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A_2-PROOF OF STRUCTURE THEOREMS FOR CHEVALLEY GROUPS OF TYPE F_4

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ABSTRACT. A new geometric proof is given for the standard description of subgroups in the Chevalley group $G = G(\mathrm{F}_4, R)$ of type F_4 over a commutative ring R that are normalized by the elementary subgroup $E(\mathrm{F}_4, R)$. There are two major approaches to the proof of such results. Localization proofs (Quillen, Suslin, Bak) are based on a reduction in the dimension. The first proofs of this type for exceptional groups were given by Abe, Suzuki, Taddei and Vaserstein, but they invoked the Chevalley simplicity theorem and reduction modulo the radical. At about the same time, the first author, Stepanov, and Plotkin developed a geometric approach, decomposition of unipotents, based on reduction in the rank of the group. This approach combines the methods introduced in the theory of classical groups by Wilson, Golubchik, and Suslin with ideas of Matsumoto and Stein coming from representation theory and K-theory. For classical groups in vector representations, the resulting proofs are quite straightforward, but their generalizations to exceptional groups require an explicit knowledge of the signs of action constants, and of equations satisfied by the orbit of the highest weight vector. They depend on the presence of high rank subgroups of types A_l or D_l, such as $\mathrm{A}_5 \le \mathrm{E}_6$ and $\mathrm{A}_7 \le \mathrm{E}_7$. The first author and Gavrilovich introduced a new twist to the method of decomposition of unipotents, which made it possible to give an entirely elementary geometric proof (the proof from the Book) for Chevalley groups of types $\Phi = \mathrm{E}_6, \mathrm{E}_7$. This new proof, like the proofs for classical cases, relies upon the embedding of A_2. Unlike all previous proofs, neither results pertaining to the field case nor an explicit knowledge of structure constants and defining equations is ever used. In the present paper we show that, with some additional effort, we can make this proof work also for the case of $\Phi = \mathrm{F}_4$. Moreover, we establish some new facts about Chevalley groups of type F_4 and their 27-dimensional representation.

In the present paper, which is a sequel to the work by the first author and Mikhail Gavrilovich [12], we prove the main structure theorems for the Chevalley group $G = G(\mathrm{F}_4, R)$ of type F_4 over an arbitrary commutative ring R. More precisely, we describe subgroups of $G(\mathrm{F}_4, R)$ normalized by the elementary subgroup $E(\mathrm{F}_4, R)$. As such, this result is not new, since structure theorems are known for all Chevalley groups. For classical groups they were established in [38, 45, 82, 18, 19, 20, 3, 55, 56, 70, 72, 73, 74, 48, 49, 27, 66, 43], for exceptional ones — in [30, 31, 32, 34, 67, 68, 69, 71, 50]; see also further references in [12, 13, 42, 43, 66, 75, 80]. Thus, the gist of the present paper, as that of [12], resides not in the results themselves, but rather in the method of their proof. Namely, we propose a new geometric approach towards calculations in Chevalley groups of type F_4, which involves as little information about the group as possible.

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§1. Introduction

How does one prove results about a Chevalley group $G(\Phi, R)$? This group depends on two parameters, a root system Φ and a commutative ring R. All proofs we know today are based on reduction in the rank of the system Φ, reduction in the dimension of the ring R, or a combination of both. Usually, the proofs in terms of the dimension of the ring are called *localization proofs*. They are covered in a *huge* number of publications; see references in [39, 40, 41, 42, 43, 52, 53, 54, 66].

On the other hand, the proofs based on reduction to groups of smaller rank usually do not invoke anything apart from the geometry of the underlying module. This is why they are often called *geometric proofs*. Let us briefly recapitulate the evolution of such proofs.

• A classical group preserves an invariant of degree 2 and can be defined by equations of degree 2. Over fields, the geometry of the corresponding modules has been studied for several millenia, and quite officially, under this name, for more than a century, at least starting with the books by Camille Jordan and Leonard Dickson. It constitutes the subject of what is known as *geometric algebra* [2] or *geometry of classical groups* [21]. In the 1940s and the 1950s, an absolutely outstanding role in the development of this direction was played by the work by Jean Dieudonné. Actually, starting with the 1960s, an overwhelming majority of results concerning classical groups over rings were first proved by exactly these methods. A systematic account of this theory can be found in the remarkable book by Alexander Hahn and Timothy O'Meara [51].

• Exceptional groups cannot be described in terms of invariants of degree 2. In fact, in minimal representations they preserve an invariant of degree 3 or 4; see, for instance, [35, 36, 47, 61, 75, 77], where one can find references to classical works. For example, in [14, 15] one can find an explicit form of equations that determine whether a matrix belongs to the group $G(\mathrm{E}_6, R)$ in the 27-dimensional representation, as well as many further references. Over fields, the *geometry of exceptional groups* was anticipated by Dickson, and has been studied in depth starting with the 1950s, especially by the Dutch and Belgian schools, most notably by such remarkable mathematicians as Hans Freudenthal, Jacques Tits, Tonny Springer, Ferdinand Veldkamp, and others. Over the last two decades, this area got an entirely new flourish in connection with the study of subgroups of exceptional groups, most prominently in the works by Michael Aschbacher, Arjeh Cohen, Bruce Cooperstein, and many others. At the same time, over rings, the overall progress has been much more modest than that; see [75, 77, 14]. In any case, until very recently, no proof of the main structure theorems for *arbitrary* commutative rings has been completed along these lines.

• A considerable conceptual advance in the study of exceptional groups over rings was constituted by the work by Alexei Stepanov, the first author, and Eugene Plotkin [17, 75, 80, 66], who noticed that, by using some representation theory, it is possible to reduce calculations necessary to prove structure theorems to elementary and stable ones. The technology of stable calculations was developed in the foundational papers by Hideya Matsumoto [57] and Michael Stein [63] in connection with applications to K-theory. In [17], a specific method that reduces calculations necessary to prove structure theorem to elementary and stable ones was baptised *decomposition of unipotents*. A drastic simplification was caused by the fact that this approach only invokes *quadratic* equations on an individual column, or an individual row of a matrix, rather than the actual equations of degree 3 or 4 defining the group — and moreover, only a portion of those quadratic equations; see [9, 11]!

For classical groups, at least in vector representations, a similar program was immediately implemented in [26, 7]. Detailed proofs were published in [75, 66, 42, 43], and even in a much larger generality: the ground rings were not necessarily commutative, the groups were not necessarily split, etc. At the same time, for exceptional groups, many technical aspects turned out *terribly* much harder than it seemed when [17] was conceived. It turned out that, as opposed to classical cases, the proofs for exceptional groups engage huge classical embeddings, such as $\mathrm{A}_5 \leq \mathrm{E}_6$ and $\mathrm{A}_7 \leq \mathrm{E}_7$. Moreover, the proofs vitally depended on *explicit* knowledge of the action structure constants and equations on the highest weight orbit.

The only exceptional cases for which there are published proofs by the method of decomposition of unipotents, and which do not depend on explicit computer calculations, are (E_6, ϖ_1), (E_7, ϖ_7), and (E_6, ϖ_2); see [77, 11]. The proofs for adjoint representations, and for the case of multiply laced systems, turned out to be technically much more demanding. For example, in the 1989 work by the first author and Eugene Plotkin, the following elements of root type

$$\begin{aligned}
&x_{\beta_1}(\pm g^2_{\lambda_1\mu})x_{\beta_2}(\pm g^2_{\lambda_2\mu})x_{\beta_3}(\pm g^2_{\lambda_3\mu})\\
&\quad\cdot x_{\beta_4}(\pm g_{\lambda_1\mu}g_{\lambda_2\mu})x_{\beta_5}(\pm g_{\lambda_1\mu}g_{\lambda_3\mu})x_{\beta_6}(\pm g_{\lambda_2\mu}g_{\lambda_3\mu})\\
&\quad\cdot x_{\beta_7}(\pm g_{\lambda_1\mu}g_{\nu_1\mu} \pm g_{\lambda_2\mu}g_{\nu_2\mu} \pm g_{\lambda_3\mu}g_{\nu_2\mu})\\
&\quad\cdot x_{\beta_8}(\pm g_{\lambda_1\mu}g_{\rho_1\mu} \pm g_{\lambda_2\mu}g_{\rho_2\mu} \pm g_{\lambda_3\mu}g_{\rho_3\mu})
\end{aligned}$$

were used to stabilize columns of a matrix $g \in G(\mathrm{F}_4, R)$ in the 26-dimensional representation, for several dozens of various configurations of roots β_i and weights $\lambda_i, \mu, \nu_i, \rho_i$. A glimpse of this expression suffices to apprehend why the details of the calculations remain unpublished.

• In the paper [12] by the first author and Mikhail Gavrilovich, a further improvement of this method was launched, for groups of types E_6 and E_7. Rather straightforward group-theoretic arguments allowed us to replace a reference to the presence of huge classical embeddings by the reference to the rank 2 embeddings $\mathrm{A}_2 \leq \mathrm{E}_6, \mathrm{E}_7$. Even more remarkably, we could avoid any reference to the explicit knowledge of structure constants and quadratic equations, or, for that matter, to any nontrivial equations whatsoever. The only equations used in that proof, *the proof from the Book*, are linear equations, defining the Lie algebra of the group G.

The essence of the paper [12] is a new method that makes it possible to reduce structure problems to groups of smaller rank. Before [12], a similar method had been used to study classical groups for about three decades. However, for mysterious reasons it was not noted that the same idea could be applied to the exceptional groups as well! As a matter of fact, to be tractable by this method, a reductive group itself does not have to be split. Rather, it should contain a *split* subgroup of type A_2 and have a microweight representation. Microweight representations are remarkable in that all root unipotents $x_\alpha(\xi)$ act quadratically, or, what is the same, $e_\alpha^2 = 0$.

In the present paper we take a next step and show that, at the cost of some additional effort, one can further enhance this method. Namely, it suffices to require that *long* root elements $x_\alpha(\xi)$ act quadratically. Assuredly, we still require the presence of a *split* subgroup of type A_2, embedded on *long* roots. This method is both easier and more powerful than all reduction methods known today. We illustrate the occurring calculations in the example of structure theorems for groups of type F_4. Actually, the possible range of similar techniques is much wider. In particular, by this method, all results on subgroups of classical groups containing a large regularly embedded subgroup can be extended from classical cases to the case of F_4; see the references in [76, 12].

We are certain that a similar treatment of the Chevalley group of type E_8 should also be possible, and we intend to return to this problem in subsequent work.[1] However, in this case, life becomes much harder: the group of type E_8 has no microweight representations itself and is not a twisted form of a group possessing such representations. In the adjoint representation, we only have a weaker identity, $e_\alpha^3 = 0$. In fact, one of the reasons why it would be extremely interesting to obtain such proofs in the case of type E_8 is that it would be an important advance towards structure results at the level of K_2, in particular, towards the centrality of K_2. The first steps in this direction for groups of types E_6 and E_7 were taken by the first author in [79].

We cannot, and do not even try, to recall here the definitions of all necessary notions pertaining to root systems, Weyl groups, weights and representations, algebraic groups, Chevalley groups, and algebraic K-theory. As a general background, which mostly suffices to comprehend the present paper, we could cite the books [4, 5, 6, 24, 25, 28, 29, 46, 51, 60, 61, 81].

All specific facts pertaining to the theory of Chevalley groups over rings as such, as well as many further references, can be found in [1, 12, 13, 14, 15, 22, 30, 31, 32, 33, 34, 53, 57, 58, 59, 62, 63, 67, 68, 69, 71, 75, 76, 77, 78, 79, 80]. Often, we perused other works where explicit calculations in F_4 have been carried out in a similar or different context, for guidelines or tips. Apart from the papers by Michael Stein [62] and Eiichi Abe [32], this pertains, in particular, to the papers [23, 64, 65] by Vladimir Nesterov and Anja Steinbach on the geometry of root subgroups.

Many assertions established in [12] for the case of E_6 carry over to the case of F_4 virtually without any alterations, together with their proofs. In such cases we briefly sketch the proof and refer to [12] for the details. On the other hand, there are major differences between the cases of E_6 and F_4. The most important of these stem from the fact that not all roots of F_4 have the same length. As a result, in many statements we have to consider the case of long roots and that of short roots separately. Worse than that, many auxiliary results of [12] are valid, as stated, only for long roots, and that calls for tremendous additional effort in the central fragment of the proof.

The present paper is organized as follows. In §2 we state the main structure theorems for groups of type F_4 over an arbitrary commutative ring. In §§3–5 we describe our principal tools: the minimal module for E_6, the folding of E_6 to F_4, and admissible pairs. In §§6 and 7 we establish several facile results of a technical nature, pertaining to the centralizers of unipotent elements and extraction in parabolic subgroups. The technical core of the paper is §8; the proof of the *main lemma* embodies the very essence of our method. After that, the proof of the structure theorems is readily finished in §9.

