U\otimes W$ is $W^*\otimes U^*$:

$$\langle U\otimes W, W^*\otimes U^*\rangle_r := \langle W, W^*\rangle_r \langle U, U^*\rangle_r\,.$$

As a consequence, the numbering of components in a tensor power $V^{*\otimes k}$ is reverse to that in a tensor power $V^{\otimes k}$:

$$V^{*\otimes k} := V_k^*\otimes\cdots\otimes V_2^*\otimes V_1^*\,, \qquad V^{\otimes k} := V_1\otimes V_2\otimes\cdots\otimes V_k\,.$$

This peculiarity should always be kept in mind when working with operators marked by numbers of spaces where these operators act (like in formulas (1.1) and all other similar expressions).

Now we extend the braiding (2.2) to the space $V^*\otimes V^*$. Below we show that the requirement that the extended braiding be consistent with the invariance of the pairing (4.1) leads to the only choice

$$R(x^i\otimes x^j) = x^r\otimes x^s\, R_{sr}^{ji}\,. \tag{4.2}$$

Therefore, by analogy with the construction of §3, we can define the representations of the Hecke algebras $H_k(q)$, $k\in\mathbb{K}$, in the tensor powers $V^{*\otimes k}$, construct the projectors E_a^λ, and introduce the subspaces $V^*_{(\lambda,a)}\subset V^{*\otimes k}$ as images of the corresponding projectors (see (3.4)–(3.6)). Recalling the above remark on the numbering of the tensor product components, one can show that for any Young *tableau* (λ,a) there exists a unique *tableau* (λ,a') such that the spaces $V_{(\lambda,a)}$ and $V^*_{(\lambda,a')}$ are dual with respect to the form (4.1).

By definition, the class of objects of the category $\mathrm{SW}(V_{(m|n)})$ consists of all direct sums of spaces $V_\lambda\otimes V_\mu^*$ and $V_\mu^*\otimes V_\lambda$, where λ and μ are partitions of nonnegative integers. The zero partition corresponds to the basic space $V_0 := V$ or to its dual space $V_0^* := V^*$. The ground field $\mathbb{K}$ is treated as the unit object of the category $\mathrm{SW}(V_{(m|n)})$:

$$\mathbb{K}\otimes V = V = V\otimes\mathbb{K}.$$

Now we define the class of morphisms of $\mathrm{SW}(V_{(m|n)})$. First, we should define the set of braiding morphisms $R_{U,W}$ that realizes the isomorphisms $U\otimes W\cong W\otimes U$ for any two objects U and W. The braidings R_{V_λ,V_μ} and $R_{V_\lambda^*,V_\mu^*}$ are completely determined by $R_{V,V}$ and R_{V^*,V^*} given by (2.2) and (4.2). Therefore, we only need consistent definitions of R_{V,V^*} and $R_{V^*,V}$, because the braidings R_{V_λ,V_μ^*} and $R_{V_\lambda^*,V_\mu}$ can then be constructed by standard methods (see, e.g., [GLS1]). The consistency condition is the following requirement. Having defined the four braidings mentioned above, we get a linear operator on $(V\oplus V^*)^{\otimes 2}$. Our definitions are consistent if this operator satisfies the Yang–Baxter

[7]Recall that a (quasi)tensor category of vector spaces is *rigid* if, with any of its objects U, a dual object U^* is associated so that the maps $U\otimes U^*\to\mathbb{K}$ and $U^*\otimes U\to\mathbb{K}$ are categorical morphisms.

equation. This problem is solved in the next proposition. The main idea of such a construction belongs to V. Lyubashenko (see [LS] and the references therein).

Proposition 9. *Let Ψ be the skew-inverse operator* (2.1) *of a skew-invertible braiding R. We extend R to the linear operator*

$$R : (V \oplus V^*)^{\otimes 2} \to (V \oplus V^*)^{\otimes 2}$$

(*we keep the same notation for the extended operator*) *that acts in accordance with the formulas*

$$\begin{aligned} &V \otimes V^* \to V^* \otimes V: && R(x_i \otimes x^j) = x^k \otimes x_l\,(R^{-1})^{lj}_{ki}\,, \\ &V^* \otimes V \to V \otimes V^*: && R(x^j \otimes x_i) = x_k \otimes x^l\,\Psi^{kj}_{li}\,, \\ &V^* \otimes V^* \to V^* \otimes V^*: && R(x^i \otimes x^j) = x^k \otimes x^l\,R^{ji}_{lk}\,, \\ &V \otimes V \to V \otimes V: && R(x_i \otimes x_j) = x_k \otimes x_l\,R^{kl}_{ij}\,. \end{aligned} \tag{4.3}$$

Then the extended operator R is a braiding, i.e., it satisfies the Yang–Baxter equation (1.1) *on the space $(V \oplus V^*)^{\otimes 3}$.*

Proof. Since R is a linear operator, it suffices to prove the proposition on the basis vectors of the space $(V \oplus V^*)^{\otimes 3}$. This space splits into the direct sum of eight subspaces (from $V \otimes V \otimes V$ to $V^* \otimes V^* \otimes V^*$) and the verification of the claim on each of these subspaces is a matter of straightforward calculations based on formulas (4.3). □

At the second step of our construction, we assume that a linear combination, the product, the direct sum, and the tensor product of a finite family of categorical morphisms is also a morphism.

Then, as in [T], we require the morphisms to be natural (or functorial). This means that

$$(g \otimes f) \circ R_{U,W} = R_{U',W'} \circ (f \otimes g),$$

where $f : U \to U'$ and $g : W \to W'$ are two categorical morphisms. As a consequence, we get a necessary condition for a map $f : U \to U'$ to be a categorical morphism:

$$(\mathrm{id}_W \otimes f) \circ R_{U,W} = R_{U',W} \circ (f \otimes \mathrm{id_W}), \quad (f \otimes \mathrm{id}_W) \circ R_{W,U} = R_{W,U'} \circ (\mathrm{id}_W \otimes f). \tag{4.4}$$

We say that a map f satisfying this condition is *R-invariant*. Thus, any categorical morphism must be R-invariant.

Proposition 10. *Provided R satisfies* (4.3)*, the following claims hold true.*

1. *The pairing* (4.1) *is R-invariant.*
2. *The linear map $\pi_r : \mathbb{K} \to V^* \otimes V$ generated by*

$$1 \overset{\pi_r}{\mapsto} \sum_{i=1}^{N} x^i \otimes x_i \tag{4.5}$$

is also R-invariant.

Proof. In proving claim 1, we can confine ourselves to the simplest case of formula (4.4) where $W = V$ or $W = V^*$. This is a consequence of the structure of objects of the category $\mathrm{SW}(V_{(m|n)})$. In other words, we must show the commutativity of the diagram

$$\begin{array}{ccc} (V \otimes V^*) \otimes V^{\#} & \overset{(4.3)}{\longleftrightarrow} & V^{\#} \otimes (V \otimes V^*) \\ \langle\, ,\rangle_r \otimes \mathrm{id} \downarrow & & \downarrow \mathrm{id} \otimes \langle\, ,\rangle_r \\ \mathbb{K} \otimes V^{\#} & = & V^{\#} \otimes \mathbb{K}, \end{array} \tag{4.6}$$

where $V^\#$ stands for V or V^*. The commutativity of the diagram immediately follows from formulas (4.3), the definitions (4.1), (2.1), and the definition of the inverse matrix R^{-1}.

Next, the same argument shows that claim 2 is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} (V^* \otimes V) \otimes V^\# & \overset{(4.3)}{\longleftrightarrow} & V^\# \otimes (V^* \otimes V) \\ \pi_r \otimes \mathrm{id} \uparrow & & \uparrow \mathrm{id} \otimes \pi_r \\ \mathbb{K} \otimes V^\# & = & V^\# \otimes \mathbb{K}, \end{array}$$

which can be proved similarly. □

Remark 11. Note that the R-invariance of the maps (4.1) and (4.5) is a motivation for the extension (4.3) of the initial braiding R. It can be shown that such an extension is unique.

In what follows, besides the right form (4.1), we also need a *left* nondegenerate bilinear form

$$\langle\,,\rangle_l : \; V^* \otimes V \to \mathbb{K},$$

with the additional requirement that the above pairing be R-invariant. This requirement prevents us from setting $\langle x^i, x_j\rangle_l = \delta^i_j$, because this is not an R-invariant pairing (a direct consequence of (4.3)).

We choose the form $\langle\,,\rangle_l$ in such a way that the following diagram be commutative:

$$\begin{array}{ccc} V^* \otimes V & \overset{(4.3)}{\longrightarrow} & V \otimes V^* \\ \langle\,,\rangle_l \downarrow & & \downarrow \langle\,,\rangle_r \\ \mathbb{K} & = & \mathbb{K}\,. \end{array} \tag{4.7}$$

A simple calculation based on (4.7) leads to the following explicit expression:

$$\langle x^i, x_j\rangle_l = B^i_{\ j}, \tag{4.8}$$

where the matrix $\|B^i_{\ j}\|$ is defined in (2.3). Such a choice guarantees the R-invariance of the left pairing $\langle\,,\rangle_l$. The commutativity of the corresponding diagram (similar to (4.6)) can be verified easily with the help of (4.3) and (2.7).

Remark 12. Note that the backward diagram

$$\begin{array}{ccc} V^* \otimes V & \overset{(4.3)}{\longleftarrow} & V \otimes V^* \\ \langle\,,\rangle_l \downarrow & & \downarrow \langle\,,\rangle_r \\ \mathbb{K} & = & \mathbb{K} \end{array} \tag{4.9}$$

is *not* commutative with the definition (4.8). In a tensor category one can define the left pairing in such a way that both diagrams (4.7) and (4.9) are commutative, while in a quasitensor category this is impossible. This is a consequence of the fact that the braiding R is not involutive: $R^2 \neq I$.

In principle, we could demand the commutativity of the above diagram, rather than of diagram (4.7). In this case, on the right-hand side of (4.8) we would obtain an additional factor of $q^{2(m-n)}$. Actually, the two variants are equivalent, and choosing between them is a matter of taste.

Now we can find another basis $\{{}^{x}0i\}_{1\le i\le N}$ of V^* that is dual to the basis $\{x_i\}_{1\le i\le n}$ with respect to the left form

$$ {}^{x}0i := q^{2(m-n)}\, C^i_{\ j}\, x^j \quad \Rightarrow \quad \langle {}^{x}0i, x_j\rangle_l = \delta^i_j. \tag{4.10}$$

The normalizing factor in the definition of the basis vector ${}^{x}0i$ is chosen in accordance with Corollary 8.

So, we have two R-invariant bilinear forms and two basic sets $\{x^i\}$ and $\{{}^{x}0i\}$ in the space V^* that are dual to the basis $\{x_i\}$ of the space V with respect to the right and left forms, respectively (see (4.1) and (4.10)). For this reason, we refer to $\{x^i\}$ (respectively, $\{{}^{x}0i\}$) as the *right* (respectively, *left*) basis of V^*.

Using (2.7), we can rewrite formulas (4.3) in terms of the left basis.

Corollary 13. *In terms of the left basis* $\{{}^{x}0i\}_{1\le i\le N}$ *of the space* V^**, the extension of the braiding* R *defined by* (4.3) *has the following form*:

$$\begin{aligned} R(x_i\otimes {}^{x}0j) &= {}^{x}0k\otimes x_l\,\Psi^{jl}_{ik},\\ R({}^{x}0j\otimes x_i) &= x_k\otimes {}^{x}0l\,(R^{-1})^{jk}_{il},\\ R({}^{x}0i\otimes {}^{x}0j) &= {}^{x}0k\otimes {}^{x}0l\,R^{ji}_{lk},\\ R(x_i\otimes x_j) &= x_k\otimes x_l\,R^{kl}_{ij}. \end{aligned} \tag{4.3′}$$

Moreover, the linear map $\pi_l:\mathbb{K}\to V\otimes V^*$ *generated by*

$$1 \overset{\pi_l}{\mapsto} \sum_{i=1}^N x_i\otimes {}^{x}0i \tag{4.11}$$

is R*-invariant.*

Proof. We prove the first formula in (4.3′); the others are proved in the same way. Using the definition of the left basis (4.10) and the first formula in (4.3), we get (recall that summation over repeated indices is assumed)

$$R(x_i\otimes {}^{x}0j) = x^u\otimes x_l\, q^{2(m-n)}\,C^j_{\ s}(R^{-1})^{ls}_{ui} = {}^{x}0k\otimes x_l\, q^{2(m-n)} C^j_{\ s}(R^{-1})^{ls}_{ui}B^u_{\ k},$$

where in the last identity we come back from the right basis to the left by the formula inverse to (4.10),

$$x^u = B^u_{\ k}\,{}^{x}0k.$$

Then, from the formulas in the second line of (2.7) and Corollary 8 we deduce the relation

$$q^{2(m-n)}C_1R^{-1}_{21}B_2 = \Psi_{12},$$

which allows us to make the following substitution in the above line of transformations:

$$q^{2(m-n)}C^j_{\ s}(R^{-1})^{ls}_{ui}B^u_{\ k} = \Psi^{jl}_{ik}.$$

So, finally we get

$$R(x_i\otimes {}^{x}0j) = {}^{x}0k\otimes x_l\,\Psi^{jl}_{ik},$$

which is the formula in the first line in (4.3′).

The R-invariance of the map π_l can be proved by straightforward calculations on the basis of (4.3) or (4.3′) in the same way as was done in the proof of (4.5). □

Now we are able to define the categorical morphisms of $\mathrm{SW}(V_{(m|n)})$. Together with the identity map, the list of morphisms includes (4.1), (4.5), (4.8), (4.11), and all maps (4.3) (or, equivalently, (4.3′)). Furthermore, as we have already mentioned, any linear combination, the product (successive application), the tensor product, and the direct sum of categorical morphisms is also a categorical morphism.

Remark 14. In principle, given a particular braiding R, one can compose a larger list of R-invariant maps than that mentioned above. Thus, for a superspace $V = V_0 \oplus V_1$ the projections $V \to V_0$ and $V \to V_1$ are R-invariant maps.

In what follows, we shall be especially interested in objects of the form $V^* \otimes V$ and $V \otimes V^*$, which are isomorphic to the space $\mathrm{End}(V)$ of endomorphisms of the space V. Upon fixing the basis x_i in the space V, we arrive at the standard basis $h_i^j = x_i \otimes {}^x0j$ in the space $V \otimes V^*$. Defining the action of an element $v \otimes v^* \in V \otimes V^*$ on a vector $u \in V$ by the usual rule

$$(v \otimes v^*)(u) := v\langle v^*, u\rangle_l,$$

we get the action

$$h_i^j(x_k) = \delta_k^j\, x_i$$

and the multiplication table of elements h_i^j treated as endomorphisms of the space V:

$$h_i^j \circ h_k^s = \delta_k^j\, h_i^s.$$

Fixing the right basis $\{x^i\}$ in V^*, we arrive at another basis $l_i^j = x_i \otimes x^j$ of $V \otimes V^*$ with the properties (see (4.8))

$$l_i^i(x_k) = B^j{}_k\, x_i, \qquad l_i^j \circ l_k^s = B^j{}_k\, l_i^s\,. \tag{4.12}$$

Taking (4.10) into account, we find the relationship between the two basis sets:

$$h_i^j = q^{2(m-n)}\, C^j{}_k l_i^k\,. \tag{4.13}$$

Now we introduce the linear map $\mathrm{Tr}_R : \mathrm{End}(V) \to \mathbb{K}$ via the categorical morphism (4.1),

$$\mathrm{Tr}_R(l_j^i) = \langle x_j, x^i\rangle_r = \delta_j^i. \tag{4.14}$$

This map is called the *R-trace* in what follows. By (4.13), the R-trace of an operator $\mathsf{F} \in \mathrm{End}(V)$ is given by

$$\mathrm{Tr}_R(\mathsf{F}) = q^{2(m-n)}\ \mathrm{Tr}(F \cdot C)\,, \tag{4.15}$$

where F is the matrix of the operator F with respect to basis $\{x_i\}$.

To complete the section, we calculate the R-dimension of the objects V_λ of our category. By definition, the R-dimension of an object $V_\lambda \subset V^{\otimes k}$, $\lambda \vdash k$, is given by

$$\dim_R V_\lambda := \mathrm{Tr}_R(\mathrm{id}_{V_\lambda}) = q^{2k(m-n)}\, \mathrm{Tr}_{(1\dots k)}(C_1 \cdots C_k E_a^\lambda)\,. \tag{4.16}$$

With the help of (A.2), one can prove that the above definition does not depend on the value of a. Moreover, like the classical dimension, the R-dimension is an additive-multiplicative functional:

$$\dim_R(U \otimes W) = \dim_R U \dim_R W\,, \qquad \dim_R(U \oplus W) = \dim_R U + \dim_R W\,.$$

We introduce the R-analogs $Q_\pm(t)$ of the HP series $P_\pm(t)$ (see (3.2)) by the relation

$$Q_\pm(t) = \sum_{k \ge 0} t^k\, \dim_R \Lambda_\pm^k(V)\,.$$

Then the following proposition holds true.

Proposition 15. *Given a skew-invertible Hecke symmetry with birank $(m|n)$, we find the following properties of the series $Q_\pm$:*

1) *if $m - n = 0$, then $\dim_R V_\lambda = 0$ for any $\lambda \neq 0$, and therefore $Q_+(t) = Q_-(t) = 1$;*

2) *if* $m-n>0$, *then*

$$\dim_R V_\lambda = \dim_R {V_\lambda}^* = s_\lambda(q^{m-n-1}, q^{m-n-3}, \dots, q^{1-m+n}),$$

and therefore

$$Q_-(t) = \sum_{k=0}^{m-n} \binom{m-n}{k}_q t^k\,, \qquad \binom{p}{k}_q := \frac{p_q(p-1)_q\cdots(p-k+1)_q}{k_q(k-1)_q\cdots 2_q 1_q}\,;$$

3) *if* $m-n<0$, *then*

$$\dim_R V_\lambda = \dim_R {V_\lambda}^* = s_{\lambda^*}(q^{n-m-1}, q^{n-m-3}, \dots, q^{1-n+m}),$$

where λ^* *is the conjugate partition, and therefore*

$$Q_+(t) = \sum_{k=0}^{n-m} \binom{n-m}{k}_q t^k\,.$$

Proof. The proposition is proved by direct calculations based on the definition (4.16). The calculations are similar to those in the even case (see, e.g., [GLS1]). □

We emphasize that the $Q_\pm(t)$ depend only on the birank of the given Hecke symmetry R, whereas the corresponding HP series $P_\pm(t)$ depend substantially on a specific form of R.

§5. mREA: Definition and deformation properties

If R is an involutive ($R^2 = I$) skew-invertible symmetry, then the space $\mathrm{End}(V)$ can be endowed with the structure of a *generalized Lie algebra* (see [G1, G3]). The corresponding enveloping algebra $U_R(\mathrm{End}(V))$ is defined as the following quotient:

$$U_R(\mathrm{End}(V)) = T(\mathrm{End}(V))/\langle \mathcal{J}_R\rangle, \tag{5.1}$$

where $\langle \mathcal{J}_R\rangle$ is a two-sided ideal of the free tensor algebra $T(\mathrm{End}(V))$ generated by the subset $\mathcal{J}_R \subset T(\mathrm{End}(V))$ of the form

$$\mathcal{J}_R = \{X\otimes Y - R_{\mathrm{End}}(X\otimes Y) - X\circ Y + \circ R_{\mathrm{End}}(X\otimes Y) \mid X, Y \in \mathrm{End}(V)\}\,. \tag{5.2}$$

Here $\circ$ is the product in $\mathrm{End}(V)$ viewed as an associative algebra of linear operators on V, and the linear operator $R_{\mathrm{End}} : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)^{\otimes 2}$ is an extension of the braiding R to the space $\mathrm{End}(V)^{\otimes 2}$. Its explicit form can be obtained by using (4.3).

Namely, choosing the basis $l_j^i = x_j \otimes x^i$ in the space $\mathrm{End}(V)$ and applying the corresponding formulas from the list (4.3), we find

$$R_{\mathrm{End}}(l_j^i \otimes l_s^k) = l_{b_1}^{a_1} \otimes l_{b_2}^{a_2}\,(R^{-1})^{b_2c_1}_{a_1c_2}\, R^{b_1c_2}_{jr_1}\, R^{kr_2}_{a_2c_1}\, \Psi^{r_1 i}_{r_2 s}\,. \tag{5.3}$$

In order to present this formula in a more transparent form, we introduce a matrix notation that will be useful in what follows. Let L be the $N\times N$-matrix (recall that $N = \dim V$) with the matrix elements

$$L_i^{\,j} = l_i^j, \tag{5.4}$$

where the subscript enumerates the rows and the superscript enumerates the columns of L. Then, denoting by $\bar R$ the transpose of R,

$$\bar R_{i_1i_2}^{\,j_1j_2} = R^{j_1j_2}_{\,i_1i_2}\,,$$

we put

$$L_{\bar 1} = L\otimes I, \quad L_{\bar 2} = \bar R_{12} L_{\bar 1} \bar R_{12}^{-1}. \tag{5.5}$$

Now, multiplying the two sides of (5.3) by R and R^{-1} and recalling the definition of Ψ (see (2.1)), we represent formula (5.3) in the equivalent form

$$R_{\mathrm{End}}(L_{\bar 1} \otimes L_{\bar 2}) = L_{\bar 2} \otimes L_{\bar 1} \tag{5.6}$$

(summation over the corresponding matrix indices is assumed).

Note that a direct generalization of (5.1)–(5.2) with the definition (5.6) from the involutive symmetry to the Hecke symmetry case leads to an algebra which possesses bad deformation properties and a poor representation theory. Fortunately, for any skew-invertible Hecke symmetry R there exists another generalization of the enveloping algebra $U_R(\mathrm{End}(V))$ (5.1) that has good deformation properties (see Proposition 20 below) and coincides with the enveloping algebra $U_R(\mathrm{End}(V))$ when R is involutive.

Definition 16. The associative algebra generated by the unit element $e_{\mathcal{L}}$ and the indeterminates l_i^j, $1 \le i, j \le N$, subject to the system of relations

$$R^{kl}_{ij}\, l^m_k\, R^{pq}_{ml}\, l^r_p - l^a_i\, R^{bc}_{aj}\, l^d_b\, R^{rq}_{dc} - \hbar(R^{aq}_{ij}\, l^r_a - l^b_i\, R^{rq}_{bj}) = 0 \tag{5.7}$$

is called the reflection equation algebra (REA) (see [KS]) and is denoted by $\mathcal{L}(R_q)$ if $\hbar = 0$, and it is called the modified reflection equation algebra (mREA) and is denoted by $\mathcal{L}(R_q, \hbar)$ if $\hbar \neq 0$.

The defining relations (5.7) can be presented in a compact form in terms of the matrix L (see (5.4)) and the transpose $\bar R$:

$$\bar R_{12} L_1 \bar R_{12} L_1 - L_1 \bar R_{12} L_1 \bar R_{12} - \hbar(\bar R_{12}\, L_1 - L_1\, \bar R_{12}) = 0\,. \tag{5.8}$$

Remark 17. Note that, making a linear transformation of generators $l_i^j \mapsto m_i^j$ (for $q \neq \pm 1$)

$$M = I e_{\mathcal{L}} - \omega \hbar^{-1} L, \qquad \omega = q - q^{-1}\,, \quad M = \|m_i^j\|\,,$$

we arrive at the following form of commutation relations (5.8):

$$\bar R_{12} M_1 \bar R_{12} M_1 - M_1 \bar R_{12} M_1 \bar R_{12} = 0\,. \tag{5.9}$$

This means that the algebras $\mathcal{L}(R_q, \hbar)$ and $\mathcal{L}(R_q)$ are isomorphic for all $q \neq \pm 1$. The basis of mREA generators with commutation relations (5.8) is more suitable for treating this algebra as an analog of the universal enveloping algebra $U(gl(m|n))$.

Now we prove that the commutation relations (5.8) are consistent with the structure of the category $\mathrm{SW}(V_{(m|n)})$ in the following sense. We treat the mREA as a quotient of the tensor algebra $T(V \otimes V^*)$ over the two-sided ideal generated by relations (5.8) or, equivalently, (5.9). These relations are consistent with the structure of the category if the corresponding two-sided ideal is invariant with respect to braidings of the category, or, in other words, if the mREA commutation relations are *R-invariant.*

Proposition 18. *The commutation relations* (5.8) *are R-invariant.*

Proof. To prove the proposition, it suffices to show that the commutation relations (5.8) are preserved under commuting with V or V^* with respect to the braidings of the category $\mathrm{SW}(V_{(m|n)})$. This can be done by straightforward calculations based on formulas (4.3) and $l_i^j = x_i \otimes x^j$. To simplify calculations, working with the generators m_i^j as in (5.9) is more convenient.

For example, taking a basis vector $x_i \in V$ and using (4.3), we get

$$\mathsf{R}(x_{i_1} \otimes m_{i_2}^{j_2}) = \bar R_{i_1 i_2}^{\;a_1 a_2} m_{a_1}^{b_1} (\bar R^{-1})_{b_1 a_2}^{\;c_1 j_2} \otimes x_{c_1} \quad \text{or} \quad \mathsf{R}(x_1 \otimes M_2) = \bar R_{12} M_1 \bar R_{12}^{-1} \otimes x_1\,,$$

where R is the general notation for the corresponding braiding, $\mathsf{R} = R_{V,V\otimes V^*}$ in the above formulas. Now we can directly get the desired result:

$$x_1 \otimes (\bar{R}_{23}M_2\bar{R}_{23}M_2 - M_2\bar{R}_{23}M_2\bar{R}_{23}) \stackrel{\mathsf{R}}{\to} \bar{R}_{12}\bar{R}_{23}(\bar{R}_{12}M_1\bar{R}_{12}M_1 - M_1\bar{R}_{12}M_1\bar{R}_{12})\bar{R}_{23}^{-1}\bar{R}_{12}^{-1} \otimes x_1\,.$$

The commutativity with V^* is verified similarly. □

Proposition 19. *Let R be an involutive skew-invertible symmetry. Then the commutation relations among the generators $\{l_i^j\}$ of the algebra $U_R(\mathrm{End}(V))$ (see (5.1)) are equivalent to (5.8) with $\hbar = 1$. Therefore, in accordance with Definition 16, the algebra (5.1) coincides with mREA $\mathcal{L}_q(R, 1)$.*

Proof. In the involutive case, we have $R = R^{-1}$ by definition. Therefore, the matrix $L_{\overline{2}}$ defined in (5.5) can be written as $L_{\overline{2}} = \bar{R}_{12}L_1\bar{R}_{12}$. This leads to the following action of R_{End} (see (5.6)):

$$R_{\mathrm{End}}(L_1 \otimes \bar{R}_{12}L_1\bar{R}_{12}) = \bar{R}_{12}L_1\bar{R}_{12} \otimes L_1.$$

Now, setting $X = L_1$ and $Y = \bar{R}_{12}L_1\bar{R}_{12}$ in (5.2), and taking into account the multiplication table (4.12) for the generators l_i^j, we get

$$X \circ Y = L_1R_{12}\,, \qquad \circ R_{\mathrm{End}}(X \otimes Y) = R_{12}L_1\,.$$

Together with the above form of the action of R_{End}, this allows us to represent the set $\mathcal{J}_R$ (see (5.2)) in the form (5.8) with $\hbar = 1$. □

The main deformation property of the mREA is given by the following proposition.

Proposition 20. *Let R be a skew-invertible involutive Hecke symmetry, $R^2 = I$, and let $U \subset \mathbb{K}$ be a neighborhood of $1 \in \mathbb{K}$. Consider a family of skew-invertible Hecke symmetries R_q analytically depending on $q \in U$ and satisfying the condition $R_1 = R$. Denote by $\mathcal{L}^{(k)}(R_q)$ the homogeneous component of $\mathcal{L}(R_q)$ of the kth order. Then, provided q is generic, the following claims hold true.*

1. $\dim \mathcal{L}^{(k)}(R_q) = \dim \mathcal{L}^{(k)}(R)$ *for all* $k \geq 0$.
2. $\mathrm{Gr}\,\mathcal{L}(R_q, \hbar) \cong \mathcal{L}(R_q)$, *where* $\mathrm{Gr}\,\mathcal{L}(R_q, \hbar)$ *is the graded algebra associated with the filtrated algebra* $\mathcal{L}(R_q, \hbar)$.