§2. Structure theorems

The state of description of normal subgroups in groups of type F_4 is somewhat different from that for groups of types E_6 and E_7. As we started to work on the present paper, we were positive that the existing literature contained a *complete* answer to the problem of description of normal subgroups in all Chevalley groups of rank at least 2 over commutative rings. This, indeed, turned out to be the case for all types *except* for the type F_4. In fact, Abe's paper [32] gives a remarkably ingenious description of the subgroups in $G(F_4, R)$ normalized by $E(F_4, R)$, for an arbitrary commutative ring R, in terms of admissible pairs of ideals (A, B). Yet, that paper, like all other published works, contains no general level reduction techniques that would work in the case of

[1] *Note added in proof.* This is indeed the case. An A_2-proof for this case has been produced by the first author and Alexander Luzgarev.

F_4. We were amazed to discover that answers to the most immediate questions concerning relative groups defined by admissible pairs were not readily available in the existing literature.

First, we state the principal result, due to Eiichi Abe, concerning the description of subgroups in the Chevalley group of type F_4 that are normalized by the elementary subgroup. The definitions of admissible pairs and of relative elementary groups, used in the following statement, are recalled in §5.

Theorem 1. *For an arbitrary commutative ring R, the standard description of the subgroups normalized by the elementary group $E(\mathrm{F}_4, R)$ holds true in the Chevalley group $G(\mathrm{F}_4, R)$. Namely, for any such subgroup H there exists a unique admissible pair of ideals (A, B) in the ring R such that*

$$E(\mathrm{F}_4, R, A, B) \leq H \leq C(\mathrm{F}_4, R, A, B).$$

Here the full congruence subgroup $C(\mathrm{F}_4, R, A, B)$ is defined not in terms of the reduction homomorphism but internally, as the transporter of $E(\mathrm{F}_4, R)$ into $E(\mathrm{F}_4, R, A, B)$. We do not try to come up with new proofs for those results by Abe, which are established by *elementary* calculations. Observe, by the way, that some of those calculations were contained already in [62]. Instead, we propose a new approach towards the central fragment of the proof of Theorem 1. Namely, the proof in the works by Vaserstein and Abe is based upon the localization and patching method, proposed by Quillen and Suslin, and thus, in the final analysis, upon reduction to local rings.

In the present paper, which is *essentially* an expanded version of the Diploma paper by the second author, done under the supervision of the first author, we propose an entirely different proof of this theorem. Our proof, as that in [3, 66, 12], is based upon reduction to groups of smaller rank, in this particular case, to those of types B_3 and C_3.

The first unpleasant surprise was the absense of a working technique for level reduction in the published papers on the normal structure of Chevalley groups. Namely, [34, 71, 31, 32] are based on reduction to local rings, whereas for local rings, level reduction boils down to reduction modulo the Jacobson radical. This means that, out of hand, our main lemma implies, via reduction to smaller rank, not the above structure theorem in full force, but merely the following *weak* structure theorem.

Theorem 2. *Let R be an arbitrary commutative ring, and let H be a subgroup of the Chevalley group $G(\mathrm{F}_4, R)$ normalized by the elementary group $E(\mathrm{F}_4, R)$. Then there exists a unique ideal $A \trianglelefteq R$ such that*

$$E(\mathrm{F}_4, R, A, A_2) \leq H \leq C(\mathrm{F}_4, R, A).$$

Here A_2 (not to be confused with A_2!) denotes the ideal generated by all 2ξ and ξ^2 for $\xi \in A$; see §5. Obviously, if $A_2 = A$ for all ideals, for instance, when $2 \in R^*$, this is precisely the usual structure theorem, stated in terms of a *single* ideal. This is exactly how it was presented in the Diploma paper of the second author, and in §8 of [13].

However, subsequently we decided that it would be wholesome to completely clarify the situation and prove Abe's theorem as is, without any restrictions on the ground ring, by our method. For classical groups, level reduction techniques independent of the invertibility of 2 were developed in Chapter IV of Tony Bak's Thesis [38]; see also [43, 54]. However, to implement this reduction, one must start with resolving several natural questions that remained open for the group of type F_4. Here are the most immediate ones.

- Is the group $E(\mathrm{F}_4, R, A, B)$ normal in $G(\mathrm{F}_4, R)$?
- What is the principal congruence subgroup $G(\mathrm{F}_4, R, A, B)$?

• Do different definitions of the full congruence subgroup $C(\mathrm{F}_4, R, A, B)$ coincide?

In the literature we could find answers to these questions only under various simplifying assumptions, typically something like $2 \in R^*$, which guarantee that we fall into the classical situation when $A = B = A_2$. Let us state the general answer to these questions.

Theorem 3. *For any admissible pair (A, B), the subgroup $E(\mathrm{F}_4, R, A, B)$ is normal in $G(\mathrm{F}_4, R)$, and we have equalities*

$$\begin{aligned} C(\mathrm{F}_4, R, A, B) &= \{g \in G(\mathrm{F}_4, R) \mid [g, E(\mathrm{F}_4, R)] \leq E(\mathrm{F}_4, R, A, B)\} \\ &= \{g \in G(\mathrm{F}_4, R) \mid [g, E(\mathrm{F}_4, R)] \leq G(\mathrm{F}_4, R, A, B)\} \\ &= \{g \in G(\mathrm{F}_4, R) \mid [g, G(\mathrm{F}_4, R)] \leq G(\mathrm{F}_4, R, A, B)\} \\ &= G(\mathrm{F}_4, R, A, B). \end{aligned}$$

We see the following possible approaches to the proof of Theorem 3.

• Stein's relativization with two parameters, in the spirit of §§4 and 5 of [42] or §1 of [40].

• A group-theoretic proof in the spirit of that proposed by Alexei Stepanov in [27], where the necessary argument was hidden behind the words "immediately follows", in the proof of Corollary 3.4.

• A localization proof that imitates the proof in [41, 68, 53, 40].

The first two approaches would look congenial in the context of the main part of the present paper. However, as of yet, we have not succeeded in guessing an appropriate analog of the construction from [42]. On the other hand, a group-theoretic proof is based on the presence of *some* subgroup normal in G between $E(\mathrm{F}_4, R, A, B)$ and $C(\mathrm{F}_4, R, A, B)$. In the classical situations such a subgroup is well known; this is the principal congruence subgroup $G(\Phi, R, I)$. In our case, $G(\mathrm{F}_4, R, A, B) = C(\mathrm{F}_4, R, A, B)$. As a matter of fact, we know how to define $G(\mathrm{F}_4, R, A, B)$ explicitly as a matrix group of degree 27 by congruences modulo various ideals, in the spirit of Borevich's net subgroups. For Chevalley groups, the corresponding construction was carried through in [16]. However, a straightforward verification of the normality of $G(\mathrm{F}_4, R, A, B)$ in $G(\mathrm{F}_4, R)$ leads to rather onerous calculations.[2]

This is why, after serious doubts, we decided to omit this proof,[3] and refer to the localization proof in [40]. To deduce Theorem 1 from Theorem 2 we use, instead of Theorem 3, another difficult external fact, namely the standard description of the normal subgroups of $G(\mathrm{D}_4, R)$. This standard description was first established by Igor Golubchik in 1975, see [19], and was repeatedly generalized and proved afresh since then; see [20, 56, 72, 66, 43]. See [13] for a more complete account of the historical development. In particular, the *original* proof by Golubchik was precisely an A_2-proof in the sense of [12], as opposed to the A_3-proofs in [66] and [43]. Thus, it contained a germ of the main idea of the proof from the Book.

§3. The 27-dimensional module

The principal tool used in the present paper, as in [77, 12, 14, 79], is the 27-dimensional module $V = V(\varpi_1)$ of the Chevalley group $G = G(\mathrm{E}_6, R)$ of type E_6 with the highest weight $\omega = \varpi_1$. By $\Lambda = \Lambda(\omega)$ we denote the set of weights of the module V. Since this is

[2]It was precisely to avoid such matrix calculations, in the *much easier* case of the unitary group, that we developed a version of relativization with two parameters in [42].

[3]After the present paper had been submitted, another proof of Theorem 3 was discovered, based on a completely different idea. That proof, published in [83], does not require any arduous calculation whatsoever.

a microweight representation, we have $\Lambda = W(\mathrm{E}_6)\omega$. In the sequel we fix a crystal base v^λ, $\lambda \in \Lambda$, of the module V. This means the following.

• The base v^λ consists of weight vectors.

• The base v^λ is admissible, or, in other words, for any root $\alpha \in \Phi$ and any $\xi \in R$ we have

$$x_\alpha(\xi)v^\lambda = v^\lambda + c_{\lambda\alpha}\xi v^{\lambda+\alpha},$$

and all action structure constants $c_{\lambda\alpha}$ are equal to ± 1.

• The structure constants $c_{\lambda\alpha}$ are equal to $+1$ for both the fundamental and the negative fundamental roots, i.e., $c_{\lambda\alpha} = +1$ whenever $\alpha \in \pm\Pi$.

The existence of such a base is classically known; in [77] and [8] one can find two elementary proofs, and in [15] one can find explicit sign tables for the structure constants in a crystal base. However, in the present paper we do not refer to such explicit knowledge of structure constants.

Another extremely important structure related to this module is the invariant cubic form. There is a *vast* literature devoted to the study of this form, initiated by the early twentieth century work by Leonard Dickson and Élie Cartan. Later, in the 1950s and 1960s, this form became a principal tool in the work on the geometry of exceptional groups [61]. In modern times this form became a major instrument in the study of subgroups of $G(\mathrm{E}_6, R)$ and $G(\mathrm{F}_4, R)$; see, in particular, [35, 36, 47], as well as further references in [75, 77, 15, 14].

We mention some of our recent papers related to the study of the cubic form. In [15], one can find explicit coordinate expressions of the form, for various choices of a base. In [14], there is an explicit list of equations a matrix $g \in \mathrm{GL}(27, R)$ should satisfy to belong to the normalizer of the group $G(\mathrm{E}_6, R)$. Finally, in [44], a proper definition of cubic forms over rings was discussed, similar to Bak's definition of quadratic forms. It is remarkable, and completely unexpected, that the proofs in the present paper make no reference whatsoever to the existence of this form, or, in fact, to any nontrivial equations on the entries of matrices from $G(\mathrm{E}_6, R)$ or $G(\mathrm{F}_4, R)$!

In Figure 1 we reproduce the *weight diagram*, or, what is the same in this case, the crystal graph of this module. Recall that the vertices of this diagram are indexed by the weights of the module V, whereas the marks of the bonds correspond to the fundamental roots. Our numbering of the fundamental roots follows [5]. In [15] one can find a thorough computer study of this module, in particular, lists of weights of this representation in various realizations, its structure constants, the images of root elements, the invariant cubic form and its derivatives, as well as the codes to calculate all this stuff, in `Mathematica`.

As in other works on structure theory where representations are used, we conceive a vector $a \in V$, $a = \sum a_\lambda v^\lambda$, as a coordinate *column* $a = (a_\lambda)$, $\lambda \in \Lambda$. In accordance with that, an element b of the contragradient module V^* should be thought of as a *row* $b = (b_\lambda)$, $\lambda \in \Lambda$. Obviously, with respect to the weights Λ^* of the contragradient module V^*, the perspective should be reversed: the elements of V^* should be viewed as *columns* $b = (b_\lambda)$, $\lambda \in \Lambda^*$, and the elements V as *rows* $a = (a_\lambda)$, $\lambda \in \Lambda^*$.

Furthermore, usually an element $g \in G(\mathrm{E}_6, R)$ is identified with its image in the representation π and is viewed as a (27×27) matrix $g = (g_{\lambda\mu})$, $\lambda, \mu \in \Lambda$, with respect to the base v^λ. As usual, the columns of this matrix are the coordinate columns of the vectors gv^μ, $\mu \in \Lambda$, with respect to the base v^λ, $\lambda \in \Lambda$. We systematically use the following notation: the μth column of a matrix g will be denoted by $g_{*\mu}$, whereas its λth row will be denoted by $g_{\lambda *}$. The inverse of g is denoted by $g^{-1} = (g'_{\lambda\mu})$, $\lambda, \mu \in \Lambda$.

We emphasize that in the present paper, as in [12], both columns and rows of matrices are indexed by the weights $\lambda, \mu \in \Lambda$ of the module V itself. This is precisely why the

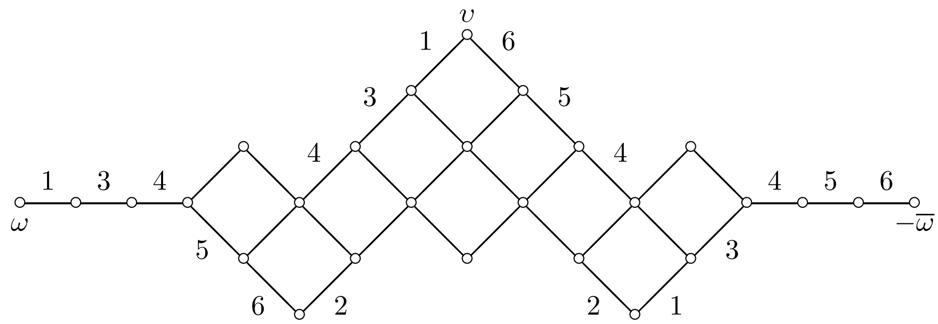

FIGURE 1. (E_6, ϖ_1).

coordinates of a vector in V^* are indexed by the weights of the module V and are written as *rows*. As a rule, in representation theory they are indexed by the weights of the module V^* and written as columns. For exactly this reason, the formulas describing the action of elements of the group G on columns and on rows appear differently. We reproduce these formulas in the next lemma. In fact, it is merely a paraphrase of the definition of an admissible base.

Lemma 1. *For any $g \in \mathrm{GL}(27, R)$, $\alpha \in \Phi$, and $\xi \in R$, we have the following formulas*:

$$(x_\alpha(\xi)g)_{\lambda\mu} = g_{\lambda\mu} \pm \xi g_{\lambda-\alpha,\mu}, \qquad (gx_\alpha(\xi))_{\lambda\mu} = g_{\lambda\mu} \pm \xi g_{\lambda,\mu+\alpha}.$$

As has already been mentioned, one can also specify the signs in the above formulas, but we do not need this in the present paper.

Below we collect some obvious properties of weights of the representation $(\mathrm{E}_6, V(\varpi_1))$. All these statements hold true in greater generality, at least for microweight representations of simply laced root systems, as was established in the papers cited above. However, in the present paper we only use them for the case of $\Phi = \mathrm{E}_6$, $\Lambda = \Lambda(\varpi_1)$. Recall that in this case the orbits of the Weyl group on pairs of weights (λ, μ), $\lambda, \mu \in \Lambda$, are distinguished by a single invariant, namely, by the distance $d(\lambda, \mu)$ in the *weight graph*. As opposed to the weight diagram, a weight graph has edges marked by *all* positive roots, not only by the fundamental ones. Thus, for E_6 this distance takes three possible values. It is equal to 0 if $\lambda = \mu$, it is equal to 1 if $\lambda - \mu \in \Phi$, and, finally, it is equal to 2 if $\lambda \neq \mu$ and the difference $\lambda - \mu$ is not a root.

The following claim is Lemma 3 of [77].

Lemma 2. *Suppose $\lambda \in \Lambda$, and $\alpha, \beta, \alpha + \beta \in \Phi$. If $\lambda + \alpha + \beta \in \Lambda$, then exactly one of the sums $\lambda + \alpha$ or $\lambda + \beta$ belongs to Λ.*

The next claim is Proposition 2 of [12], after correction of an obvious mistake. Namely, in [12], we omitted the condition $\alpha \neq \lambda - \rho$; compare with Lemma 3 of the same paper.

Lemma 3. *If $\lambda - \alpha, \rho + \alpha \in \Lambda$, and $\alpha \neq \lambda - \rho$, then $d(\lambda, \rho) = 2$.*

§4. F_4 AS TWISTED E_6

In the present paper, we use the following two realizations of the roots of $\Phi = \mathrm{F}_4$. First, this is the usual realization in the 4-dimensional Euclidean space,

$$\mathrm{F}_4 = \Big\{ \pm e_i,\ \pm e_i \pm e_j,\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \Big\},$$

where $1 \leq i \neq j \leq 4$, and the signs are chosen independently. The roots of the form $\pm e_i \pm e_j$ are long, and in the sequel the set of long roots will be denoted by Φ_l. All remaining roots are short, and the set of short roots is denoted by Φ_s.

As usual, we choose

$$\alpha_1 = e_2 - e_3, \qquad \alpha_2 = e_3 - e_4, \qquad \alpha_3 = e_4, \qquad \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$$

as fundamental roots. Then $\delta = e_1 + e_2$ is the maximal root with respect to this order, while $\rho = e_1$ is the dominant short root. Usually we write the root $\alpha = p\alpha_1 + q\alpha_2 + r\alpha_3 + s\alpha_4 \in F_4$ in the Dynkin form as $pqrs$. For example, in this notation, $\delta = 2342$, whereas $\rho = 1232$.

TABLE 1. Long roots of F_4

$e_2 - e_3 = 1000 = \begin{smallmatrix}00000\\1\end{smallmatrix}$	$e_3 - e_4 = 0100 = \begin{smallmatrix}00100\\0\end{smallmatrix}$
$e_2 - e_4 = 1100 = \begin{smallmatrix}00100\\1\end{smallmatrix}$	$e_3 + e_4 = 0120 = \begin{smallmatrix}01110\\0\end{smallmatrix}$
$e_2 + e_4 = 1120 = \begin{smallmatrix}01110\\1\end{smallmatrix}$	$e_2 + e_3 = 1220 = \begin{smallmatrix}01210\\1\end{smallmatrix}$
$e_1 - e_2 = 0122 = \begin{smallmatrix}11111\\0\end{smallmatrix}$	$e_1 - e_3 = 1122 = \begin{smallmatrix}11111\\1\end{smallmatrix}$
$e_1 - e_4 = 1222 = \begin{smallmatrix}11211\\1\end{smallmatrix}$	$e_1 + e_4 = 1242 = \begin{smallmatrix}12221\\1\end{smallmatrix}$
$e_1 + e_3 = 1342 = \begin{smallmatrix}12321\\1\end{smallmatrix}$	$e_1 + e_2 = 2342 = \begin{smallmatrix}12321\\2\end{smallmatrix}$

TABLE 2. Short roots of F_4

$$e_4 = 0010 = \left\{\begin{smallmatrix}01000\\0\end{smallmatrix}, \begin{smallmatrix}00010\\0\end{smallmatrix}\right\}$$

$$\tfrac{1}{2}(e_1 - e_2 - e_3 - e_4) = 0001 = \left\{\begin{smallmatrix}10000\\0\end{smallmatrix}, \begin{smallmatrix}00001\\0\end{smallmatrix}\right\}$$

$$e_3 = 0110 = \left\{\begin{smallmatrix}01100\\0\end{smallmatrix}, \begin{smallmatrix}00110\\0\end{smallmatrix}\right\}$$

$$\tfrac{1}{2}(e_1 - e_2 - e_3 + e_4) = 0011 = \left\{\begin{smallmatrix}11000\\0\end{smallmatrix}, \begin{smallmatrix}00011\\0\end{smallmatrix}\right\}$$

$$e_2 = 1110 = \left\{\begin{smallmatrix}01100\\1\end{smallmatrix}, \begin{smallmatrix}00110\\1\end{smallmatrix}\right\}$$

$$\tfrac{1}{2}(e_1 - e_2 + e_3 - e_4) = 0111 = \left\{\begin{smallmatrix}11100\\0\end{smallmatrix}, \begin{smallmatrix}00111\\0\end{smallmatrix}\right\}$$

$$\tfrac{1}{2}(e_1 + e_2 - e_3 - e_4) = 1111 = \left\{\begin{smallmatrix}11100\\1\end{smallmatrix}, \begin{smallmatrix}00111\\1\end{smallmatrix}\right\}$$

$$\tfrac{1}{2}(e_1 - e_2 + e_3 + e_4) = 0121 = \left\{\begin{smallmatrix}11110\\0\end{smallmatrix}, \begin{smallmatrix}01111\\0\end{smallmatrix}\right\}$$

$$\tfrac{1}{2}(e_1 + e_2 - e_3 + e_4) = 1121 = \left\{\begin{smallmatrix}11110\\1\end{smallmatrix}, \begin{smallmatrix}01111\\1\end{smallmatrix}\right\}$$

$$\tfrac{1}{2}(e_1 + e_2 + e_3 - e_4) = 1221 = \left\{\begin{smallmatrix}11210\\1\end{smallmatrix}, \begin{smallmatrix}01211\\1\end{smallmatrix}\right\}$$

$$\tfrac{1}{2}(e_1 + e_2 + e_3 + e_4) = 1231 = \left\{\begin{smallmatrix}12210\\1\end{smallmatrix}, \begin{smallmatrix}01221\\1\end{smallmatrix}\right\}$$

$$e_1 = 1232 = \left\{\begin{smallmatrix}12211\\1\end{smallmatrix}, \begin{smallmatrix}11221\\1\end{smallmatrix}\right\}$$

Secondly, we realize F_4 as the twisted root system of type E_6. Namely, consider the root system E_6 and a fundamental root system $\Pi = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$ therein. Recall that we always use the same numbering of the fundamental roots as in [5], so that, for example, β_2 is orthogonal to all fundamental roots apart from β_4. Consider the outer automorphism $\alpha \mapsto \bar{\alpha}$ of order 2 of the system E_6 that permutes β_1 with β_6 and β_3 with β_5, leaving β_2 and β_4 invariant. We can conceive the roots of F_4 as the orbits of this automorphism.

In the sequel we interpret the root elements of F_4 inside the Chevalley group of type E_6 as follows:

$$\begin{aligned} x_{\alpha_1}(\xi) &= x_{\beta_2}(\xi),\\ x_{\alpha_2}(\xi) &= x_{\beta_4}(\xi),\\ x_{\alpha_3}(\xi) &= x_{\beta_3}(\xi)x_{\beta_5}(\xi),\\ x_{\alpha_4}(\xi) &= x_{\beta_1}(\xi)x_{\beta_6}(\xi). \end{aligned}$$

Note the unusual choice of signs, which differs from that in [25]! In general, in this realization every root element of the group $G(\mathrm{F}_4, R)$ has the form $x_\beta(\xi)$, where $\bar{\beta} = \beta$ (long root elements), or the form $x_\beta(\xi)x_{\bar{\beta}}(\pm\xi)$, where $\bar{\beta} \neq \beta$ (short root elements). Although we do not need this in the present paper, for future reference we reproduce an explicit choice of signs in the expression of short root elements, which corresponds to a positive Chevalley base:

$$\begin{aligned}
x_{0001}(\xi) &= x_{\substack{10000\\0}}(\xi)x_{\substack{00001\\0}}(\xi), & x_{0010}(\xi) &= x_{\substack{01000\\0}}(\xi)x_{\substack{00010\\0}}(\xi),\\
x_{0011}(\xi) &= x_{\substack{11000\\0}}(\xi)x_{\substack{00011\\0}}(-\xi), & x_{0110}(\xi) &= x_{\substack{01100\\0}}(\xi)x_{\substack{00110\\0}}(-\xi),\\
x_{1110}(\xi) &= x_{\substack{01100\\1}}(\xi)x_{\substack{00110\\1}}(-\xi), & x_{0111}(\xi) &= x_{\substack{11100\\0}}(\xi)x_{\substack{00111\\0}}(\xi),\\
x_{1111}(\xi) &= x_{\substack{11100\\1}}(\xi)x_{\substack{00111\\1}}(\xi), & x_{0121}(\xi) &= x_{\substack{11110\\0}}(\xi)x_{\substack{01111\\0}}(-\xi),\\
x_{1121}(\xi) &= x_{\substack{11110\\1}}(\xi)x_{\substack{01111\\1}}(-\xi), & x_{1221}(\xi) &= x_{\substack{11210\\1}}(\xi)x_{\substack{01211\\1}}(-\xi),\\
x_{1231}(\xi) &= x_{\substack{12210\\1}}(\xi)x_{\substack{01221\\1}}(-\xi), & x_{1232}(\xi) &= x_{\substack{12211\\1}}(\xi)x_{\substack{11221\\1}}(\xi).
\end{aligned}$$

Next, we reproduce the weight diagram for the restriction of the 27-dimensional representation of $G(\mathrm{E}_6, R)$ to $G(\mathrm{F}_4, R)$, in this embedding. Now, the marks on the edges correspond to the usual numbering of the fundamental roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, the roots α_1, α_2 being long.

Whereas the Weyl group $W(\mathrm{E}_6)$ acts on Λ transitively, for $W(\mathrm{F}_4)$ this is not the case. Namely, the representation $(\mathrm{E}_6, \varpi_1) \downarrow \mathrm{F}_4$ is reducible and is the direct sum of the 26-dimensional representation on short roots and the trivial 1-dimensional representation. Thus, $W(\mathrm{F}_4)\omega$ consists of 24 weights corresponding to the short roots of F_4. In the sequel they will be indexed by these short roots.

When describing specific calculations with weights, it is convenient to use branching $(\mathrm{E}_6, \varpi_1) \downarrow \mathrm{D}_4$. Since D_4 is regularly embedded in E_6, to obtain this branching it suffices to erase the edges marked by 1 or 6 in the diagram (E_6, ϖ_1). Passing to F_4, this means that we focus on its restriction to B_3 obtained by erasing the edges marked by 4. However, one of the resulting 8-dimensional representations is reducible. Thus, this representation is decomposed into a direct sum of three 1-dimensional representations corresponding to the weights $\omega = 1232$, υ, and $-\bar{\omega} = -1232$, and three 8-dimensional representations. Let

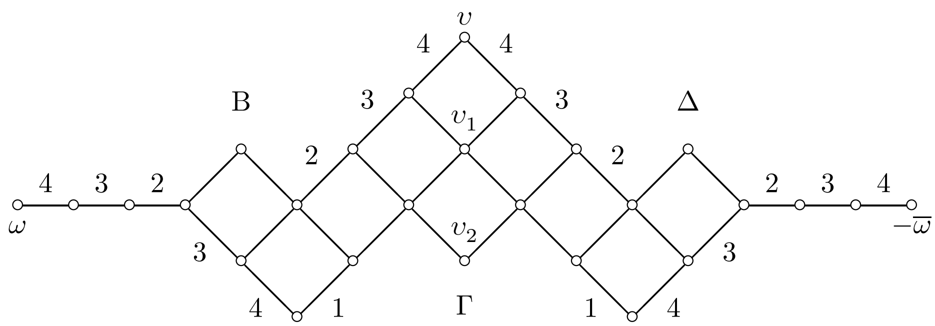

FIGURE 2. $(E_6, \varpi_1) \downarrow F_4$.