Proof. The verification of claim 1 is based on the following observations. Below we construct a projector $(\mathrm{Span}(l_i^j))^{\otimes 3} \to \mathcal{L}^{(3)}(R_q)$. The explicit form of that projector allows us to conclude that its rank is constant for generic $q \in U$. Therefore,

$$\dim \mathcal{L}^{(3)}(R_q) = \dim \mathcal{L}^{(3)}(R). \tag{5.10}$$

For an involutive R, the algebra $\mathcal{L}(R)$ is the symmetric algebra of the linear space $\mathrm{Span}(l_i^j)$ equipped with the involutive braiding R_{End}. This algebra is a Koszul one. (For the definition of this notion the reader is referred to [PP].) The Koszul property of $\mathcal{L}(R)$ easily follows from the exactness of the second kind Koszul complex constructed in [G3].

Now we apply the result of [PP] (generalizing [Dr]) that asserts that the Koszul property of $\mathcal{L}(R)$ and relation (5.10) imply claim 1 of our proposition. Moreover, it can be shown that for a generic $q \in U$ the algebra $\mathcal{L}(R_q)$ is also a Koszul algebra.

In order to prove claim 2 of the proposition, we consider the map $[\ ,\ ]$ that sends the left-hand side of (5.7) to its right-hand side. As was shown in [G4], this map satisfies the Jacobi relation in the form suggested in [PP]. Then, by the generalization of the PBW theorem given in [PP] (see also [BG]), we arrive at claim 2. □

Remark 21. Note that the skew-invertible Hecke symmetries with nonclassical HP series $P_-(t)$, constructed by methods of [G3], depend analytically on q in a neighborhood of 1.

Now, we pass to a construction of the projector mentioned in the proof of Proposition 20. We represent the commutation relations for the REA $\mathcal{L}(R_q)$ ((5.8) at $\hbar = 0$) in the equivalent form

$$\bar{R}_{12} L_{\bar{1}} L_{\bar{2}} - L_{\bar{1}} L_{\bar{2}} \bar{R}_{12} = 0. \tag{5.11}$$

Consider the unital associative algebra $\mathfrak{L}$ over $\mathbb{K}$ freely generated by N^2 generators l_i^j,

$$\mathfrak{L} = \mathbb{K}\langle l_i^j \rangle, \quad 1 \le i, j \le N.$$

The algebra $\mathcal{L}(R_q)$ is the quotient of $\mathfrak{L}$ over the two-sided ideal $\langle \mathcal{I}_- \rangle$ generated by the left-hand side of (5.11):

$$\mathcal{L}(R_q) = \mathfrak{L}/\langle \mathcal{I}_- \rangle, \qquad \mathcal{I}_- = L_{\bar{1}} L_{\bar{2}} - \bar{R}_{12} L_{\bar{1}} L_{\bar{2}} \bar{R}_{12}^{-1}\,. \tag{5.12}$$

As a vector space, the algebra $\mathfrak{L}$ can be decomposed into a direct sum of homogeneous components,

$$\mathfrak{L} = \bigoplus_{k \ge 0} \mathfrak{L}_k, \quad \mathfrak{L}_0 \cong \mathbb{K},$$

where each $\mathfrak{L}_k$ is the linear span of the kth order monomials in the generators l_i^j. The following basis turns out to be convenient:

$$\mathfrak{L}_k = \operatorname{Span}[L_{\bar{1}} L_{\bar{2}} \cdots L_{\bar{k}}]. \tag{5.13}$$

This notation means that $\mathfrak{L}_k$ is spanned by the matrix elements of the right-hand side matrix. The matrices $L_{\overline{m}}$ are defined by the recurrence rule

$$L_{\bar{1}} = L \otimes I, \qquad L_{\overline{k+1}} = \bar{R}_k L_{\bar{k}} \bar{R}_k^{-1}, \quad k \ge 1, \tag{5.14}$$

where the shorthand notation $R_k := R_{k,k+1}$ is used; this notation will be used systematically in the sequel.

Remark 22. Note that, due to the definitions (5.11), (5.14), and the Yang–Baxter equation for R, the following relation holds true:

$$\bar{R}_k L_{\bar{k}} L_{\overline{k+1}} = L_{\bar{k}} L_{\overline{k+1}} \bar{R}_k, \quad k \ge 1. \tag{5.15}$$

Relation (5.15) is typical of the so-called quantum matrix algebras, the REA being a particular case of them. For a detailed treatment of this issue the reader is referred to [IOP].

For the algebra $\mathcal{L}(R_q)$, we have a similar vector space decomposition

$$\mathcal{L}(R_q) = \bigoplus_{k \ge 0} \mathcal{L}_k, \quad \mathcal{L}_0 \cong \mathbb{K}, \quad \mathcal{L}_k \subset \mathfrak{L}_k.$$

Let us try to describe the subspaces $\mathcal{L}_k$ explicitly. In other words, we want to find a series of projection operators $\mathcal{S}_k : \mathfrak{L}_k \to \mathfrak{L}_k$ with the property

$$\operatorname{Im} \mathcal{S}_k = \mathcal{L}_k \subset \mathfrak{L}_k\,.$$

Here we construct such projectors for the second and third order components $\mathcal{L}_k$, $k = 2, 3$.

We introduce a linear operator $Q : \mathfrak{L}_2 \to \mathfrak{L}_2$ by the formula

$$Q(L_{\bar{1}} L_{\bar{2}}) := \bar{R}_1 L_{\bar{1}} L_{\bar{2}} \bar{R}_1^{-1}, \tag{5.16}$$

or symbolically $Q = \bar{R}_1 \circ \bar{R}_1^{-1}$. Using the Yang–Baxter equation for $\bar{R}$, we easily see that Q also satisfies the Yang–Baxter equation

$$Q_1 Q_2 Q_1 = Q_2 Q_1 Q_2\,, \tag{5.17}$$

where $Q_1 = Q \otimes \mathrm{id}$ and $Q_2 = \mathrm{id} \otimes Q$ are obvious extensions of Q to the space $\mathfrak{L}_3$. Moreover, using the fact that R is a Hecke symmetry, we can find a minimal polynomial of the operator Q,

$$(Q + q^2\,\mathbf{I})(Q + q^{-2}\,\mathbf{I})(Q - \mathbf{I}) = 0, \quad \mathbf{I} := I \circ I\,. \tag{5.18}$$

This implies that the semisimple operator Q has three eigenvalues on the space $\mathfrak{L}_2$. In obvious notation, we get the vector space decomposition

$$\mathfrak{L}_2 = \mathfrak{L}^{(-q^2)} \oplus \mathfrak{L}^{(1)} \oplus \mathfrak{L}^{(-q^{-2})}\,.$$

The Hecke condition (1.2) allows us to find the expressions for the corresponding projectors:

$$\begin{aligned} \mathcal{P}^{(-q^2)} &= P_+(R) \circ P_-(R),\\ \mathcal{P}^{(-q^{-2})} &= P_-(R) \circ P_+(R),\\ \mathcal{P}^{(1)} &= P_+(R) \circ P_+(R) + P_-(R) \circ P_-(R)\,, \end{aligned} \tag{5.19}$$

where

$$P_\pm(R) = \frac{q^{\mp 1} I \pm \bar{R}}{2_q}.$$

Indeed, a direct calculation shows that

$$Q\mathcal{P}^{(a)} = \mathcal{P}^{(a)} Q = a\,\mathcal{P}^{(a)}, \quad a = -q^{\pm 2},\, 1,$$

and on the other hand, the operators $\mathcal{P}^{(a)}$ form a complete set of orthonormal projectors on $\mathfrak{L}_2$:

$$\mathcal{P}^{(a)}\mathcal{P}^{(b)} = \delta^{ab}\,\mathcal{P}^{(a)}, \quad \mathcal{P}^{(-q^2)} + \mathcal{P}^{(1)} + \mathcal{P}^{(-q^{-2})} = \mathbf{I}.$$

Now we use the fact that (5.11) is equivalent to

$$(Q - \mathbf{I})(L_{\overline{1}} L_{\overline{2}}) = 0.$$

This means that the second order component $\mathcal{L}_2$ of the REA $\mathcal{L}(R_q)$ coincides (as a vector space) with the subspace $\mathfrak{L}^{(1)} \subset \mathfrak{L}_2$. Consider the following pair of orthonormal projection operators on $\mathfrak{L}_2$:

$$\mathcal{S} := \mathcal{P}^{(1)}, \quad \mathcal{A} := \mathcal{P}^{(-q^2)} + \mathcal{P}^{(-q^{-2})}, \quad \mathcal{S}\mathcal{A} = \mathcal{A}\,\mathcal{S} = 0, \quad \mathcal{S} + \mathcal{A} = \mathbf{I}.$$

These operators can be expressed in terms of Q:

$$\begin{aligned} \mathcal{S} &= \frac{1}{2_q^2}\left((q^2 + q^{-2})\,\mathbf{I} + Q + Q^{-1}\right),\\ \mathcal{A} &= \frac{1}{2_q^2}\left(2\,\mathbf{I} - Q - Q^{-1}\right)\,, \end{aligned} \tag{5.20}$$

where the inverse operator Q^{-1} can be obtained from (5.18),

$$Q^{-1} = Q^2 + (q^2 - 1 + q^{-2})Q - (q^2 - 1 + q^{-2})\mathbf{I}\,.$$

It can be shown that

$$\mathrm{Span}(\mathcal{I}_-) = \mathrm{Im}\,\mathcal{A},$$

as vector subspaces in $\mathfrak{L}_2$, where $\mathcal{I}_-$ is defined in (5.12).

The above considerations prove the following proposition.

Proposition 23. *The second order homogeneous component $\mathcal{L}_2$ of the REA $\mathcal{L}(R_q)$ coincides with the image of the q-symmetrizer $\mathcal{S}$,*

$$\mathcal{L}_2 = \mathrm{Im}\,\mathcal{S} = \mathfrak{L}_2 / \mathrm{Im}\,\mathcal{A}. \tag{5.21}$$

Now we pass to the third order homogeneous component $\mathfrak{L}_3$ and find the projector to the corresponding component $\mathcal{L}_3 \subset \mathfrak{L}_3$ of the REA $\mathcal{L}(R_q)$.

Extend the action of $\mathcal{S}$ and $\mathcal{A}$ to the subspace $\mathfrak{L}_3$. With the projector $\mathcal{S}$, we associate two operators $\mathcal{S}_1$ and $\mathcal{S}_2$ in accordance with the rule (see the definition (5.19))

$$\mathcal{S}_1 := \mathcal{P}^{(1)}(R_1), \qquad \mathcal{S}_2 := \mathcal{P}^{(1)}(R_2)\,,$$

which means that

$$\mathcal{S}_1(xyz) := (\mathcal{S}(xy))z, \qquad \mathcal{S}_2(xyz) := x(\mathcal{S}(yz)), \qquad xyz \in \mathfrak{L}_3\,.$$

The formulas for $\mathcal{A}$ are similar.

At this point we can see an advantage of using the basis (5.13). Indeed, the *quadratic* homogeneous component $\mathcal{L}_2$ can be embedded into $\mathfrak{L}_3$ in different ways, the following two being most important in the sequel:

$$\mathcal{L}_2 \cdot \mathfrak{L}_1 \subset \mathfrak{L}_3 \quad \text{and} \quad \mathfrak{L}_1 \cdot \mathcal{L}_2 \subset \mathfrak{L}_3\,.$$

Formulas (5.11), (5.15) and Proposition 23 show that these embeddings can be identified with the images of the operators $\mathcal{S}_1$ and $\mathcal{S}_2$:

$$\mathcal{L}_2 \cdot \mathfrak{L}_1 = \mathcal{S}_1(\mathfrak{L}_3)\,, \qquad \mathfrak{L}_1 \cdot \mathcal{L}_2 = \mathcal{S}_2(\mathfrak{L}_3).$$

The following technical lemma plays a crucial role in the further considerations.

Lemma 24. *The q-symmetrizer $\mathcal{S}$ obeys the following fifth order relation on the subspace $\mathfrak{L}_3$:*

$$\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1 - a\,\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1 + b\,\mathcal{S}_1 = \mathcal{S}_2\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2 - a\,\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2 + b\,\mathcal{S}_2, \tag{5.22}$$

where

$$a = (q^4 + q^2 + 4 + q^{-2} + q^{-4})/2_q^4, \quad b = 4_q^2/2_q^8\,. \tag{5.23}$$

Proof. The lemma is proved by a direct calculation, which can be considerably simplified if for $\mathcal{S}$ one uses (5.20) instead of the initial definition (5.19). □

Consider the operator $\mathcal{S}^{(3)} : \mathfrak{L}_3 \to \mathfrak{L}_3$ defined by

$$\mathcal{S}^{(3)} = \frac{2_q^6}{4\cdot 3_q^2}\,(\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1 - a\,\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1 + b\,\mathcal{S}_1)\,, \tag{5.24}$$

where a and b are as in (5.23). By (5.22), there exists an equivalent form of the above operator:

$$\mathcal{S}^{(3)} = \frac{2_q^6}{4\cdot 3_q^2}\,(\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2 - a\,\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2 + b\,\mathcal{S}_2)\,. \tag{5.25}$$

In fact, the operator $\mathcal{S}^{(3)}$ is precisely the projector onto $\mathcal{L}_3 \subset \mathfrak{L}_3$ we are looking for.

Proposition 25. *The third order homogeneous component $\mathcal{L}_3$ of the REA $\mathcal{L}(R_q)$ is the image of the projection operator $\mathcal{S}^{(3)}$ under its action on $\mathfrak{L}_3$,*

$$\mathcal{L}_3 = \mathrm{Im}\;\mathcal{S}^{(3)}\,.$$

Proof. The fact that $(\mathcal{S}^{(3)})^2 = \mathcal{S}^{(3)}$ can be verified by a straightforward calculation.

Consider now the projection of relation (5.12) to the third order homogeneous component:

$$\mathcal{L}_3 = \mathfrak{L}_3/\langle \mathcal{I}_- \rangle_3, \qquad \langle \mathcal{I}_- \rangle_3 = \mathfrak{L}_1 \cdot \mathrm{Im}\;\mathcal{A}_2 \cup \mathrm{Im}\;\mathcal{A}_1 \cdot \mathfrak{L}_1\,. \tag{5.26}$$

As can be seen from (5.24) and (5.25), we have $\langle \mathcal{I}_- \rangle_3 \subseteq \mathrm{Ker}\,\mathcal{S}^{(3)}$, and therefore, $\mathrm{Im}\;\mathcal{S}^{(3)} \subseteq \mathcal{L}_3$.