B, Γ, Δ be the sets of weights of these representations, in accordance with the decreasing height of their highest weight. Denoting weights as short roots of F_4, we have

$$B = \{1231, 1221, 1121, 0121, 1111, 0111, 0011, 0001\},$$

whereas $\Delta = -B$ consists of opposite roots. On the other hand, Γ consists of the roots $1110, 0110, 0010$, their opposites, and two more *zero* weights, υ_1 and υ_2.

§5. ADMISSIBLE PAIRS

Here we recall the precise definition of the standard description of normal subgroups in the Chevalley group of type F_4. Clearly, since F_4 has two different root lengths, such a description should be given in terms of *pairs* of ideals in R, not merely one ideal! Obviously, under various simplifying assumptions such as, for instance, $2 \in R^*$, this description reduces to the classical standard description in terms of one ideal.

In the sequel, $\Phi = F_4$. We denote by $\Phi_l = D_4$ the set of long roots of the system Φ, and by Φ_s its set of short roots.

The following notion was introduced by Abe in [30, 31, 32] and by Abe–Suzuki in [34]. Let A be an ideal in R. Denote by A_2 the ideal generated by 2ξ and ξ^2 for all $\xi \in A$. A pair of ideals (A, B) of the ring R is said to be *admissible* if $A_2 \le B \le A$.

Under the simplifying assumption that every element ξ of the ring R belongs to the ideal generated by 2ξ and ξ^2, we have $A_2 = A$. Thus, under this assumption, the admissible pairs (A, B) correspond bijectively to the ideals $I = A = B$ of the ring R, and we again return to the usual notion of standardness, in terms of a single ideal.

With any admissible pair (A, B), we can associate the corresponding *relative elementary subgroup* $E(\Phi, R, A, B)$ of level (A, B); by definition, this is the smallest *normal* subgroup of $E(\Phi, R)$ generated by the elementary root unipotents of level (A, B). In other words,

$$E(\Phi, R, A, B) = \big\langle x_\alpha(\xi), \alpha \in \Phi_s, \xi \in A;\ x_\beta(\zeta), \beta \in \Phi_l, \zeta \in B \big\rangle^{E(\Phi, R)}.$$

In the case where $A = B = I$, this notion specializes to the usual notion of the relative elementary group

$$E(\Phi, R, I) = \langle x_\alpha(\xi), \alpha \in \Phi, \xi \in I \rangle^{E(\Phi, R)}.$$

The next lemma follows immediately from the definition of $E(\Phi, R, A, B)$ and the fact that every root of the root system F_4 embeds into a subsystem of type A_2.

Lemma 4. *For any admissible pair* (A, B)*, we have*

$$[E(\Phi, R), E(\Phi, R, A, B)] = E(\Phi, R, A, B).$$

Next, we describe generators of $E(\Phi, R, A, B)$ *as a subgroup*. For a root $\alpha \in \Phi$ and any two ring elements $\xi, \zeta \in R$, we set

$$z_\alpha(\xi, \zeta) = x_\alpha(\zeta) x_{-\alpha}(\xi) x_\alpha(-\zeta).$$

Remark. Our definition differs from that used by Abe, as far as the sign of the root α is concerned. Moreover, the definition of the element $z_\alpha(\xi, \zeta)$ in Subsection 2.4 of [32] contains a mistake in the order of the arguments. The following definitions show that the intended order of the arguments is the same as ours.

The Chevalley commutator formula immediately implies that the group $E(\Phi, R, I)$ is generated by the elements $z_\alpha(\xi, \zeta)$ contained in it; see, *for example*, [71, Theorem 2]. A similar result holds true also for the group $E(\Phi, R, A, B)$, but the proof in this case requires a much more serious effort; see [32, Proposition 2.4] (see also [42, Proposition 5.1]).

Lemma 5. *The group $E(\Phi, R, A, B)$ is generated by the elements of the form $z_\alpha(\xi, \zeta)$ contained in it. In other words,*

$$E(\Phi, R, A, B) = \Big\langle z_\alpha(\xi, \eta), \alpha \in \Phi_s, \xi \in A, \eta \in R; z_\beta(\zeta, \eta), \beta \in \Phi_l, \zeta \in B, \eta \in R \Big\rangle.$$

This implies the following result, which will be used in the proof of Theorem 1.

Lemma 6. *For any admissible pair we have*

$$E(\mathrm{F}_4, R, A) = E(\mathrm{F}_4, R, A, B) E(\mathrm{D}_4, R, A).$$

Proof. Indeed, the group $E(\mathrm{F}_4, R, A, B)$ is normal in $E(\mathrm{F}_4, R)$ and, *a fortiori*, in a smaller group $E(\mathrm{F}_4, R, A)$, so that the product on the right-hand side is a subgroup containing all the generators of the subgroup on the left-hand side. □

In the sequel we shall denote by H a subgroup of $G(\Phi, R)$ normalized by $E(\Phi, R)$. With every such subgroup, we can associate an admissible pair (A, B), naturally interpreted as the *lower level* of this subgroup. The following result is a special case of Theorems 1 and 2 in [32].

Lemma 7. *For every root $\alpha \in \mathrm{F}_4$, the set*

$$I_\alpha = \{\xi \in R \mid x_\alpha(\xi) \in H\}$$

is an ideal in R. This ideal depends not on the root α itself, but only on its length. Setting $A = I_\alpha$ for short roots α and $B = I_\beta$ for long roots β yields an admissible pair (A, B). We have

$$E(\Phi, R, A, B) \leq H,$$

and furthermore, (A, B) is the largest admissible pair with this property.

The admissible pair constructed in this lemma is called the admissible pair *associated* with the subgroup H. Under the simplifying assumption we mentioned above, one has $A = B$, and we return to the usual definition of the lower level as the largest ideal $I \trianglelefteq R$ such that $E(\Phi, R, I) \leq H$.

Next, we define the *full congruence subgroup* $C(\Phi, R, A, B)$ of level (A, B) as the transporter of the absolute elementary group to the relative elementary group of level (A, B):

$$C(\Phi, R, A, B) = \{g \in G(\Phi, R) \mid [g, E(\Phi, R)] \leq E(\Phi, R, A, B)\}.$$

As usual, in the case where $A = B = I$, we denote the corresponding full congruence subgroup simply by $C(\Phi, R, I)$. By definition, $E(\Phi, R, A, B)$ is normal in the absolute elementary group $E(\Phi, R)$. Unfortunately, the existing literature does not answer the question whether this group is normal in the Chevalley group $G(\Phi, R)$ for $\Phi = \mathrm{F}_4$. At the same time, this fact and other similar commutator formulas are absolutely unavoidable to carry through a proof of the structure theorems via level reduction, with a subsequent adjustment of the admissible pair, as in [38].

For $A = B = I$ this is indeed the case. The following result is obtained easily from the normality of the elementary subgroup established by Giovanni Taddei [69] via relativization. See, for example, [71, Theorem 1].

Lemma 8. *For every irreducible root system of rank at least* 2*, the relative elementary subgroup* $E(\Phi, R, I)$ *is normal in the Chevalley group* $G(\Phi, R)$*. Moreover,*

$$[E(\Phi, R), C(\Phi, R, I)] \leq E(\Phi, R, I).$$

Combining this with the main result of Abe–Hurley [33], one easily obtains the following result; see [32, Corollary 4].

Lemma 9. *If, moreover, the group* $E(\Phi, R)$ *is perfect, then for any ideal* $I \trianglelefteq R$ *we have*

$$\begin{aligned} C(\Phi, R, I) &= \{g \in G(\Phi, R) \mid [g, G(\Phi, R)] \leq G(\Phi, R, I)\} \\ &= \{g \in G(\Phi, R) \mid [g, E(\Phi, R)] \leq G(\Phi, R, I)\} \\ &= \{g \in G(\Phi, R) \mid [g, E(\Phi, R)] \leq E(\Phi, R, I)\}. \end{aligned}$$

As is well known, see [62], among groups of rank at least 2, the only cases where the elementary group $E(\Phi, R)$ is not perfect are the groups of types B_2 and G_2 over a commutative ring R that possesses residue fields $\mathbb{F}_2$ of two elements.

§6. Centralizers of unipotent root elements

In the present section we prove several auxiliary results that will be used in extraction from parabolic subgroups and in the proof of the main lemma. We start with calculating the centralizer of an elementary root element $x_\alpha(\xi)$. Since the proofs in the present paper are not likely to excite someone who has not already read [12], we presume that the reader is familiar with that paper. In particular, we do not reproduce §6 thereof, summarizing the generalities on the structure of parabolic subgroups, and limit ourselves to an absolute minimum of indispensable notation.

For any $1 \leq i \leq l$, we denote by P_i the standard parabolic subgroup obtained by erasing the negative root $-\alpha_i$ for some fundamental $\alpha_i \in \Pi$. By L_i and U_i we denote its Levi subgroup and its unipotent radical, respectively, and by U_i^- we denote the unipotent radical of the opposite parabolic subgroup P_i^- with the same Levi subgroup, obtained by erasure of α_i. By Δ_i we denote the root system of L_i, and by Σ_i the set of roots of U_i. The commutator subgroups, the central series, and all other group-theoretic constructions are always interpreted in the sense of the theory of algebraic groups.

The following result is Lemma 8 of [12]. In the sequel we sometimes use the notation $g_{\lambda\mu}$ even when λ and/or μ are not roots. In such cases $g_{\lambda\mu}$ is assumed to be equal to 0.

Lemma 10. *If an element* $g \in \mathrm{GL}(n, R)$ *commutes with a root element* $x_\alpha(\xi)$ *for some root* $\alpha \in \Phi$*, then*:

$$\begin{aligned} &\xi g_{\lambda\mu} = 0 \quad \textit{if} \quad \lambda + \alpha \in \Lambda, \quad \textit{but} \quad \mu + \alpha \notin \Lambda; \\ &\xi g_{\lambda\mu} = 0 \quad \textit{if} \quad \mu - \alpha \in \Lambda, \quad \textit{but} \quad \lambda - \alpha \notin \Lambda; \\ &\xi(g_{\lambda\mu} - g_{\lambda+\alpha,\mu+\alpha}) = 0 \quad \textit{if} \quad \lambda + \alpha,\ \mu + \alpha \in \Lambda. \end{aligned}$$

Essentially, the following result coincides with [12, Proposition 4]. Admittedly, it was stated there in a slightly less precise form. However, exactly this was proved there, not merely for long roots of F_4, but rather for all roots of E_6.

Proposition 1. *If $[g, x_\alpha(1)] = 1$ for some $g \in \mathrm{GL}(27, R)$ and some long root $\alpha \in \mathrm{F}_4$, then g lies in a parabolic subgroup $P \le \mathrm{GL}(27, R)$ of type $(6, 15, 6)$ whose intersection with the Chevalley group of type F_4, in the embedding discussed in §3, is a parabolic subgroup of type P_1 thereof.*

An analog of this proposition for the case of short roots follows roughly the same pattern. However, there is a substantial difference between these two results. Namely, in the case of long roots the centralizer is a parabolic subgroup, up to a one-dimensional torus. More precisely, P_1 is the normalizer of the root subgroup X_δ. For short roots this is not the case, because now not all elements of the unipotent radical fall in the centralizer. However, since in the proof of the main lemma we need to be able to cope with arbitrary elements of P_4, for further reductions it is only important that for a short root the centralizer of $x_\alpha(1)$ *be contained* in P_4. Surely, this is well known when the ground ring is a field, and for an arbitrary commutative ring nothing changes, because we only use it for the value of the parameter equal to 1. On the other hand, below we establish a slightly more general result on the centralizers of short root elements in $\mathrm{GL}(27, R)$.

Proposition 2. *If $[g, x_\alpha(1)] = 1$ for some $g \in \mathrm{GL}(27, R)$ and some short root $\alpha \in \mathrm{F}_4$, then g lies in a parabolic subgroup $P \le \mathrm{GL}(27, R)$ of type $(9, 9, 9)$ whose intersection with the Chevalley group of type F_4, in the embedding discussed in §3, is a parabolic subgroup of type P_4 thereof.*

Proof. Without loss of generality, we may assume that $\alpha = \rho = 2132$. We fix the following order of weights: the highest weight ω, then the weights of B in the order

$$\mathrm{B}_1 = \{1231, 1111, 0111, 0011\}, \qquad \mathrm{B}_2 = \{1221, 1121, 0121, 0001\},$$

then the weights 1110, 0110, 0010, then zero weights v_1, v, v_2, then the weights -0010, -0110, -1110, then the weights of $-\mathrm{B}_2$ and $-\mathrm{B}_1$ in the opposite order, and, finally, the lowest weight $-\omega^*$. A glance at the tables of [20] (for example, see the explicit shape of the elements $e_{\substack{12211\\1}}$ and $e_{\substack{11221\\1}}$ in Table 10 therein) convinces us that in such a base $x_\rho(1)$ has the form

$$x_\rho(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & e_4 & 0 & 0 \\ 0 & 0 & e_4 & 0 & 0 & 0 & 0 & 0 & 0 & -e_4 & 0 \\ 0 & 0 & 0 & e_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, to prove the statement on the centralizer of $x_\rho(1)$ in $\mathrm{GL}(27, R)$, it remains, as in [12], to refer to Lemma 7. On the other hand, in F_4 (and even in E_6) there are no additions between v and the other 8 central weights. Combined with the absence of downward additions between the nines, this shows that, upon restriction to F_4, the diagram is cut through the 5 central edges with mark 1 as well as the 5 symmetric edges with mark 6. Obviously, then it should be cut through the remaining outer edges with marks 1 and 6. This shows that in the first column there are no downward additions, so that all of its nondiagonal components are equal to 0, and we fall into P_4. $\square$

Remark. The intersection of a subgroup of type $(9, 9, 9)$ with the Chevalley group of type E_6, embedded in this way, is a *submaximal* parabolic subgroup with the Levi subgroup of type D_4.

The next analog of [12, Lemma 11] immediately follows from the Chevalley commutator formula.

Lemma 11. *The elementary group $E(\mathrm{F}_4, R)$ is generated by the unipotent radicals of two opposite parabolic subgroups U_1 and U_1^-, or else U_4 and U_4^-, respectively.*

Proof. Any long/short root $\gamma \in \Delta_i$ is the difference of two long/short roots $\alpha, \beta \in \Sigma_i$. It follows that

$$x_\gamma(\xi) = [x_a(\pm\xi), x_{-\beta}(1)] \in \langle U_i, U_i^- \rangle.$$

□

By Theorem 2 in [37], the next result follows in exactly the same way as [12, Proposition 7]. In the Diploma paper of the second author, there is a direct elementary proof of this result in the style of §8 of [12], which does not involve results on the structure of U_i as an L_i-module. We do not reproduce it. As we observed in [12], such a proof is merely an exercise in patience.

Proposition 3. *Let $i = 1, 4$. If an element $z \in L_i$ commutes with all $x_\alpha(1)$, $\alpha \in \Sigma_i$, then it is central.*

§7. Extraction from parabolic subgroups

In the present section we keep the assumption that H is a subgroup in $G(\mathrm{F}_4, R)$ normalized by $E(\mathrm{F}_4, R)$. Recall that the center of the group $G(\mathrm{F}_4, R)$ is equal to the center of the group $E(\mathrm{F}_4, R)$ and is equal to 1. It is our intention to prove that if H has a nontrivial element contained in a proper parabolic subgroup, then H contains a nontrivial root element. Although this result is true in general, we prove it only in the following two cases where we actually use it.[4]

- Parabolic subgroup P_1 with Levi subgroup of type C_3. The unipotent radical U_1 is extraspecial. In other words, $[U_1, U_1] = X_\delta = C(U_1)$, and the factor-group $U_1/[U_1, U_1]$ is the 14-dimensional fundamental module of the group $\mathrm{Sp}(6, R)$.
- Parabolic subgroup P_4 with Levi subgroup of type B_3. In this case $[U_4, U_4]$ is the natural 7-dimensional module of the group $\mathrm{Spin}(7, R)$, whereas $U_4/[U_4, U_4]$ is its 8-dimensional spin module.

Proposition 4. *If H contains a nontrivial element lying in a proper parabolic subgroup P_1 or P_4, then H contains a nontrivial root unipotent.*

Proof. Let $g = zv \in P_i$, where $i = 1, 4$, $z \in L_i$, $v \in U_i$, and, moreover, $z \neq \varepsilon e$. Take a root $\alpha \in \Sigma_i$ and consider the commutator

$$u = [g, x_\alpha(1)] = [zv, x_\alpha(1)] = {}^z[v, x_\alpha(1)][z, x_\alpha(1)].$$

The first commutator on the right-hand side lies in $[U_i, U_i]$, whereas the second lies in U_i. By Proposition 3, there exists a root α such that $u \neq e$. Our first objective is to beget, when possible, a nontrivial element already in the *commutator subgroup* of the unipotent radical $[U_i, U_i]$. In the case of P_1 this will finish the proof.

[4] *Note added in proof.* In [84], we give a very short proof of the parabolic reduction in the general case. Proposition 4 below is completely superseded by [84, Theorem 1].

Namely, in the case of P_1 we consider the expansion $u = \prod x_\alpha(u_\alpha)$, $\alpha \in \Sigma_1$. If $u_\alpha \neq 0$ for some *long* root $\alpha \neq \delta$, or else $2u_\alpha \neq 0$ for some *short* root α, then the element

$$[u, x_{\delta-\alpha}(1)] = x_\delta(\pm c_\alpha u_\alpha) \in H,$$

where $c_\alpha = 1$ or $c_\alpha = 2$, depending on whether α is long or short, is precisely the required nontrivial [long] root element. Thus, we may assume that $c_\alpha u_\alpha = 0$ for all roots $\alpha \in \Sigma_1$ except, maybe, for δ. In this case the parameter u_δ does not depend on the order of roots in Σ_1, and if $u_\delta \neq 0$, then

$$y = [u, x_{-\alpha_1}(1)] \in H \cap U_1.$$

Moreover, we have $y_{\delta-\alpha_1} = \pm u_\delta \neq 0$ in the expansion $y = \prod x_\alpha(y_\alpha)$, $\alpha \in \Sigma_1$, so that this case reduces to that considered previously. Thus, it only remains to settle the case where $c_\alpha u_\alpha = 0$ for *all* roots $\alpha \in \Sigma_1$, but $u_\alpha \neq 0$ for some short root α. Clearly, without loss of generality we may assume that $u_{1110} \neq 0$. Since $2u_\alpha = 0$ for all short roots, all factors $x_\alpha(u_\alpha)$ in the expansion of u commute, and

$$[u, x_{\alpha_4}(1)] = x_{1111}(\pm u_{1110}) x_{1232}(\pm u_{1231}) \in H.$$

Passing to another commutator, we see that

$$[[u, x_{\alpha_4}(1)], x_{\alpha_3}(1)] = x_{1121}(\pm u_{1110}) \in H$$

is a nontrivial [short] root element in H.

In the case of P_4, all 6 long roots of Σ_4 have coefficient 2 in α_4 and thus fall into $\Sigma_4(2)$. Let $u = \prod x_\alpha(u_\alpha)$, $\alpha \in \Sigma_1$, and observe that the parameters u_α in short roots $\alpha \neq \rho$ are uniquely determined. If $2u_\alpha \neq 0$ for some of these short roots, then, taking an appropriate root $\beta \in \Sigma_1$, $\beta \neq \rho$, we get a nontrivial element $[u, x_\beta(1)] \in [U_4, U_4]$. Indeed, in this case there is no loss of generality in assuming that already $2u_{\alpha_4} \neq 0$, so that $\alpha = \alpha_4$. Taking $\beta = 0121$, we get

$$[u, x_{0121}(1)] = x_{0122}(\pm 2u_{\alpha_4}) y \in H \cap [U_4, U_4],$$

where y is the product of $x_\gamma(*)$ over roots of Σ_1 larger than 0122, namely, over 1122, 1222, $\rho = 1232$, 1242, 1342, and $\delta = 2342$. Thus, we have found a nontrivial element in $H \cap [U_4, U_4]$.

On the other hand, if $2u_\alpha = 0$ for all short roots $\alpha \in \Sigma_1$, $\alpha \neq \rho$, but $u_\alpha \neq 0$ for some of these roots, then

$$[u, x_{\rho-\alpha}(1)] = x_\rho(\pm u_\alpha) \in H$$

is a nontrivial [short] root element in H. Thus, in the sequel we may assume that $u_\alpha = 0$ for all short roots $\alpha \in \Sigma_1$, $\alpha \neq \rho$, so that in this case the element $u \in H \cap [U_4, U_4]$ is itself nontrivial.

First, let $u_\gamma \neq 0$ for some *long* root $\gamma \in \Sigma_1$; without loss of generality we may assume that already $u_{0122} \neq 0$, so that $\gamma = 0122$. In this case

$$[u, x_{\alpha_1}(1)] = x_{1122}(\pm u_{0122}) x_\delta(*) \in H,$$

and thus, finally,

$$[[u, x_{\alpha_1}(1)], x_{\alpha_2}(1)] = x_{1222}(\pm u_{0122}) \in H$$

is the required nontrivial [long] root element. This shows that $u_\gamma = 0$ for all long roots $\gamma \in \Sigma_1$, so that, from the very start, $u = x_\rho(u_\rho) \in H$ was a nontrivial [short] root element in H. This finishes the proof of the proposition. □

Now we deduce two corollaries from this proposition. Their proofs coincide with the proofs of the corresponding corollaries in [12] almost word for word. First, observe that, since the center of the group $G(\mathrm{F}_4, R)$ is trivial, here we do not require any analog of

[12, Proposition 5]. This is precisely why here we have no need to stipulate that the roots α and β are long.

Corollary 1. *If $x_\alpha(\xi)x_\beta(\zeta)$ stabilizes the first column of a matrix $g \in H$, but does not commute with g, then H contains a nontrivial root element.*

Proof. By assumption, $z = [g, x] \in H$ lies in a parabolic subgroup P_4. Since x does not commute with g, the commutator z is nontrivial, and then the subgroup H contains a nontrivial root element by Proposition 4. $\square$

Corollary 2. *If for a nontrivial element $g \in H$ its commutator $[g, x_\alpha(1)]$ with a root element $x_\alpha(1)$ is central for some root $\alpha \in \Phi$, then H contains a nontrivial elementary root element.*

Proof. Indeed, from Proposition 1 when α is long, or from Proposition 2 when α is short, it follows that the element g is contained in a parabolic subgroup of type P_1 or P_4, respectively. In any case, we can apply Proposition 4 to this element. $\square$

§8. The main lemma

We come up to the most important auxiliary result of the present paper. Its proof constitutes the gist of our approach, where the main computational trick resides. We keep the assumption that $\omega = \varpi_1$ is the first fundamental weight of the root system E_6.

Main lemma. *Let us fix a weight $\lambda \in W(F_4)\omega$. Assume that for a nontrivial element $g \in H$ we have $g'_{\rho\lambda} = 0$ for all weights $\rho \in \Lambda$, $d(\lambda, \rho) = 2$. Then H contains a nontrivial root element.*

Proof. The proof starts in exactly the same way as in the case of E_6. Obviously, now the key computation can be carried through only for long roots, so that it does not suffice to require that a difference of two weights is a root, it is necessary that it is a *long* root. This introduces substantial additional technical strain in the rest of the proof, as compared with [12]. The first (difficult!) step consists in proving that either H contains a nontrivial root element, or the *square* of the ideal I in R generated by the entries $g_{\mu\lambda} = 0$, $\mu \neq \lambda$, equals 0. After that it is relatively easy to finish the proof. As a next approximation we shall prove that either H contains a nontrivial root element, or the ideal I itself equals 0. However, in the latter case, the element g is contained in a proper parabolic subgroup and we can invoke Proposition 4.

• Replacing g by its conjugate under the action of an element of $N(F_4, R)$, without loss of generality we can assume that $\lambda = \omega$. This does not influence general arguments essentially, but as soon as we start actual calculations, it is usually noticeably easier to perform them for the first column of the matrix g.

• It is easily seen that for any $\lambda \in W(F_4)\omega$ there exist six weights $\mu \in \Lambda$ such that $\lambda - \mu$ is a long root. Obviously, all of them lie in $W(F_4)\omega$. For example, for $\lambda = \omega$ these are precisely the weights $1110, 0110, 0010$ lying in Γ, and their opposites. Clearly, these weights form a single orbit under the action of the Weyl group $W(B_3)$, stabilizing ω.