On the other hand, since $\mathcal{S}+\mathcal{A}=\mathbf{I}$, the subspace $\mathcal{L}_3$ given in (5.26) can be presented as

$$\mathcal{L}_3=\mathfrak{L}_1\cdot\operatorname{Im}\mathcal{S}_2\cap\operatorname{Im}\mathcal{S}_1\cdot\mathfrak{L}_1\,.$$

Comparing this form of $\mathcal{L}_3$ with the structure of $\mathcal{S}^{(3)}$ given in (5.24) and (5.25), we see that $\operatorname{Im}\mathcal{S}^{(3)}\supseteq\mathcal{L}_3$. Together with the reverse inclusion obtained above, this completes the proof. $\square$

§6. The braided bialgebra structure and representation theory

In this section we consider finite-dimensional representations of the mREA (5.8) in the category $\mathrm{SW}(V_{(m|n)})$. Note that the class of all finite-dimensional representations of the algebra in question is wider; for instance, it includes a large number of one-dimensional representations. For the particular case of the $U_q(sl(m))$ R-matrix, all such representations were classified in [Mu1]. Next, the finite-dimensional representations of an mREA can be constructed on the basis of the $U_q(sl(m))$ representations, because the REA $\mathcal{L}(R_q)$ can be embedded (as an algebra) into the quantum group in this particular case.

The representation theory developed below does not depend on a particular choice of the R-matrix and works well in the general situation where the quantum group does not exist. Moreover, an important property of the theory suggested is the equivariance of the representations with which we are dealing. By definition, a representation ρ_U of mREA in a space U is *equivariant* if the map

$$\mathrm{End}(V)\to\mathrm{End}(U):\quad l_i^j\mapsto\rho_U(l_i^j)$$

is a morphism of the category $\mathrm{SW}(V_{(m|n)})$.

This property has an important consequence, which will be used below. Namely, for any mREA module W with equivariant representation $\rho_W:\mathcal{L}(R_q,1)\to\mathrm{End}(W)$, the diagram

$$\begin{array}{ccc} U\otimes(A\otimes W) & \stackrel{\mathsf{R}}{\longleftrightarrow} & (A\otimes W)\otimes U\\ \downarrow \mathrm{id}\otimes\rho_W & & \rho_W\otimes\mathrm{id}\downarrow\\ U\otimes W & \stackrel{\mathsf{R}}{\longleftrightarrow} & W\otimes U \end{array}\tag{6.1}$$

is commutative for any object U of the category $\mathrm{SW}(V_{(m|n)})$ and any subspace $A\subset\mathcal{L}(R_q,1)$. The equivariance condition allows us to define an mREA representation in the tensor product of mREA modules.

For the particular case of an even Hecke symmetry of rank $(m|0)$, the equivariant representation theory of the associated mREA turns out to be similar to the representation theory of the algebra $U(sl(m))$. At the beginning of §4, we mentioned the specific peculiarity of the even case. Namely, in the corresponding category of the mREA representations, the space V^* can be identified with the object $V_{(1^{m-1})}$, and for constructing the complete representation theory it suffices, in fact, to define the mREA-module structure on the space V and on its tensor powers $V^{\otimes k}$. Any tensor product $V^{\otimes k}$ is a reducible mREA-module and expands into the direct sum (3.5) of mREA-invariant subspaces V_λ (for the details, see [GS2, S]).

However, the construction in the papers cited is insufficient for the treatment of the general case of the birank $(m|n)$. The reason is that in the general case we must construct representations in the tensor products $V^{*\otimes k}$ independently of those in the tensor products $V^{\otimes k}$, and the central problem here consists in extending the mREA-module structure to the tensor product $V_\lambda\otimes V_\mu^*$ of modules.

In the present section we suggest a regular procedure for constructing the mREA representations, which works well independently of the birank of the Hecke symmetry. Our construction is based on the braided bialgebra structure in the mREA. Throughout this section, we set $\hbar = 1$ in the mREA commutation relations (5.8).

The main component of the braided bialgebra is the coproduct Δ, which is a homomorphism of the mREA $\mathcal{L}(R_q, 1)$ into some associative braided algebra $\mathbf{L}(R_q)$; the latter is defined as follows.

- As a vector space over the field $\mathbb{K}$, the algebra $\mathbf{L}(R_q)$ is isomorphic to the tensor product of two copies of mREA:

$$\mathbf{L}(R_q) = \mathcal{L}(R_q, 1) \otimes \mathcal{L}(R_q, 1)\,.$$

- The product $\star : \mathbf{L}(R_q)^{\otimes 2} \to \mathbf{L}(R_q)$ is defined by the rule

$$(a_1 \otimes b_1) \star (a_2 \otimes b_2) := a_1 a_2' \otimes b_1' b_2\,, \qquad a_i \otimes b_i \in \mathbf{L}(R_q)\,, \tag{6.2}$$

where $a_1 a_2'$ and $b_1 b_2'$ are the usual products of elements of mREA, while a_1' and b_1' result from the action of the braiding R_{End} (see (5.6)) on the tensor product $b_1 \otimes a_2$,

$$a_2' \otimes b_1' := R_{\mathrm{End}}(b_1 \otimes a_2)\,. \tag{6.3}$$

We must verify that the product (6.2) is indeed associative. For this, we need the following lemma.

Lemma 26. *Consider the copies* (5.14) *of the matrix L. Then*

$$R_{\mathrm{End}}(L_{\overline{k}} \otimes L_{\overline{p}}) = L_{\overline{p}} \otimes L_{\overline{k}} \quad \textit{for all} \quad k < p,\ k, p \in \mathbb{N}\,. \tag{6.4}$$

Proof. The proof consists in a straightforward calculation on the basis of relation (5.6) rewritten in a slightly modified form,

$$R_{\mathrm{End}}(L_1 \bar{R}_{12} \otimes L_1) = \bar{R}_{12} L_1 \bar{R}_{12}^{-1} \otimes L_1 \bar{R}_{12}\,,$$

and on the Yang–Baxter equation (1.1), which allows us to interchange the chains of R-matrices forming the copies $L_{\overline{k}}$ and $L_{\overline{p}}$. $\square$

Now the associativity of (6.2) can be proved easily for the matrices $X^i_{r,s}$ whose components are homogeneous monomials in generators of $\mathcal{L}(R_q, 1)$:

$$X^i_{r,s} := L_{\overline{i}} \cdots L_{\overline{i+r-1}} \otimes L_{\overline{i+r}} \cdots L_{\overline{i+r+s-1}}\,,$$

where we represent the homogeneous components of mREA as a linear span of elements similar to those in (5.13).

Note that for any triple $X^{i_1}_{r_1,s_1}$, $X^{i_2}_{r_2,s_2}$, and $X^{i_3}_{r_3,s_3}$ we can always choose i_1, i_2, and i_3 so that

$$i_3 \geq i_2 + r_2 + s_2 \geq i_1 + r_1 + s_1.$$

Then, the associativity condition

$$(X^{i_1}_{r_1,s_1} \star X^{i_2}_{r_2,s_2}) \star X^{i_3}_{r_3,s_3} = X^{i_1}_{r_1,s_1} \star (X^{i_2}_{r_2,s_2} \star X^{i_3}_{r_3,s_3})$$

is an immediate consequence of Lemma 26. Since any element of $\mathbf{L}(R_q)$ can be presented as a linear combination of matrix elements of some $X^{(i)}_{r,s}$, we conclude that (6.2) determines an associative product in $\mathbf{L}(R_q)$.

Note that the mREA is isomorphic to any of the two subalgebras of $\mathbf{L}(R_q)$ obtained by the following embeddings:

$$a \mapsto e_{\mathcal{L}} \otimes a \quad \text{or} \quad a \mapsto a \otimes e_{\mathcal{L}}\,,$$

where $e_{\mathcal{L}}$ is the unit of mREA $\mathcal{L}(R_q,1)$. This can easily be deduced from the fact that the unit element $e_{\mathcal{L}}$ trivially commutes with any $a \in \mathcal{L}(R_q,1)$ with respect to the braiding R_{End}. As a consequence, we have

$$(e_{\mathcal{L}} \otimes a_1) \star (e_{\mathcal{L}} \otimes a_2) = (e_{\mathcal{L}} \otimes a_1 a_2) \quad \text{and} \quad (a_1 \otimes e_{\mathcal{L}}) \star (a_2 \otimes e_{\mathcal{L}}) = (a_1 a_2 \otimes e_{\mathcal{L}})\,.$$

Let a linear map $\Delta : \mathcal{L}(R_q,1) \to \mathbf{L}(R_q)$ be defined by the following rules:

$$\begin{aligned} &\Delta(e_{\mathcal{L}}) := e_{\mathcal{L}} \otimes e_{\mathcal{L}}, \\ &\Delta(l_i^j) := l_i^j \otimes e_{\mathcal{L}} + e_{\mathcal{L}} \otimes l_i^j - (q-q^{-1}) \sum_k l_i^k \otimes l_k^j, \\ &\Delta(ab) := \Delta(a) \star \Delta(b) \qquad \forall\, a,b \in \mathcal{L}(R_q,1). \end{aligned} \tag{6.5}$$

In addition to (6.5), we introduce a linear map $\varepsilon : \mathcal{L}(R_q,1) \to \mathbb{K}$ such that

$$\begin{aligned} &\varepsilon(e_{\mathcal{L}}) := 1, \\ &\varepsilon(l_i^j) := 0, \\ &\varepsilon(ab) := \varepsilon(a)\varepsilon(b), \qquad a,b \in \mathcal{L}(R_q,1)\,. \end{aligned} \tag{6.6}$$

The next proposition establishes an important property of the maps Δ and ε.

Proposition 27. *The maps Δ and ε given in* (6.5) *and* (6.6) *are* (*respectively*) *the coproduct and the counit of the braided bialgebra structure on the mREA $\mathcal{L}(R_q,1)$.*

Proof. First, we prove that the map Δ gives rise to an algebra homomorphism $\mathcal{L}(R_q,1) \to \mathbf{L}(R_q)$. It is convenient to work with the generators m_i^j introduced in Remark 17. In terms of these generators, the map Δ is expressed as

$$\Delta(m_i^j) = \sum_s m_i^s \otimes m_s^j\,, \tag{6.7}$$

or, in the matrix form,

$$\Delta(M_{\overline{k}}) = M_{\overline{k}} \otimes M_{\overline{k}}, \qquad k \geq 1\,.$$

Taking definitions (6.2) and (5.6) into account, we find

$$\Delta(M_{\overline{1}} M_{\overline{2}}) = \Delta(M_{\overline{1}}) \star \Delta(M_{\overline{2}}) = M_{\overline{1}} M_{\overline{2}} \otimes M_{\overline{1}} M_{\overline{2}}\,.$$

Comparing this with (5.9), we finally get

$$R_{12} \Delta(M_{\overline{1}} M_{\overline{2}}) = \Delta(M_{\overline{1}} M_{\overline{2}}) R_{12},$$

which means that the map Δ is an algebra homomorphism. Note that the braided coproduct (6.7) was suggested in [M].

The interrelation between ε and Δ

$$(\mathrm{id} \otimes \varepsilon)\,\Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id})\,\Delta$$

is verified trivially. □

Now we ask the following question: what representations of $\mathbf{L}(R_q)$ can be constructed by using mREA representations? Given two equivariant mREA modules U and W with representations $\rho_U : \mathcal{L}(R_q,1) \to \mathrm{End}(U)$ and $\rho_W : \mathcal{L}(R_q,1) \to \mathrm{End}(W)$, we construct the map $\rho_{U\otimes W} : \mathbf{L}(R_q) \to \mathrm{End}(U \otimes W)$ by the following rule:

$$\rho_{U\otimes W}(a \otimes b) \triangleright (u \otimes w) = (\rho_U(a) \triangleright u') \otimes (\rho_W(b') \triangleright w)\,, \qquad a \otimes b \in \mathbf{L}(R_q), \tag{6.8}$$

where the symbol $\triangleright$ stands for the action of the corresponding operator, and the element(s) b' and vector(s) u' result from the action of the corresponding braiding (depending on b and u) of the category $\mathrm{SW}(V_{(m|n)})$,

$$u' \otimes b' := \mathsf{R}(b \otimes u)\,.$$

By Proposition 18, the definition (6.8) is self-consistent, because the map $b \mapsto \rho_W(b')$ is also a representation of mREA $\mathcal{L}(R_q, 1)$.

Proposition 28. *The action* (6.8) *determines a representation of the algebra* $\mathbf{L}(R_q)$.

Proof. Consider two arbitrary elements

$$X_i = (a_i \otimes b_i) \in \mathbf{L}(R_q), \quad i = 1, 2\,,$$

of the algebra $\mathbf{L}(R_q)$. We prove that for all $u \otimes w \in U \otimes W$ the following relation holds true:

$$\rho_{U\otimes W}(X_1 \star X_2) \triangleright (u \otimes w) = \rho_{U\otimes W}(X_1) \triangleright (\rho_{U\otimes W}(X_2) \triangleright (u \otimes w))\,. \tag{6.9}$$

The left- and the right-hand side of (6.9) are actually the maps sending the element

$$(a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \otimes (u \otimes w)$$

to a vector in the space $U \otimes W$. We prove that the results of applying these maps to the above element are the same.