Take any two distinct weights $\mu, \nu \in \Lambda$ such that $\alpha = \lambda - \mu$ and $\beta = \lambda - \nu$ are long roots, and consider the root element

$$x = x(\mu, \nu) = x_\alpha(g'_{\nu\lambda})x_\beta(\varepsilon g'_{\mu\lambda}).$$

An explicit choice of the sign $\varepsilon = \varepsilon(\alpha, \beta) = \pm 1$ is described below. Now, as in [12], we form the commutator

$$z = z(\mu, \nu) = [x, g] \in H.$$

Since the roots α, β are *long*, nothing changes in our computation, as compared with the case of E_6. Namely, take any weight $\rho \neq \mu, \nu$. Now, by Lemma 3, if $\rho + \alpha \in \Lambda$ or $\rho + \beta \in \Lambda$, then $d(\lambda, \rho) = 2$. However, $g'_{\rho\lambda} = 0$ for all such ρ by assumption. Thus, in the λth column of the matrix g^{-1}, multiplication by x^{-1} produces exactly two additions, both to the diagonal entry $g'_{\lambda\lambda}$. Moreover, the entry $g'_{\nu\lambda}$ is added with the coefficient $\pm g'_{\mu\lambda}$, whereas the entry $g'_{\nu\lambda}$ is added with the coefficient $\pm\varepsilon g'_{\mu\lambda}$.

Clearly, choosing $\varepsilon = \pm 1$ we can secure the mutual cancellation of these terms. Since there are only two additions, in order to achieve this, we do not even need to explicitly know the structure constants $c_{\mu\alpha}$ and $c_{\nu\beta}$. Upon such a choice, the λth column of the matrix g^{-1} does not change at all; in other words,

$$(x^{-1}g^{-1})_{*\lambda} = x^{-1}g'_{*\lambda} = g'_{*\lambda}.$$

But then, of course,

$$(gx^{-1}g^{-1})_{*\lambda} = gg'_{*\lambda} = v^{\lambda}.$$

Since the representation (E_6, ϖ_1) is microweight, the sums $\lambda + \alpha, \lambda + \beta$ are not weights, and thus, $xv^{\lambda} = v^{\lambda}$. But then

$$z_{*\lambda} = (xgx^{-1}g^{-1})_{*\lambda} = xv^{\lambda} = v^{\lambda},$$

so that for all such μ, ν the element $z = z(\mu, \nu)$ falls into a proper parabolic subgroup of type P_1 of the Chevalley group $G(\mathrm{E}_6, R)$, and thereby also in a proper parabolic subgroup of type P_4 of the Chevalley group $G(\mathrm{F}_4, R)$.

If at least one of these elements is nontrivial, we can resort to parabolic reduction. In this case, Proposition 4 gives us a nontrivial root unipotent.

• This means that in the sequel we can limit ourselves to the case where *all* such z are trivial. Now, we try to find out what this means for the matrix g itself. In other words, we assume that g^{-1} commutes with all $x = x(\mu, \nu)$. Let us calculate the entries of the matrices $g^{-1}x$ and xg^{-1} at the position (ρ, σ):

$$(g^{-1}x)_{\rho\sigma} = g'_{\rho\sigma} \pm g'_{\nu\lambda}g'_{\rho,\sigma+\alpha} \pm g'_{\mu\lambda}g'_{\rho,\sigma+\beta} \pm g'_{\mu\lambda}g'_{\nu\lambda}g'_{\rho,\sigma+\alpha+\beta},$$

$$(xg^{-1})_{\rho\sigma} = g'_{\rho\sigma} \pm g'_{\nu\lambda}g'_{\rho-\alpha,\sigma} \pm g'_{\mu\lambda}g'_{\rho-\beta,\sigma} \pm g'_{\mu\lambda}g'_{\nu\lambda}g'_{\rho-\alpha-\beta,\sigma}.$$

By Lemma 2, for each position, out of the three new summands, not more than two can appear simultaneously in each of these expressions.

In the first place, we are interested in the case where a *single* new summand appears on the right-hand sides of these two expressions. By comparing them, we see that in this case this new summand equals 0.

• First, here we take $\sigma = \mu$, where, as above, $\mu \in \Lambda$ is a weight such that $\alpha = \lambda - \mu \in \Phi_l$. Further, let ρ be a weight at distance 1 of λ such that $\rho - \alpha \notin \Lambda$. In this case there exists a weight $\nu \in \Lambda$ such that $\beta = \lambda - \nu \in \Phi_l$, and moreover, $\mu + \beta, \mu + \alpha + \beta, \rho - \beta \notin \Lambda$. To somewhat simplify the notation, from this moment on we assume that $\lambda = \omega$, $\mu = 1110$. As we mentioned at the very beginning of the proof, this does not restrict generality. In this case the weights ρ such that $\rho - \alpha$ is a weight are precisely $\omega, 0121, 0111, 0011, 0001$, and -1110. We claim that for all other weights we can find ν as required. Indeed, in this case the condition that $\mu + \beta$ is not a weight is fulfilled automatically, because all allowable ν are smaller than μ. To assure that $\rho - \beta$ is not a weight either, it suffices to observe that for $\rho = 1231, 1221$, and also for $\rho = 0010, \upsilon_1, \upsilon_2$, or -0010 we can take $\nu = 0110$. For $\rho = 1121$ we can take $\nu = 0110$ or -0110. Finally, for $\rho = 1111$ we can take $\nu = -0010$.

Substituting in the expressions from the previous item first $\sigma = \mu$, and then $\sigma = \nu$, we see that $g'_{\rho\lambda}g'_{\mu\lambda} = 0$ for all such μ and ρ. In the case of E_6, the proof essentially terminates here, because in this case for each weight $\mu \neq \lambda$ at least one of the following assertions is true: $\lambda - \mu$ is a long root, or $g'_{\mu\lambda} = 0$. As opposed to that, for the case of F_4, the proof merely commences here. As yet, we have not completed the proof of the relations $g'_{\rho\lambda} g'_{\mu\lambda} = 0$ for the weights μ such that $\lambda - \mu$ is a long root. Worse than that: we have not even *started* the analysis of the relations $g'_{\rho\lambda} g'_{\mu\lambda} = 0$ for the case where both roots $\lambda - \mu, \lambda - \rho$ are short!

• As above, let $\mu \in \Lambda$ be a weight such that $\alpha = \lambda - \mu \in \Phi_l$. As a next step, we verify that $g'_{\rho\lambda} g'_{\mu\lambda} = 0$ for all $\rho \neq \lambda$. For this, we argue as follows. Take any root γ — possibly a short one! — and consider the commutator

$$z = z(\mu, \gamma) = [x_\gamma(g'_{\mu\lambda}), g] \in H.$$

Multiplication by a root element $x_\gamma(g'_{\mu\lambda})$ amounts to six or thirteen additions in the λth column of the matrix g^{-1}, depending on whether γ is a long root, or a short root. We try to figure out when these additions do not change the λth column of g^{-1}. Clearly, this amounts to the condition $g'_{\mu\lambda} g'_{\rho-\gamma,\lambda} = 0$ for all $\rho \in \Lambda$ such that $\rho - \gamma \in \Lambda$. In this case precisely the same calculation as that performed above for the matrices $z(\mu, \nu)$ shows that $z_{*\lambda} = v^\lambda$, so that $z = z(\mu, \gamma)$ falls into a parabolic subgroup of type P_1 of the Chevalley group $G(\mathrm{E}_6, R)$. Thus, once again we can conclude that H contains a nontrivial root unipotent, or else that the element $z = z(\mu, \gamma)$ is equal to 1.

At the next step of the proof we undertake the search for roots γ such that for any $\rho \neq \lambda$ at least one of the three following assertions is true:

∘ $\rho - \gamma \notin \Lambda$,

∘ $d(\rho - \gamma, \lambda) = 2$,

∘ at the preceding steps it has already been proved that $g'_{\mu\lambda} g'_{\rho-\gamma,\lambda} = 0$.

The proof in the next item will run in several passes and is organized roughly as follows. From the preceding argument we know that each time we unearth a root γ with this property, we face the following customary alternative: either H contains a nontrivial root unipotent, or g commutes with $x_\gamma(g'_{\mu\lambda})$. In the latter case we get

$$g'_{\rho\sigma} \pm g'_{\mu\lambda} g'_{\rho-\gamma,\sigma} = (x_\gamma(g'_{\mu\lambda}) g^{-1})_{\rho\sigma} = (g^{-1} x_\gamma(g'_{\mu\lambda}))_{\rho\sigma} = g'_{\rho\sigma} \pm g'_{\mu\lambda} g'_{\rho,\sigma+\gamma}.$$

In its turn, this last relation implies new identities of the form $g'_{\mu\lambda} g'_{\rho\lambda} = 0$, which can be used in the search for further roots γ with this property.

• Again, to simplify notation, from now on we assume that $\lambda = \omega$, $\mu = 1110$. Let ρ be a weight; we must show that $g'_{\mu\omega} g'_{\rho\omega} = 0$ whenever $\rho - \alpha$ is a weight.

We claim that the short root element $x_{0001}(g'_{\mu\omega})$ stabilizes the first column of g^{-1}. Indeed, 8 out of 13 additions in the first column (namely, 5 additions by $x_{\beta_1}(*)$, 2 additions by $x_{\beta_6}(*)$, as well as the only addition over the zero weight) originate from positions (ρ, ω) with $d(\rho, \omega) = 2$, whence $g'_{\rho\omega} = 0$. The 5 remaining additions originate from positions (ρ, ω) with $\rho = 1231, 1110, 0110, 0010$, or v_1. For all of those 5, at one of the preceding steps we have already established that $g'_{\mu\lambda} g'_{\rho,\lambda} = 0$. It follows that this addition actually stabilizes the first column of the matrix g^{-1}. This means that, without loss of generality, in the sequel we can assume that g commutes with $x_{0001}(g'_{\mu\omega})$. But then, substituting $\rho = 0121$ and $\sigma = \mu$ in the formula in the preceding item, we get a new relation $g'_{\mu\omega} g'_{\rho\omega} = 0$. Obviously, the same argument with a different choice of γ applies to all weights ρ that can be transformed to 0121 by an element of $W(\mathrm{F}_4)$ stabilizing ω and μ, i. e., $\rho = 0111, 0011, 0001$. Namely, for these cases we can take $\gamma = 0011, 0111, 0121$, respectively.

This means that, out of all cases displayed in the previous item, only the case where $\mu = 1110$, $\rho = -1110$ remains open. Here we argue in exactly the same way as for other weights, but with a different choice of γ. Namely, we claim that a short root unipotent

$x_{1232}(g'_{\mu\omega})$ stabilizes the first column of the matrix g^{-1}. Indeed, 11 out of 13 additions in the first column (namely, 5 additions by $x_{\substack{11221\\1}}(*)$ and $x_{\substack{12211\\1}}(*)$ each, as well as the only addition over the zero weight) originate from positions (ρ, ω) with $d(\rho, \omega) = 2$, and thus $g'_{\rho\omega} = 0$. The 2 remaining additions come from the zero weights $\rho = \upsilon_1, \upsilon_2$. With regard to the latter additions, we have already established that $g'_{\mu\omega} g'_{\rho,\omega} = 0$. Thus, without loss of generality we can assume once again that g commutes with $x_{1232}(g'_{\mu\omega})$. But then, substituting $\rho = -1110$ and $\sigma = 1110$ in the formula from the preceding item, we get the relation $g'_{\mu\omega} g'_{\rho\omega} = 0$ in the last case where it was still open.

• Summarizing the above, we can conclude that $g'_{\mu\lambda} g'_{\rho\lambda} = 0$ for all $\rho, \mu \in \Lambda$ such that $\lambda - \mu \in \Phi_l$. To consider the remaining cases, it suffices to observe that in the expression $(x_\gamma(\xi) g^{-1})_{*\lambda} = g^{-1}_{*\lambda}$ we could replace ξ by $g'_{\mu\lambda}$ for any μ. The first column is stabilized whenever $g'_{\mu\lambda} g'_{\rho-\gamma,\lambda} = 0$ for all $\rho \in \Lambda$. Already the above relation (with σ replaced by $\lambda - \gamma$) gives us $g'_{\mu\lambda} g'_{\sigma\lambda} = 0$ whenever μ, σ are such that there exists a root γ for which $x_\gamma(\xi)$ stabilizes the first column of g^{-1}, and moreover, $\tau - \gamma \notin \Lambda$, $\lambda - \gamma \in \Lambda$. For every pair of weights it is very easy to find such γ. In fact, virtually any long root γ will do. It suffices to require that $\tau - \gamma \notin \Lambda$, and then the relations we obtained in the preceding item ensure that the first column is stabilized automatically (the difference of $\rho - \gamma$ and a long root is manifestly distinct from υ_1 and υ_2).

• Thus, we have shown that $g'_{\mu\lambda} g'_{\nu\lambda} = 0$ for all $\mu, \nu \in \Lambda$. In other words, if A is an ideal in R generated by the elements $g'_{\mu\lambda}$, $\mu \neq \lambda$, then $A^2 = 0$. Moreover, reduction modulo A shows that $g_{\mu\lambda} \in A$ for all $\mu \neq \lambda$.