We introduce the shorthand notation

$$\mathsf{R}(b_1 \otimes a_2) = a_2' \otimes b_1'\,, \quad \mathsf{R}(b_2 \otimes u) = u' \otimes b_2'\,, \quad \mathsf{R}(b_1' \otimes u') = u'' \otimes b_1''\,.$$

The definitions (6.2) and (6.8) allow us to represent the left-hand side of (6.9) as the composition of the following morphisms:

$$\begin{aligned}(a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \otimes (u \otimes w) &\mapsto (a_1 \otimes a_2') \otimes (b_1' \otimes b_2) \otimes (u \otimes w)\\ &\mapsto (a_1 a_2' \otimes b_1' b_2) \otimes (u \otimes w) \mapsto (a_1 a_2' \otimes u'') \otimes (b_1'' b_2' \otimes w)\\ &\mapsto (\rho_U(a_1 a_2') \triangleright u'') \otimes (\rho_W(b_1'' b_2') \triangleright w)\,.\end{aligned}$$

Now, we take into account the equivariance condition (6.1) for the representations ρ_U and ρ_W. This condition means that under the action of the categorical braidings the vector $\rho_U(a) \triangleright u$ commutes with any object in the same way as the element $a \otimes u$ does. Therefore, the right-hand side of (6.9) can be represented as the following composition:

$$\begin{aligned}&(a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \otimes (u \otimes w) \mapsto (a_1 \otimes b_1) \otimes (a_2 \otimes u') \otimes (b_2' \otimes w)\\ &\quad\mapsto (a_1 \otimes b_1) \otimes (\rho_U(a_2) \triangleright u' \otimes \rho_W(b_2') \triangleright w) \mapsto (a_1 \otimes \rho_U(a_2') \triangleright u'') \otimes (b_1'' \otimes \rho_W(b_2') \triangleright w)\\ &\mapsto (\rho_U(a_1) \triangleright \rho_U(a_2') \triangleright u'') \otimes (\rho_W(b_1'') \triangleright \rho_W(b_2') \triangleright w) \mapsto (\rho_U(a_1 a_2') \triangleright u'') \otimes (\rho_W(b_1'' b_2') \triangleright w)\,.\end{aligned}$$

So, having started with the same initial element, the maps on the left- and the right-hand side of (6.9) give the same resulting vector in $U \otimes W$. Therefore, these maps coincide. □

Corollary 29. *Let* U *and* W *be two* $\mathcal{L}(R_q, 1)$*-modules with equivariant representations* ρ_U *and* ρ_W*. Then the equivariant representation* $\mathcal{L}(R_q, 1) \to \mathrm{End}(U \otimes W)$ *is given by the rule*

$$a \mapsto \rho_{U\otimes W}(\Delta(a))\,, \qquad a \in \mathcal{L}(R_q, 1)\,, \tag{6.10}$$

where the coproduct Δ *and the map* $\rho_{U\otimes W}$ *are given* (*respectively*) *by formulas* (6.5) *and* (6.8)*.*

Proof. This corollary is a direct consequence of Propositions 27 and 28. □

As was mentioned at the beginning of this section, equivariant representations of mREA in spaces V_λ, $\lambda \vdash k \in \mathbb{N}$, were constructed in [GS2, S]. By the same method we can define mREA representations in spaces V_λ^*. Then, formula (6.10) allows us to define the mREA-module structure on any object of the category $\mathrm{SW}(V_{(m|n)})$.

To complete the picture, we briefly outline the main ideas of [GS2, S], and moreover, prove the equivariance of the representation in the space V^*. Contrary to the even case, for the Hecke symmetry of general birank, the equivariance of representations in the dual spaces V_λ^* should be established independently of that in the spaces V_λ.

The basic left representation of mREA $\mathcal{L}(R_q,1)$ in the space V is defined in terms of matrix elements of the operator B,

$$\rho_1(l_i^j)\triangleright x_k = B^j_{\;k} x_i. \tag{6.11}$$

As was shown in [S], the map $\rho_2 : \mathcal{L}(R_q,1)\to \mathrm{End}(V^{\otimes 2})$ defined by

$$\rho_2(l_i^j)\triangleright(x_{k_1}\otimes x_{k_2}) = (\rho_1(l_i^j)\triangleright x_{k_1})\otimes x_{k_2} + \Big(R^{-1}\circ(\rho_1(l_i^j)\otimes I)\circ R^{-1}\Big)\triangleright(x_{k_1}\otimes x_{k_2})$$

is a representation. An extension of the basic representation up to higher representations $\rho_p : \mathcal{L}(R_q,1)\to\mathrm{End}(V^{\otimes p})$, $p\ge 3$, is defined in a similar way. It can be shown by direct calculations that these extensions coincide with the universal recipe (6.10).

The representations of the above form are completely reducible — the space $V^{\otimes p}$ expands into the direct sum of invariant subspaces V_λ labeled by partitions $\lambda\vdash p$. The restriction of the representation ρ_p to the subspaces V_λ is obtained by the action of the orthogonal projectors E_a^λ (see (3.4) and (3.5)),

$$\rho_{\lambda,a} = E_a^\lambda\circ\rho_p\circ E_a^\lambda\,, \tag{6.12}$$

the modules with different a being equivalent.

The basic representation $\rho_1^* : \mathcal{L}(R_q,1)\to\mathrm{End}(V^*)$ is given by

$$\rho_1^*(l_i^j)\triangleright x^k = -x^r\,\bar R^{\,kj}_{ri}\,. \tag{6.13}$$

To prove the equivariance of this representation, we need the following lemma.

Lemma 30. *Let R be a skew-invertible Hecke symmetry. Then the map*

$$V\otimes V^*\to V^*\otimes V\;:\quad x_i\otimes x^j\mapsto x^k\otimes x_l R^{lj}_{ki} \tag{6.14}$$

is a categorical morphism.

Proof. We use the fact that for a Hecke symmetry we have $R = R^{-1} + (q-q^{-1})I$ (see (1.2)). Substituting this in (6.14),

$$x_i\otimes x^j\mapsto x^k\otimes x_l\,(R^{-1})^{lj}_{ki} + (q-q^{-1})\,\delta_i^j\,x^k\otimes x_k,$$

we find that the map in question is a linear combination of a categorical morphism from the list (4.3) and the map

$$x_i\otimes x^j\mapsto\delta_i^j\,x^k\otimes x_k.$$

The latter map can be presented as a composition of categorical morphisms,

$$x_i\otimes x^j \overset{\langle\,,\rangle_r}{\mapsto} \delta_i^j\,1 \overset{\pi_r}{\mapsto} \delta_i^j\,x^k\otimes x_k,$$

where the categorical morphisms (4.1) and (4.5) were used consecutively.

So, we conclude that the initial map (6.14) is a linear combination of categorical morphisms; therefore, it is a categorical morphisms itself by definition. □

Proposition 31. *The representation* (6.13) *of the algebra $\mathcal{L}(R_q,1)$ in the space V^* is equivariant.*

Proof. To prove the equivariance of ρ_1^*, we must show that the map $\rho_1^* : \mathcal{L}(R_q,1)\to\mathrm{End}(V^*)$ is a categorical morphism.

Identifying l_i^j with $x_i\otimes x^j$, we can treat any left equivariant action of l_i^j on a basis vector $x^k\in V^*$ as a categorical morphism

$$V\otimes V^*\otimes V^*\to V^*.$$

We construct such an action as the following composition of morphisms:

$$V\otimes V^*\otimes V^* \overset{(6.14)\otimes I}{\longrightarrow} V^*\otimes V\otimes V^* \overset{I\otimes\langle\,,\,\rangle_r}{\longrightarrow} V^*\otimes\mathbb{K}\cong V^*,$$

which gives explicitly

$$l_i^j\triangleright x^k = R^{kj}_{ri}\,x^r = x^r\,\bar R^{\,kj}_{ri}.$$

Up to a sign, this categorical morphism coincides with the left representation (6.13). $\square$

Now, we consider a particular example of the "adjoint" mREA representation acting in the linear span of the generators l_i^j. Since

$$\mathrm{Span}(l_i^j)\cong V\otimes V^*\,,$$

the representation involved is constructed by the general formula (6.10), which now reads

$$l_i^j\mapsto \rho_{V\otimes V^*}(\Delta(l_i^j))\,,$$

where we should take the basic representations (6.11) and (6.13) as $\rho_V(l_i^j)$ and $\rho_{V^*}(l_i^j)$, respectively. Omitting straightforward calculations, we write the final result in the compact matrix form:

$$\rho_{V\otimes V^*}(L_{\overline{1}})\triangleright L_{\overline{2}} = L_1R_{12}-R_{12}L_1. \tag{6.15}$$

Applying the coproduct Δ (see (6.5)), we extend this representation to any homogeneous component of the mREA.

Note that the above action (6.15) is an L-linear part of the defining commutation relations of the mREA (5.8) if we rewrite them in the equivalent form

$$L_{\overline{1}}L_{\overline{2}}-R_{12}^{-1}L_{\overline{1}}L_{\overline{2}}R_{12} = L_1R_{12}-R_{12}L_1\,.$$

In this sense, the action (6.15) is similar to the adjoint action of a Lie algebra $\mathfrak{g}$ on its universal enveloping algebra $U(\mathfrak{g})$, which is also determined by the linear part of the Lie bracket and then is extended from the Lie algebra to the higher components of $U(\mathfrak{g})$ via the standard coproduct operation.

To conclude the section, we consider the question of "sl-reduction", that is, the passage from mREA $\mathcal{L}(R_q,1)$ to the quotient algebra

$$\mathcal{S}L(R_q):=\mathcal{L}(R_q,1)/\langle\mathrm{Tr}_R\,L\rangle\,,\qquad \mathrm{Tr}_R\,L:=\mathrm{Tr}(CL)\,. \tag{6.16}$$

The element $\ell:=\mathrm{Tr}_R\,L$ is central in mREA, which can easily be proved by calculating the R-trace in the second space from the matrix relation (5.8). This is done with the help of formulas (2.8).

To describe the quotient algebra $\mathcal{S}L(R_q)$ explicitly, we pass to the new set of generators $\{f_i^j,\ell\}$ related to the initial set by a linear transformation,

$$l_i^j = f_i^j+(\mathrm{Tr}(C))^{-1}\delta_i^j\,\ell\quad\text{or}\quad L=F+(\mathrm{Tr}(C))^{-1}I\,\ell\,, \tag{6.17}$$

where $F=\|f_i^j\|$. Obviously, $\mathrm{Tr}_R\,F=0$. Note that, by (3.10), the above shift is possible if and only if $m\neq n$.

In terms of the new generators, the commutation relations of mREA read

$$\begin{cases}\bar R_{12}F_1\bar R_{12}F_1-F_1\bar R_{12}F_1\bar R_{12} = (e_{\mathcal{L}}-\dfrac{\omega}{\mathrm{Tr}(C)}\,\ell)(\bar R_{12}F_1-F_1\bar R_{12}),\\[2mm] \ell\,F=F\,\ell\,,\qquad \mathrm{Tr}_R\,F=0\,,\end{cases}$$

where $\omega=q-q^{-1}$. Now, the quotient (6.16) can easily be described. The matrix $F=\|f_i^j\|$ of $\mathcal{S}L(R_q)$ generators satisfy the same commutation relations (5.8) as the matrix L,

$$\bar R_{12}F_1\bar R_{12}F_1-F_1\bar R_{12}F_1\bar R_{12} = \bar R_{12}F_1-F_1\bar R_{12}\,,\quad \mathrm{Tr}_R\,F=0\,, \tag{6.18}$$

but the generators f_i^j are linearly dependent, due to the relation $\mathrm{Tr}_R F = \mathrm{Tr}(CF) = 0$.

It is not difficult to rewrite the representation (6.15) in terms of the generators f_i^j and ℓ. Taking (6.17) into account, after a short calculation we obtain

$$\begin{aligned}
&\rho_{V\otimes V^*}(\ell) \triangleright \ell = 0, \qquad \rho_{V\otimes V^*}(F_1) \triangleright \ell = 0\,,\\
&\rho_{V\otimes V^*}(\ell) \triangleright F_1 = -\omega\ \mathrm{Tr}(C)\, F_1,\\
&\rho_{V\otimes V^*}(F_{\overline{1}}) \triangleright F_{\overline{2}} = F_1\bar{R}_{12} - \bar{R}_{12}F_1 + \omega \bar{R}_{12}F_1\bar{R}_{12}^{-1}\,.
\end{aligned} \tag{6.19}$$

Note that relation (6.19) defines the "adjoint" representation of the quotient algebra $\mathcal{S}L(R_q)$, but, contrary to the mREA $\mathcal{L}(R_q, 1)$, this representation is not given by the linear part of the quadratic-linear commutation relations (6.18).

In general, given a representation $\rho : \mathcal{L}(R_q, 1) \to \mathrm{End}(U)$ such that the element ℓ is a multiple of the unit operator (for example, an irreducible representation),

$$\rho(\ell) = \chi\, I_U\,, \qquad \chi \in \mathbb{K},$$

we can construct the corresponding representation $\tilde{\rho} : \mathcal{S}L(R_q) \to \mathrm{End}(U)$ by the formula [S]

$$\tilde{\rho}(f_i^j) = \frac{1}{\xi}\left(\rho(l_i^j) - (\mathrm{Tr}(C))^{-1}\rho(\ell)\,\delta_i^j\right), \qquad \xi = 1 - (q - q^{-1})(\mathrm{Tr}(C))^{-1}\chi\,. \tag{6.20}$$

Finally, we note that the REA (5.9) admits a series of automorphisms $M \mapsto zM$ with nonzero $z \in \mathbb{K}$. At the level of mREA representations, these automorphisms look as follows (recall that $\hbar = 1$):

$$\rho_U(l_i^j) \mapsto \rho_U^z(l_i^j) = z\rho_U(l_i^j) + \delta_i^j(1 - z)(q - q^{-1})^{-1}\, I_U\,.$$

By using (6.20), it can be shown that the corresponding representation $\tilde{\rho}_U$ of the algebra $\mathcal{S}L(R_q)$ constructed from ρ_U^z does not depend on z; in other words, the entire class of mREA representations ρ_U^z connected by the above automorphisms gives one and the same representation of the quotient algebra $\mathcal{S}L(R_q)$.

Remark 32. In this connection, we would like to discuss the problem of a suitable definition of braided (quantum, generalized) Lie algebras. For the first time, such an object was introduced in [G1] as certain data $(\mathfrak{g}, \sigma, [\ ,\ ])$, where $\mathfrak{g}$ is a vector space, $\sigma : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 2}$ is an involutive symmetry, and $[\ ,\ ] : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ is an operator ("braided Lie bracket") such that

1) $[\ ,\ ]\sigma = -[\ ,\ ]$;
2) $\sigma[\ ,\ ]_{23} = [\ ,\ ]_{12}\sigma_{23}\sigma_{12}$;
3) $[\ ,\ ][\ ,\ ]_{23}(I + \sigma_{12}\sigma_{23} + \sigma_{23}\sigma_{12}) = 0$.

Note that the third relation can be presented as follows:

$$[\ ,\ ][\ ,\ ]_{12} = [\ ,\ ][\ ,\ ]_{23}(I - \sigma_{12})\,. \tag{6.21}$$

A typical example is

$$\mathfrak{g} = \mathrm{End}(V)\,, \quad \sigma = R_{\mathrm{End}}\,, \quad [\ ,\ ] = \circ(I - \sigma)$$

(in the setting of §5). Another example can be obtained by restricting the above operators to the subspace of traceless elements of the algebra $\mathrm{End}(V)$. The enveloping algebras of both braided Lie algebras can be defined by (5.1).