It only remains to consider the commutator $z = [g, x_\alpha(1)]$, $\alpha \in \Phi_l$. The column $v = x_\alpha(1) g^{-1}_{*\lambda}$ differs from $g^{-1}_{*\lambda}$ only at the λth component, $v_\lambda = g'_{\lambda\lambda} + g'_{\mu\lambda}$; all other components coincide. It follows that $gxg^{-1}_{*\lambda}$ is proportional to the base column v^λ, and z lies in a parabolic subgroup of type P_1. Now, either z is nontrivial and we can use parabolic reduction, or z is trivial. In the latter case, g itself lies in a proper parabolic subgroup by Proposition 1, and once again we can invoke parabolic reduction. In either of these cases H contains a nontrivial root unipotent. □

Remark. In the Russian original, but not in the English translation, in §8 of [13] there was a systematic misprint in the exposition of the proof of this lemma. Namely, as a result of an erroneous interpretation of a TEX macro, the coefficient in α_2 in the Dynkin form of the roots of E_7 is repeated *twice*: once in its legitimate place in the second row, and once more as the second component of the first row.

§9. Standard description

In the present section we finish the proof of Theorems 1 and 2. Obviously, all these labors are necessary only because we refuse to impose simplifying assumptions such as $2 \in R^*$. Under this assumption, Theorem 2 with $A = B$ *immediately* follows from our main lemma by level reduction with two parameters, in the form it was used in [38, 43]. Thereby, one has no need to use any commutator formulas, apart from Lemma 7.

Proof of Theorem 2. Let, as usual, H be a nontrivial subgroup in $G = G(\mathrm{F}_4, R)$, normalized by $E(\mathrm{F}_4, R)$. First, we verify that H contains a nontrivial root unipotent. Indeed, let $g \in H$, $g \neq 1$. If g commutes with a long root unipotent $x_\alpha(1)$, then H contains a nontrivial root unipotent by Corollary 2.

On the other hand, if g does not commute with a *long* root unipotent $x_\alpha(1)$, then, replacing if necessary g by $[g, x_\alpha(1)] = (g x_\alpha(1) g^{-1}) x_\alpha(-1)$, we can assume from the very start that $g \in H$ is a nontrivial element of the following form: the product of a long root unipotent by an elementary long root unipotent. The element g differs from a root unipotent in at most 6 columns. This means that even after throwing out the 3 zero columns, we still have at least 18 columns with respect to which g satisfies conditions of the main lemma. This shows that in this case we can also conclude that H contains a nontrivial root unipotent.

Now, the proof of Theorem 2 can be concluded by a standard argument known as level reduction. Namely, let B denote the lower level of the subgroup H, i.e., the largest ideal in R such that $E(\mathrm{F}_4, R, B) \leq H$. As is well known, see [62, 32], the ideal B can be defined as the set of $\xi \in R$ such that $x_\alpha(\xi) \in H$ for some (= any) *long* root α. Consider the image $\bar{H} \leq G(\mathrm{F}_4, R/B)$ of the subgroup H under the reduction homomorphism $\pi_B : G(\mathrm{F}_4, R) \longrightarrow G(\mathrm{F}_4, R/B)$. Since $\bar{H}$ is normalized by $E(\mathrm{F}_4, R/B) = \pi_B(E(\mathrm{F}_4, R))$, the above implies that if $\bar{H} \neq 1$, then $\bar{H}$ contains a nontrivial root unipotent $x_\alpha(\bar{\xi}) = x_\alpha(\xi + B)$ for some $\xi \notin B$.

This means that H contains an element of the form $h = x_a(\xi)g$, for some $g \in G(\mathrm{F}_4, R, B)$. Now, let β be a root *of the same length* as α and forming an angle of $2\pi/3$ with it. Then

$$[h, x_\beta(1)] = {}^{x_\alpha(\xi)}[g, x_\beta(1)][x_\alpha(\xi), x_\beta(1)] \in H.$$

Since the first factor belongs to $E(\mathrm{F}_4, R, B)$ by Lemma 9, the second factor $x_{\alpha+\beta}(\pm\xi)$ also belongs to H.

• If the root α (and thus also the root $\alpha + \beta$) is long, we get a contradiction with the definition of the lower level.

• If the root α is short, then the results of [62, 32], summarized in Lemma 6, imply that $2\xi, \xi^2 \in B$.

Denoting by A the upper level of the subgroup H, i.e., the set of $\xi \in R$ such that $x_\alpha(\xi) \in H$ for some (= any) *short* root α, we get $A_2 \leq B \leq A$. The product $H \cdot E(\mathrm{F}_4, R, A)$ contains no additional *short* root unipotents; thus, its lower level equals its upper level equals A. If the reduction $\bar{\bar{H}} = \pi_A(H) \leq G(\mathrm{F}_4, R/A)$ is still nontrivial, then exactly the same argument as before gives a nontrivial root unipotent $x_\alpha(\xi) \in H \cdot E(\mathrm{F}_4, R, A)$ for some $\xi \notin A$. That contradicts the definition of the level.

Thus, $\bar{\bar{H}} = 1$, or, what is the same, $H \cdot E(\mathrm{F}_4, R, A) \leq C(\mathrm{F}_4, R, A)$. This shows that the upper level we have defined coincides indeed with the upper level in the usual sense! Summarizing the above we can conclude that

$$E(\mathrm{F}_4, R, A, A_2) \leq E(\mathrm{F}_4, R, A, B) \leq H \leq H \cdot E(\mathrm{F}_4, R, A) \leq C(\mathrm{F}_4, R, A),$$

as claimed. □

In its turn, now Theorem 1 purely formally follows from Theorem 2 and the standard description of the normal subgroups in $G(\mathrm{D}_4, R)$. Of course, we could prove Theorem 1 right away, but we preferred to detach Theorem 2, to show that its proof, as opposed to the proof of Theorem 1!, does not depend on any external considerations *whatsoever*.

Proof of Theorem 1. Let $H \leq G(\mathrm{F}_4, R)$ be a subgroup normalized by $E(\mathrm{F}_4, R)$. By Theorem 2, there exists a lower level B and an upper level A such that $E(\mathrm{F}_4, R, A, B) \leq H \leq C(\mathrm{F}_4, R, A)$. We wish to prove that then in fact $H \leq C(\mathrm{F}_4, R, A, B)$, or, what is the same, $[H, E(\mathrm{F}_4, R)] \leq E(\mathrm{F}_4, R, A, B)$. Recalling Lemmas 6 and 8, we see that our assumption implies that

$$E(\mathrm{F}_4, R, A, B) \leq [H, E(\mathrm{F}_4, R)] \leq E(\mathrm{F}_4, R, A) = E(\mathrm{F}_4, R, A, B)E(\mathrm{D}_4, R, A).$$

Thus, if $[H, E(\mathrm{F}_4, R)] \neq E(\mathrm{F}_4, R, A, B)$, then the group $[H, E(\mathrm{F}_4, R)]$ contains a nontrivial element of $E(\mathrm{D}_4, R, A) \setminus E(\mathrm{D}_4, R, B)$. Now, to finish the proof, it only remains to refer to the standard description of normal subgroups in $G(\mathrm{D}_4, R)$. Indeed, this description implies that $[H, E(\mathrm{F}_4, R)] \le H$ contains a long root unipotent $x_\alpha(\xi)$ for some $\xi \in A \setminus B$. But this contradicts the definition of the lower level! Thus, $[H, E(\mathrm{F}_4, R)] = E(\mathrm{F}_4, R, A, B)$, as claimed. □

Remark. Our proof of Theorem 1 follows the same guidelines as the proof of Proposition 3.3 in Chapter IV of the Thesis by Tony Bak [38, pages 4.22–4.27]. There, a similar argument was carried through for the unitary group. Namely, it was observed that, for a group G, the strong structure theorem follows from the weak structure theorem for the group itself and the strong structure theorem for an appropriate subgroup. Understandably, the subgroup in [38] was constructed in a completely different way, by using stability conditions. Unfortunately, that thesis was never published, and even those who do have the text have seldom reached the middle of Chapter IV. This is why the exact relationship between the weak and the strong structure theorems, and even the fact that without assumption $2 \in R^*$ the standard description should be stated in terms of form ideals ($\approx$ admissible pairs), are known only to a limited number of experts. In fact, the outline of the proof by Abe in [32, 33] was completely different. Namely, he reduced the proof to local rings, and for local rings he invoked reduction modulo the Jacobson radical, with a subsequent reference to the Chevalley–Tits theorem for the field case. Thus, level reduction was not used there at all. As a curiosity, we observe that even in the paper [74], which nominally acknowledges that the standard description should be given in terms of form ideals, the relative form parameter is missing from the notation. As noted in [54], this leads to a direct mistake. For classical groups of rank at least 3 the problems arising were discussed in all possible detail in [43]. For the group $\mathrm{Sp}(4, R)$, the situation is *dramatically* more complicated and was completely clarified in the lambent paper by Douglas Costa and Gordon Keller [49].

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References

[1] E. Abe, *Automorphisms of Chevalley groups over commutative rings*, Algebra i Analiz **5** (1993), no. 2, 74–90; English transl., St. Petersburg Math. J. **5** (1994), no. 2, 287–300. MR1223171 (94e:20062)

[2] E. Artin, *Geometric algebra*, Intersci. Publ., Inc., New York–London, 1957. MR0082463 (18:553e)

[3] Z. I. Borevich and N. A. Vavilov, *Arrangement of subgroups in the general linear group over a commutative ring*, Algebraic Geometry and its Applications, Trudy Mat. Inst. Steklov. **165** (1984), 24–42; English transl. in Proc. Steklov Inst. Math. **1985**, no. 3. MR0752930 (86e:20052)

[4] A. Borel, *Properties and linear representations of Chevalley groups*, Seminar on Algebraic Groups and Related Finite Groups (Inst. Adv. Study, Princeton, NJ, 1968/69), Lecture Notes in Math., vol. 131, Springer-Verlag, Berlin–New York, 1970, pp. 1–55. MR0258838 (41:3484)

[5] N. Bourbaki, *Lie groups and Lie algebras*. Chapters 4–6, Springer-Verlag, Berlin, 2002. MR1890629 (2003a:17001)

[6] ———, *Lie groups and Lie algebras*. Chapters 7–9, Springer-Verlag, Berlin, 2005. MR2109105 (2005h:17001)

[7] N. A. Vavilov, *Subgroups of split classical groups*, Doctor. Diss., Leningrad. Gos. Univ., Leningrad, 1987. (Russian)

[8] ———, *Can one see the signs of structure constants?* Algebra i Analiz **19** (2007), no. 4, 34–68; English transl., St. Petersburg Math. J. **19** (2008), no. 4, 519–543. MR2381932 (2009b:20087)

[9] ———, *Numerology of quadratic equations*, Algebra i Analiz **20** (2008), no. 5, 9–40; English transl., St. Petersburg Math. J. **20** (2009), no. 5. MR2492358

[10] ———, *Calculations in exceptional groups*, Vestnik Samarsk. Univ. **2007**, no. 7, 11–24. (Russian)