Now, observe that relation (6.21) takes the form (6.15) if we put

$$\mathfrak{g} = \mathrm{Span}(l_i^j)\,, \quad \sigma(L_{\overline{1}}L_{\overline{2}}) = R_{12}^{-1}L_{\overline{1}}L_{\overline{2}}R_{12}\,, \quad [L_{\overline{1}}, L_{\overline{2}}] = L_{\overline{1}}R_{12} - R_{12}L_{\overline{1}}\,. \tag{6.22}$$

So, if we define a braided Lie algebra with such $\mathfrak{g}$, σ, and $[\ ,\ ]$, then the third axiom of the above list (in the form (6.21)) will be satisfied. By contrast, relations 1) and 2) fail and must be modified. Thus, an analog of 1) can be presented in the form

$$[\ ,\ ]\mathcal{S} = 0\,,$$

where $\mathcal{S}$ is as in (5.20). The verification of this relation is straightforward and is left to the reader. In 2), the map σ must be replaced by R_{End}. This is a consequence of the fact that the bracket $[\ ,\ ]$ in (6.22) is a categorical morphism. But if we restrict ourselves to the traceless part of the space $\mathfrak{g}$, then relation (6.21) also fails.

So, it is somewhat contradictory to define a braided Lie algebra in the space $\mathrm{End}(V)$ (where the space V is equipped with a skew-invertible Hecke symmetry) with the use of the above three axioms. However, in many papers (cf. [Wo, GM]) the braided (quantum) Lie algebras related to noninvolutive braidings were introduced via these axioms or their slight modifications. Recalling our observation (see the Introduction) on "braided Lie algebras" related to braidings of the Birman–Murakami–Wenzl type, we can conclude that there are no "braided Lie algebras" satisfying the above list of axioms and, at the same time, such that their enveloping algebras possess good deformation properties.

§7. Quantization with a deformed trace

In this section we consider the semiclassical structures arising from the mREA $\mathcal{L}(R_q,\hbar)$ (see (5.8)), provided that R is the standard $U_q(sl(m))$ Hecke symmetry (1.3). In this case the mREA $\mathcal{L}(R_q,\hbar)$ is treated as a two-parameter deformation of the commutative algebra $\mathbb{K}[gl(m)^*]$. We clarify the role of the corresponding Poisson brackets in defining the quantum homogeneous spaces. At the end of the section we study the infinitesimal counterpart of the deformed R-trace.

Given a Hecke symmetry R as in (1.3), a straightforward calculation allows us to find a Poisson pencil that is the semiclassical counterpart of the two-parameter algebra $\mathcal{L}(R_q,\hbar)$. Indeed, setting $q=1$ in (1.3), we pass from $\mathcal{L}(R_q,\hbar)$ to the algebra $U(gl(m)_\hbar)$. Therefore, the Poisson bracket corresponding to the deformation described by the parameter $\hbar$ is the linear Poisson–Lie bracket on the space of functions on $gl(m)^*$.

In order to find the second generating bracket of the Poisson pencil, we put $\hbar=0$ in (5.8), arriving thereby at the nonmodified REA. Introducing the matrix $\mathcal{R}=RP$,

$$\mathcal{R} = \sum_{i,j}^{m} q^{\delta_{ij}} h_i^i \otimes h_j^j + (q-q^{-1}) \sum_{i<j}^{m} h_i^j \otimes h_j^i\,,$$

we transform the commutation relations of the REA to

$$\bar{\mathcal{R}}_{12}\, L_1\, \bar{\mathcal{R}}_{21}\, L_2 - L_2\, \bar{\mathcal{R}}_{12}\, L_1\, \bar{\mathcal{R}}_{21} = 0 \tag{7.1}$$

(we recall that the bar over the symbol of a matrix means transposition).

Setting $q=e^\nu$, $\nu\in\mathbb{K}$, and noting that $\mathcal{R}=I$ at $q=1$, we arrive at the following expansion of $\mathcal{R}$ in a ν-series: $\mathcal{R}=1+\nu\,\mathsf{r}+O(\nu^2)$, where

$$\mathsf{r} = \sum_{i=1}^{m} h_i^i \otimes h_i^i + 2\sum_{i<j}^{m} h_i^j \otimes h_j^i \tag{7.2}$$

is the classical $sl(m)$ r-matrix,

$$[\mathsf{r}_{12},\mathsf{r}_{13}] + [\mathsf{r}_{12},\mathsf{r}_{23}] + [\mathsf{r}_{13},\mathsf{r}_{23}] = 0.$$

Now, the part of the commutation relation (7.1) that is linear in ν represents the second Poisson bracket in $\mathbb{K}[gl(m)^*]$:

$$\{L_1,L_2\}_r = L_2L_1\bar{\mathsf{r}}_{21} - \bar{\mathsf{r}}_{12}L_1L_2 + L_2\bar{\mathsf{r}}_{12}L_1 - L_1\bar{\mathsf{r}}_{21}L_2\,. \tag{7.3}$$

This formula is defined on the matrix elements l_i^j of the matrix L that form a basis of linear functions on the space $gl(m)^*$. The extension of the bracket (7.3) from the generators l_i^j to arbitrary functions (polynomials in generators) is described in terms of vector fields on $gl(m)^*$. In order to obtain such an extension, we introduce the matrices

$$\mathsf{r}_\pm = \frac{1}{2}\,(\mathsf{r}_{12} \pm \mathsf{r}_{21}).$$

Formula (7.2) shows that the above matrices are the images of

$$(7.4)\quad r_- = \sum_{i<j}^{m}(e_i^j \otimes e_j^i - e_j^i \otimes e_i^j) := \sum_{i<j}^{m} e_i^j \wedge e_j^i\,, \quad r_+ = \sum_{i,j}^{m} e_i^j \otimes e_j^i\,, \quad r_\pm \in gl(m)^{\otimes 2},$$

under the fundamental vector representation $e_i^j \mapsto h_i^j$, the elements e_i^j being the standard basis of $gl(m)$,

$$[e_i^j, e_k^r] = \delta_k^j\, e_i^r - \delta_i^r\, e_k^j.$$

Now, consider the actions of $gl(m)$ on $\mathbb{K}[gl(m)^*]$ by the left, right, and adjoint vector fields

$$(7.5)\qquad e_i^j \triangleright l_k^s := \delta_k^j\, l_i^s\,, \qquad l_k^s \triangleleft e_i^j := \delta_i^s\, l_k^j\,, \qquad \mathrm{ad}\, e_i^j(l_k^s) = e_i^j \triangleright l_k^s - l_k^s \triangleleft e_i^j\,,$$

which are extended to any polynomial in $\mathbb{K}[gl(m)^*]$ by the Leibniz rule. Then, taking the above definitions of $r_\pm$ into account, we can rewrite (7.3) in the general form

$$(7.6)\qquad \{f,g\}_r = \circ\, r_+^{\mathrm{l,r}}(f\otimes g) - \circ\, r_+^{\mathrm{r,l}}(f\otimes g) - \circ\, r_-^{\mathrm{ad,ad}}(f\otimes g), \qquad f,g \in \mathbb{K}[gl(m)^*].$$

Here $\circ : \mathbb{K}[gl(m)^*]^{\otimes 2} \to \mathbb{K}[gl(m)^*]$ stands for the commutative pointwise product of functions on $gl(m)^*$, and the superscripts of $r_\pm$ denote the following actions:

$$(7.7)\qquad r_-^{\mathrm{ad,ad}}(f\otimes g) := \sum_{i<j}^{m} \mathrm{ad}\, e_i^j(f) \wedge \mathrm{ad}\, e_j^i(g), \quad r_+^{\mathrm{l,r}}(f\otimes g) := \sum_{i,j}(e_i^j \triangleright f)\otimes(g \triangleleft e_j^i)\,.$$

Note that the brackets

$$\{f,\, g\}_- = \circ\, r_-^{\mathrm{ad,ad}}(f\otimes g) \quad \text{and} \quad \{f,g\}_+ = \circ\, r_+^{\mathrm{l,r}}(f\otimes g) - \circ\, r_+^{\mathrm{r,l}}(f\otimes g)$$

are *not* Poisson, since they do not obey the Jacobi identity.

The bracket $\{\,\}_+$ is $gl(m)$-covariant, that is,

$$\mathrm{ad}\, X(\{f,g\}_+) = \{\mathrm{ad}\, X(f), g\}_+ + \{f, \mathrm{ad}\, X(g)\}_+, \qquad f,g \in \mathbb{K}[gl(m)^*], \quad X \in gl(m).$$

The bracket (7.6) restricts to any $GL(m)$-orbit $\mathcal{O} \subset gl(m)^*$ because for any $f \in I_{\mathcal{O}}$, where $I_{\mathcal{O}}$ is an ideal of functions vanishing on this orbit, and for any $g \in \mathbb{K}[gl(m)^*]$, we have

$$\{f,g\}_r \in I_{\mathcal{O}}\,.$$

This property is evident for the component $\{\,\}_-$ because the operation in question is defined via adjoint vector fields. The proof for the component $\{\,\}_+$ was given in [D]. In particular, the bracket (7.6) can be restricted to the variety $c_1 := \sum l_i^i = c$, where $c \in \mathbb{K}$ is a constant. Setting $c = 0$, we get a Poisson bracket on the algebra $\mathbb{K}[sl(m)^*]$.

Remark 33. Being restricted to $\mathbb{K}[\mathfrak{g}^*]$, where $\mathfrak{g} = sl(m)$, the bracket $\{\,\}_+$ admits the following interpretation [G4, D]. Consider the space $\mathfrak{g}^{\otimes 2}$ as an adjoint $\mathfrak{g}$-module. It decomposes into the direct sum of submodules

$$\mathfrak{g}^{\otimes 2} = \mathfrak{g}_s \oplus \mathfrak{g}_a,$$

where $\mathfrak{g}_s(\mathfrak{g}_a)$ is the symmetric (skew-symmetric) subspace of $\mathfrak{g}^{\otimes 2}$. For $m > 2$, there exist subspaces $\mathfrak{g}_+ \subset \mathfrak{g}_s$ and $\mathfrak{g}_- \subset \mathfrak{g}_a$ that are isomorphic to $\mathfrak{g}$ itself as adjoint $\mathfrak{g}$-modules. Therefore, there is a unique (up to a factor) nontrivial $\mathfrak{g}$-covariant morphism $\beta : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 2}$ sending $\mathfrak{g}_-$ to $\mathfrak{g}_+$.

Since the space of linear functions on $\mathfrak{g}^*$ with the Poisson–Lie bracket is isomorphic to $\mathfrak{g}$ as a Lie algebra, the morphism β can also be defined on the entire algebra $\mathbb{K}[\mathfrak{g}^*]$ (via the Leibniz rule), giving a bracket that coincides (up to a factor) with $\{\,\}_+$. Note that for $m = 2$ the component $\mathfrak{g}_+$ vanishes, so that the bracket $\{\,\}_+$ also vanishes.

Proposition 34. *The bracket* (7.6) *is compatible with the Poisson–Lie bracket* (*their Schouten bracket vanishes*), *and therefore, any bracket of the pencil* (1.4) *is Poisson.*

This claim is an immediate consequence of the fact that the family of algebras $\mathcal{L}(R_q, \hbar)$ is a two-parameter deformation of the commutative algebra $\mathbb{K}[gl(m)^*]$, but this can also be verified by direct calculations [D].

Now, we consider the case of $m = 2$ in more detail. As we have noticed before, in this case the component $\{\,\}_+$ of the Poisson bracket $\{\,\}_r$ vanishes and we have $\{\,\}_r = \{\,\}_-$. Let $\{H, E, F\}$ be the Cartan–Chevalley generators of $sl(2)$. Then $r_- = E \wedge F$ (see (7.4)) and the Poisson bracket (7.6) reads

$$\{a, b\}_r = -\operatorname{ad} E(a)\ \operatorname{ad} F(b) + \operatorname{ad} F(a)\ \operatorname{ad} E(b).$$

Take the generators $\{e, f, h\}$ of $\mathbb{K}[sl(2)^*]$ that correspond to the Cartan–Chevalley generators under the isomorphism of $sl(2)$ and the Lie algebra of linear functions on $sl(2)^*$. A simple calculation on the basis of (7.5) gives

$$\{h, e\}_r = -2eh\,, \quad \{h, f\}_r = 2fh\,, \quad \{e, f\}_r = -h^2.$$

Note that this differs from the Poisson–Lie bracket only by the factor $-h$; therefore, each leaf of the bracket $\{\,\}_r$ lies in a leaf of the Poisson–Lie bracket. Moreover, the element $c_2 = h^2/2+2ef$ is central with respect to both brackets; hence, the corresponding Poisson pencil can be restricted to the quotient $\mathbb{K}[sl(2)^*]/\langle c_2 - c\rangle$ for any $c \in \mathbb{K}$.

Let $\mathbb{K} = \mathbb{C}$, and let the elements

$$x = \frac{1}{2}\,(e - f), \quad y = \frac{i}{2}\,(e + f), \quad z = \frac{i}{2}\,h$$

be the generators of $su(2)$. In these generators, we get the Poisson pencil $\{\,\}_{PL,r}$, where

$$\begin{aligned} &\{x, y\}_{PL} = z, && \{y, z\}_{PL} = x, && \{z, x\}_{PL} = y,\\ &\{x, y\}_r = z^2, && \{y, z\}_r = xz, && \{z, x\}_r = yz. \end{aligned}$$

Here we have renormalized the bracket $\{\,\}_r$ since this does not affect the Poisson pencil. The quadratic central element takes the form $c_2 = x^2 + y^2 + z^2$.

A particular bracket of this Poisson pencil (namely, $\{\,\}_{PL} - \{\,\}_r$) appeared in [Sh] (see the Appendix by J.-H. Lu and A. Weinstein) in studying a semiclassical counterpart of the quantum sphere. In the paper cited, the quantum sphere was represented as an operator algebra. This approach is based on the paper [P], where the quantum sphere was treated as a C^*-algebra, and its irreducible representations (as a C^*-algebra) were classified. As a result, in [Sh] the quantum sphere was presented in terms of functional analysis.

Our method of constructing the quantum sphere (or the quantum hyperboloid, which is the same over the field $\mathbb{K} = \mathbb{C}$) is completely different. First, we quantize the Kirillov–Kostant–Souriau (KKS) bracket[8] on the sphere and realize the resulting quantum algebra as the quotient

$$U(su(2)_\hbar)/\langle x^2 + y^2 + z^2 - c\rangle\,, \quad c \in \mathbb{K}\,,\ c \neq 0. \tag{7.8}$$

[8]Recall that the KKS bracket is the restriction of the Poisson–Lie bracket to a coadjoint orbit of the Lie group in the space dual to its Lie algebra.

We are interested in finite-dimensional representations of this algebra. There exists a set of negative values $c^{(k)} = -\hbar^2 k(k+2)/4$, $k \in \mathbb{N}$, of the parameter c such that the quotient algebra (7.8) admits a finite-dimensional representation if $c = c^{(k)}$ for some $k \in \mathbb{N}$.

Returning to the generators of the algebra $sl(2)$, we obtain a one-parameter family of algebras

$$\mathcal{S}L^c(\hbar) = U(sl(2)_\hbar)/\langle \frac{1}{2}\, H^2 + EF + FE - c\rangle.$$

Any algebra $\mathcal{S}L^c(\hbar)$ in this family, as well as $\mathcal{S}L^c = \mathbb{K}[sl(2)^*]/\langle \frac{h^2}{2} + ef + fe - c\rangle$, being equipped with the $sl(2)$-action, can be expanded into a multiplicity-free direct sum of $sl(2)$-modules

$$\mathcal{S}L^c \cong \bigoplus_{k\geq 0} V_k\,.$$

Let $\alpha : \mathcal{S}L^c \to \mathcal{S}L^c(\hbar)$ be an $sl(2)$-invariant map sending the highest weight elements $e^{\otimes k} \in \mathcal{S}L^c$ to $E^{\otimes k} \in \mathcal{S}L^c(\hbar)$. This requirement determines the map α completely. Now, the commutative algebra $\mathcal{S}L^c$ can be equipped with a new noncommutative product $\star$ coming from the algebra $\mathcal{S}L^c(\hbar)$,

$$f \star_\hbar g = \alpha^{-1}(\alpha(f) \circ \alpha(g)), \qquad f, g \in \mathcal{S}L^c, \tag{7.9}$$

where $\circ$ is the product in the algebra $\mathcal{S}L^c(\hbar)$. Thus, we have quantized the KKS bracket on the hyperboloid $c_2 = c$ in the spirit of deformation quantization scheme, by introducing a new product in the initial space of commutative functions. Note that, in accordance with [R], our algebraic quantization cannot be extended on the function space $C^\infty[S^2]$.

Now, we deform the algebra $\mathbb{K}[sl(2)^*]$ in the q and $\hbar$ "directions" simultaneously. We get the mREA (5.8) with R-matrix given in (1.3), where we should set $m = 2$. Extracting the R-traceless elements from the set of four mREA generators, we arrive at a unital associative algebra generated by three linearly independent elements $\{\widehat{H}, \widehat{E}, \widehat{F}\}$ subject to the system of commutation relations

$$\begin{aligned}
&q^2 \widehat{H}\widehat{E} - \widehat{E}\widehat{H} = 2_q \hbar\, \widehat{E}\,,\\
&\widehat{H}\widehat{F} - q^2 \widehat{F}\widehat{H} = -2_q \hbar\, \widehat{F}\,,\\
&q(\widehat{E}\widehat{F} - \widehat{F}\widehat{E}) = \widehat{H}\Big(\hbar - \frac{(q^2-1)}{2_q}\,\widehat{H}\Big).
\end{aligned}$$

We denote this algebra by $\mathcal{S}L(q,\hbar)$. The element $C_q = \frac{\widehat{H}^2}{2_q} + q^{-1}\widehat{E}\widehat{F} + q\widehat{F}\widehat{E}$ is central and is called the *braided Casimir element*. We put

$$\mathcal{S}L^c(q,\hbar) = \mathcal{S}L(q,\hbar)/\langle C_q - c\rangle.$$

We call this algebra the quantum hyperboloid or (considering it over the field $\mathbb{K} = \mathbb{C}$) the quantum sphere. It is a two-parameter deformation of the initial commutative algebra $\mathcal{S}L^c$. For a generic value of q, it is possible to define a map $\alpha_q : \mathcal{S}L^c \to \mathcal{S}L^c(q,\hbar)$ similar to α (but without the equivariance property) and represent the product in $\mathcal{S}L^c$ in the spirit of relation (7.9).

As in the case of the algebra (7.8), there exists a series of values $c = c_k$ such that the corresponding quotient algebra $\mathcal{S}L^{c_k}(q,\hbar)$ has a finite-dimensional equivariant representation. Its construction was described in §6. Letting $q \to 1$, we get a representation of the algebra $\mathcal{S}L^{c_k}(\hbar)$. By contrast, the representation theory of the quantum sphere suggested in [P] has nothing in common with the theory of finite-dimensional representations of $sl(2)$ (or $su(2)$).

In general, by quantizing the KKS bracket on a semisimple orbit, we represent the quantum algebra as an appropriate quotient of the enveloping algebra $U(\mathfrak{g}_\hbar)$ with

$\mathfrak{g} = gl(m)$ or $sl(m)$. (Note that if such an orbit is not generic, then the problem of finding defining relations of the corresponding "quantum orbit" is somewhat subtle; cf. [DM].) Finally, we additionally deform this quotient in the "q-direction" and get some quotient of the algebra $\mathcal{L}(q, \hbar)$.

Observe that on a generic orbit in $\mathfrak{g}^*$, where $\mathfrak{g}$ is a simple Lie algebra, there exists a family of nonequivalent Poisson brackets that give rise to $U_q(\mathfrak{g})$-covariant algebras. One of them is the reduced Sklyanin bracket. It is often described in terms of the Bruhat decomposition (see [LW]). The classification of all these brackets and their deformation quantization are given in [DGS] and [D]. The reduced Sklyanin bracket can also be quantized in terms of the so-called Hopf–Galois extension (see [DGH]). But only the bracket (7.6), restricted to a semisimple orbit, is compatible with the KKS bracket, and the quantization of the corresponding Poisson pencil can be realized in the spirit of affine algebraic geometry, i.e., via generators and relations among them.

Note that on the sphere (hyperboloid), the reduced Sklyanin bracket coincides with one of the brackets in the Poisson pencil $\{\,,\}_{\mathrm{KKS},r}$ (this is also true for any symmetric orbit). So, the Sklyanin bracket can be quantized via different approaches. However, for $m > 2$ and for higher-dimensional orbits the notion of quantum orbits must be specified depending on the bracket to be quantized.

As to the other classical simple algebras $\mathfrak{g}$ of B, C, or D series, there is no two-parameter deformation of the algebra $\mathbb{K}[\mathfrak{g}^*]$ (see [D]). Though a quadratic-linear algebra (similar to $\mathcal{L}(R_q, \hbar)$) can be constructed in this case (see [DGG] for the details), we note that neither this algebra nor the associated quadratic algebra is a deformation of its classical counterpart.

We complete this section with considering a semiclassical analog of the quantum trace in the spirit of [G2]. Poisson pencils considered in that paper are similar to the above ones, but they were generated by triangular classical r-matrices, which give rise to involutive braidings. The main difference is that the result of the "double quantization" of the Poisson pencil from [G2] was treated as the enveloping algebra of a generalized Lie algebra, and its finite-dimensional representations formed a tensor (rather than a quasitensor) category.

As is known, on any symplectic variety there is a Liouville (or invariant, or symplectic) measure $d\mu$ with the basic property $\int \{f, g\}\, d\mu = 0$. In the framework of the deformation quantization, this measure gives rise to a trace with usual properties (see [GR]). It is merely the case of the KKS bracket on a semisimple orbit. For nonsymplectic Poisson brackets, one usually tries to describe its symplectic leaves and to quantize them separately, i.e., to associate an operator algebra with each of the leaves. In the framework of our approach, we are not dealing with quantizing leaves of the bracket $\{\,,\}_r$ or any bracket from the Poisson pencil $\{\,,\}_{\mathrm{KKS},r}$, but we quantize this Poisson pencil as a whole. In other words, we simultaneously q-deform all algebras arising from "$\hbar$-quantization" and arrive at operator algebras with deformed traces.

Consider the Poisson pencil $\{\,,\}_{\mathrm{KKS},r}$ on a semisimple orbit $\mathcal{O} \subset su(m)^*$. Since the bracket $\{\,,\}_r$ is not symplectic, the pencil involved has no Liouville measure on the entire orbit (a similar case was considered in [G2]). Nevertheless, the following proposition holds true independently of a specific form of the matrix r.

Proposition 35. *Let $\{\,,\}_{\mathrm{KKS},r}$ be the Poisson pencil on a semisimple orbit $\mathcal{O} \subset \mathfrak{g}^*$, where $\mathfrak{g} = su(m)$ (or its complexification) and $d\mu$ is the Liouville measure for the bracket $\{\,,\}_{\mathrm{KKS}}$. Then the quantity*

$$\langle a, b\rangle = \int_{\mathcal{O}} \{a, b\}_r \, d\mu \tag{7.10}$$

is a cocycle with respect to the bracket $\{\,,\}_{\rm KKS}$*, i.e.*,

$$\langle a, \{b,c\}_{\rm KKS}\rangle + \langle b, \{c,a\}_{\rm KKS}\rangle + \langle c, \{a,b\}_{\rm KKS}\rangle = 0.$$

This statement is a simple consequence of the fact that the brackets $\{\,,\}_{\rm KKS}$ and $\{\,,\}_r$ are compatible. The cocycle (7.10) is treated as an infinitesimal term of the deformation of the pairing $a\otimes b \mapsto \int_{\mathcal O} ab\, d\mu$ [G2].

In a similar way, we consider an infinitesimal term of the deformation of the pairing $A\otimes B \mapsto {\rm Tr}(A\circ B)$. For this, we use the relation

$$\mathrm{Tr}_R \circ(\mathcal{R}_{12}L_1\mathcal{R}_{21}L_2 - L_2\mathcal{R}_{12}L_1\mathcal{R}_{21}) = 0\,,$$

where the entries of the matrices L_1 and L_2 belong to ${\rm End}(V)$, the symbol $\circ$ stands for the product (4.12) in this algebra, and the operation ${\rm Tr}_R$ is applied to each entry. The above relation is true because ${\rm Tr}_R\, l_i^j = \delta_i^j$.

Then, expanding the R-matrix and the R-trace into a ν-series,

$$\mathcal{R} = I + \nu\, \mathsf{r} + O(\nu^2)\,, \qquad \mathrm{Tr}_R = \mathrm{Tr} + \nu\, b + O(\nu^2)$$

(r is given by (7.2)), we get the explicit form of the operation $b\,\circ$ on the skew-symmetric subspace $\wedge^2({\rm End}(V))$:

$$b\circ(L_1\otimes L_2 - L_2\otimes L_1) = -\,\mathrm{Tr}\circ(\mathsf{r}_{12}L_1L_2 + L_1\mathsf{r}_{21}L_2 - L_2\mathsf{r}_{12}L_1 - L_2L_1\mathsf{r}_{21})\,.$$

Having defined the operation $b\,\circ$ on the basis elements in this way, we directly get the general expression (see (7.4)–(7.7) for the notation)

(7.11)

$$b\circ(A\otimes B - B\otimes A) = \mathrm{Tr}\circ(-r_-^{\rm ad,ad}(A\otimes B) - r_+^{\rm r,l}(A\otimes B) + r_+^{\rm l,r}(A\otimes B)), \quad A,B\in \mathrm{End}(V)\,.$$

Thus, we obtain the skew-symmetrized linear term of deformation of the pairing

$$A\otimes B\mapsto \mathrm{Tr}(A\circ B)\,.$$

Proposition 36. *The quantity* $\langle A,B\rangle = -b\circ(A\otimes B - B\otimes A)$ *is a cocycle on the Lie algebra* $gl(m)$*, i.e.*,

$$\langle A,[B,C]\rangle + \langle B,[C,A]\rangle + \langle C,[A,B]\rangle = 0.$$

This cocycle reduces to the Lie algebra $sl(m)$*.*

It is not difficult to write an explicit form of the cocycle $\langle A,B\rangle$. Indeed, it can be shown that the second and third terms on the right-hand side of (7.11) give no contribution to this cocycle; using the cyclic property of the usual trace, we get

$$\langle A,B\rangle = \mathrm{Tr}\,\Big([A,B]\circ\sum_{\alpha>0}[X_\alpha, X_{-\alpha}]\Big) = \mathrm{Tr}\,\Big([A,B]\circ\sum_{\alpha>0}H_\alpha\Big),$$

where the sum is over the set of all positive roots.

Appendix

In this section we collect some facts and definitions pertaining to the theory of the A_{k-1} series Hecke algebras $H_k(q)$, used in the main text of the paper. For a detailed survey of the subject the reader is referred to [OP1]. Throughout this section we use the definitions and notation of that paper. At the end of the section we prove Proposition 7, stated in §2.

By definition, a *Hecke algebra of A_{k-1} series* is a unital associative algebra $H_k(q)$ over a field $\mathbb{K}$ generated by elements σ_i, $1 \le i \le k-1$, subject to the following commutation relations:

$$
\begin{aligned}
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}, \sigma_i\sigma_{i+1}, && 1 \le i \le k-2;\\
\sigma_i\sigma_j &= \sigma_j\sigma_i, && |i-j| \ge 2;\\
\sigma_i^2 &= 1_H - (q - q^{-1})\,\sigma_i, && 1 \le i \le k-1.
\end{aligned}
$$

Here 1_H is the unit of the algebra, and $q \in \mathbb{K}$ is a nonzero element of the ground field. Below we assume $\mathbb{K}$ to be the field of complex numbers $\mathbb{C}$ or the field $\mathbb{C}(q)$ of rational functions of a formal variable q.

At a generic value of q, the Hecke algebra $H_p(q)$ is semisimple and isomorphic to the group algebra of the kth order symmetric group $\mathbb{K}[\mathfrak{S}_k]$; see [We]. Therefore, being viewed as a regular two-sided $H_k(q)$-module, the Hecke algebra $H_k(q)$ can be presented as a direct sum of simple ideals (the Wedderburn–Artin theorem)

$$H_k(q) = \bigoplus_{\lambda\vdash k} M^\lambda$$

labeled by partitions λ of the integer k. Under the left (right) action of the Hecke algebra, the submodules M^λ are reducible and can be further decomposed into the direct sum of the corresponding equivalent one-sided (left or right) submodules

$$M^\lambda = \bigoplus_{a=1}^{d_\lambda} M_{l(r)}^{(\lambda,a)},$$

where d_λ is the number of the standard Young *tableaux* (λ, a) corresponding to the partition λ; see [Mac]. The index a enumerates the standard *tableaux* in accordance with some ordering (say, lexicographical).

In each ideal M^λ, we can fix a linear basis of "matrix units" e_{ab}^λ with the multiplication table

$$e_{ab}^\lambda\, e_{cd}^\mu = \delta^{\lambda\mu}\delta_{bc}\, e_{ad}^\mu\,.$$

A subset e_{ab}^λ, $1 \le b \le d_\lambda$ (with a fixed value of the first index), forms a basis of the right module $M_r^{(\lambda,a)}$, while fixing the second index gives a basis of the left module $M_l^{(\lambda,b)}$.