[11] ———, *Decomposition of unipotents in adjoint representation of a Chevalley group of type* E_6, Algebra i Analiz (to appear). (Russian)
[12] N. A. Vavilov and M. R. Gavrilovich, A_2*-proof of structure theorems for Chevalley groups of types* E_6 *and* E_7, Algebra i Analiz **16** (2004), no. 4, 54–87; English transl., St. Petersburg Math. J. **16** (2005), no. 4, 649–672. MR2090851 (2005m:20115)
[13] N. A. Vavilov, M. R. Gavrilovich, and S. I. Nikolenko, *The structure of Chevalley groups: A proof from the Book*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **330** (2006), 36–76; English transl., J. Math. Sci. (N.Y.) **140** (2007), no. 5, 626–645. MR2253566 (2007k:20108)
[14] N. A. Vavilov and A. Yu. Luzgarev, *Normalizer of the Chevalley group of type* E_6, Algebra i Analiz **19** (2007), no. 5, 35–62; English transl. in St. Petersburg Math. J. **19** (2008), no. 5. MR2381940 (2008m:20077)
[15] N. A. Vavilov, A. Yu. Luzgarev, and I. M. Pevzner, *A Chevalley group of type* E_6 *in the* 27*-dimensional representation*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **338** (2006), 5–68; English transl., J. Math. Sci. (N.Y.) **145** (2007), no. 1, 4697–4736. MR2354606 (2009b:20022)
[16] N. A. Vavilov and E. B. Plotkin, *Net subgroups of Chevalley groups*. I, II, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **94** (1979), 40–49; **114** (1982), 62–76; English transl. in J. Soviet Math. **19** (1982), no. 1; **27** (1984), no. 4. MR0571514 (81f:20059); MR0669560 (83k:20051)
[17] N. A. Vavilov, E. B. Plotkin, and A. V. Stepanov, *Calculations in Chevalley groups over commutative rings*, Dokl. Akad. Nauk SSSR **307** (1989), no. 4, 788–791; English transl., Soviet Math. Dokl. **40** (1990), no. 1, 145–147. MR1020667 (90j:20093)
[18] I. Z. Golubchik, *The full linear group over an associative ring*, Uspekhi Mat. Nauk **28** (1973), no. 3, 179–180. (Russian) MR0396783 (53:643)
[19] ———, *Normal subgroups of the orthogonal group over the associative ring with involution*, Uspekhi Mat. Nauk **30** (1975), no. 6, 165. (Russian)
[20] ———, *Normal subgroups of the linear and unitary groups over associative rings*, Spaces over Algebras, and some Problems in the Theory of Nets, Bashkir. Gos. Ped. Inst., Ufa, 1985, pp. 122–142. (Russian) MR0975035
[21] J. Dieudonné, *La géométrie des groupes classiques*, Ergeb. Math. Grenzgeb. (N.F.), Bd. 5, Springer-Verlag, Berlin–New York, 1971. MR0310083 (46:9186)
[22] A. Yu. Luzgarev, *On overgroups of* $E(E_6,\mathrm{R})$ *and* $E(E_7,\mathrm{R})$ *in minimal representations*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **319** (2004), 216–243; English transl., J. Math. Sci. (N.Y.) **134** (2006), no. 6, 2558–2571. MR2117858 (2006k:20098)
[23] V. V. Nesterov, *Generation of pairs of short root subgroups in Chevalley groups*, Algebra i Analiz **16** (2004), no. 6, 172–208; English transl., St. Petersburg Math. J. **16** (2005), no. 6, 1051–1077. MR2117453 (2006b:20070)
[24] T. A. Springer, *Linear algebraic groups*, Itogi Nauki i Tekhniki Sovrem. Probl. Mat. Fundam. Napravleniya, vol. 55, Algebraic Geometry. IV, VINITI, Moscow, 1989, pp. 5–136; English transl., Encyclopaedia Math. Sci., vol. 55, Springer-Verlag, Berlin, 1994, pp. 1–121. MR1100484 (92g:20061); MR1309681 (95g:14002)
[25] R. Steinberg, *Lectures on Chevalley groups*, Yale Univ., New Haven, Conn., 1968. MR0466335 (57:6215)
[26] A. V. Stepanov, *Stability conditions in the theory of linear groups over rings*, Candidate Diss., Leningrad. Gos. Univ., Leningrad, 1987. (Russian)
[27] ———, *On the normal structure of the general linear group over a ring*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **236** (1997), 166–182; English transl., J. Math. Sci. (N.Y.) **95** (1999), no. 2, 2146–2155. MR1754458 (2001a:20080)
[28] J. E. Humphreys, *Linear algebraic groups*, Grad. Texts in Math., No. 21, Springer-Verlag, New York–Heidelberg, 1975. MR0396773 (53:633)
[29] ———, *Introduction to Lie algebras and representation theory*, Grad. Texts in Math., No. 9, Springer-Verlag, New York–Berlin, 1978. MR0499562 (81b:17007)
[30] E. Abe, *Chevalley groups over local rings*, Tôhoku Math. J. (2) **21** (1969), no. 3, 474–494. MR0258837 (41:3483)
[31] ———, *Chevalley groups over commutative rings*, Radical Theory (Sendai, 1988), Uchida Rokakuho, Tokyo, 1989, pp. 1–23. MR0999577 (91a:20047)
[32] ———, *Normal subgroups of Chevalley groups over commutative rings*, Algebraic K-Theory and Algebraic Number Theory (Honolulu, HI, 1987), Contemp. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1989, pp. 1–17. MR0991973 (91a:20046)
[33] E. Abe and J. Hurley, *Centers of Chevalley groups over commutative rings*, Comm. Algebra **16** (1988), no. 1, 57–74. MR0921942 (90e:20040)

[34] E. Abe and K. Suzuki, *On normal subgroups of Chevalley groups over commutative rings*, Tôhoku Math. J. (2) **28** (1976), no. 2, 185–198. MR0439947 (55:12828)

[35] M. Aschbacher, *The 27-dimensional module for* E_6. I – IV, Invent. Math. **89** (1987), no. 1, 159–195; J. London Math. Soc. **37** (1988), 275–293; Trans. Amer. Math. Soc. **321** (1990), 45–84; J. Algebra **131** (1990), 23–39. MR0892190 (88h:20045); MR0928524 (89a:20041); MR0986684 (90m:20044); MR1054997 (91f:20049)

[36] ———, *Some multilinear forms with large isometry groups*, Geom. Dedicata **25** (1988), no. 1–3, 417–465. MR0925846 (89c:20067)

[37] H. Azad, M. Barry, and G. M. Seitz, *On the structure of parabolic subgroups*, Comm. Algebra **18** (1990), 551–562. MR1047327 (91d:20048)

[38] A. Bak, *The stable structure of quadratic modules*, Thesis, Columbia Univ., 1969.

[39] ———, *Nonabelian K-theory: the nilpotent class of* K_1 *and general stability*, K-Theory **4** (1991), no. 4, 363–397. MR1115826 (92g:19001)

[40] A. Bak, R. Hazrat, and N. Vavilov, *Localization-completion strikes again: Relative* K_1 *are nilpotent by abelian*, K-Theory (to appear).

[41] A. Bak and N. Vavilov, *Normality for elementary subgroup functors*, Math. Proc. Cambridge Philos. Soc. **118** (1995), no. 1, 35–47. MR1329456 (96d:20046)

[42] ———, *Structure of hyperbolic unitary groups.* I. *Elementary subgroups*, Algebra Colloq. **7** (2000), no. 2, 159–196. MR1810843 (2002b:20070)

[43] A. Bak, R. Hazrat, and N. Vavilov, *Structure of hyperbolic unitary groups.* II. *Normal subgroups*, Algebra Colloq. (to appear).

[44] A. Bak and N. Vavilov, *Cubic form parameters* (to appear).

[45] H. Bass, *Unitary algebraic K-theory*, Algebraic K-Theory. III: Hermitian K-Theory and Geometric Applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Lecture Notes in Math., vol. 343, Springer, Berlin, 1973, pp. 57–265. MR0371994 (51:8211)

[46] R. Carter, *Simple groups of Lie type*, Pure Appl. Math., vol. 28, John Wiley and Sons, London, 1972. MR0407163 (53:10946)

[47] A. M. Cohen and B. N. Cooperstein, *The* 2*-spaces of the standard* $\mathrm{E}_6(q)$*-module*, Geom. Dedicata **25** (1988), no. 1–3, 467–480. MR0925847 (89c:51013)

[48] D. L. Costa and G. E. Keller, *The* $E(2,A)$ *sections of* $\mathrm{SL}(2,A)$, Ann. of Math. (2) **134** (1991), no. 1, 159–188. MR1114610 (92f:20047)

[49] ———, *Radix redux: Normal subgroups of symplectic groups*, J. Reine Angew. Math. **427** (1992), no. 1, 51–105. MR1162432 (93h:20053)

[50] ———, *On the normal subgroups of* $\mathrm{G}_2(A)$, Trans. Amer. Math. Soc. **351** (1999), no. 12, 5051–5088. MR1487611 (2000c:20070)

[51] A. J. Hahn and O. T. O'Meara, *The classical groups and K-theory*, Grundlehren Math. Wiss., vol. 291, Springer-Verlag, Berlin, 1989. MR1007302 (90i:20002)

[52] R. Hazrat, *Dimension theory and non-stable* K_1 *of quadratic modules*, K-Theory **27** (2002), 293–328. MR1962906 (2004a:19005)

[53] R. Hazrat and N. Vavilov, K_1 *of Chevalley groups are nilpotent*, J. Pure Appl. Algebra **179** (2003), 99–116. MR1958377 (2004i:20081)

[54] ———, *Bak's work on K-theory of rings* (with an appendix by Max Karoubi), Preprint no. 5, Queen's Univ., Belfast, 2008, pp. 1–60.

[55] Fu An Li, *The structure of symplectic groups over arbitrary commutative rings*, Acta Math. Sinica (N. S.) **3** (1987), no. 3, 247–255. MR0916269 (88m:20098)

[56] ———, *The structure of orthogonal groups over arbitrary commutative rings*, Chinese Ann. Math. Ser. B **10** (1989), no. 3, 341–350. MR1027673 (90k:20084)

[57] H. Matsumoto, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 1–62. MR0240214 (39:1566)

[58] E. B. Plotkin, *On the stability of the* K_1*-functor for Chevalley groups of type* E_7, J. Algebra **210** (1998), 67–85. MR1656415 (99k:20099)

[59] E. B. Plotkin, A. A. Semenov, and N. A. Vavilov, *Visual basic representations: An atlas*, Internat. J. Algebra Comput. **8** (1998), no. 1, 61–95. MR1492062 (98m:17010)

[60] T. A. Springer, *Linear algebraic groups*, Progr. Math., vol. 9, Birkhäuser, Boston, MA, 1981. MR0632835 (84i:20002)

[61] T. A. Springer and F. D. Veldkamp, *Octonions, Jordan algebras and exceptional groups*, Springer-Verlag, Berlin, 2000. MR1763974 (2001f:17006)

[62] M. R. Stein, *Generators, relations and coverings of Chevalley groups over commutative rings*, Amer. J. Math. **93** (1971), no. 4, 965–1004. MR0322073 (48:437)

[63] ———, *Stability theorems for K_1, K_2 and related functors modeled on Chevalley groups*, Japan. J. Math. (N.S.) **4** (1978), no. 1, 77–108. MR0528869 (81c:20031)

[64] A. I. Steinbach, *Groups of Lie type generated by long root elements in* $F_4(K)$, Habilitationsschrift, Gießen, 2000.

[65] ———, *Subgroups of the Chevalley groups of type* F_4 *arising from a polar space*, Adv. Geom. **3** (2003), 73–100. MR1956589 (2004e:20083)

[66] A. V. Stepanov and N. A. Vavilov, *Decomposition of transvections: A theme with variations*, K-Theory **19** (2000), 109–153. MR1740757 (2000m:20076)

[67] K. Suzuki, *On normal subgroups of twisted Chevalley groups over local rings*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **13** (1977), no. 366–382, 238–249. MR0447426 (56:5738)

[68] ———, *Normality of the elementary subgroups of twisted Chevalley groups over commutative rings*, J. Algebra **175** (1995), no. 2, 526–536. MR1339654 (96m:20077)

[69] G. Taddei, *Normalité des groupes élémentaires dans les groupes de Chevalley sur un anneau*, Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Part II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 693–710. MR0862660 (88a:20054)

[70] L. N. Vaserstein, *On the normal subgroups of* GL_n *over a ring*, Algebraic K-Theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980), Lecture Notes in Math., vol. 854, Springer, Berlin–New York, 1981, pp. 456–465. MR0618316 (83c:20058)

[71] ———, *On normal subgroups of Chevalley groups over commutative rings*, Tôhoku Math. J. (2) **38** (1986), no. 2, 219–230. MR0843808 (87k:20081)

[72] ———, *Normal subgroups of orthogonal groups over commutative rings*, Amer. J. Math. **110** (1988), no. 5, 955–973. MR0961501 (89i:20071)

[73] ———, *Normal subgroups of symplectic groups over rings*, K-Theory **2** (1989), no. 5, 647–673. MR0999398 (90f:20064)

[74] L. N. Vaserstein and Hong You, *Normal subgroups of classical groups over rings*, J. Pure Appl. Algebra **105** (1995), no. 1, 93–105. MR1364152 (96k:20096)

[75] N. A. Vavilov, *Structure of Chevalley groups over commutative rings*, Nonassociative Algebras and Related Topics (Hiroshima, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 219–335. MR1150262 (92k:20090)

[76] ———, *Intermediate subgroups in Chevalley groups*, Groups of Lie Type and their Geometries (Como, 1993), London Math. Soc. Lecture Note Ser., vol. 207, Cambridge Univ. Press, Cambridge, 1995, pp. 233–280. MR1320525 (96c:20085)

[77] ———, *A third look at weight diagrams*, Ren. Sem. Mat. Univ. Padova **104** (2000), 201–250. MR1809357 (2001i:20099)

[78] ———, *Do it yourself structure constants for Lie algebras of type E_l*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **281** (2001), 60–104; English transl., J. Math. Sci. (N.Y.) **120** (2004), no. 4, 1513–1548. MR1875718 (2002k:17022)

[79] ———, *An* A_3*-proof of structure theorems for Chevalley groups of types* E_6 *and* E_7, Internat. J. Algebra Comput. **17** (2007), no. 5–6, 1283–1298. MR2355697

[80] N. A. Vavilov and E. B. Plotkin, *Chevalley groups over commutative rings.* I. *Elementary calculations*, Acta Appl. Math. **45** (1996), 73–115. MR1409655 (97h:20056)

[81] W. C. Waterhouse, *Introduction to affine group schemes*, Grad. Texts in Math., vol. 66, Springer-Verlag, New York–Berlin, 1979. MR0547117 (82e:14003)

[82] J. S. Wilson, *The normal and subnormal structure of general linear groups*, Proc. Cambridge Philos. Soc. **71** (1972), 163–177. MR0291304 (45:398)

[83] R. Hazrat, V. Petrov, and N. Vavilov, *Relative subgroups in Chevalley groups*, K-Theory (to appear).

[84] N. A. Vavilov and A. K. Stavrova, *Basic reductions in the description of normal subgroups*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **349** (2007), 30–52; English transl., J. Math. Sci. (N.Y.) **151** (2008), no. 3, 2949–2960.

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