The diagonal elements e_{aa}^λ, denoted briefly by e_a^λ, form the set of primitive idempotents of the Hecke algebra $H_k(q)$. The idempotents e_a^λ are constructed explicitly as some polynomials in the Jucys–Murphy elements $\mathcal{J}_p$, $1 \le p \le k$ (see [OP1] for the details), which are defined by the iterative rule

$$\mathcal{J}_1 = 1_H, \quad \mathcal{J}_{p+1} = \sigma_p\, \mathcal{J}_p\, \sigma_p.$$

The set of Jucys–Murphy elements is a basis of the maximal commutative subalgebra of $H_k(q)$. An important property of these elements is as follows:

$$\mathcal{J}_p\, e_a^\lambda = e_a^\lambda\, \mathcal{J}_p = j_p(\lambda,a) e_a^\lambda, \quad j_p(\lambda,a) = q^{2(c_p - r_p)} \in \mathbb{K}. \tag{A.1}$$

Here the positive integers c_p and r_p are the number of the column and the number of the row of the Young *tableau* (λ, a) that contain the box with integer p. Below we give a simple example of a Young *tableau* of the partition $\lambda = (3, 2, 1)$:

$$
\begin{array}{|c|c|c|}
\hline
1 & 3 & 4 \\
\hline
2 & 6 & \multicolumn{1}{c}{} \\
\cline{1-2}
5 & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array}
\quad \Rightarrow \quad
\begin{array}{ll}
j_1 = 1, & j_4 = q^4, \\
j_2 = q^{-2}, & j_5 = q^{-4}, \\
j_3 = q^2, & j_6 = 1.
\end{array}
$$

Any two idempotents e_a^λ and e_b^λ corresponding to different *tableaux* (λ, a) and (λ, b) of a partition $\lambda \vdash k$ can be transformed into each other by the two-sided action of some invertible elements of the Hecke algebra $H_k(q)$ [OP1]:

$$e_a^\lambda = x_{ab}^\lambda \, e_b^\lambda \, y_{ab}^\lambda \,, \qquad x_{ab}^\lambda,\, y_{ab}^\lambda \in H_k(q). \tag{A.2}$$

Now, consider a Hecke symmetry $R : V^{\otimes 2} \to V^{\otimes 2}$, and define a special representation ρ_R of a Hecke algebra $H_k(q)$ in the tensor product $V^{\otimes p}$, $p \geq k$, by the rule

$$\begin{aligned} &\rho_R(1_H) = \mathrm{id}_V^{\otimes p}, \\ &\rho_R(\sigma_i) = \mathrm{id}_V^{\otimes(i-1)} \otimes R_i \otimes \mathrm{id}_V^{\otimes(p-i-1)}, \qquad 1 \leq i \leq k-1, \\ &\rho_R(xy) = \rho_R(x)\rho_R(y), \quad x, y \in H_k(q) \end{aligned} \tag{A.3}$$

(we recall that $R_i := R_{ii+1}$). The fact that ρ_R is a representation follows immediately from (1.1) and (1.2).

Suppose the birank of R is $(m|n)$, i.e., the corresponding HP series $P_-(t)$ is of the form (3.3). Consider the partitions

$$\begin{aligned} \lambda_{m,n} &:= ((n+1)^{m+1}), \qquad \lambda_{m,n} \vdash (m+1)(n+1), \\ \lambda_{m,n}^- &:= ((n+1)^m, n), \qquad \lambda_{m,n}^- \vdash mn + m + n\,. \end{aligned} \tag{A.4}$$

In a graphic form, the partition $\lambda_{m,n}$ is represented by a rectangular diagram with $m+1$ rows of length $n+1$, while the diagram of $\lambda_{m,n}^-$ is obtained by removing one box in the right lower corner of the rectangle. Note that $\lambda_{m,n}^- \in \mathsf{H}(m,n)$, while the partition $\lambda_{m,n}$ is the minimal partition not belonging to the hook $\mathsf{H}(m,n)$ (see Definition 5).

Below we list the properties of representations ρ_R that follow immediately from Proposition 6.

i) The images $E_a^\lambda = \rho_R(e_a^\lambda)$ are nonzero, $e_a^\lambda \in H_k(q)$, for all $2 \leq k < (m+1)(n+1)$.

ii) The representation ρ_R of $H_{(m+1)(n+1)}(q)$ possesses a kernel generated by

$$\rho_R(e_a^{\lambda_{m,n}}) = 0, \quad 1 \leq a \leq d_{\lambda_{m,n}},$$

and $\rho_R(e_a^\mu) \neq 0$ for all $\mu \vdash (m+1)(n+1)$, $\mu \neq \lambda_{m,n}$.

iii) For any integer $p \geq (m+1)(n+1)$ and any partition $\nu \vdash p$ we have

$$\rho_R(e_a^\nu) = 0 \quad \Leftrightarrow \quad \lambda_{m,n} \subset \nu,$$

where the inclusion $\mu = (\mu_1, \mu_2, \dots) \subset \nu = (\nu_1, \nu_2, \dots)$ means that $\mu_i \leq \nu_i$ for all i.

Proof of Proposition 7. We denote $p := (m+1)(n+1)$ to obtain more compact formulas. In the Hecke algebra $H_p(q)$, we take the Hecke subalgebra $H_{p-1}(q) \subset H_p(q)$ generated by $\sigma_i \in H_p(q)$, $1 \leq i \leq p-2$. Fix a standard Young *tableau* $(\lambda_{m,n}, a)$ (see (A.4)) and consider the idempotents $e_{a^-}^{\lambda_{m,n}^-} \in H_{p-1}(q)$ and $e_a^{\lambda_{m,n}} \in H_p(q)$. Here the notation $(\lambda_{m,n}^-, a^-)$ refers to a special choice of the corresponding Young *tableau*: it is properly included into the Young *tableau* $(\lambda_{m,n}, a)$. In other words, the integers from 1 to $p-1$ occupy the same positions in the *tableau* $(\lambda_{m,n}^-, a^-)$ as they do in the *tableau* $(\lambda_{m,n}, a)$. Note that, since we consider the standard Young *tableaux*, the only possible position for the number p is the box in the right lower corner of the rectangular *tableau* $(\lambda_{m,n}, a)$.

Now we apply the map $\rho_R : H_p(q) \to \mathrm{End}(V^{\otimes n})$ to the relation (see [OP1])

$$e_a^{\lambda_{m,n}} = e_{a^-}^{\lambda_{m,n}^-} \, \frac{(\mathcal{J}_p - q^{2(n+1)} 1_H)}{(q^{2(n-m)} - q^{2(n+1)})} \, \frac{(\mathcal{J}_p - q^{-2(m+1)} 1_H)}{(q^{2(n-m)} - q^{-2(m+1)})}.$$

Denoting $\rho_R(\mathcal{J}_k) := J_k$ and using property ii), we get the identity

$$0 = E_{a^-}^{\lambda^-_{m,n}} \frac{(J_p - q^{2(n+1)} I)}{(q^{2(n-m)} - q^{2(n+1)})} \frac{(J_p - q^{-2(m+1)} I)}{(q^{2(n-m)} - q^{-2(m+1)})},$$

where $E_{a^-}^{\lambda^-_{m,n}} \neq 0$ by i), and the letter I stands for the identity operator on the space $V^{\otimes p}$.

We calculate the trace tr of the above identity in the last (pth) component of the tensor product $V^{\otimes p}$, where tr coincides up to a factor with the categorical R-trace (4.15),

$$\mathsf{tr}(X) := \mathrm{Tr}\,(C \cdot X).$$

It is clear that $\mathsf{tr}(I) = \mathrm{Tr}\; C$ is the object we are interested in.

Since the matrix $E_{a^-}^{\lambda^-_{m,n}}$ is a polynomial in J_k with $k < p$, it can be factored out of the trace operation in the pth space, and we arrive at the relation

$$0 = E_{a^-}^{\lambda^-_{m,n}} \mathsf{tr}_{(p)}\Big(J_p^2 - (q^{2(n+1)} + q^{-2(m+1)})J_p + q^{2(n-m)} I\Big). \tag{A.5}$$

We consider the traces of the terms of the above identity separately. Introducing the auxiliary shorthand notation $\omega := q - q^{-1}$, we find

$$\mathsf{tr}_{(p)}(J_p) = \mathsf{tr}_{(p)}\left(R_{p-1} J_{p-1}(R_{p-1}^{-1} + \omega I)\right) = \omega J_{p-1} + I_{p-1}\, \mathsf{tr}_{(p-1)}(J_{p-1}).$$

In this line of transformations we have used the iterative definition of the Jucys–Murphy element, the Hecke condition for R, and properties (2.4) and (2.8) of tr listed in §2. Since the trace in (A.5) is multiplied by an idempotent, we can replace the Jucys–Murphy element J_{p-1} by the corresponding "eigenvalue" j_{p-1} defined in (A.1):

$$E_{a^-}^{\lambda^-_{m,n}} \mathsf{tr}_{(p)}(J_p) = E_{a^-}^{\lambda^-_{m,n}} \left(\omega j_{p-1} + \mathsf{tr}_{(p-1)}(J_{p-1})\right). \tag{A.6}$$

To simplify the formulas, we omit the symbol of the idempotent and perform all calculations bearing in mind the possibility to replace each free of trace Jucys–Murphy element J_k by the corresponding number j_k.

Thus, the calculation of $\mathsf{tr}_{(p)}(J_p)$ is completed by straightforward induction, with the use of (A.6):

$$\mathsf{tr}_{(p)}(J_p) = \omega \sum_{k=1}^{p-1} j_k + \mathsf{tr}(I)$$

(recall that $J_1 = I$ by definition).

Now, we transform the term with the second power of J_p:

$$\begin{aligned}
\mathsf{tr}_{(p)}(J_p^2) &= \mathsf{tr}_{(p)}\left(R_{p-1} J_{p-1} R_{p-1}(R_{p-1}^{-1} + \omega I) J_{p-1} R_{p-1}\right) \\
&= \mathsf{tr}_{(p)}\left(R_{p-1} J_{p-1}^2 (R_{p-1}^{-1} + \omega I)\right) + \omega\, J_{p-1} \mathsf{tr}_{(p)}\left(J_p(R_{p-1}^{-1} + \omega I)\right) \\
&= 2\omega\, J_{p-1}^2 + \omega^2\, J_{p-1} \mathsf{tr}_{(p)}(J_p) + \mathsf{tr}_{(p-1)}(J_{p-1}^2) \\
&= 2\omega\, j_{p-1}^2 + \omega^2\, j_{p-1} \mathsf{tr}_{(p)}(J_p) + \mathsf{tr}_{(p-1)}(J_{p-1}^2).
\end{aligned}$$

Substituting the value of $\mathsf{tr}_{(p)}(J_p)$, we get the base for the inductive calculation,

$$\mathsf{tr}_{(p)}(J_p^2) = 2\omega\, j_{p-1}^2 + \omega^2 j_{p-1}\, \mathsf{tr}(I) + \omega^3 j_{p-1} \sum_{k=1}^{p-1} j_k + \mathsf{tr}_{(p-1)}(J_{p-1}^2).$$

This leads to the following expression:

$$\mathsf{tr}_{(p)}(J_p^2) = 2\omega \sum_{k=1}^{p-1} j_k^2 + \omega^3 \sum_{k=1}^{p-1} j_k \sum_{s=1}^{k} j_s + \Big(1 + \omega^2 \sum_{k=1}^{p-1} j_k\Big)\, \mathsf{tr}(I).$$

Substituting all the calculated components in identity (A.5) and using the fact that $E^{\lambda^-_{m,n}}_{a^-} \neq 0$, we find the following linear equation for $\mathsf{tr}(I)$:

$$\alpha\, \mathsf{tr}(I) + \beta = 0$$

with

$$\alpha = 1 + q^{2(n-m)} - q^{2(n+1)} - q^{-2(m+1)} + \omega^2 \sum_{k=1}^{p-1} j_k,$$

$$\beta = \omega\left(2 + \frac{\omega^2}{2}\right)\sum_{k=1}^{p-1} j_k^2 + \frac{\omega^3}{2}\Bigl(\sum_{k=1}^{p-1} j_k\Bigr)^2 - \omega\Bigl(q^{2(n+1)} + q^{-2(m+1)}\Bigr)\sum_{k=1}^{p-1} j_k\,;$$

to find the coefficient β we have used the identity

$$\sum_{k=1}^{p-1} j_k \sum_{s=1}^{k} j_s = \frac{1}{2}\sum_{k=1}^{p-1} j_k^2 + \frac{1}{2}\Bigl(\sum_{k=1}^{p-1} j_k\Bigr)^2.$$

Taking into account the definition of j_k and the form of the diagram $\lambda^-_{m,n} = ((n+1)^m, n)$, we can easily calculate the sum of the eigenvalues j_k:

$$\sum_{k=1}^{p-1} j_k = \sum_{k=1}^{p-1} q^{2(c_k - r_k)} = (1 + q^2 + \dots + q^{2n})(1 + q^{-2} + \dots + q^{-2m}) - q^{2(n-m)}$$
$$= q^{n-m}(n+1)_q(m+1)_q - q^{2(n-m)},$$

whence

$$\sum_{k=1}^{p-1} j_k^2 = \sum_{k=1}^{p-1} (q^2)^{2(c_k - r_k)} = q^{2(n-m)}(n+1)_{q^2}(m+1)_{q^2} - q^{4(n-m)}\,.$$

Now by a short calculation we simplify the coefficient α to the form

$$\alpha = -\omega^2\, q^{2(n-m)}.$$

The transformation of β is more involved though also straightforward; it is useful to invoke the identity

$$k_{q^2} = \frac{q^{2k} - q^{-2k}}{q^2 - q^{-2}} = \frac{(q^k - q^{-k})}{(q - q^{-1})}\,\frac{(q^k + q^{-k})}{(q + q^{-1})} = k_q\,\frac{q^k + q^{-k}}{2_q}.$$

Omitting routine calculations, we present the final result:

$$\beta = \omega^2\, q^{3(n-m)}(m-n)_q.$$

So, we finally get

$$\mathsf{tr}(I) = \mathrm{Tr}\, C = -\frac{\beta}{\alpha} = q^{n-m}(m-n)_q.$$

This completes the proof. $\square$

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